Block-Diagonal Boundary Conditions in the Degenerate Parent-Hamiltonian Ground Space

2026-06-11

Verdict: open-gap (resolved).

1 Notation

For a tensor $A = (A_i)_{i=1}^d$ with bond dimension D, write

$$\Gamma_N^A(X)(i_1, \ldots, i_N) = \mathrm{tr}(A_{i_1} \cdots A_{i_N} X) \qquad (X \in M_D(\mathbb{C})). \tag{1}$$

Set

$$\mathcal{G}_N^A = \{\Gamma_N^A(X) : X \in M_D(\mathbb{C})\}, \qquad V^{(N)}(A) = \Gamma_N^A(I). \tag{2}$$

For the ring with periodic boundary conditions, with $\mathrm{res}_{s,L}$ denoting restriction to the length-L interval beginning at $s \in \mathbb{Z}/N\mathbb{Z}$, the local ground-space condition is

$$\mathcal{K}_{N,L}(A) = \bigcap_{s \in \mathbb{Z}/N\mathbb{Z}} \{\psi : \mathrm{res}_{s,L}(\psi) \in \mathcal{G}_L^A\}. \tag{3}$$

When weights are present, write

$$\widehat{A}_i = \bigoplus_{j=1}^{b} \mu_j A_i^j, \qquad \mu_j \neq 0, \tag{4}$$

for the corresponding block-diagonal tensor. The letters g and b both denote the number of blocks below: g follows the review's block-injective canonical form notation, while b follows [PGVWC07, Theorem 12].

2 The source assertion

Cirac et al. describe the degenerate parent-Hamiltonian ground space for a matrix product state with more than one block in canonical form in Section IV.C of [CPGSV21].[1] The source's block-injective canonical form is a block-diagonal physical–virtual correspondence. If

$$\widetilde{A}^i = \bigoplus_{j=1}^{g} A_j^i, \tag{5}$$

then

$$\forall X = (X_j)_j \in \bigoplus_{j=1}^{g} M_{D_j}(\mathbb{C}), \qquad \exists (c_i(X))_i, \qquad X = \sum_i c_i(X)\widetilde{A}^i. \tag{6}$$

After blocking, the corresponding word-span statement is

$$\exists m_0 \quad \forall m \geq m_0, \qquad \operatorname{span}\{(A_1^\omega, \ldots, A_g^\omega) : |\omega| = m\} = \prod_{j=1}^{g} M_{D_j}(\mathbb{C}). \tag{7}$$

The Lean development separates the product-span input from the finite blocking hypothesis. First, it proves the one-step intersection identity from [PGVWC07, Theorem 12] under the displayed product-span equation. Then the normalized basis-of-normal-tensors (BNT) theorems derive that product span from the injectivity of

$$\Gamma_{L_0}^{A_j} : M_{D_j}(\mathbb{C}) \to (\mathbb{C}^d)^{\otimes L_0}, \tag{8}$$

together with the separated normalized block hypotheses. For $b \geq 2$, the simultaneous products span the product algebra at every length $n \geq 3(b-1)(L_0+1)$. Thus the Lean statements no longer replace Condition C1 by a length-one span condition in the boundary-decomposition path. The boundary conditions are compared after a global change of cut, using only the simultaneous product span on the long complementary segment of length $N - L_0$.

The resulting parent-Hamiltonian theorem in [CPGSV21, Section IV.C] is

$$N \geq L_0 + 1 \quad \Longrightarrow \quad \ker(H_N) = \operatorname{span}\{V^{(N)}(A_j) : j = 1, \ldots, g\}. \tag{9}$$

The sentence preceding the theorem restricts the proof to block-diagonal boundary conditions. After opening the periodic boundary at the chosen cut, this means boundary matrices of the form

$$X = \bigoplus_{j=1}^{g} X_j, \qquad Y = \bigoplus_{j=1}^{g} Y_j \qquad (X_j, Y_j \in M_{D_j}(\mathbb{C})). \tag{10}$$

