Normal Parent-Hamiltonian Range Reduction in Cirac–Perez-Garcia–Schuch–Verstraete 2021

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Verdict: scope-restriction (resolved).

1 The assertion

The range-reduction step for normal matrix product states appears in Section IV.C of Cirac–Pérez-García–Schuch–Verstraete [CPGSV21].[1]

Let A be a normal MPS tensor, and let $L_0 > 0$ be its injectivity length: after blocking L_0 consecutive sites, the resulting tensor is injective. The source first proves an intersection step for two overlapping local Hamiltonian terms of range $L_0 + 1$. These terms act on sites $1, \dots, L_0 + 1$ and $2, \dots, L_0 + 2$. The source denotes the three regions by A, B, and C: the first site, the middle region consisting of sites $2, \dots, L_0 + 1$, and the last site. The tensor associated to the middle region B is injective. For a state in the intersection

$$\mathcal{G}_{1,\dots,L_0+1} \cap \mathcal{G}_{2,\dots,L_0+2},$$

the argument inverts the tensor associated with the first-site region A, restores the middle region B, and then uses injectivity of the enlarged region to invert the joint region. It concludes that every vector in the intersection already lies in $\mathcal{G}_{1,\dots,L_0+2}$; the reverse inclusion is immediate. [2]

The source then says that this argument can be iterated. Hence the ground space obtained from all length-$L_0 + 1$ terms agrees with the ground space obtained from all longer length-$L_0 + k$ terms. When $k = L_0$, the proof either appeals to the previous $2L_0$-range uniqueness theorem or applies the same kind

[1] For traceability, this note corresponds to issue #1042. The principal range-reduction comparison is recorded in issue #588; the periodic-boundary comparison is recorded in issue #2368; its current coordinate form was tracked by issue #2405, now closed after the length-L_0 trace-probe step isolated in issue #2929; and the parent-Hamiltonian consequences are recorded in issue #460.

[2] Source location: `Papers/2011.12127/TN-Review-main.tex`:2049–2077, especially the middle-region argument around the displayed figures following `eq:4:initiate-intersection-proof`.

of argument at the periodic boundary. The resulting theorem states that a normal MPS which becomes injective after blocking L_0 sites has a unique ground state for the parent Hamiltonian whose interaction range is $L_0 + 1$. [3]

2 The formal statement

The corresponding formal statement now keeps the source endpoint $L_0+1 \leq N$. It assumes that A is normal, that A is injective after blocking a positive number L_0 of sites, and that the ring has length $N \geq 2$ with an interaction range L satisfying $L_0 < L \leq N$. The desired conclusion is that the periodic-boundary ground space is the line spanned by the periodic MPS vector $\Omega_N(A)$:

$$\mathcal{G}_{N,L}(A) = \mathbb{C}\,\Omega_N(A).$$

In the formal statement this is the equality between the periodic-boundary ground space and the subspace $\mathbb{C}\,\Omega_N(A)$. The hard inclusion into $\mathbb{C}\,\Omega_N(A)$ combines the complementary-word proof for $L_0 + 1 < N$ with a full-ring cyclic change of cut for $N = L_0 + 1$. [4]

The blueprint records the corresponding closure-property step in `blueprint/src/chapter/ch13_parent_hamiltonian_normal_closing_reverse.tex`, labelled `thm:normal_range_reduction_containment`. Its proof sketch separates the open-chain intersection iteration from the comparison at the periodic boundary, matching the source distinction between the intersection property and the closure property.

The first mismatch is that the available intersection theorem is not a literal expression of the proof through the region B. That theorem was written for a tensor that is already injective at one site. Its proof decomposes the identity as a linear combination of the one-site matrices A_j and then iterates this single-letter intersection property through contiguous windows. [5] For a normal tensor with injectivity length L_0, the available inverse is the inverse for words of length L_0, not a one-site inverse. This is not a contradiction in the source argument; it is a mismatch between the diagrammatic region-by-region proof and the available formal statements for intersection properties.

