The SAL to Eta-Local Structure Step in Cirac–Perez-Garcia–Schuch–Verstraete 2017 Appendix C.2

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Verdict: open-gap (resolved).

1 Source statement

Appendix C.2 of Cirac–Pérez-Garcia–Schuch–Verstraete proves that a simple MPDO satisfying the strong area law has the commuting nearest-neighbor product form

$$\sigma^{(N)}(\mathcal{K}) \propto \prod_{n=1}^{N} B_{n,n+1},$$

where $B_{n,n+1} \geq 0$ and the translated two-site operators commute [CPGSV16, Appendix C.2, Proposition C.8]. In the local source file this is Proposition `3to4`, `Papers/1606.00608/MPDO-22-12-17-2.tex:1571--1593`.

Let $\tilde{\sigma}^{(N)}(\mathcal{K}) = U^{\dagger\otimes N}\sigma^{(N)}(\mathcal{K})U^{\otimes N}$ denote the density operator after the local unitary from Lemma `propSN`. The proof uses the sector decomposition obtained earlier in Appendix C.2:

$$\tilde{\sigma}^{(N)}(\mathcal{K}) = \bigoplus_{k_1,\ldots,k_N} \bigotimes_{n=1}^{N} \eta_{k_n,k_{n+1}},$$

with $\eta_{k_n,k_{n+1}} \geq 0$ acting between the right subspace of site n and the left subspace of site $n+1$. It then defines

$$B_{n,n+1} = \sum_{k_n,k_{n+1}} \mathrm{Id}_{H_{L,n}^{(k_n)}} \otimes \eta_{k_n,k_{n+1}} \otimes \mathrm{Id}_{H_{R,n+1}^{(k_{n+1})}},$$

regarded as the identity away from sites $n, n+1$. With this definition, $\tilde{\sigma}^{(N)}(\mathcal{K}) = \prod_n B_{n,n+1}$ and the commutation of the translated B-operators follows from their sector-local action.

2 Current formalization

The entropy implication at the beginning of Lemma C.2 is formalized. For $N \geq 4$, saturation gives $I_1 = I_2$, and hence

$$S_{N-1} + S_1 = S_2 + S_{N-2}.$$

This is equality in strong subadditivity for the normalized $(N-1)$-site marginal divided into three consecutive regions of lengths 1, 1, and $N-3$. The Hayashi characterization gives a direct-sum tensor-product decomposition on the middle site. Splitting the third region into its first site and its remaining $N-4$ sites and tracing the latter preserves this decomposition. Consequently the first-three-site marginal has the form asserted in Lemma `Lsigma3` for every $N \geq 4$; the decomposition is chosen separately for each N. [1]

The desired commuting product form is currently recorded as an eta-local structure for K.[2] It contains the assembled positive translation-invariant two-site bond and its realization of the finite-chain MPO. From this structure, the development proves the commuting-form property and, after ZCL is assumed separately, the GSNNCH–ZCL consequence.

Earlier versions also introduced a structure combining unrelated local, commuting-form, and doubled-index hypotheses, followed by deductions from the doubled-index condition to a fusion formulation of the RFP condition. These declarations did not express the source's structural fixed-point condition and have been removed. In particular, they no longer obscure the missing construction of the operators $B_{n,n+1}$ from SAL.

3 Missing step

After the entropy equality above, the source applies the inverse tensor $\mathcal{K}^{-1}$ to the Hayashi decomposition and proves Lemma `propSN`. The algebraic part of this passage is now formalized: the concrete tensor $\mathcal{K}$ has a sector factorization by tensors r_k and l_k, the neighboring operators $\eta_{k,h} = r_k l_h$ are defined without a positivity assertion, and the resulting cyclic contraction satisfies

$$\tilde{\sigma}^{(N)}(\mathcal{K}) = \bigoplus_{k_1,\ldots,k_N} \bigotimes_{n=1}^{N} \eta_{k_n,k_{n+1}}$$

in sector-adapted coordinates for every nonempty chain length.

For the four-site chain, the virtual contraction of the traced fourth site is now identified before applying the inverse tensor. Writing

$$R_4 = \operatorname{tr}\Big[\rho^{(4)}(\mathcal{K})\Big]^{-1} \sum_i \mathcal{K}^{i,i},$$

[1] The formal statements are `MPOTensor.isSSAEquality_tripartite_of_isSAL` and `MPOTensor.exists_etaStructure_reducedBlockState_three_of_isSAL`; the partial-trace preservation result is `Entropy.exists_quantumMarkovDecomposition_rightMarginalAlong`.