[1] For traceability, the boundary-crossing comparison has a formal span-dependent form on the main branch (`TNLean/MPS/ParentHamiltonian/BNTBlockDiagonalCrossingTrace.lean` and `TNLean/MPS/ParentHamiltonian/BNTBlockDiagonalPGVWCComparison.lean`). The short crossing-tail span hypothesis is removed by the global change-of-cut comparison in `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_global_cut_bnt_c1_pgwvc07_of_dualFixedPoint`; this resolves issue #3597 at the length bound of the source. The removal of the former extra (L_0+1) in the separation constant resolves issue #4195 (the earlier tracker issue #2971, which tracked proving the boundary-crossing comparison itself, is now closed). Pull request #2724 records the earlier conditional reduction on the main branch, reducing the unconditional target to boundary-crossing comparison identities. Earlier reductions were tracked by issue #2641, issue #905, and issue #870. The relevant source lines are `Papers/2011.12127/TN-Review-main.tex`:2112–2128.

3 Normalization scope

The PGVWC07 source normalization used in the block-intersection calculation is

$$\sum_a A_a^j (A_a^j)^\dagger = I. \tag{11}$$

For the dual transfer operator, the general canonical form has a positive definite fixed point Λ_j satisfying

$$\sum_a (A_a^j)^\dagger \Lambda_j A_a^j = \Lambda_j. \tag{12}$$

The unrestricted source-length declarations assume these source identities and positive dual fixed-point data; unlike the diagonal-coordinate choice made in PGVWC07, the proofs do not require the matrices Λ_j to be diagonal. A temporary square-root gauge makes each block left-canonical only inside the fixed-length simultaneous product-span proof; gauge invariance transports that span back to the source representatives. Long-word propagation, the global change-of-cut comparison, the block-diagonal boundary calculation, and the parent-Hamiltonian kernel argument all remain in the original source-unital gauge. In particular, the formalization does not assert that a nonunitary gauge preserves both normalization identities.

The declarations ending in `_pgvwc07_of_dualFixedPoint` give the unrestricted chain-space, boundary-closing, kernel-containment, and exact kernel-equality conclusions of Theorem 12. The older declarations ending simply in `_pgvwc07` remain available as the doubly normalized specialization $\Lambda_j = I$; their main chain-space capstone is proved by applying the unrestricted theorem to the identity dual fixed point. This resolves the normalization restriction tracked by issue #5751, with the source-range tuple-span layer supplied by issue #5764 and the global-cut/kernel threading by issue #5770.

4 The block-diagonal intersection identity

The older representation paper [PGVWC07, Theorem *Degeneracy of the ground space v2*] gives the finite-chain identity used by the source argument.[2] It fixes the block count and local length by

$$b = g, \qquad L \geq 3(b-1)(L_0+1)+1. \tag{13}$$

Thus

$$m \geq L \tag{14}$$

[2]Source location: `Papers/quant-ph_0608197/MPSarchive.tex`:1424–1456.

is the range used in the intersection step:

$$\mathbb{C}^d \otimes \left(\bigoplus_j \mathcal{G}_m^{A_j} \right) \cap \left(\bigoplus_j \mathcal{G}_m^{A_j} \right) \otimes \mathbb{C}^d = \bigoplus_j \mathcal{G}_{m+1}^{A_j}. \tag{15}$$

For a vector ψ in the left-hand side,

$$\begin{aligned} \psi &= \sum_j \sum_{i_1,\dots,i_{m+1}} \operatorname{tr}\Big(A^j_{i_{m+1}} C^j_{i_1} A^j_{i_2} \cdots A^j_{i_m} \Big) |i_1 \cdots i_{m+1}\rangle \\ &= \sum_j \sum_{i_1,\dots,i_{m+1}} \operatorname{tr}\Big(D^j_{i_{m+1}} A^j_{i_1} A^j_{i_2} \cdots A^j_{i_m} \Big) |i_1 \cdots i_{m+1}\rangle. \end{aligned} \tag{16}$$

