

The Interaction Range in the Block-Injective Parent-Hamiltonian Gap

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Verdict: scope-restriction (resolved).

1 The range condition

The gap theorem in Ref. [CPGSV21, Section IV.C, Gaps], local source lines 2183–2187, asserts a uniform positive gap for MPS parent Hamiltonians, including those of block-injective tensors. The assertion has no interaction-range qualifier and fails at some shorter ranges; an explicit canonical counterexample with three blocks at range two is documented in `docs/paper-gaps/cpgsv21_short_range_parent_gap.tex`. This note resolves the comparison at the sufficient ranges below, without narrowing the source definition of a parent interaction. For at least two canonical blocks, Ref. [PGVWC07, Theorem 12] gives the explicit interaction condition

$$R \geq 3(r-1)(L_0+1)+1, \tag{1}$$

where the blocks satisfy condition C1 at the common positive length L_0 . The local source is `MPSarchive.tex`, lines 1424–1458.

Let $A^0, \dots, A^{r-1}$ have positive bond dimensions and be pairwise inequivalent under virtual conjugation and nonzero scalar multiplication. For nonzero coefficients μ_j , write $B = \bigoplus_j \mu_j A^j$ and let $H_{R,N}$ be its canonical periodic parent Hamiltonian. The overlap argument initially constructs a gapped range $R = 2p$ after choosing p sufficiently large. This alone would not prove a gap at every prescribed range satisfying (1). The comparison below removes this restriction, as developed under issue #7676.

2 Comparison of two interaction ranges

For $W \in \mathbb{N}$, let $G_W(B)$ be the local MPS space of B on W sites, the span of the vectors $\sigma \mapsto \text{tr}(B^\sigma X)$ for $\sigma \in \{0, \dots, d-1\}^W$ and virtual matrices X . For $R \leq W$, let $K_{R,W}$ be the open-chain parent Hamiltonian of range R on W sites, the sum over the $W - R + 1$ initial sites i of the orthogonal projection onto

$G_R(B)^\perp$ acting on sites $i, \dots, i + R - 1$. Fix $R > 0$, and choose $W \geq 2R$ such that $K_{R,W}$ has kernel $G_W(B)$ and the periodic range- W Hamiltonians have a uniform gap $\gamma_W > 0$ for all sufficiently large volumes. Set $h_W = I - P_{G_W(B)}$. Finite dimensionality and positivity give constants $\kappa, C > 0$ such that

$$\kappa h_W \leq K_{R,W} \leq C h_W. \quad (2)$$

The lower bound follows from injectivity of $K_{R,W}$ on its kernel complement. For the upper bound one may take $C = \|K_{R,W}\| + 1$.

Embed (2) at every cyclic initial site of a sufficiently long periodic chain. Each range- R term appears exactly $m = W - R + 1$ times. Thus

$$\kappa H_{W,N} \leq m H_{R,N} \leq C H_{W,N}. \quad (3)$$

Both periodic operators have the same kernel: the quadratic form of a positive operator vanishes precisely on its kernel, and the two inequalities give the two inclusions. The range- W gap therefore implies

$$\frac{\kappa \gamma_W}{m} \|v\| \leq \|H_{R,N} v\| \quad (v \perp \ker H_{R,N}). \quad (4)$$

All constants are independent of N . To include the finitely many excluded lengths, restrict each corresponding Hamiltonian to its kernel complement. This restriction is injective and finite-dimensional, so it has a positive norm lower bound. Taking the minimum of these finitely many bounds and the uniform eventual bound gives one positive constant for every N . The comparison uses every cyclic initial site and imposes no divisibility restriction on the chain length. It never identifies the open-chain kernel with the periodic kernel.

3 The simultaneous-injectivity range

Suppose instead that a positive integer S is supplied for which the length- S word tuples of the blocks span the full product matrix algebra. This is the simultaneous block-injectivity condition in Ref. [CPGSV21, Section IV.C], local source lines 2114–2129. Then the uniform gap holds at every range

$$R \geq S + 1. \quad (5)$$

The number of blocks need not be at least two, and no normalization or separate inequivalence assumption is imposed on the original blocks.