The formal argument therefore separates the proof into two parts instead of reproducing the compressed source proof verbatim. The open-chain component

[3] Source location: `Papers/2011.12127/TN-Review-main.tex`:2078–2079 for the iteration and periodic-boundary closure-property sentence, and `Papers/2011.12127/TN-Review-main.tex`:2087–2090 for the theorem labeled `thm:4:unique-gs-L0_plus_1`.

[4] Formal locations: `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:885 for `MPSTensor.chainGroundSpace_le_mpvSubmodule_of_normal_range_reduction`, and `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:923 for `MPSTensor.chainGroundSpace_eq_mpvSubmodule_normal`. The formal hypotheses are named `IsNormalA` and `IsNBlkInjectiveAL0`, together with `0<L0`, `2<=N`, `L0+1<=N`, and `L0<L<=N`.

[5] Formal locations: `TNLean/MPS/ParentHamiltonian/IntersectionProperty.lean`:281–284 for `MPSTensor.groundSpace_intersection`, `TNLean/MPS/ParentHamiltonian/IntersectionProperty.lean`:341–364 for the use of `decompositionMaphA1`, and `TNLean/MPS/ParentHamiltonian/CyclicWindow.lean`:153–163 for the contiguous iteration using that theorem.

has been proved by a block-injective range-reduction lemma: if every contiguous window of length L_0+1 lies in the corresponding local MPS ground space, then the full open chain lies in the full MPS ground space. Combining this with cyclic window monotonicity gives an open-chain consequence from the cyclic length-L constraints. This supplies the formal analog of the iterated open-boundary part of the source proof, without expressing the inverse for the region B through the older single-site intersection lemma. [6]

3 The boundary block-window identity

After the open-boundary range reduction, a periodic-boundary ground state can be written in trace form with a matrix X. The boundary comparison uses the block-window identity

$$XA^{\alpha}A^{\nu} = A^{\alpha}Y_{\nu}$$

for every length-L_0 word α and every nonempty complementary word ν. The block-injective commutation lemma then gives $XA^k = A^kX$ for every one-site matrix of A. Block injectivity finally forces X to be scalar, and hence the state lies in $\mathbb{C}\,\Omega_N(A)$.

This is why the open-chain containment cannot simply be reversed. Let $\Gamma_N(X)$ denote the length-N trace-form vector $\Gamma_N(X)_\omega = \operatorname{tr}(A^{\omega}X)$. A vector $\Gamma_N(X) \in G_N(A)$ need not lie in the periodic-boundary ground space. For a boundary-crossing window, the restriction has trace coefficients with X between the two parts of the window word,

$$\operatorname{tr}(A^{\omega_{\mathrm{tail}}}XA^{\omega_{\mathrm{head}}}A^{\rho})\,, \tag{1}$$

whereas membership in $G_L(A)$ would require the same coefficients to be written as

$$\operatorname{tr}(A^{\omega_{\mathrm{tail}}}A^{\omega_{\mathrm{head}}}Y_{\rho}) \tag{2}$$

for a matrix Y_ρ depending only on the labels outside the window. The closure-property comparison supplies the movement of X through the boundary-crossing word.

In the coordinate form used here, μ denotes the nonempty word placed on the sites outside the two cyclic windows and η is the letter used on the remaining outside sites. This is a coordinate form of the periodic-boundary closure-property sentence in Section IV.C, lines 2078–2079. The reduction isolates the movement of X as an identity between boundary-restriction matrices in the closure-property comparison, which implies the one-site commutation

[6] Formal locations: `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:266 for `MPSTensor.contiguous_mem_groundSpace_of_isNBlkInjective`, and `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:328 for `MPSTensor.chainGroundSpace_le_groundSpace_of_isNBlkInjective`.

identity. After choosing boundary-restriction matrices for the two boundary-crossing restrictions, the verified one-sided boundary-product lemma gives, for every physical letter j,

$$W_\eta A^j = A^\mu A^j X, \qquad X A^j A^\mu = A^j V_\eta.$$

Here W_η is the boundary-restriction matrix for the boundary-crossing restriction beginning at the last site, and V_η is the boundary-restriction matrix for the second boundary-crossing restriction. [7] This local matrix comparison gives the following commutation identity through the block-injective commutation lemma: for every physical letter k,

$$XA^k = A^k X.$$

This is the one-site commutation step in the periodic-boundary closure-property argument.[8]

For an N-site vector ψ, $\mathrm{Res}^{\tau}_{i,L}(\psi)$ denotes the L-site vector whose σ-coordinate is the coordinate of ψ obtained by placing σ in the cyclic window of length L beginning at i, and by using τ on the remaining sites.