The direct-sum separation lemma and the one-block injectivity condition identify, for every j, i_1, i_{m+1},

$$A^j_{i_{m+1}} C^j_{i_1} = D^j_{i_{m+1}} A^j_{i_1}. \tag{17}$$

The source line writes $E^j = \sum_k C^j_k A^j_k$, without an adjoint on the second factor. The normalization used in the next line is $\sum_k A^j_k A^{j\dagger}_k = I$, so the calculation requires the adjointed matrix

$$E^j = \sum_k C^j_k A^{j\dagger}_k. \tag{18}$$

Together with the normalization

$$\sum_{i_1} A^j_{i_1} A^{j\dagger}_{i_1} = I, \tag{19}$$

the displayed matrix identity gives

$$\begin{aligned} A^j_{i_{m+1}} E^j A^j_{i_1} &= \sum_k A^j_{i_{m+1}} C^j_k A^{j\dagger}_k A^j_{i_1} \\ &= D^j_{i_{m+1}} \left(\sum_k A^j_k A^{j\dagger}_k \right) A^j_{i_1} \\ &= D^j_{i_{m+1}} A^j_{i_1} = A^j_{i_{m+1}} C^j_{i_1}, \end{aligned} \tag{20}$$

which puts the vector in $\bigoplus_j \mathcal{G}_{m+1}^{A_j}$.

5 The boundary condition and the global change of cut

The present Lean statements prove the one-step identity under the product-span hypothesis

$$\operatorname{span}\{(A^1_w, \dots, A^b_w) : |w| = m\} = \prod_{j=1}^{b} M_{D_j}(\mathbb{C}), \tag{21}$$

and, in the normalized BNT case, give the internal direct sums

$$\bigvee_j \mathcal{G}_m^{A_j} = \bigoplus_j \mathcal{G}_m^{A_j}, \qquad \bigvee_j \mathcal{G}_{m+1}^{A_j} = \bigoplus_j \mathcal{G}_{m+1}^{A_j}. \tag{22}$$

The proved one-step conclusion is therefore

$$\mathbb{C}^d \otimes \left(\bigoplus_j \mathcal{G}_m^{A_j}\right) \cap \left(\bigoplus_j \mathcal{G}_m^{A_j}\right) \otimes \mathbb{C}^d = \bigoplus_j \mathcal{G}_{m+1}^{A_j} \tag{23}$$

for all sufficiently large m satisfying the product-span hypotheses. The tuple-span, internal-sum, and one-step intersection statements now hold under the source canonical data: for each block j, let Λ^j be a positive definite matrix satisfying

$$\sum_a A_a^j A_a^{j\dagger} = I, \qquad\qquad \sum_a A_a^{j\dagger} \Lambda^j A_a^j = \Lambda^j. \tag{24}$$

These statements require neither diagonality of Λ^j nor the additional identity $\sum_a A_a^{j\dagger} A_a^j = I$ on the same representative. The proof uses the positive dual fixed point only to prepare a temporary left-canonical gauge for the sharp base separation theorem, transports the tuple span back, and then propagates it with the source unital identity.[3] The earlier doubly-normalized declarations remain available as special-purpose siblings.

The normalization relaxation is now threaded through the open-boundary containment, unique block sum, block-diagonal boundary representation, global change of cut, periodic chain-space equality, kernel containment, and exact kernel equality. In particular,

$$\mathcal{K}_{N,L}(\widehat{A}) = \bigvee_{j=1}^{b} \mathcal{K}_{N,L}(A_j), \quad \ker H_L^{(N)}(\widehat{A}) = \operatorname{span}\{V^{(N)}(A_j) : 1 \le j \le b\}, \tag{25}$$

under the source unital and positive dual fixed-point hypotheses, with no source left-canonical assumption.[4]

[3]Lean declarations: `MPSTensor.wordTupleSpanTop_threeBlock_mul_pred_of_blocksNotGaugePhaseEquiv_c1_of_dualFixedPoint`, `MPSTensor.wordTupleSpanTop_of_ge_of_bnt_directSum_unital_c1_pgvwc07_of_dualFixedPoint`, `MPSTensor.groundSpace_iSupIndep_of_ge_of_bnt_directSum_unital_c1_pgvwc07_of_dualFixedPoint`, and `MPSTensor.pgvwc07_iSup_restriction_intersection_of_ge_of_bnt_directSum_unital_c1_pgvwc07_of_dualFixedPoint`.

