

The Periodic-Sector Argument in Proposition 4.13

2026-07-13

Verdict: local-correction (resolved).

1 The printed argument

The proof of Proposition 4.13 of Cirac, Pérez-García, Schuch, and Verstraete [CPGSV16, Proposition 4.13] excludes nontrivial periodic sectors of the vertically viewed tensor. The local source is `Papers/1606.00608/MPD0-22-12-17-2.tex`, lines 1888–1893. It states that a nontrivial period supplies an orthogonal projector Q such that, for every N ,

$$Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1, \quad Q_1 [H^{(N+1)}]^p = [H^{(N+1)}]^p Q_1,$$

where $Q_1 = Q \otimes \mathbb{1}^{\otimes N}$. Since $H^{(N+1)}$ is positive semidefinite, the second equality implies $Q_1 H^{(N+1)} = H^{(N+1)} Q_1$, contradicting the first.

2 The quantifier correction

The cyclic decomposition of the peripheral spectrum gives a projector Q which is invariant under every word of one full period and is displaced by one vertical layer,

$$Q\widetilde{M} \neq Q\widetilde{M}Q.$$

The representative-grouped form of Lemma L proves the contrapositive statement

$$Q\widetilde{M} \neq Q\widetilde{M}Q \implies \exists N \geq 0 : Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1.$$

It does not imply the stronger inequality for every N printed at source line 1889. That stronger assertion is unnecessary: the positivity argument at lines 1890–1893 is pointwise in N , so the same existential witness suffices.

This is a local correction of the intermediate statement. The corrected periodic-sector implication is now proved both from literal CPSV canonical form and from the stronger normalized BNT-refined horizontal form. It closes the periodic-exclusion step used in Proposition 4.13. The later vertical phase-class grouping, gauge normalization, and coisometry construction are recorded in `cpgsv17_vertical_cf_grouping.tex`.

3 Formal proof

The literal proof establishes the corrected implication in `MPSTensor.IsCPSV CanonicalForm.exists_not_commute_of_displaced` in `TNLean/MPS/MPDO/CPSVPeriodicExclusion`. Suppose, to the contrary, that Q_1 commutes with $H^{(N+1)}$ for every N . Idempotence of Q gives

$$Q_1 H^{(N+1)} = Q_1 H^{(N+1)} Q_1.$$

After writing these as first-site contractions of the doubled-index tensor, the original-space form of Lemma L yields $Q\widetilde{M} = Q\widetilde{M}Q$, contradicting displacement. The stronger normalized BNT-refined horizontal-form variant is `MPOTensor.IsHorizontalCF.exists_not_commute_of_displaced` in `TNLean/MPS/MPDO/HorizontalBNT`.

The cyclic decomposition and its letter shift are formalized in `TNLean/MPS/MPDO/CyclicProjector.lean`. They give $p \geq 1$, an orthogonal projector Q , full-period word invariance, and displacement. Choose the noncommuting length $N + 1$ above. Full-period invariance transfers to the p -fold vertically stacked tensor, so

$$Q_1 [H^{(N+1)}]^p = [H^{(N+1)}]^p Q_1.$$

The positive-semidefinite power-commutation theorem then gives commutation with $H^{(N+1)}$, a contradiction. The literal conclusion is `MPOTensor.hasNoPeriodicVectors_verticalTensor_of_cpsvCanonicalForm` in `TNLean/MPS/MPDO/CPSVPeriodicExclusion`; the stronger-hypothesis variant is `MPOTensor.hasNoPeriodicVectors_verticalTensor_of_horizontalCF` in `TNLean/MPS/MPDO/CyclicProjector`.

Neither proof introduces an assumption beyond its stated canonical-form and MPDO hypotheses.

4 Status

The periodic-sector step is complete under literal CPSV canonical form and under the stronger normalized BNT-refined hypothesis `MPOTensor.IsHorizontalCF`. The earlier all-length condition remains only as a stronger conditional auxiliary statement and is not used by either proof.¹

Combining literal periodic exclusion with literal invariant-projector closure gives the normal vertical corners with positive weights and exact letterwise reconstruction in `MPSTensor.IsCPSVCanonicalForm.exists_normal_verticalBlockDecomp_with_isometry`. The subsequent vertical phase-class grouping, normalization of the grouped representatives, and final coisometry are also complete. Together they give the literal theorem `MPOTensor.verticalCF_of_cpsvCanonicalForm` in `TNLean/MPS/MPDO/CPSVVerticalCanonicalForm`. The stronger theorem from the normalized BNT-refined horizontal form remains available separately.

¹Formal predicate: `MPOTensor.NoninvariantProjectorNoncommuting`.

This completion concerns Proposition 4.13 only. It neither proves the positive-fusion statement in Appendix C.4 nor the renormalization fixed-point characterization in Theorem 4.14.

Verdict. The printed all-length inequality is replaced by the existence of one noncommuting length. This local correction suffices for the periodic-sector step, and literal Proposition 4.13 is now complete by the independent grouping, normalization, and coisometry results.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.