Let $V^{(S)}(A^i)$ denote the length- S periodic matrix product vector generated by the block A^i , obtained by closing the length- S matrix product on the identity boundary matrix. Indeed, projecting the simultaneous span onto any component proves injectivity of that block at length S , so each periodic vector $V^{(S)}(A^i)$ is nonzero. The simultaneous span also makes the local ground spaces $G_S(A^i)$ independent, and this separates distinct blocks. Suppose that $i \neq j$ and $A_a^i = \zeta X A_a^j X^{-1}$ for every physical index a , with $\zeta \neq 0$ and X invertible. Then

$$V^{(S)}(A^i) = \zeta^S V^{(S)}(A^j) \in G_S(A^i) \cap G_S(A^j) = \{0\}, \quad (6)$$

contradicting $V^{(S)}(A^i) \neq 0$. Independent primitive normalization preserves the supplied length S , simultaneous spanning, and all local ground spaces. In the normalized family, simultaneous spanning propagates to every larger length. The open-chain intersection theorem therefore gives $\ker K_{R,W} = G_W(B)$ for every $W \geq R$ as soon as (5) holds. Choose a gapped range $W \geq 2R$ and apply the two-sided comparison above. The finite minimum over the remaining chain lengths again gives one positive gap for every N .

For an original canonical tensor with repeated multiplicity copies, the local ground spaces equal those of its chosen normal representatives at every positive length. Hence the canonical parent interactions, and then every periodic parent Hamiltonian, agree exactly. The same uniform gap transfers to the original tensor. The representatives may be chosen once, before specifying S or R .

4 General positive parent interactions

The definition in Ref. [CPGSV21, Section IV.C], local source lines 1996–1999, permits any fixed positive local operator h with $\ker h = G_R(A)$. Let $h_R = I - P_{G_R(A)}$ be the orthogonal projection onto $G_R(A)^\perp$. Finite dimensionality gives constants $\kappa, C > 0$ such that

$$\kappa h_R \leq h \leq C h_R. \quad (7)$$

If $h = 0$, then $h_R = 0$ and any positive constants suffice. Otherwise one may use the least and greatest positive eigenvalues of h .

Tensoring with the identity, conjugating by cyclic translations, and summing over initial sites preserve (7). Thus, with the same constants for every chain length $N \geq R$,

$$\kappa H_{R,N}(A) \leq H_N(h) \leq C H_{R,N}(A). \quad (8)$$

The two Hamiltonians have the same kernel. A uniform gap γ for the canonical projection Hamiltonian therefore gives a uniform gap $\kappa\gamma$ for $H_N(h)$. This is the projection reduction stated in the review at local source lines 2170–2172. In particular, it preserves the interaction range $R \geq S + 1$ established above.

The interaction h is fixed before varying N . Equality of local kernels alone would not suffice for an arbitrary volume-dependent choice: replacing h by h/N preserves each kernel but need not preserve a positive uniform gap.

5 Recovering the printed threshold

The simultaneous word tuples span the full product matrix algebra at length $3(r-1)(L_0+1)$. Trace-preserving normalization propagates this span to every larger length: represent $M_j(A_a^j)^\dagger$ by word tuples, append A_a^j , and sum over a , using $\sum_a (A_a^j)^\dagger A_a^j = I$. The intersection property then identifies the open range- R kernel with $G_W(B)$ for every $W \geq R$ whenever (1) holds. The overlap

construction supplies a gapped W above any prescribed bound, so the preceding comparison applies to every such R .

Finally, independent primitive normalization of the blocks preserves injectivity at the specified length L_0 , pairwise inequivalence, and all local ground spaces. It therefore preserves every parent Hamiltonian exactly. No larger C1 length is substituted during this step. The formalized conclusion is consequently a uniform periodic gap over every chain length at each range in (1), with no normalization hypothesis on the original blocks. The earlier chosen-range restriction is resolved.

6 An automatic sufficient range

For a canonical tensor of positive bond dimension D , quantum Wielandt and block separation supply $S > 0$ with $S + 1 \leq 3D^5$ for its distinct normal representatives. Thus every range $R \geq 3D^5$ has a positive periodic gap uniform over all chain lengths, with no separately supplied simultaneous span. The same conclusion holds for every fixed positive parent interaction at that range. This is a sufficient dimension bound; it does not replace the smaller admissible threshold $S + 1$ when S is known. The declarations are in `TNLean/MPS/ParentHamiltonian/Martingale/CanonicalGapAtBoundedRange.lean`.

References

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.
- [PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.