

Injective PEPS Parent Interactions: the Physical Reconstruction Condition

2026-10-02

Verdict: local-correction (resolved).

1 The printed construction

Section IV.C.1 of [CPGSV21] constructs an injective PEPS from virtual entangled pairs by a product of site maps A . The equation labelled `eq:4:deformed-parent-1` transports a virtual parent interaction h by a chosen site left inverse:

$$h' = (A^{-1} \otimes A^{-1})^\dagger h (A^{-1} \otimes A^{-1}). \quad (1)$$

The following sentence asserts that its kernel is the regional PEPS space.¹ This kernel assertion needs a physical reconstruction condition when the injective site maps are rectangular.

2 A nondegenerate example

Consider two vertices joined by a bond of dimension $D = 1$ and take $A : \mathbb{C} \rightarrow \mathbb{C}^2$, $A(z) = (z, 0)$, at each vertex. The bond space is nonzero and both site maps are injective. The normalized virtual Bell vector is the scalar 1, so its complement projector is $h = 0$. Consequently (1) gives $h' = 0$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Its kernel is four-dimensional, whereas the actual two-site PEPS space is $\text{span}\{e_0 \otimes e_0\}$. Thus the failure is present for positive bond dimension and an injective, nonzero tensor; it is not a zero-dimensional convention.²

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2031–2037.

²Results: `TNLean.PEPS.InjectiveParentReconstructionCounterexample.bareKernel_n`, `e_productSiteRange`, `TNLean.PEPS.InjectiveParentReconstructionCounterexample.finrank_productSiteRange`, and `TNLean.PEPS.InjectiveParentReconstructionCounterexample.finrank_bareKernel`.

3 The corrected interaction

Let $F : \mathcal{V} \rightarrow \mathcal{H}$ be the product site map on a finite region, and let $L : \mathcal{H} \rightarrow \mathcal{V}$ be a left inverse, so $LF = I_{\mathcal{V}}$. For any positive semidefinite virtual interaction H , put

$$K = L^{\dagger}HL + (I_{\mathcal{H}} - FL)^{\dagger}(I_{\mathcal{H}} - FL). \quad (2)$$

Both summands are positive semidefinite. Their sum annihilates ψ precisely when $HL\psi = 0$ and $FL\psi = \psi$. Indeed,

$$\langle \psi, K\psi \rangle = \|H^{1/2}L\psi\|^2 + \|(I_{\mathcal{H}} - FL)\psi\|^2.$$

It follows that $\ker K = F(\ker H)$. In contrast, the bare interaction satisfies $\ker(L^{\dagger}HL) = F(\ker H) \oplus \ker L$. The additional summand in (2) removes exactly the unused physical directions. It vanishes if F is bijective, and does not alter the quadratic form on $F(\mathcal{V})$. If $\ker H$ is the actual regional virtual bond space, the corrected kernel is exactly the regional physical PEPS space.³

4 Scope of the correction

This corrects the displayed interaction, not the uniqueness conclusion for genuine parent interactions with the prescribed exact local kernels. For injective tensors, uniqueness of the ground state of genuine nearest-neighbour parent interactions follows from those kernels and the virtual Bell constraints. Unused physical directions must first be excluded. Every vertex must be covered by a parent region; isolated vertices require their own single-site conditions. The reconstruction summand acts on the chosen region. Its existence does not by itself establish a decomposition into smaller local terms, nor any infinite-volume spectral-gap statement.

References

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

³Results: `TNLean.PEPS.injectiveRegionParentTransport_posSemidef`, `TNLean.PEPS.injectiveRegionParentTransport_mulVec_eq_zero_iff`, `TNLean.PEPS.ker_injectiveRegionParentTransport`, and `TNLean.PEPS.isRegionParentInteraction_injectiveRegionParentTransport`.