

Diagonal Restriction and the Vertical Canonical Form in Proposition 4.13

2026-07-11

Verdict: unfaithful (resolved).

1 The source argument

Proposition 4.13 of Cirac, Pérez-García, Schuch, and Verstraete concerns a tensor in horizontal canonical form that generates positive semidefinite density operators at every system size. Its conclusion is that the same tensor, now viewed in the vertical direction, admits a vertical canonical form [CPGSV16, Proposition 4.13]. The local source is `Papers/1606.00608/MPD0-22-12-17-2.tex`, lines 1863–1921.

The proof does not restrict each horizontal normal tensor from physical pairs (i, j) to diagonal pairs (i, i) and then assert that the restricted tensor is normal. Instead, it constructs a new canonical form in the vertical direction. Positivity and the first-site insertion lemma rule out a one-sided invariant projection at lines 1873–1887. Positivity then excludes nontrivial periodic sectors at lines 1889–1893. Only after these two steps does the proof invoke a canonical-form decomposition in the vertical direction. The remainder of the argument, lines 1895–1921, proves positivity of its multiplicity weights and reduces the corresponding similarities to unitary changes of basis.

The block-injectivity proposition at lines 340–345 has a different role. It separates the normal blocks belonging to one fixed canonical-form decomposition. It does not compare horizontal blocks with their diagonal restrictions, and it does not imply that injectivity survives deletion of all off-diagonal physical matrices.

2 A finite-dimensional obstruction

The stronger assertion is false under the stated horizontal canonical-form hypotheses.¹ On bond space $\mathbb{C}^2$, let

$$A^{(a,b)} = c_a E_{ab}, \quad c_0 = \frac{3}{5}, \quad c_1 = \frac{4}{5}, \quad a, b \in \{0, 1\}.$$

¹The correction is tracked in issue 2537.

The four matrices are nonzero scalar multiples of the matrix units, hence span $M_2(\mathbb{C})$. They also satisfy

$$\sum_{a,b} (A^{(a,b)})^* A^{(a,b)} = (c_0^2 + c_1^2)I = I.$$

Taking a single horizontal block makes the simultaneous block-separation condition identical to ordinary injectivity. Thus this example satisfies the injectivity, left-canonicity, nonzero-weight, and finite-length block-separation assumptions in the older per-copy `HorizontalCFData` structure.

Its diagonal restriction is

$$A^{\text{diag}}(0) = \frac{3}{5}E_{00}, \quad A^{\text{diag}}(1) = \frac{4}{5}E_{11}.$$

Every word in these two matrices is diagonal. Consequently, no blocking length can span $M_2(\mathbb{C})$, so the restricted tensor is not normal. This example is formalized in `TNLean/MPS/MPDO/BiCFDerivation/DiagonalRestrictionCounterexample.lean`.

The example is not presented as a counterexample to Proposition 4.13. The formal horizontal structure records properties of a block decomposition but does not assume that the associated tensor generates positive semidefinite operators at every length. That positivity hypothesis is essential in the source proof. More importantly, even with positivity available, the proposition constructs a new vertical basis of normal tensors rather than reusing the diagonal restrictions of the horizontal basis.

3 The corrected formal theorem

The literal theorem now proved assumes the CPSV canonical form of Proposition 4.13 together with the MPDO positivity hypothesis and concludes vertical canonical form.² A separate theorem retains the normalized BNT-refined horizontal hypothesis `MPOTensor.IsHorizontalCF` for arguments that already use that stronger form.³

The proof follows the three stages visible in the source:

1. use positivity and the first-site insertion lemma to rule out one-sided invariant projections in the vertical direction;
2. use positivity to rule out nontrivial vertical periods;
3. obtain the vertical canonical-form blocks independently, and prove the positivity and unitary normalization of their multiplicity coefficients.

²Formal declaration: `MPOTensor.verticalCF_of_cpsvCanonicalForm` in `TNLean/MPS/MPDO/CPSVVerticalCanonicalForm`.

³Formal declaration: `MPOTensor.verticalCF_of_horizontalCF` in `TNLean/MPS/MPDO/VerticalCanonicalForm`.

The target statement is now in place: the vertical canonical form is stated on the vertically viewed tensor, whose letters are indexed by the horizontal bond pair and whose matrices act on the physical space, in the rectangular orientation of the source. In that orientation the displayed map is a coisometry onto the retained sectors; an earlier statement on the diagonal tensor $i \mapsto M^{ii}$ has been removed. The definitional realignment is recorded in `cpgsv17_vertical_tensor_definition.tex`.

The already formalized equality of matrix product vectors after diagonal restriction remains useful as an auxiliary identity. It cannot supply the normal blocks of the vertical canonical form. In particular, Newton–Girard identities for scalar weights address only the multiplicity coefficients; they cannot restore matrix directions discarded by the restriction $(i, j) \mapsto (i, i)$. The vertical phase-class grouping and the resulting positive diagonal coefficients are described in `cpgsv17_vertical_cf_grouping.tex`; they do not arise from diagonal restriction.

The first operator identity at lines 1873–1887 is now formalized: if $H \geq 0$, $P = P^*$, and $PH = PHP$, then

$$HP = (PH)^* = (PHP)^* = PHP, \quad (I - P)HP = 0.$$

The doubled-index forms of the three first-site contractions are now explicit. For doubled indices $p = (a, b)$ and $q = (c, e)$ they are

$$(Y_L)_{p,q} = P_{a,c}\delta_{b,e}, \quad (Y_R)_{p,q} = \delta_{a,c}P_{e,b}, \quad (Y_C)_{p,q} = P_{a,c}P_{e,b}.$$

Their contractions with the horizontal matrix product vector are proved to be the entries of $PH^{(N)}$, $H^{(N)}P$, and $PH^{(N)}P$, respectively. Consequently, the corresponding two applications of the first-site insertion lemma are formalized without abstract placeholder matrices: the two entrywise operator identities force equality of the opposite-corner insertions on every horizontal canonical-form block.