[4]Lean declarations: `MPSTensor.chainGroundSpace_toTensorFromBlocks_le_iSup_groundSpace_of_ge_of_bnt_directSum_unital_c1_pgvwc07_of_dualFixedPoint`, `MPSTensor.exists_unique_sum_groundSpace_of_chainGroundSpace_toTensorFromBlocks_of_bnt_unital_c1_pgvwc07_of_dualFixedPoint`, `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_global_cut_bnt_c1_pgvwc07_of_dualFixedPoint`, and `MPSTensor.ker_parentHamiltonian_toTensorFromBlocks_eq_bntMPSVectorSpan_of_global_cut_bnt_c1_pgvwc07_of_dualFixedPoint`.

Earlier conditional reductions isolated the source comparison in the coordinates obtained by opening the periodic boundary for boundary-crossing windows. Suppose

$$\psi = \Gamma_N^{\widehat{A}}\left(\bigoplus_{j=1}^{b} X_j\right) \in \mathcal{K}_{N,L}(\widehat{A}). \tag{26}$$

For a cyclic interval beginning at i and crossing the boundary cut, put $a = i + L - N$. The word β records the part of the interval before the chosen cut, while ρ records the outside word of length $N - L$. These are the blocked words obtained by opening the source boundary-index comparison. The conditional theorem assumes that, for every block j and every such outside word ρ, there is a matrix $E_{j,i,\rho}$ such that for every word β of length a,

$$\mu_j^N X_j A_\beta^j A_\rho^j = A_\beta^j E_{j,i,\rho}. \tag{27}$$

Under the usual block-injectivity and normalization hypotheses this assumption implies

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{K}_{N,L}(A_j) \qquad (1 \leq j \leq b). \tag{28}$$

Thus the trace representation of ψ is not the obstruction. The periodic component theorem proves the displayed conclusion from these boundary-crossing coordinate identities. The global change-of-cut proof avoids assuming those identities separately: it translates the whole periodic state past the final L_0 sites and compares the two open-boundary decompositions against the long complementary segment.[5]

The live blueprint records the block-diagonal periodic-boundary ground space, the forward inclusion, the block-diagonal intersection identity from [PGVWC07, Theorem 12], the parent-ground-space containment, and the final span statement with explicit citations. It also separates these opened-boundary coordinates from the paper's boundary indices, and records the unconditional global-cut conclusion at the source length bound. With this notation,

$$\sum_{j=1}^{b} \mathcal{K}_{N,L}(A_j) = \left\{ \sum_{j=1}^{b} \psi_j : \psi_j \in \mathcal{K}_{N,L}(A_j) \right\}. \tag{29}$$

[5]Formal declarations: `MPSTensor.block_boundary_intertwines_of_cyclicTranslate_sum_groundSpaceMap_eq`, `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_global_cut_bnt_c1_pgvwc07_of_dualFixedPoint`, and `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_of_global_cut_bnt_c1_pgvwc07_of_dualFixedPoint`. The earlier conditional declarations are `MPSTensor.blockDiagonal_boundary_component_chainGroundSpace_of_complementary_word_identities`, `MPSTensor.blockDiagonal_boundary_component_chainGroundSpace_of_complementary_word_identities_of_injective`, `MPSTensor.blockDiagonal_boundary_component_chainGroundSpace_of_pgvwc_comparison_of_injective`, `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_complementary_identities_bnt_c1`, and `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_and_iSupIndep_of_complementary_identities`.

The source-range conclusion now proved is

$$[b \geq 2 \ \wedge \ L_0 \geq 1 \ \wedge \ (\forall j,\ \mu_j \neq 0) \ \wedge \ L_0 < L \ \wedge \ L \geq 3(b-1)(L_0+1)+1 \ \wedge \ N \geq L + L_0] \Longrightarrow \mathcal{K}_{N,L}(\widehat{A}) = \sum_{j=1}^{b} \mathcal{K}_{N,L}(A_j). \tag{30}$$

