The Vertical Direction in Proposition 4.13 Is the Index-Exchanged Tensor

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Verdict: unfaithful (resolved).

1 Source definition of the vertical direction

Proposition 4.13 (source label `Prop:IV.12`) of [CPGSV16] is a statement about the tensor viewed in the vertical direction. The source defines that direction explicitly: the tensor is considered “as generating MPV by concatenating it vertically, and thus by exchanging the notion of physical and virtual indices” (`Papers/1606.00608/MPDO-22-12-17-2.tex`, line 943). The vertically viewed tensor $\widetilde{M}$ therefore has its letters indexed by the horizontal bond pair (a, b), each letter being the operator on the physical space with matrix elements

$$(\widetilde{M}_{ab})_{ij} = M_{ab}^{ij}. \tag{1}$$

The conclusion of Proposition 4.13 (restated at lines 1863–1870, equation `UMU-appendix`) is that for a canonical-form tensor generating matrix product density operators there is an isometry U on the physical space with

$$U\widetilde{M}U^\dagger = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha, \tag{2}$$

where the μ_α are positive diagonal matrices and $\{M_\alpha\}_\alpha$ is a basis of normal tensors for $\widetilde{M}$. The multiplicity coefficients $m_\alpha = \mathrm{tr}(\mu_\alpha)$ of this decomposition are consumed by the renormalization fixed-point characterization (Theorem 4.14, line 1956).

2 The former diagonal-tensor surrogate

Until this realignment, the formal vertical canonical-form predicate stated the conclusion of Proposition 4.13 as a property of the *diagonal* MPS tensor $i \mapsto M^{ii}$: the diagonal tensor was required to generate the same matrix product vector family as a block-diagonal tensor obtained by repeating basis blocks with positive scalar weights. Both sides of that comparison live on the physical alphabet $\{0,\dots,d-1\}$ and have matrices acting on the horizontal bond space — the opposite of (1), where the letters range over bond pairs and the matrices act on the physical space.

The surrogate also cannot carry the multiplicity coefficients of (2). Take $d=2$, $D=1$, and $M^{ij}=\delta_{ij}/2$, a matrix product density operator in canonical form. Its vertically viewed tensor is the single physical-space letter $\widetilde{M}_{11}=\frac{1}{2}\mathbb{1}_2$, whose decomposition (2) has one block with $\mu=\mathrm{diag}(\frac{1}{2},\frac{1}{2})$: the multiplicity is 2 and $m=\mathrm{tr}(\mu)=1$. The diagonal tensor, by contrast, has the scalar family $V^{(N)}=2^{-N}$, and a two-weight flattening would require positive reals with

$$\omega_1^N+\omega_2^N=2^{-N}\qquad (N\geq 1);$$

the cases $N=1$ and $N=2$ force $\omega_1\omega_2=0$, a contradiction. The surrogate is therefore satisfiable only with multiplicity 1 for this tensor, so the coefficients m_α needed at line 1956 are lost. Consequently the diagonal-tensor statement is *not* a corollary of Proposition 4.13, and a proof of the surrogate would not formalize the proposition.

3 The realigned formal statement

The realignment (GitHub issue #3712) introduces the vertically viewed tensor (1) as a formal object and restates the vertical canonical-form predicate on it: there are a basis of normal tensors $\{M_\alpha\}_\alpha$ for $\widetilde{M}$, positive diagonal matrices $\mu_\alpha=\mathrm{diag}(\omega_{\alpha,0},\dots,\omega_{\alpha,r_\alpha-1})$ with $r_\alpha\geq 1$, and a coisometry U onto the retained nonzero sector space satisfying (2) letter by letter over the horizontal bond pairs. It satisfies $UU^\dagger=\mathbb{1}$ on that retained space; its adjoint is the isometric inclusion into the original physical space. The source's general canonical-form construction allows $\sum_k D_k<D$ and hence permits a zero orthogonal complement (lines 214–225). Thus $U^\dagger U$ is in general the support projection rather than the identity. This correction is documented in `cpgsv17_vertical_isometry_zero_sector.tex`. The proposition does not display the reverse identity separately. It follows from the preceding assertion that the vertical tensor is itself in canonical form: it reconstructs the original tensor from the displayed direct sum and requires the omitted complement to be a zero sector. Since each μ_α is diagonal, the summand $\mu_\alpha\otimes M_\alpha$ is the direct sum $\bigoplus_{k=0}^{r_\alpha-1}\omega_{\alpha,k}M_\alpha$, so the right-hand side of (2) is a block-diagonal tensor over the pairs (α,k) in which both the multiplicities r_α and the diagonal entries $\omega_{\alpha,k}$ remain explicit; the coefficients $m_\alpha=\mathrm{tr}(\mu_\alpha)=\sum_{k=0}^{r_\alpha-1}\omega_{\alpha,k}$ are recovered from them. The one-site actions used in the invariant-projection step are identified with left and right multiplication on the letters of $\widetilde{M}$.

The diagonal-tensor identities remain in the development as auxiliary results: the matrix product vectors of $i\mapsto M^{ii}$ are the diagonal entries $\langle\sigma|\rho^{(N)}(M)|\sigma\rangle$ of the density operator, which is the positivity input to the proof of Proposition 4.13. They no longer state its conclusion.

4 Relation to the neighboring notes

`cpgsv17_vertical_cf_grouping.tex` records that the positive diagonal coefficients in (2) arise from the vertical canonical decomposition, not from flattening horizontal weights; `cpgsv17_vertical_diagonal_restriction.tex` records that diagonal restriction of horizontal blocks does not preserve normality. Neither note documented the definitional mismatch addressed here: the former predicate placed the conclusion of Proposition 4.13 on the wrong tensor. The literal horizontal-to-vertical implication is now proved by `MPOTensor.verticalCF_of_cpsvCanonicalForm` in `TNLean/MPS/MPDO/CPSVVerticalCanonicalForm.lean`. It assumes the literal CPSV canonical form and MPDO positivity of Proposition 4.13. The earlier theorem `MPOTensor.verticalCF_of_horizontalCF` in `TNLean/MPS/MPDO/VerticalCanonicalForm.lean` remains as a separate result under the stronger normalized BNT-refined hypothesis `MPOTensor.IsHorizontalCF`.

Verdict. This is a corrected definitional mismatch. The former predicate placed the conclusion of Proposition 4.13 on the diagonal tensor $i \mapsto M^{ii}$ rather than on the vertically viewed tensor $\widetilde{M}$ defined at source line 943, and the $d = 2$, $D = 1$ example shows the diagonal surrogate cannot carry the multiplicity coefficients $m_\alpha = \mathrm{tr}(\mu_\alpha)$ of (2); a proof of the surrogate would therefore not have formalized the proposition. The realignment restores the source-faithful vertical tensor definition. The later correction in `cpgsv17_vertical_isometry_zero_sector.tex` expresses the source's one-sided condition as a coisometry onto the retained nonzero sectors and retains exact reconstruction. The horizontal-to-vertical implication is proved with this corrected definition under the literal CPSV canonical-form and MPDO hypotheses, and the stronger normalized BNT-refined theorem remains available separately. The phase-class grouping and positive diagonal coefficients used in the proof are detailed in `cpgsv17_vertical_cf_grouping.tex`. This completes Proposition 4.13 but does not prove the positive-fusion statement in Appendix C.4 or the renormalization fixed-point characterization in Theorem 4.14.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.