Vertical Canonical-Form Grouping in Proposition 4.13

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Verdict: clarification (resolved).

1 Source statement and proof order

Proposition 4.13 (source label `Prop:IV.12`) of [CPGSV16] states that a canonical-form tensor generating matrix product density operators is also in canonical form in the vertical direction. With the rectangular map called an isometry in that proposition, the tensor has the form

$$U\widetilde{M}U^\dagger = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha, \tag{1}$$

where each μ_α is positive and diagonal and the tensors M_α form a basis of normal tensors. Here $\widetilde{M}$ is the tensor viewed in the vertical direction, with letters indexed by the horizontal bond pair and matrices acting on the physical space. In the rectangular orientation of this display, U maps the physical space onto the retained sector space and therefore satisfies

$$UU^\dagger = I. \tag{2}$$

The exact vertical canonical-form statement also contains the reverse identity

$$\widetilde{M} = U^\dagger \left(\bigoplus_\alpha \mu_\alpha \otimes M_\alpha \right) U. \tag{3}$$

The source calls U an isometry and prints only (1); the orientation (2) and reverse reconstruction (3) are the source-faithful rectangular form recorded in `cpgsv17_vertical_tensor_definition.tex`.

The proof has a definite order. It first uses positivity of the generated operators and Lemma L to exclude one-sided transitions and nontrivial periodic sectors. Only after vertical canonical form has been obtained does it write the vertical decomposition as

$$\bigoplus_{\alpha,k} \mu_{\alpha,k} X_{\alpha,k} M_\alpha X_{\alpha,k}^{-1}. \tag{4}$$

The remainder of the proof shows that $\mu_{\alpha,k} > 0$ and that each gauge $X_{\alpha,k}$ is proportional to a unitary. The copy index k in (4) is then the diagonal index of μ_α in (1).

Thus the positive diagonal coefficients in the conclusion are obtained from the *vertical* canonical decomposition. They are not obtained by flattening the scalar weights of a previously chosen horizontal decomposition while retaining the diagonal restrictions of its blocks.

The relevant local source passage is `Papers/1606.00608/MPDO-22-12-17-2.tex:1861--1922`.

2 Exact finite-sum statement

Suppose an indexed family of blocks $(B_k)_{k\in I}$ has already been grouped into copies of representative blocks $(A_\alpha)_{\alpha\in J}$. Write r_α for the number of copies, the size of the diagonal matrix μ_α; the trace coefficient $m_\alpha = \operatorname{tr}\mu_\alpha$ is a distinct quantity (see `cpgsv17_vertical_tensor_definition.tex`). Let

$$e : I \simeq \{(\alpha, j) : \alpha \in J, 0 \leq j < r_\alpha\} \tag{5}$$

be the grouping bijection. If

$$\begin{aligned} \dim B_k &= \dim A_{e(k)_1}, & (6)\\ \nu_k &= \omega_{e(k)}, & (7)\\ V^{(N)}(B_k) &= V^{(N)}(A_{e(k)_1}) \qquad (N \geq 1), & (8) \end{aligned}$$

then finite-sum reindexing gives

$$V^{(N)}\left(\bigoplus_{k\in I} \nu_k B_k\right) = V^{(N)}\left(\bigoplus_{\alpha\in J}\bigoplus_{j=0}^{r_\alpha-1} \omega_{\alpha,j} A_\alpha\right) \tag{9}$$

for every N, including $N = 0$. The equality at $N = 0$ follows from the pointwise dimension equalities in (8). Summing these equalities along the bijection (5) gives the total bond-dimension equality.

This statement is formalized by `MPOTensor.sameMPV2_toTensorFromBlocks_verticalAssembledTensor_of_equiv`. In this displayed name, the plain digit 2 denotes the subscripted character used in the source declaration.

3 Boundary of the present result

The normalized BNT-refined horizontal form `MPOTensor.IsHorizontalCF` is strictly stronger than the literal CPSV canonical form of Proposition 4.13: its retained sectors fill the horizontal bond space, every repeated-copy weight is nonzero, and its normal blocks are already grouped over BNT representatives. The stronger horizontal theorem is preserved, but the literal implication is now proved independently. Under literal CPSV canonical form, periodic exclusion, the normal vertical block decomposition, finite-chain separation, vertical phase-class grouping with positive effective coefficients, actual-grouped Figure 8, unitary gauge normalization, normalized physical sector maps, and the final rectangular coisometry are all complete.

