The Vertical Canonical Form Retains Only the Nonzero Physical Sectors

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Verdict: local-correction (resolved).

1 The dimension allowed by canonical form

The canonical-form construction of [CPGSV16] writes a tensor as a direct sum of nonzero irreducible blocks of dimensions D_k and states explicitly that

$$\sum_k D_k \le D, \tag{1}$$

because zero blocks may remain in the original space (source lines 214–225). Proposition 4.13 later applies this construction to the vertically viewed tensor and obtains

$$U\widetilde{M}U^\dagger = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha, \tag{2}$$

where the matrices μ_α are positive and diagonal and the tensors M_α form a basis of normal tensors (source lines 943–951 and 1863–1870).

Let d be the original physical dimension and let s be the dimension of the right-hand side of (2). When the generic canonical-form dimension bound (1) is applied vertically, it reads $s \le d$. In the displayed orientation, $U : \mathbb{C}^d \to \mathbb{C}^s$ maps the physical space onto the retained nonzero sector space. Consequently

$$UU^\dagger = I_s, \qquad U^\dagger U = P, \tag{3}$$

where P is the support projection of the nonzero sectors. Equality $U^\dagger U = I_d$ holds only when there is no zero complement. It is not part of the hypotheses or conclusion of Proposition 4.13. The source does not print a reverse identity separately. Its preceding assertion that $\widetilde{M}$ is in canonical form says that the discarded complement is a zero sector, which is recorded explicitly by the reconstruction identity

$$\widetilde{M} = U^\dagger \left(\bigoplus_\alpha \mu_\alpha \otimes M_\alpha \right) U. \tag{4}$$

Without (4), equation (2) alone is only a compression and does not exclude nonzero entries in the discarded or off-diagonal corners.

2 A zero-sector example

Take horizontal bond dimension one and physical dimension two, and define the local tensor by

$$M^{00} = 1, \qquad M^{01} = M^{10} = M^{11} = 0. \tag{5}$$

It is a one-block horizontal normal tensor. At every positive length it generates the positive operator

$$\rho^{(N)}(M) = |0 \cdots 0\rangle\langle 0 \cdots 0|. \tag{6}$$

The vertically viewed tensor has one letter,

$$\widetilde{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Its nonzero canonical part is the one-dimensional tensor with letter 1. Equation (2) holds with

$$U = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad UU^\dagger = 1, \qquad U^\dagger U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

A square unitary formulation can describe this tensor if it retains the explicit zero block [0]. The earlier formal predicate had no such block: its right-hand side contained only strictly positive weights multiplying normal tensors. Within that positive-block-only assembly, a unitary preserves the two-dimensional space while the second one-dimensional corner of (7) necessarily has weight zero. Treating the whole matrix as one block would not give a normal tensor. Thus the combination of a two-sided unitary and the positive-block-only assembly would make Proposition 4.13 false even in this elementary case.

3 Trace loss under transported sector maps

Appendix C.4 transports the physical refinement and coarse-graining maps to the retained one-site and two-site vertical coordinates (source lines 1955–1980). Let U_1 and U_2 be the corresponding coisometries and write

$$P_1 = U_1^\dagger U_1, \qquad P_2 = U_2^\dagger U_2. \tag{9}$$

The normalized diagonal embedding and the sectorwise partial trace preserve the sum of the sector traces. The physical maps also preserve trace. Thus, if Z_T is the positive matrix immediately before the final U_2 compression in the transported refinement map, then

$$\mathrm{Tr}_{\mathrm{sec}}(\widetilde{T}(X)) = \mathrm{tr}(U_2 Z_T U_2^\dagger) = \mathrm{tr}(P_2 Z_T) \le \mathrm{tr}(Z_T) = \mathrm{Tr}_{\mathrm{sec}}(X). \tag{10}$$

For a positive matrix, equality holds if and only if $P_2 Z_T = Z_T$. The same argument for the transported coarse-graining map gives

$$\mathrm{Tr}_{\mathrm{sec}}(\widetilde{S}(Y)) \le \mathrm{Tr}_{\mathrm{sec}}(Y), \qquad \mathrm{Tr}_{\mathrm{sec}}(\widetilde{S}(Y)) = \mathrm{Tr}_{\mathrm{sec}}(Y) \iff P_1 Z_S = Z_S, \tag{11}$$

where Z_S is the matrix before the final U_1 compression.

Consequently the transported maps need not be trace preserving on arbitrary elements of the full finite products merely because the physical maps are trace preserving. Such a conclusion would require the missing image-preservation statements

$$P_2 Z_T = Z_T, \qquad P_1 Z_S = Z_S. \tag{12}$$

The displayed identities of Appendix C.4 establish the required action on the weighted sector operators arising from the canonical forms; they do not, by themselves, establish (12) for every element of the finite products. For every virtual bond matrix X, the source-generated families now satisfy (12): the normalized embedding gives the contracted weighted direct sum, the two reconstruction identities identify the precompression matrix with $U_i^\dagger B_i(X) U_i$, and $U_i U_i^\dagger = 1$ gives the support equation. Cyclicity of trace then yields trace equality on these families. No global trace-preserving extension, complete-positivity assertion for the finite products, or inverse relation is asserted.