[2] The formal structure is `MPOTensor.EtaLocalStructureData`.

the normalized three-site marginal satisfies

$$\sigma_3^{(4)}(\mathcal{K})_{u,v} = \mathrm{tr}(\mathcal{K}^{u,v} R_4)\,.$$

If the four-site trace is nonzero, then $R_4 \neq 0$, and hence some entry $(R_4)_{\beta,\alpha}$ is nonzero. Thus the boundary matrix and the nonzero entry selected before equation `formK` are available without adding SAL or injectivity as separate hypotheses.[3]

The double inverse-map contraction preceding the sector factorization is also formalized. If R denotes the virtual contraction of the remaining sites, then

$$\sum_{p_1,p_3} (\mathcal{K}^{-1})^{\alpha_1,\beta_1}_{p_1} (\mathcal{K}^{-1})^{\alpha_3,\beta_3}_{p_3} \,\mathrm{tr}(\mathcal{K}^{p_1}\mathcal{K}^{p_2}\mathcal{K}^{p_3}R) = \mathcal{K}^{p_2}_{\beta_1,\alpha_3} R_{\beta_3,\alpha_1}\,.$$

This is the calculation at lines 1415–1438 of Appendix C.2 in arXiv:1606.00608.[4]

The source display at line 1427 writes the remaining scalar as m_{β_3,α_3}. This repeats the third-site left index and loses the first-site left index. Contracting the two matrix units gives

$$\mathrm{tr}(E_{\alpha_1,\beta_1}\mathcal{K}^{p_2}E_{\alpha_3,\beta_3}R) = \mathcal{K}^{p_2}_{\beta_1,\alpha_3} R_{\beta_3,\alpha_1}\,,$$

so the scalar is m_{β_3,α_1}. The diagram defining $m_{\beta,\alpha}$ and the subsequent choice of a nonzero pair (α,β) are consistent with this correction. Thus the repeated α_3 is a local index typo, not a different mathematical hypothesis.

The comparison with the Hayashi factors is now formalized. For a three-site closure against R, let $A^{(k)}_{\alpha_1,\beta_1}$ and $B^{(k)}_{\alpha_3,\beta_3}$ be obtained by contracting the inverse tensor against $\rho^{(k)}_{Ab_1}$ and $\rho^{(k)}_{b_2c}$, respectively. In the basis selected by the middle-site unitary, every diagonal sector satisfies

$$R_{\beta_3,\alpha_1}\, \widetilde{\mathcal{K}}^{(k)}_{\beta_1,\alpha_3}((l,r),(l',r')) = p_k A^{(k)}_{\alpha_1,\beta_1}(l,l') B^{(k)}_{\alpha_3,\beta_3}(r,r').$$

The factor p_k is explicit because the formal Hayashi decomposition stores normalized left and right density operators. The proof commutes the two outer contractions through the unitary change of basis and applies the double inverse-map contraction above. [5]

The tensor factorization is now formalized. The three-site marginal of the normalized four-site chain is a three-site closure against R_4, the off-diagonal Hayashi sectors of the transformed physical slice vanish, and dividing the diagonal comparison by a chosen nonzero entry R_{β_3,α_1} produces sector tensors l_k and r_k with

$$U_B\, \mathcal{K}^{\cdot,\cdot}_{\beta_1,\alpha_3}\, U_B^\dagger = \bigoplus_k (l_k)_{\beta_1} \otimes (r_k)_{\alpha_3}$$

[3] The formal declarations are `MPOTensor.normalizedFourSiteTail`, `MPOTensor.reducedBlockState_four_three_apply`, `MPOTensor.normalizedFourSiteTail_ne_zero`, and `MPOTensor.exists_normalizedFourSiteTail_entry_ne_zero`.

[4] The formal declarations are `MPOTensor.inverseMapThreeSiteContraction` and `MPOTensor.inverseMapThreeSiteContraction_eq`.