An earlier auxiliary route compared one scalar weight μ_α with several weights $\omega_{\alpha,j}$ under the additional identities

$$\mu_\alpha^N = \sum_j \omega_{\alpha,j}^N, \qquad N \geq 1. \tag{10}$$

Although this gives a correct conditional positive-length comparison, (10) is not the flattening argument in Proposition 4.13. The corresponding terminal lemmas have therefore been removed from the development. In the source, the several weights are already the weights of the several vertical sectors in (4); no single horizontal coefficient is replaced by their power sum.

Diagonal restriction also does not preserve injectivity of a horizontal block. For $d = D \geq 2$, let the doubled-index matrices be

$$B^{(i,j)} = d^{-1/2} E_{ij}.$$

They span $M_D(\mathbb{C})$ and satisfy $\sum_{i,j} (B^{(i,j)})^\dagger B^{(i,j)} = I$. Their diagonal restriction is $B^{(i,i)} = d^{-1/2} E_{ii}$, whose span is only the diagonal subalgebra. Thus the BNT grouping in the conclusion must be obtained from the new vertical canonical decomposition, not by declaring the diagonal restrictions of the horizontal blocks to be normal blocks.

The exact reindexing step (9) is therefore complete once the bijection and the pointwise block identifications are known. Under the normalized BNT-refined hypothesis, the one-sided invariant-matrix calculation from source lines 1874–1887 gives invariant-projector closure for the vertically viewed tensor (`MPOTensor.IsHorizontalCF.hasInvariantProjectorClosure_verticalTensor`). For literal CPSV canonical form, the same conclusion is now proved by `MPSTensor.IsCPSVCanonicalForm.hasInvariantProjectorClosure_verticalTensor`. The general decomposition under this closure condition produces a complete orthogonal family of irreducible reducing corners, with isometries, both intertwining identities, exact compression, and equality of matrix product vectors at every length (`MPOTensor.IsHorizontalCF.exists_irreducible_verticalBlockDecomp_with_isometry`). Its range projections are complete, pairwise orthogonal, and commute with all vertical letters (`MPOTensor.IsHorizontalCF.exists_complete_verticalReducingProjectors`). The nonzero corners can moreover be spectrally normalized without losing their physical embeddings: the resulting blocks are normal, their weights are positive, both intertwining identities and exact compression remain, and the retained sectors reconstruct every vertical letter literally (`MPOTensor.IsHorizontalCF.exists_normal_verticalBlockDecomp_with_isometry` in `TNLean/MPS/MPDO/VerticalSpectral.lean`). The complement of their range projections consists precisely of the omitted zero corners, so no false completeness assertion is made after this removal.

The same normal-corner conclusion is now complete under literal CPSV canonical form. The theorem `MPSTensor.IsCPSVCanonicalForm.exists_normal_verticalBlockDecomp_with_isometry` in `TNLean/MPS/MPDO/CPSVPeriodicExclusion.lean` combines literal invariant-projector closure with literal periodic exclusion. It produces positive-dimensional normal blocks B_k, positive weights ν_k, and pairwise orthogonal isometries V_k satisfying

$$\widetilde{M}_v V_k = V_k(\nu_k B_k^v), \tag{11}$$

$$V_k^\dagger \widetilde{M}_v = (\nu_k B_k^v) V_k^\dagger, \tag{12}$$

$$\nu_k B_k^v = V_k^\dagger \widetilde{M}_v V_k, \tag{13}$$

$$\widetilde{M}_v = \sum_k V_k(\nu_k B_k^v) V_k^\dagger. \tag{14}$$

The omitted orthogonal complement is an all-zero sector. This theorem alone does not partition the B_k into gauge-equivalent representative families; the subsequent grouping, normalization, and coisometry results complete Proposition 4.13.

