

A Gapless Short-Range Parent Hamiltonian of a Canonical MPS

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Verdict: false-source (resolved).

1 The unrestricted source assertion

The parent-interaction definition in [CPGSV21, Section IV.C] permits any positive operator on L sites whose kernel is the local matrix product space G_L . The subsequent gap theorem states that all MPS parent Hamiltonians have a positive gap uniform in the chain length. There is no interaction-range qualification in that theorem. The local source locations are `Papers/2011.12127/TN-Review-main.tex:1996–1999` and `2183–2187`. Comparable uniqueness and ground-space statements explicitly specify their interaction lengths, so a sufficient-range hypothesis cannot be inserted silently into this assertion.

The example below has a nonzero range-two parent interaction with exactly the kernel required by the source definition. Its tensor is in canonical form, with three inequivalent normalized normal blocks of positive bond dimension and nonzero weights. Nevertheless its positive excitation energies approach zero as the ring length tends to infinity. The counterexample is verified by the declarations listed below: the local parent kernel, the normalized normal-block data, the periodic Fourier eigenvector, and failure of an eventual uniform gap are all proved.

2 Three normal product-state blocks

Let $\alpha = 1/\sqrt{2}$, and define the two physical matrices

$$\begin{aligned} B^0 &= \text{diag}(1, 0, \alpha), \\ B^1 &= \text{diag}(0, 1, \alpha). \end{aligned} \tag{1}$$

Thus $B = A^0 \oplus A^1 \oplus A^+$ has three bond-one blocks $A^0 = (1, 0)$, $A^1 = (0, 1)$, and $A^+ = (\alpha, \alpha)$, each with weight one. Their product vectors are $|0\rangle^{\otimes N}$, $|1\rangle^{\otimes N}$, and $|+\rangle^{\otimes N}$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

For each block, the sum of squared moduli of its two letters is one. Its transfer map on $\mathbb{C}$ is therefore the identity, which has spectral radius one and is primitive on the cone of nonnegative scalars. The block is normal and normalized. Similarity is trivial in bond dimension one, and no pair of these two-component vectors differs by a phase. All weights are nonzero. Thus (1) is a canonical tensor of bond dimension three, with no zero or empty summand.

The simultaneous word tuples at length two include

$$\begin{aligned} w_{00} &= (1, 0, 1/2), \\ w_{01} &= (0, 0, 1/2), \\ w_{11} &= (0, 1, 1/2). \end{aligned} \tag{2}$$

They span $\mathbb{C}^3$, since the coordinate vectors are $e_0 = w_{00} - w_{01}$, $e_1 = w_{11} - w_{01}$, and $e_2 = 2w_{01}$. At length one there are only two word tuples, so the simultaneous span cannot have dimension three. The simultaneous injectivity length is therefore $S = 2$.

3 The exact two-site parent interaction

For an arbitrary boundary matrix $X \in M_3(\mathbb{C})$, the diagonal form in (1) gives

$$\Gamma_2^B(X) = X_{00}|00\rangle + X_{11}|11\rangle + X_{22}|++\rangle. \tag{3}$$

It follows that

$$G_2(B) = \text{span}\{|00\rangle, |11\rangle, |++\rangle\} = \text{Sym}^2(\mathbb{C}^2). \tag{4}$$

Indeed, $2|++\rangle - |00\rangle - |11\rangle = |01\rangle + |10\rangle$. The orthogonal complement is the singlet line generated by $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.

Let F exchange the two physical sites. The canonical parent interaction is the nonzero positive projection

$$h_2 = |\psi^-\rangle\langle\psi^-| = \frac{I - F}{2} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Its kernel is exactly (4). In terms of spin-1/2 operators, $h_2 = \frac{1}{4}I - \mathbf{S}_1 \cdot \mathbf{S}_2$. Consequently its periodic sum is the isotropic ferromagnetic Heisenberg chain with ground energy zero.

For comparison with the primary literature, the isotropic case of Koma-Nachtergaele [KN97], equations (2.1)–(2.3), has this local singlet projection. Their Lemma 4, equation (3.13), gives the same interior one-magnon recurrence. Their finite-volume boundary convention is open; the periodic formula below is derived directly.

4 Positive one-magnon eigenvalues on the ring

Fix $N \geq 3$ and index the sites by $\mathbb{Z}/N\mathbb{Z}$. Define $H_N = \sum_j (h_2)_{j,j+1}$. Let $|j\rangle$ denote the computational basis vector with a single spin in state 1 at site j and every other spin in state 0. These N vectors are orthonormal.