The following implication is not available. The trace-form space $\mathcal{G}_N^A$ is not contained in the periodic-boundary ground space $\mathcal{K}_{N,L}(A)$ for an arbitrary boundary matrix X. If the cyclic window crosses the chosen virtual boundary, then a vector $\Gamma_N^A(X)$ has coefficients of the form

$$\operatorname{tr}(A^{\omega_{\text{tail}}} X A^{\omega_{\text{head}}} A^{\rho})\,, \tag{31}$$

where $\omega_{\text{tail}}\omega_{\text{head}}$ is the word seen in the cyclic window and ρ is the outside word. This is not, for general X, of the form

$$\operatorname{tr}(A^{\omega_{\text{tail}}} A^{\omega_{\text{head}}} Y_{\rho}) \tag{32}$$

with one matrix Y_ρ independent of the window word. The missing step is therefore a boundary-condition comparison, not the inclusion $\mathcal{G}_N^A \subseteq \mathcal{K}_{N,L}(A)$. The source theorem uses the multi-block hypothesis, the positive common injectivity length, the nonzero-weight hypothesis, and the length range of [PGVWC07, Theorem 12]:

$$b \geq 2, \qquad L_0 \geq 1, \qquad \forall j,\ \mu_j \neq 0, \qquad L_0 < L, \qquad L \geq 3(b-1)(L_0+1)+1, \qquad N \geq L + L_0. \tag{33}$$

For $b \geq 2$ and $L_0 \geq 1$, the source length bound implies $L \geq 3L_0+4 > L_0$; the inequality $L_0 < L$ is displayed separately because the one-block uniqueness input is used in that range. The one-block case is the injective parent-Hamiltonian theorem rather than the degenerate block-injective boundary theorem. The former proof required the more conservative range $L \geq (L_0+1)+3(b-1)(L_0+1)+1$. This extra (L_0+1) is no longer present. Once the simultaneous tuples span at $q = 3(b-1)(L_0+1)$, the unital normalization propagates the homogeneous tuple span from q to every larger length: a target tuple $(M_j)_j$ is obtained at length $q+1$ by representing $((A_a^j)^\dagger M_j)_j$ at length q, prefixing A_a^j, and summing over a. The periodic-boundary comparison reductions now separate into three forms: a decomposition into blockwise vectors with periodic boundary conditions, a block-diagonal boundary representation whose components already belong to the corresponding blockwise periodic-boundary spaces, and the boundary-crossing coordinate formulation displayed above.[6] The global change-of-cut theorem discharges the boundary-crossing comparison hypothesis at the

[6] Lean declarations: `MPSTensor.chainGroundSpace_toTensorFromBlocks_le_iSup_of_boundary_decomposition`, `MPSTensor.chainGroundSpace_toTensorFromBlocks_le_iSup_of_blockDiagonal_boundary_groundSpaceMap`, `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_and_iSupIndep_of_bnt_c1_boundary_decomposition`, `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_and_iSupIndep_of_bnt_c1_blockBoundary`, and `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_and_iSupIndep_of_complementary_identities`.

source length constant. For

$$X = \bigoplus_j X_j, \tag{34}$$

the boundary-matrix formula is

$$\begin{aligned} \Gamma_N^{\widehat{A}}(X)(i_1,\dots,i_N) &= \sum_j \mu_j^N \operatorname{tr}(A_{i_1}^j \cdots A_{i_N}^j X_j) \\ &= \sum_j \mu_j^N \Gamma_N^{A_j}(X_j)(i_1,\dots,i_N). \end{aligned} \tag{35}$$

For the next display, abbreviate

$$\Gamma_j(X) = \Gamma_N^{A_j}(X), \qquad V_j = V^{(N)}(A_j). \tag{36}$$

Thus the desired proof with periodic boundary conditions is the composition of the two implications

$$\begin{aligned} \psi \in \mathcal{K}_{N,L}(\widehat{A}) \Longrightarrow \exists (X_j)_{j=1}^b, \quad \psi = \sum_{j=1}^b \Gamma_j(X_j), \\ \forall\, 1 \le j \le b, \quad \Gamma_j(X_j) \in \mathcal{K}_{N,L}(A_j), \end{aligned} \tag{37}$$

and, for every $X_1,\dots,X_b$,

$$\begin{aligned} \forall\, 1 \le j \le b, \quad \Gamma_j(X_j) \in \mathcal{K}_{N,L}(A_j) \Longrightarrow \exists (c_j)_{j=1}^b, \\ \forall\, 1 \le j \le b, \quad \Gamma_j(X_j) = c_j V_j. \end{aligned} \tag{38}$$