The literal sector-compression separation is also complete in `MPSTensor.IsCPSVCanonicalForm.exists_sectorCompression_ne_zero_of_corner` from `TNLean/MPS/MPDO/CPSVVerticalBNT.lean`. For every matrix P,

$$\left(\exists v,\ P\widetilde{M}_v P \neq 0\right) \Longrightarrow \left(\exists N \geq 0,\ P_1 H^{(N+1)} P_1 \neq 0\right). \tag{15}$$

If all the compressions vanished, the two-sided first-site insertion by P would agree with zero on every positive-length matrix product vector. Original-space Lemma L would then make the inserted tensor, and hence every corner $P\widetilde{M}_v P$, vanish. This implication uses neither positivity nor vertical phase-class grouping. It completes only the separation clause at source line 1898; the later trace argument cannot be applied to the source sectors until those sectors have actually been grouped.

The normalized corners are now grouped by matrix-product-vector phase class under the literal CPSV hypothesis in `MPSTensor.IsCPSVCanonicalForm.exists_verticalBNTGrouping_with_isometry` (`TNLean/MPS/MPDO/CPSVVerticalBNT.lean`). The normalized BNT-refined horizontal wrapper remains available as `MPOTensor.IsHorizontalCF.exists_verticalBNTGrouping_with_isometry`. One normal representative is chosen from each class. Every copy has the same bond dimension as its representative and is a unit-modulus scalar times an invertible conjugate of that representative; the scalar of the chosen representative is one. The representatives are pairwise gauge-phase distinct and form a basis of normal tensors for the weighted block tensor. The physical isometries are not changed: both intertwining identities, exact compression, and the literal reconstruction of each vertical letter survive the regrouping. For every grouped sector, its physical range projector satisfies the representative-loop identity and gives a nonzero compression at some finite length. The MPDO trace argument then proves that every grouped coefficient is positive. This is the phase-class decomposition and positivity argument of source lines 1898–1902, obtained by combining the BNT characterization at lines 1135–1148 with the preceding isometry-preserving vertical sector theorem. The subsequent gauge normalization and coisometry construction turn this grouped decomposition into the positive-diagonal form asserted at lines 1895–1921. Equality of the bond dimensions within each phase class is obtained from non-decay of the positive-length rectangular overlap, as in Lemma `equalMPS`; the empty-word coefficient is not used in this comparison.

The periodic-sector exclusion is also complete under both hypotheses. Under literal CPSV canonical form, `MPSTensor.IsCPSVCanonicalForm.exists_not_commute_of_displaced` proves that every displaced idempotent fails to commute with the generated density operator at some chain length. The theorem `MPOTensor.hasNoPeriodicVectors_verticalTensor_of_cpsvCanonicalForm` then combines the displaced cyclic projector of a nontrivial period, its full-period word invariance, the stacked-layer power identity, and positivity to exclude periodic vectors of $\widetilde{M}$. The source prints noncommutation at every length, whereas this argument requires and proves only the existence of one noncommuting length. The same conclusion under the normalized BNT-refined horizontal hypothesis is `MPOTensor.hasNoPeriodicVectors_verticalTensor_of_horizontalCF`. The quantifier correction is recorded in `docs/paper-gaps/cpgsv17_periodic_sector_projector.tex`.

Under both the literal CPSV and normalized BNT-refined horizontal hypotheses, the positive-weight step of source lines 1898–1902 is complete. For each grouped sector, write $\nu_{\alpha,k}$ for the raw sector weight, $\zeta_{\alpha,k}$ for its phase relative to the chosen representative, and $c_{\alpha,k} = \nu_{\alpha,k}\zeta_{\alpha,k}$ for the grouped coefficient. Insertion of the physical range projector $P_{\alpha,k}$ into the single-site vertical loop selects the contribution $c_{\alpha,k}E_\alpha$, where $(E_\alpha)_{ab} = \mathrm{tr}(M_\alpha^{(a,b)})$. This formula is independent of the copy index because cyclicity of trace removes the internal similarity. Sector-compression separation supplies a nonzero compression; MPDO positivity then makes its trace positive. Since the trace is $c_{\alpha,k}d_\alpha$, one obtains $d_\alpha > 0$ and $c_{\alpha,k} > 0$ under the normalization $c_{\alpha,1} = \nu_{\alpha,1} > 0$. Thus the coefficient denoted by $\mu_{\alpha,k}$ in the source is $\mu_{\alpha,k} = c_{\alpha,k}$ (`TNLean/MPS/MPDO/VerticalBNT.lean`, `TNLean/MPS/MPDO/CPSVVerticalBNT.lean`, and `TNLean/MPS/MPDO/SectorTrace.lean`). For the unitary gauges, the pairwise Gram-conjugation identity of the second displayed diagram at source lines 1914–1919 is proved in `TNLean/MPS/MPDO/FigureEightPairwise.lean`. The dependent identification with the actual grouped sector and its distinguished copy is proved under the normalized BNT-refined hypothesis in `TNLean/MPS/MPDO/GroupedFigure8.lean` and under literal CPSV canonical form in `TNLean/MPS/MPDO/CPSVGroupedFigureEight.lean`. Neither argument identifies either corner separately with the raw reflected adjoint. The scalar commutant of the normal tensor M_α (`MPSTensor.IsNormal.eq_smul_one_of_commute`) makes the relative Gram ratio scalar. Positivity of the two Gram matrices then gives