The first stage is now complete. Writing $P_1 = P \otimes I^{\otimes(N-1)}$, the vertically viewed relation $PM = PMP$ gives

$$P_1 H^{(N)} = P_1 H^{(N)} P_1 = H^{(N)} P_1,$$

and the representative-grouped Lemma L transfers this equality through every repeated weighted sector and the direct sum of the copy gauges. Thus the original tensor satisfies the literal equality $MP = PMP$, equivalently $(I - P)MP = 0$. The formal theorem assumes only $P = P^*$; the source specializes to an orthogonal projector.

For the periodic-sector stage, the final positive-operator implication is also formalized:

$$Q[H^{(N)}]^p = [H^{(N)}]^p Q \implies QH^{(N)} = H^{(N)}Q.$$

A nontrivial vertical period now yields the displaced cyclic projector and its word invariance. The horizontal canonical-form version of Lemma L supplies a chain length at which the displaced projector fails to commute; positivity at

that length gives the contradiction. Thus nontrivial vertical periods are excluded without the stronger all-length inequality printed in the source.

Invariant-projector closure is also established for the vertically viewed tensor, and it gives a complete orthogonal family of irreducible reducing corners with exact physical isometries (`TNLean/MPS/MPDO/VerticalReduction.lean`). After discarding precisely the zero corners, the remaining corners are spectrally normalized to normal tensors while retaining those isometries, both intertwining identities, exact compression, and literal reconstruction of every vertical letter (`TNLean/MPS/MPDO/VerticalSpectral.lean`).

The positive-weight part of the third stage is complete. The physical range projector of every grouped sector has the required loop identity, with $(E_\alpha)_{ab} = \text{tr}(M_\alpha^{(a,b)})$, and horizontal separation supplies a nonzero compression. Its trace is the sector weight $\mu_{\alpha,k}$ times a sector trace d_α , so MPDO positivity forces $d_\alpha > 0$ and $\mu_{\alpha,k} > 0$ under the normalization $\mu_{\alpha,1} > 0$ (`TNLean/MPS/MPDO/VerticalBNT.lean` and `TNLean/MPS/MPDO/SectorTrace.lean`). The pairwise Gram-conjugation identity of the second displayed diagram at lines 1914–1919 is proved in `TNLean/MPS/MPDO/FigureEightPairwise.lean`, and its dependent transport to the actual grouped sectors is proved in `TNLean/MPS/MPDO/GroupedFigure8.lean`. Normal-tensor rigidity then shows that the Gram matrix of each sector gauge $X_{\alpha,k}$ is a positive real multiple of the reference Gram matrix,

$$X_{\alpha,k}^\dagger X_{\alpha,k} = \omega_{\alpha,k} X_{\alpha,1}^\dagger X_{\alpha,1}.$$

The relative gauge $X_{\alpha,k} X_{\alpha,1}^{-1}$ is therefore proportional to a unitary; the grouping chooses $X_{\alpha,1} = \mathbf{1}$. The positive scalar identity and the unitary normalization of the actual grouped gauge are proved in `TNLean/MPS/MPDO/GroupedGramNormalization.lean`, with the source passage at lines 1903–1921 recorded in both theorem docstrings. Projector closure and the absence of periodic vectors already give an abstract positive-length normal-block decomposition (`TNLean/MPS/CanonicalForm/ProjectorClosureSpectral.lean`). The literal isometry-preserving refinement is now supplied by `TNLean/MPS/MPDO/VerticalSpectral.lean`. The normal blocks are now grouped into pairwise gauge-phase-distinct BNT representatives in `TNLean/MPS/MPDO/VerticalBNT.lean`. Every copy is expressed by a positive coefficient, a unit-modulus phase, and an invertible bond-space gauge, while the physical isometries and the literal vertical reconstruction are retained. The unitary gauge normalizations are transported to the representative bond spaces and composed with the physical sector isometries in `TNLean/MPS/MPDO/NormalizedGroupedSectors.lean`. The algebraic basis-of-normal-tensors property is established in `TNLean/MPS/MPDO/VerticalBNTConstruction.lean`, using exact reconstruction at positive lengths and a retained positive-dimensional representative at length zero. The resulting orthogonal sector maps are assembled into the final coisometry in `TNLean/MPS/MPDO/VerticalCanonicalForm.lean`. Together with the MPDO positivity hypothesis, the normalized BNT-refined horizontal form therefore gives the displayed vertical identity and exact reconstruction. The source prints the former; the latter is implicit in its assertion that the vertical tensor is itself

in canonical form and is not displayed as a separate formula.

The corresponding constructions under literal CPSV canonical form are summarized in `cpgsv17_vertical_cf_grouping.tex`. They give `MPOTensor.verticalCF_of_cpsvCanonicalForm` without using diagonal restriction.

Verdict. Literal Proposition 4.13 is complete, and the stronger theorem under `MPOTensor.IsHorizontalCF` remains available separately. Neither proof uses diagonal restriction to obtain the normal vertical representatives; that route remains false. Proposition 4.13 does not settle the positive-fusion problem of Appendix C.4 or the renormalization fixed-point characterization of Theorem 4.14.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.