The composition gives

$$\begin{aligned} &\psi = \sum_{j=1}^b \Gamma_j(X_j) = \sum_{j=1}^b c_j V_j, \\ &\psi \in \operatorname{span}\{V_j : j = 1,\dots,b\}. \end{aligned} \tag{39}$$

6 Conclusion

This is not a source-error claim. The formal development proves the block-diagonal periodic-boundary comparison without a short-tail span or an external boundary-comparison hypothesis in the source range

$$b \ge 2, \qquad L_0 \ge 1, \qquad L \ge 3(b-1)(L_0+1)+1, \qquad N \ge L + L_0. \tag{40}$$

For every block j one has

$$\psi \in \mathcal{K}_{N,L}(\widehat{A}) \quad \Longrightarrow \quad \Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{K}_{N,L}(A_j). \tag{41}$$

The earlier conditional reduction recorded by pull request #2724 and the boundary-trace implication remain useful intermediate statements. The latter records the following assertion. Assume the common word-span property for the middle-word length m. If, for every boundary-crossing interval, word β before the cut, outside word ρ, and middle word w of length m, the two boundary trace expressions from [PGVWC07, Theorem 12] satisfy

$$\sum_j \operatorname{tr}\Big(A^j_\beta C^j_\rho A^j_w\Big) = \sum_j \operatorname{tr}\Big((\mu_j^N X_j A^j_\beta) A^j_\rho A^j_w\Big)\,, \tag{42}$$

then

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{K}_{N,L}(A_j) \tag{43}$$

for every block j in the block-injective range.[7] There is also an explicit-boundary version: once $\psi = \Gamma_N^{\widehat{A}}(\bigoplus_j X_j)$ is fixed, the same equality of boundary trace expressions gives

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{K}_{N,L}(A_j) \tag{44}$$

for every block j, without using any span assumption at the short boundary-crossing tail length $N-i$.[8] At the fixed wrapped-tail length supplied by a single boundary-crossing cyclic window, the boundary trace expressions are now derived from the local constraint once the block-diagonal boundary representation is fixed: for $N < i+L$, outside word ρ of length $N-L$, and wrapped tail word α of length $N-i$, there are matrices $C^j_{i,\rho}$ such that

$$\sum_j \operatorname{tr}\Big(A^j_\beta C^j_{i,\rho} A^j_\alpha\Big) = \sum_j \operatorname{tr}\Big((\mu_j^N X_j A^j_\beta) A^j_\rho A^j_\alpha\Big)\,. \tag{45}$$

This fixed-tail statement is recorded as `MPSTensor.blockDiagonal_boundary_crossing_trace_decomposition_of_boundary`. It should not be read as the middle-word trace hypothesis at a common product-spanning length: the fixed cyclic window gives the wrapped tail length $N-i$, whereas the separating trace-decomposition implication is stated at a common product-span length m.

In the normalized BNT source range, this common middle-word span is no longer an external assumption. If

$$m = 3(b-1)(L_0+1), \qquad L \geq m+1, \tag{46}$$

then the normalized block-separation theorem gives

$$\operatorname{span}\{(A^1_w,\ldots,A^b_w) : |w| = m\} = \prod_j M_{D_j}(\mathbb{C}). \tag{47}$$

[7] Lean declaration: `MPSTensor.blockDiagonal_boundary_component_chainGroundSpace_of_trace_decomposition_of_injective`.

[8] Lean declaration: `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_trace_decomposition_of_boundary`.