Once this identity is available, the boundary-restriction equality lemma proves that the two boundary-restriction matrices have the same first-letter products:

$$W_\eta A^j = V_\eta A^j,$$

for every η and j. The first-letter restriction lemma then identifies the two cyclic restrictions

$$\mathrm{Res}^{\tau^+_\eta(\mu)}_{M,L_0+1}(\psi) = \mathrm{Res}^{\tau^-_\eta(\mu)}_{M+1-L_0,L_0+1}(\psi).$$

Thus the equality of boundary restrictions is no longer the primitive open comparison: it is a checked consequence of the one-sided products and the commutation identity.[9]

The block-injective commutation lemma reduces the commutation identity itself to this local matrix equation. It states that, if A is L_0-block-injective and there are matrices Y_ν such that, for every length-L_0 word α and every complementary word ν,

$$XA^\alpha A^\nu = A^\alpha Y_\nu,$$

[7] Formal locations: `TNLean/MPS/ParentHamiltonian/BoundaryClosingStripping.lean` for `MPSTensor.closure_property_boundary_restrictions_eq_of_groundSpaceMap`, and `TNLean/MPS/ParentHamiltonian/BoundaryClosing.lean` for `MPSTensor.closure_property_boundary_one_sided_products_of_groundSpaceMap`.

[8] In the formal proof of `MPSTensor.closure_property_boundary_restrictions_eq_of_groundSpaceMap`, the commutation identity is obtained from `MPSTensor.boundary_matrix_commutes_of_isNBlkInjective_of_block_matEq`.

[9] Formal locations: `TNLean/MPS/ParentHamiltonian/BoundaryMatrixBlock.lean` for `MPSTensor.boundary_restrictions_eq_of_commutes_and_one_sided`, and `TNLean/MPS/ParentHamiltonian/BoundaryClosingStripping.lean` for its use in `MPSTensor.closure_property_boundary_restrictions_eq_of_groundSpaceMap`.

then $XA^k = A^k X$ for every physical letter k.[10] The trace rotation at the last cyclic window has now been isolated as the identity

$$\operatorname{tr}\left(A^j X A^\alpha A^\nu\right) = \operatorname{tr}\left(A^j A^\alpha Y_\nu\right),$$

recorded as `MPSTensor.closure_property_boundary_block_window_trace_eq_of_groundSpaceMap`. These identities pair the matrix difference only with the single letters A^j, and therefore do not separate matrices for a general L_0-block-injective tensor. The corresponding trace identities against all length-L_0 word products A^β are now obtained from the same boundary-crossing cyclic-window comparison; those products span the full matrix algebra by block injectivity.[11] The trace-separation theorem records the algebraic consequence: these length-L_0 trace identities give $XA^\alpha A^\nu = A^\alpha Y_\nu$.[12] The equation is now obtained from the cyclic-window constraints. The block-injective commutation lemma then supplies the one-site commutation identity, the boundary-restriction equality lemma supplies the equality of the two boundary restrictions, and the periodic-boundary comparison follows. [13]

Equivalently, the same boundary phenomenon can be recognized in coordinate terms. For an N-site vector ψ, the two restrictions are obtained from the two boundary-crossing words

$$\tau_\eta^+(\mu)_i = \begin{cases} \mu_{i-L_0}, & L_0 \le i < N-1, \\ \eta, & \text{otherwise}, \end{cases} \qquad \tau_\eta^-(\mu)_i = \begin{cases} \mu_{i-1}, & 1 \le i < N-L_0, \\ \eta, & \text{otherwise}. \end{cases}$$

The direct comparison of these two restrictions, and the left-multiplied coordinate comparison obtained after taking first-letter restrictions, are consequences of the one-site commutation identity $XA^k = A^k X$.