$$X_{\alpha,k}^\dagger X_{\alpha,k} = \omega_{\alpha,k} X_{\alpha,1}^\dagger X_{\alpha,1}, \qquad \omega_{\alpha,k} > 0.$$

The relative gauge $X_{\alpha,k}X_{\alpha,1}^{-1}$ becomes unitary after division by $\sqrt{\omega_{\alpha,k}}$. The grouping theorem now exposes its choice $X_{\alpha,1} = \mathbb{1}$, giving the normalization printed by the source under the normalized BNT-refined hypothesis by `MPOTensor.IsMPDO.grouped_sector_gram_eq_pos_smul_one` and `MPOTensor.IsMPDO.grouped_sector_exists_unitary_normalization` in `TNLean/MPS/MPDO/GroupedGramNormalization.lean`. The literal counterparts `MPSTensor.IsCPSVCanonicalForm.grouped_sector_gram_eq_pos_smul_one` and `MPSTensor.IsCPSVCanonicalForm.grouped_sector_exists_unitary_normalization` are proved in `TNLean/MPS/MPDO/CPSVGroupedGramNormalization.lean`.

The reflected marked-chain, two-sector separation, and dependent-dimension identification of source lines 1909–1919 are therefore complete, as is the positive scalar Gram identity and unitary normalization of every grouped gauge. For the normalized BNT-refined hypothesis, the normalized gauges are composed with the physical sector isometries in `TNLean/MPS/MPDO/NormalizedGroupedSectors.lean`. The literal counterpart `MPSTensor.IsCPSVCanonicalForm.exists_normalized_grouped_sector_maps` is proved in `TNLean/MPS/MPDO/CPSVNormalizedGroupedSectors.lean`. In both cases the resulting maps $W_{\alpha,q}$ have mutually orthogonal isometric ranges, satisfy the representative intertwining identities, and reconstruct every vertical letter exactly.

The representatives are shown to form an algebraic basis of normal tensors for the original vertical tensor in `TNLean/MPS/MPDO/VerticalBNTConstruction.lean`. At positive chain length this follows from the exact reconstruction and cyclicity of trace. At length zero, one retained representative of positive bond dimension supplies the dimension identity required by the definition of a basis of normal tensors.

Finally, `MPOTensor.verticalCF_of_horizontalCF` in `TNLean/MPS/MPDO/VerticalCanonicalForm.lean` and the independent literal theorem `MPOTensor.verticalCF_of_cpsvCanonicalForm` in `TNLean/MPS/MPDO/CPSVVerticalCanonicalForm.lean` apply the shared orthogonal-sector coisometry theorem. They identify the dependent sector pairs (α, q) with the standard finite direct-sum coordinates and prove (2), (1), and (3). The reverse reconstruction is not printed separately in the proposition. The original formulation of GitHub issue #2537, interpreted as a direct conversion of the horizontal scalar weights and diagonal restrictions, remains a false route rather than a consequence of the proposition.

