

Pure-Exponential Correlator Expansion and Jordan Blocks in CPGSV21

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Verdict: false-source (open).

1 Source statement

Section II.B.3 of Cirac–Pérez-García–Schuch–Verstraete [1] notes that the transfer matrix E need not be Hermitian and “may have some Jordan blocks”, and assumes a unique eigenvalue of largest modulus which is algebraically simple. It then states that a connected correlator at distance n has the form

$$C(X, Y; n) = \sum_{i=2}^{D^2} c_{XY}(i) \lambda_i^n, \quad (1)$$

a sum of $D^2 - 1$ pure exponentials, with decay rate set by $|\lambda_2|$.¹

2 The deviation

In trace-preserving gauge with fixed state ρ , the traceless matrices are invariant. Equation (1) holds when the transfer map, restricted to the traceless matrices, is diagonalizable. If that restriction has a Jordan block $J = \lambda I + N$ of size $k + 1$, then

$$J^n = \sum_{j=0}^{\min(k,n)} \binom{n}{j} \lambda^{n-j} N^j, \quad (2)$$

and a correlator whose insertions detect N is a polynomial-exponential sequence rather than a sum of pure exponentials. Likewise, a uniform bound $|C(X, Y; n)| \leq C_{XY} |\lambda_2|^n$ can fail at a defective subleading eigenvalue. A bound $C_{XY} r^n$ still holds for every rate r strictly above the spectral radius of the complementary map, by the Gelfand formula.

¹Source location: `Papers/2011.12127/TN-Review-main.tex:433–440`.

Explicit Hermitian physical example. Let

$$P = \frac{1}{16} \begin{pmatrix} 9 & 5 & 2 \\ 3 & 7 & 6 \\ 4 & 4 & 8 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Every entry of P is strictly positive, and every row and column sums to one. Put $Q = \frac{1}{3}\mathbf{1}\mathbf{1}^\top$ and $N = uv^\top/16$. Direct calculation gives

$$P = Q + \frac{1}{4}(I - Q) + N, \quad QN = NQ = N^2 = 0.$$

Here $u^\top v = 0$, $u^\top u = 2$, and $v^\top v = 6$. Consequently, for every $n \geq 0$,

$$P^n v = 4^{-n}(v + \frac{3}{2}nu), \quad \frac{1}{3}u^\top P^n v = n4^{-n}.$$

The channel on 3×3 matrices with Kraus operators $A^{(a,b)} = \sqrt{P_{ab}}|a\rangle\langle b|$ is trace preserving and unital, with fixed state $\rho = I/3$. It annihilates off-diagonal matrices and acts as P on diagonal matrices. It is positivity improving in one step: for every nonzero positive semidefinite matrix Z , the diagonal of Z has a positive entry, so every entry of $P \text{diag}(Z)$ is positive. Moreover, its nine Kraus operators span the full matrix algebra; thus the associated tensor is already one-site injective. Its spectrum is $\{1, \frac{1}{4}, 0\}$, with a size-two Jordan block at $\frac{1}{4}$.

Define physical observables on the nine-dimensional one-site space by

$$X_{(a,b),(a,b)} = u_b, \quad Y_{(a,b),(a,b)} = v_a,$$

with all off-diagonal entries zero. These are Hermitian. Writing $\Phi_O(Z) = \sum_{a,b} P_{ab} O_{(a,b),(a,b)} Z_{bb} |a\rangle\langle a|$ for the insertion map of a diagonal observable, bistochasticity gives

$$\Phi_Y(\rho) = \frac{1}{3} \text{diag}(v), \quad \text{tr} \Phi_X(\text{diag}(z)) = u^\top z.$$

Both one-point expectations vanish. The fixed-point infinite-chain connected correlator with n unobserved intermediate sites therefore is exactly

$$\text{tr}(\Phi_X(E^n(\Phi_Y(\rho)))) - \text{tr} \Phi_X(\rho) \text{tr} \Phi_Y(\rho) = n4^{-n}.$$

The physical spacing of the one-site observables is $n+1$. This sequence cannot be a finite linear combination of pure exponentials with the transfer-map eigenvalues, nor can it satisfy a bound $C4^{-n}$ for all n . Every matrix unit occurs with a strictly positive coefficient, and the fixed state is positive definite.

Zero eigenvalues. Formula (2) is valid also at $\lambda = 0$, with the usual convention $0^0 = 1$ for integer powers. A nonzero nilpotent Jordan block contributes a finite transient. Such a contribution cannot in general be written as $p(n)0^n$: that expression vanishes for every $n > 0$, whereas the nilpotent block can remain nonzero for several positive powers. Thus an all-distance corrected expansion uses the binomial terms in (2), or separates the nonzero-eigenvalue polynomial-exponential terms from the finite nilpotent transient.

3 Corrected statements

The all-separation binomial expansion is now formalized by `MPSTensor.physicalConnectedCorrelator_exists_binomial_expansion`. Mathlib’s generalized-eigenspace decomposition supplies actual complementary transfer eigenvalues and coefficients, with no assumed expansion or diagonalizability certificate. For each eigenvalue, the cutoff is the least index at which its generalized eigenspace stabilizes, equivalently its maximal Jordan-block size.

The theorem `MPSTensor.physicalConnectedCorrelator_exists_jordan_expansion` rewrites only the nonzero eigenvalues as polynomial-times-exponential terms. The zero-eigenvalue contribution is retained as a finite transient. For a normal tensor, the complete arbitrary-gauge physical statement is `MPSTensor.IsNormalTensor.exists_physicalLeftRightCorrelation_jordan_expansion`: the nonzero frequencies are genuine eigenvalues of the original transfer map, all have modulus below one, and the transient vanishes by separation D^2 . A common trace-preserving representative supplies the exact Jordan indices and works for all finite supports. The formula includes adjacent blocks, $n = 0$.

The separately proved decay bound still uses every rate r satisfying $\text{spr}(E - P) < r < 1$. Polynomial factors need not be bounded by a constant at the spectral radius itself.

Gauge covariance. Physical insertions and the connected left-right contraction are now gauge covariant, by `MPSTensor.physicalObservableTransfer_congruence_of_gauge` and `MPSTensor.physicalLeftRightConnectedCorrelator_eq_of_gauge`. For a normal tensor, a trace-preserving gauge $B^i = UA^iU^{-1}$ with fixed state ρ of trace one gives positive definite fixed matrices $\rho^L = U^\dagger \rho U$ and $\rho^R = U^{-1} \rho U^{-\dagger}$, with $\text{tr}(\rho^L \rho^R) = 1$. The theorem `MPSTensor.IsNormalTensor.exists_physicalLeftRightCorrelation_decay` proves exponential decay in the original representation on independent finite supports at all separations, including adjacent blocks.

4 Remaining source scope

The corrected Jordan expansion, including finite nilpotent transients, is proved for normal tensors. The open verdict concerns the source’s broader class with a unique algebraically simple dominant eigenvalue and possibly singular fixed matrices. Extending the physical theorem to that class needs the left-right spectral projection directly, rather than an invertible Perron gauge. The original unqualified pure-exponential formula remains false; the proved binomial and polynomial-plus-transient formulas replace it.

References

- [1] J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Rev. Mod. Phys.*, 93:045003, 2021.