4 Resolved endpoint: the minimal ring

The source theorem includes the minimal ring $N = L_0 + 1$. This endpoint is now proved separately from the nonempty-complement argument. Write a vector in $\mathcal{G}_{L_0+1,L_0+1}(A)$ as $\Gamma_{L_0+1}(X)$. Moving the cut by L_0 sites and writing the translated state as $\Gamma_{L_0+1}(Y)$ gives

$$XA^j = A^j Y$$

for every letter j. Repeating this comparison around all L_0+1 cuts shows that X commutes with every full-ring word. Since block injectivity persists at length

[10] Formal location: `TNLean/MPS/ParentHamiltonian/BoundaryMatrixBlock.lean` for `MPSTensor.boundary_matrix_commutes_of_isNBlkInjective_of_block_matEq`.

[11] Formal statement: `MPSTensor.closure_property_boundary_block_window_trace_evalWord_mul_eq_of_groundSpaceMap`.

[12] Formal location: `TNLean/MPS/ParentHamiltonian/BoundaryClosingStripping.lean` for `MPSTensor.block_window_matrix_equation_of_trace_evalWord_mul_eq_of_isNBlkInjective`.

[13] The source-labelled statement for this step is `MPSTensor.closure_property_boundary_block_window_equation_of_groundSpaceMap`.

$L_0 + 1$, these words span the full matrix algebra, so X is scalar. Thus

$$\mathcal{G}_{L_0+1,L_0+1}(A) = \mathbb{C}\,\Omega_{L_0+1}(A).$$

The same theorem also removes the extra strict bound from the range-$2L_0$ corollary. In particular, $L_0 = 1$ and $N = 2$ is covered explicitly rather than being inferred from a nonempty complementary word.

The formal proof is `MPSTensor.full_ring_boundary_intertwines_of_chainGroundSpace` followed by `MPSTensor.chainGroundSpace_le_mpvSubmodule_of_isNBlkInjective_full_ring`. The public theorems `MPSTensor.chainGroundSpace_le_mpvSubmodule_of_normal_range_reduction`, `MPSTensor.chainGroundSpace_eq_mpvSubmodule_normal`, `MPSTensor.parentHamiltonian_unique_gs_normal`, and `MPSTensor.parentHamiltonian_unique_gs_injective` now use the non-strict source bound.

5 Conclusion

This note records two distinctions between the compressed source proof and the available formal lemmas; it is not a source-error claim. The cited Cirac–Perez-Garcia–Schuch–Verstraete 2021 argument is not refuted. The source compresses two mathematical steps which the present proof keeps separate: the B-region inverting and growing-back open-chain range reduction, and the movement of the trace-form matrix X through the boundary-crossing word in the periodic-boundary comparison.

The open-chain part has been proved under the intended block-injective hypotheses, though not by reusing the older single-site intersection theorem. For the nonempty-complement periodic-boundary comparison, the cyclic length-$(L_0 + 1)$ constraints now give the length-L_0 trace identities

$$\operatorname{tr}(A^\beta X A^\alpha A^\nu) = \operatorname{tr}(A^\beta A^\alpha Y_\nu)$$

for all length-L_0 words α, β and every nonempty complementary word ν. The trace-separation lemma then gives the matrix identity

$$X A^\alpha A^\nu = A^\alpha Y_\nu$$

for every α, ν. The block-injective commutation lemma then gives $XA^k = A^k X$ for every physical letter k, and the verified boundary-restriction equality lemma supplies the periodic-boundary comparison. At the minimal ring $N = L_0 + 1$, the separate full-ring cyclic change-of-cut argument shows that the boundary matrix is scalar and proves the same ground-space conclusion. Thus the formal theorem covers the full source range $L_0 + 1 \le N$.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.