Verdict. Proposition 4.13 is complete under its literal CPSV canonical-form hypothesis. The theorem `MPOTensor.verticalCF_of_cpsvCanonicalForm` produces a basis of normal vertical representatives, positive diagonal multiplicities, and a rectangular coisometry U satisfying $UU^\dagger = I$ together with both exact identities (1) and (3). The proof retains every inactive listed coordinate with weight zero until the vertical decomposition and does not use the unsupported diagonal-restriction statement from GitHub issue #2537. The stronger theorem for `MPOTensor.IsHorizontalCF` remains available separately. These Proposition 4.13 statements are distinct from the Appendix C.4 positive-fusion question tracked by GitHub issue #4648 and from Theorem 4.14.

The preparatory dependency tracked by GitHub issue #4956 is now complete. The structure `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement` and its constructor `MPSTensor.CPSVCanonicalFormData.exists_activeBNTRefinement` retain an explicit equivalence between phase-class copies and the original nonzero-weight listed blocks. Every active copy keeps its original coefficient and bond dimension, together with a unit phase, an invertible gauge, and the exact letterwise gauge-phase relation to its representative. The representatives form a CPSV basis of normal tensors.

The final API adjoins the inactive complement `MPSTensor.CPSVCanonicalFormData.Inactive` to the active class-copy pairs in `MPSTensor.CPSVCanonicalFormData.GroupedIndex`. The equivalence `MPSTensor.CPSVCanonicalFormData.groupedIndexEquiv` identifies this grouped order with the original listed blocks. The flattened dependent coordinates are identified with the original retained coordinates by `MPSTensor.CPSVCanonicalFormData.groupedCoordinateEquiv`; the theorem `MPSTensor.CPSVCanonicalFormData.groupedTotalDim_eq` records the resulting total-dimension equality. The underlying reusable statements are `MPSTensor.blockIndexCoordinateEquiv` and `MPSTensor.toTensorFromBlocks_eq_reindex_blockDiagonal_equiv`.

For grouped index x, let $\widehat{\mu}_x$ and $\widehat{B}_x$ denote the weight and block supplied by `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedWeight` and `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedBlocks`. An active $x = (\alpha, k)$ carries the original nonzero copy weight and the corresponding phase multiple of C_α. An inactive x carries weight zero and the original listed block. If Φ is the flattened coordinate equivalence, the grouped tensor is

$$T^i_{\mathrm{grp}} = \mathrm{reindex}_\Phi \left(\bigoplus_x \widehat{\mu}_x \widehat{B}^i_x \right). \tag{16}$$

This is `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedTensor`. It retains every zero-weight listed coordinate; no active-coordinate deletion or selecting coisometry is asserted.

If G is the block-diagonal gauge on the original listed blocks, then `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedRegroupLetterwise` proves, for every letter i,

$$\bigoplus_k \mu_k A^i_k = G T^i_{\mathrm{grp}} G^{-1}. \tag{17}$$

The intermediate identity `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.regroupedTensor_eq_groupedTensor` identifies the earlier in-place representative tensor with the explicit grouped tensor. Finally, `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.reconstructGrouped` uses the original ambient coisometry U, with $UU^\dagger = I$, to give

$$A^i = U^\dagger G T^i_{\mathrm{grp}} G^{-1} U. \tag{18}$$

Thus the class-copy-plus-inactive coordinates, the dependent coordinate permutation, the letterwise block-gauge conjugation, and the ambient reconstruction are all explicit and exact.

The active representative sector decomposition and its positive-length matrix-product-vector identity are explicit in `TNLean/MPS/CanonicalForm/ActiveBNTRefinement.lean`. Its effective copy weight is

$$\lambda_{\alpha,q} = \nu_{k(\alpha,q)}\zeta_{k(\alpha,q)}, \tag{19}$$

which is nonzero for every stored copy. The sector decomposition contains only the active class-copy pairs, while the full grouped tensor (16) continues to retain every inactive zero-weight listed coordinate. The definition `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.representativeSectorDecomposition` defines the representatives, copy numbers, and weights, and `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedTensor_sameMPVPos_representativeSectorDecomposition` identifies the matrix product vectors of the full grouped tensor and the active sector tensor at every positive length. It does not assert the empty-word identity, which would forget the inactive bond coordinates.

The simultaneous representative word-tuple span and the marked-chain consequences remain in `TNLean/MPS/MPDO/CPSVRepresentativeGroupedLemmaL.lean`. There, `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.eventuallyRepresentativeWordTupleSpan` proves simultaneous representative word-tuple span at every sufficiently large length.