Every term not adjacent to j annihilates $|j\rangle$. The two adjacent terms in (5) give

$$H_N|j\rangle = |j\rangle - \frac{1}{2}|j-1\rangle - \frac{1}{2}|j+1\rangle. \quad (6)$$

Put $\theta_N = 2\pi/N$ and $v_N = \sum_{j=0}^{N-1} e^{i\theta_N j} |j\rangle$. Its squared norm is N , so it is nonzero. The periodic boundary condition is compatible with these coefficients because $e^{i\theta_N N} = 1$. Reindexing the two shifted sums in (6) yields

$$H_N v_N = \left(1 - \frac{e^{i\theta_N} + e^{-i\theta_N}}{2}\right) v_N = \varepsilon_N v_N, \quad (7)$$

$$\varepsilon_N = 1 - \cos(2\pi/N).$$

Since $0 < 2\pi/N < \pi$, the eigenvalue is strictly positive. For $w \in \ker H_N$, self-adjointness gives $\varepsilon_N \langle w, v_N \rangle = \langle H_N w, v_N \rangle = 0$. Thus $v_N \perp \ker H_N$, without any assumption about the dimension of the ground space.

As $N \rightarrow \infty$, $2\pi/N \rightarrow 0$ and therefore $\varepsilon_N \rightarrow 0$. If a positive constant γ satisfied the source's uniform gap inequality on every ring, (7) would imply $\gamma \|v_N\| \leq \|H_N v_N\| = \varepsilon_N \|v_N\|$. Since $v_N \neq 0$, this forces $\gamma \leq \varepsilon_N$ for every $N \geq 3$, a contradiction.

5 Formal verification

The normal-block data and exact simultaneous injectivity length two are formally proved.¹ The two-site ground-space criterion is equality of the 01 and 10 coefficients, and the parent interaction is the singlet projection.²

The Fourier eigenvector, its positive eigenvalue limit, and failure of an eventual uniform gap are formally proved.³

¹Formal declarations: `MPSTensor.isCPSVBasisOfNormalTensors_shortRangeHeisenbergTensor`, `MPSTensor.wordTupleSpanTop_shortRangeHeisenbergBlock_two`, and `MPSTensor.not_wordTupleSpanTop_shortRangeHeisenbergBlock_one` in `TNLean/MPS/ParentHamiltonian/ShortRangeHeisenbergTensor`.

²Formal declarations: `MPSTensor.mem_groundSpaceES_shortRangeHeisenbergTensor_two_iff` and `MPSTensor.parentInteractionES_shortRangeHeisenbergTensor_two_apply` in the same module.

³Formal declarations: `SpinChain.oneMagnon_edge_sum`, `ZMod.cycleLaplacian_stdAddChar`, `ZMod.cycleFourierGap_pos`, `ZMod.tendsto_cycleFourierGap_zero`, `MPSTensor.parentHamiltonianES_shortRangeHeisenbergMagnon`, and `MPSTensor.not_exists_eventual_uniform_gap_shortRangeHeisenbergTensor_two`. The final two declarations are in `TNLean/MPS/ParentHamiltonian/ShortRangeHeisenbergGap`.

6 The range-qualified theorem remains valid

This example has interaction range $R = 2$ and simultaneous injectivity length $S = 2$. It does not satisfy the sufficient condition $R \geq S + 1$. It therefore leaves intact the proved gap theorem at every range $R \geq S + 1$, as well as the automatic sufficient dimension bound $R \geq 3D^5$ for canonical tensors. The corresponding prescribed-range comparison is documented in `docs/paper-gaps/cpgsv21_block_parent_interaction_range.tex`. That note resolves the passage from a chosen large range to the specified PGVWC07 and simultaneous-injectivity ranges; it does not prove the unrestricted assertion refuted here.

The earlier construction in Fernández-González et al. [FGSW⁺15], Section 2.3, Definition 3 and Remark 1, explicitly uses a jointly injective blocked tensor and specifies a sufficient interaction length. Its gap theorem does not supply the broader claim for an arbitrary shorter range. The present example uses the exact source-defined parent kernel throughout, rather than the kernel of an uncle Hamiltonian.

References

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.
- [FGSW⁺15] Carlos Fernández-González, Norbert Schuch, Michael M. Wolf, J. Ignacio Cirac, and David Pérez-García. Frustration free gapless Hamiltonians for Matrix Product States. *Communications in Mathematical Physics*, 333:299–333, 2015.
- [KN97] Tohru Koma and Bruno Nachtergaele. The spectral gap of the ferromagnetic XXZ chain. *Letters in Mathematical Physics*, 40:1–16, 1997.