4 Ambient completion in the converse construction

Definition 4.1 requires trace-preserving completely positive maps on the full one-site and two-site physical matrix algebras (source lines 638–660). In the converse implication of Theorem 4.14, Appendix C.4 instead first defines the retained-sector compositions

$$T_0 = \widetilde{R}_2 \widetilde{T} R_1, \qquad S_0 = \widetilde{R}_1 \widetilde{S} R_2 \tag{13}$$

at source lines 2065–2085. Here R_1 and R_2 begin with compression by the one-site and two-site coisometries. Consequently

$$\operatorname{tr}(T_0(X)) = \operatorname{tr}(P_1 X), \qquad \operatorname{tr}(S_0(Y)) = \operatorname{tr}(P_2 Y), \tag{14}$$

where, after the canonical relabelling of the two-site indices, $P_i = U_i^\dagger U_i$. These maps preserve trace on matrices supported in the retained sectors, but not on arbitrary ambient matrices when $P_i \neq I$.

Put $Q_i = I - P_i$. When $d > 0$, choose density matrices τ_1 and τ_2 on the one-site and two-site physical spaces. The measure-and-prepare completion

$$T(X) = T_0(X) + \operatorname{tr}(Q_1 X Q_1)\tau_2, \qquad S(Y) = S_0(Y) + \operatorname{tr}(Q_2 Y Q_2)\tau_1 \tag{15}$$

is completely positive. Since Q_i is a projection and $\operatorname{tr}(\tau_i) = 1$, equations (14) give

$$\operatorname{tr}(T(X)) = \operatorname{tr}(X), \qquad \operatorname{tr}(S(Y)) = \operatorname{tr}(Y). \tag{16}$$

When $d = 0$, both ambient physical matrix algebras contain only zero; the uncompleted maps are already completely positive and trace preserving, and no density matrix is required. The source does not specify the states τ_i or any other trace-restoring modification of the literal composites on the discarded complements. This choice is immaterial for its conclusion: the reconstruction identities give

$$P_1 M_1(Z) P_1 = M_1(Z), \qquad P_2 M_2(Z) P_2 = M_2(Z) \tag{17}$$

for every virtual matrix Z. Hence the added terms vanish on the family to which the equations at source lines 2074 and 2085 are applied.

This completion is distinct from the transported maps discussed in the preceding section. In the forward implication at source lines 1955–1980, the full physical channels are hypotheses and are compressed to retained sector coordinates; their possible trace loss is governed by (12). In the converse implication at source lines 2065–2085, the sector maps are used to construct full physical channels, and equation (15) supplies the otherwise unspecified action on the orthogonal complements.

5 Formal correction

The vertical canonical-form predicate requires $UU^\dagger = I_s$ in the orientation of (2), together with (4). This is the source-faithful conclusion compatible with (1): the displayed compression is printed in the proposition, while reconstruction records its preceding canonical-form claim. The retained part is represented rectangularly, while the omitted part is required to be zero. No full-support assumption is added to that conclusion. Literal Proposition 4.13 now uses this corrected condition in `TNLean/MPS/MPDO/CPSVVerticalCanonicalForm.lean`.[1] The theorem under the normalized BNT-refined horizontal form remains available separately in `TNLean/MPS/MPDO/VerticalCanonicalForm.lean`.[2] The vertical phase-class grouping and normalization that precede both coisometry constructions are recorded in `cpgsv17_vertical_cf_grouping.tex`. The coisometry correction itself is tracked in GitHub issue #3869.

The source-generated support identities and their trace consequences are proved in `TNLean/MPS/MPDO/VerticalSectorImagePreservation.lean`.[3]

The general measure-and-prepare completion is recorded in `TNLean/Channel/SupportCompletion.lean`.[4] Under supplied unitary conjugacies between the paired sectors, the active compositions and the completed physical channels are recorded in `TNLean/MPS/MPDO/BNTAlgebraTensorClauseConditionalPhysicalMaps.lean`.[5] The existence of these unitary conjugacies is treated separately in `cpsv16_two_site_sector_unitary_gauge_gap.tex`. The

[1] Formal declaration: `MPOTensor.verticalCF_of_cpsvCanonicalForm`.

[2] Formal declaration: `MPOTensor.verticalCF_of_horizontalCF`.

[3] The image identities are `MPOTensor.precompressionVerticalSectorT_image_preservation` and `MPOTensor.precompressionVerticalSectorS_image_preservation`; their trace consequences are `MPOTensor.transportedVerticalSectorT_trace_of_source_generated` and `MPOTensor.transportedVerticalSectorS_trace_of_source_generated`.

[4] Formal declaration: `Matrix.supportCompletion_isKrausCPTP`.

[5] Formal declaration: `MPOTensor.BNTAlgebraTensorClause.TwoSiteExactSectorGauge.UnitarySectorConjugacy.isRFPViaTS`.

complementary terms are choices on the discarded subspaces, not additional assertions of Appendix C.4. Their action on the source-generated family is fixed by (17).

Verdict. The earlier two-sided-unitary condition was absent from the source and was false when the vertically viewed tensor had a zero sector. Replacing it by the coisometry identity $UU^\dagger = I$, while retaining exact reconstruction from the nonzero sectors, corrects the conclusion without adding a full-support hypothesis. Literal Proposition 4.13 and the separate stronger horizontal theorem now prove this corrected conclusion. This completion does not make the forward transported maps trace preserving on every element of the finite products in Appendix C.4; the support conditions (12) remain necessary there. It also does not determine a preferred action on the discarded physical complements. For the converse implication, the measure-and-prepare terms in (15) give global physical channels with the same action on the source-generated family. Within this note, that construction is conditional on the unitary sector conjugacies used in (13). The later mixed one-site/two-site prefix argument proves those conjugacies from the tensor-attached BNT clause, as documented in `cpsv16_two_site_sector_unitary_gauge_gap.tex`. Consequently the zero-sector completion is used unconditionally by `MPOTensor.BNTAlgebraTensorClause.isRFPViaTS`, closing Theorem 4.14(ii)$\Rightarrow$(i) under the source's standing canonical-form and MPDO hypotheses.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.