The full-coordinate marked tensor retains the same listed coordinate space. An active copy carries the marked block $\lambda_{\alpha,q}C_\alpha$, and an inactive copy carries zero. The exact trace identity `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.trace_groupedMarkedTensor_eq_representative_markedTensor` gives

$$\operatorname{tr}(\widehat{C}^s_{\mathrm{grp}}T^w_{\mathrm{grp}}) = \sum_\alpha \left(\sum_q \lambda_{\alpha,q}^{|w|+1}\right)\operatorname{tr}(C^s_\alpha C^w_\alpha). \tag{20}$$

The exponent $|w|+1$ includes the marked site. Eventual representative span then proves the retained-coordinate marked and first-site forms of Lemma L: `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.groupedMarkedTensor_basis_eq_of_trace_agree` and `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.insertedTensor_representative_eq_of_firstSiteActionAgree`. These conclusions require no additional normalization hypothesis.

The original-space marked step is now complete in `TNLean/MPS/MPDO/CPSVOriginalSpaceLemmaL.lean`. If $A^i = V^\dagger B^i V$ and $VV^\dagger = I$, then a marked matrix $C = V^\dagger E V$ satisfies

$$\operatorname{tr}(CA^w) = \operatorname{tr}(EB^w) \tag{21}$$

for every tail word w, including the empty word. The marked matrix makes the closed chain nonempty, so (21) does not assert an unmarked length-zero matrix-product-vector equality. This is `MPSTensor.trace_marked_mul_evalWord_of_coisometry_reconstruction`; the linear-marked specialization is `MPSTensor.trace_linearMarkedTensor_mul_evalWord_of_coisometry_reconstruction`.

Linear marking of T_{grp} is exactly the full grouped marked tensor: `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.linearMarkedTensor_groupedTensor_eq_groupedMarkedTensor`. Active copies carry $\lambda_{\alpha,q}$ times the representative mark, while inactive marked blocks vanish because their raw weights are zero. Combining this identity, (21), the listed gauge, and the retained representative theorem proves literal-CF marked Lemma L on the original bond space: `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.linearMarkedTensor_eq_of_trace_agree` and `MPSTensor.IsCPSVCanonicalForm.linearMarkedTensor_eq_of_trace_agree`. No equality is asserted for the insertion into an individual inactive block.

For physical first-site actions, the preceding result gives `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.insertedTensor_eq_of_firstSiteActionAgree` and its predicate-level form `MPSTensor.IsCPSVCanonicalForm.insertedTensor_eq_of_firstSiteActionAgree`. Positivity supplies the required first-site agreement from a one-sided invariant orthogonal projection through `MPOTensor.firstSiteActionAgree_braRight_ketLeftBraRight_of_invariant`. Consequently, `MPSTensor.IsCPSVCanonicalForm.hasInvariantProjectorClosure_verticalTensor` proves invariant-projector closure of the vertical tensor under literal CPSV canonical form and `MPOTensor.IsMPDO`.

The literal pairwise Figure 8 identity is now proved in `TNLean/MPS/MPDO/CPSVFigureEight.lean` as `MPSTensor.IsCPSVCanonicalForm.gramDressing_eq_of_two_grouped_corners`. For a normal common representative, `MPSTensor.IsCPSVCanonicalForm.gram_eq_pos_smul_gram_of_two_grouped_corners` gives a real scalar $\omega > 0$ with $X^\dagger X = \omega Y^\dagger Y$. These results cover source lines 1909–1921 for any two corners satisfying the source's positive similarity equations. They preserve all inactive listed coordinates through the literal marked Lemma L argument.

Literal Proposition 4.13 is now complete. The normalized physical sector maps are furnished by `MPSTensor.IsCPSVCanonicalForm.exists_normalized_grouped_sector_maps`, and `MPOTensor.verticalCF_of_cpsvCanonicalForm` constructs the final vertical canonical form. Its rectangular map satisfies $UU^\dagger = I$, not in general $U^\dagger U = I$, and obeys both exact forward and reverse reconstruction identities. This result is independent of the Appendix C.4 positive-fusion question tracked by GitHub issue #4648.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.