Consequently the finite-range trace-decomposition implication only keeps the displayed trace equality as an external input once a block-diagonal boundary representation of the vector has been chosen.[9]

The C^j, D^j comparison after opening the boundary is closer to the proof of [PGVWC07, Theorem 12]. The words β and ρ are local word coordinates introduced after opening and blocking the cyclic boundary comparison; they reindex the source end-site comparison i_1, i_{m+1} by blocked words, and are not additional hypotheses or terminology from the paper. For a fixed boundary-crossing interval starting at site i, the local block-sum constraint and the word-tuple span of the wrapped tail word of length $N - i$ now give matrices $C^j_{i,\rho}$ satisfying

$$A^j_\beta C^j_{i,\rho} = \big((\mu_j^N X_j) A^j_\beta\big) A^j_\rho. \tag{48}$$

Once this compatibility identity is available, the normalized E^j calculation is no longer a gap: the adjoint-corrected formula

$$E^j_i = \sum_\rho C^j_{i,\rho} (A^j_\rho)^\dagger \tag{49}$$

and the identities

$$(\mu_j^N X_j) A^j_\beta = A^j_\beta E^j_i, \qquad A^j_\beta C^j_{i,\rho} = A^j_\beta E^j_i A^j_\rho \tag{50}$$

are formalized by `MPSTensor.pgvwc07_complementary_word_cde_identities_of_block_boundary_trace_decomposition`. The local boundary formula with $E_{j,i,\rho} = E^j_i A^j_\rho$ is formalized by `MPSTensor.pgvwc07_complementary_word_boundary_identities_formula_of_compatibility`; its existential consequence is `MPSTensor.pgvwc07_complementary_word_boundary_identities_of_compatibility`. This is formalized as `MPSTensor.blockDiagonal_boundary_crossing_pgvwc_comparison_of_sum_mem_iSup` and, once a block-diagonal boundary representation of ψ is fixed, into `MPSTensor.blockDiagonal_boundary_crossing_pgvwc_comparison_of_chainGroundSpace` and then into `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_crossing_pgvwc_comparison_of_boundary`. The current BNT specialization then obtains that boundary representation through `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_crossing_pgvwc_comparison_bnt_c1`. Thus the crossing comparison is not an independent trace-comparison assumption. For the earlier crossing-span route, the proof boundary was the crossing-tail word-span hypothesis and the choice of block-diagonal boundary representation for a vector $\psi \in \mathcal{K}_{N,L}(\widehat{A})$. In the normalized BNT specialization, that representation is supplied by the block-diagonal boundary theorem displayed below. This crossing-tail span hypothesis should not be confused with the large-length BNT product-span bound: if the

[9] Lean declarations: `MPSTensor.exists_blockDiagonal_boundary_chainGroundSpace_of_trace_decomposition_bnt_c1_span` and `MPSTensor.chainGroundSpace_toTensorFromBlocks_eq_iSup_and_iSupIndep_of_trace_decomposition_bnt_c1_span`.

boundary-crossing interval begins at $i = N - 1$, the required tail length is $N - i = 1$, so it is not supplied by the lower bound on L alone.

The block-diagonal boundary representation itself is not an instance of the boundary-crossing comparison. The Lean theorem `MPSTensor.exists_blockDiagonal_boundary_of_chainGroundSpace_toTensorFromBlocks_of_bnt_unital_c1_pgvwc07_of_dualFixedPoint` derives it from the source-shaped span-based one-step intersection identity `MPSTensor.pgvwc07_iSup_restriction_intersection_of_ge_of_bnt_directSum_unital_c1_pgvwc07_of_dualFixedPoint`; its block components lie in the open-boundary ground spaces $\mathcal{G}_N^{A_j}$, and its proof does not use the periodic-boundary comparison at boundary-crossing windows. The earlier conditional block-diagonal parent-space statements retain the periodic-boundary comparison as an explicit hypothesis, while the global change-of-cut theorem discharges it. Homogeneous tuple-span propagation removes the former extra $(L_0 + 1)$ from the normalized BNT product-span range.

The periodic-boundary comparison target of issue #3597 is therefore complete at the source range. The length-constant strengthening tracked by issue #4195 is also complete: the hypotheses are exactly

$$b \geq 2, \qquad L_0 \geq 1, \qquad L \geq 3(b-1)(L_0+1)+1, \qquad N \geq L + L_0. \tag{51}$$

The global-cut theorem gives

$$\mathcal{K}_{N,L}(\widehat{A}) = \sum_{j=1}^{b} \mathcal{K}_{N,L}(A_j). \tag{52}$$

The final degenerate parent-Hamiltonian ground-space equality follows by applying the one-block uniqueness theorem to each block.

References

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