[5] The formal declarations are `MPOTensor.IsThreeSiteClosure`, `MPOTensor.physicalSlice`, `MPOTensor.hayashiInverseLeft`, `MPOTensor.hayashiInverseRight`, and `MPOTensor.inverseMap_hayashi_sector_comparison`.

for all virtual indices β_1, α_3. The nonzero entry exists whenever the four-site trace is nonzero. This is the factorization displayed at lines 1434–1438. [6]

For a finite family of normal sectors, the diagonal blocks used in equation `QkKjs` are now defined by

$$O_s^{(k)}(\beta,\alpha) = \left[U_B \kappa^{(s)}_{\beta,\alpha} U_B^\dagger\right]_{k,k}.$$

Every sector of nonzero Hayashi weight occurs in at least one normal sector. Indeed, nonzero diagonal entries of the two trace-one Hayashi states give a nonzero diagonal entry of the three-site state, and the normal-sector family closure then has a nonzero summand. This existence argument requires no closing-matrix assumption.

The uniqueness cancellation is also formalized with its load-bearing hypothesis stated explicitly. If every closing matrix R_s is nonzero, the principal BNT–Markov identity shows that two distinct normal sectors cannot both have nonzero k-blocks. It therefore gives the label j_k and the projections

$$P_s = \sum_{\substack{k:p_k\neq 0\\ j_k=s}} Q_k, \qquad \sum_s P_s = \sum_{k:p_k\neq 0} Q_k.$$

These projections are mutually orthogonal.[7]

Both factors in the closing-scalar choice are now selected. Once the copy weights agree, the distinguished common weight is nonzero and

$$q_j = \sum_q \mu^N_{j,q} = r_j \mu_j^N \neq 0$$

for every chain length N. [8] Simplicity says that the physical-trace transfer $\mathcal{B}_j = \sum_i A_j^{ii}$ is not nilpotent. Hence one positive tail length L has, for every representative, a nonzero entry $\widetilde{m}_j = (\mathcal{B}_j^L)_{\alpha_j,\beta_j}$. Taking the coefficient at total chain length $L+3$ gives the closing matrix

$$R_j = q_j \mathcal{B}_j^L$$

with nonzero entry $(R_j)_{\alpha_j,\beta_j} = q_j \widetilde{m}_j \neq 0$, as in Appendix C.2, lines 1714–1718. [9] This closing-matrix bridge is now formalized with the normalization

[6]The formal declarations are `MPOTensor.isThreeSiteClosure_reducedBlockState`, `MPOTensor.inverseMap_hayashi_sector_offdiagonal`, `MPOTensor.sectorTensorL`, `MPOTensor.sectorTensorR`, `MPOTensor.physicalSlice_sector_factorization`, and `MPOTensor.exists_physicalSlice_sector_factorization`.

[7]The relevant declarations are `MPOTensor.exists_bntMarkovBlockNonzero_of_probability_ne_zero`, `MPOTensor.existsUnique_bntMarkovBlockNonzero_of_probability_ne_zero`, `MPOTensor.bntSectorProjection`, and `MPOTensor.sum_bntSectorProjection`.

[8]The relevant declarations are `MPSTensor.SectorDecomposition.commonWeight_ne_zero` and `MPSTensor.SectorDecomposition.coeff_ne_zero_of_weight_copy_independent`.

[9]The relevant declarations are `MPSTensor.SectorDecomposition.exists_common_nonzero_physicalTraceTail_entry` and `MPSTensor.SectorDecomposition.exists_common_nonzero_threeSiteClosingMatrix_entry`; the matrix form is `MPSTensor.SectorDecomposition.exists_common_threeSiteClosingMatrix_ne_zero`.

included. Equality of the complete matrix-product-vector families transports the original tensor to its horizontal BNT representation, and the normalized three-site reduced state has closing matrices

$$\widehat{R}_j = \operatorname{tr}(\rho^{(L+3)})^{-1} q_j \mathcal{B}_j^L.$$

At the four-site specialization used by the subsequent closing-matrix argument, one takes $L = 1$. SAL makes the normalization scalar nonzero, so every $\widehat{R}_j$ is nonzero. [10] Thus the explicit nonzero-closing hypothesis of the generic cancellation theorem is derived for the family closure used by the source. The simultaneous specialization is now formalized as well. After the common blocking has made the BNT representatives simultaneously one-letter spanning, biCF supplies the inverse and SAL supplies the four-site Hayashi decomposition. Every active Markov sector then has a unique BNT label, and the resulting BNT-labelled sums of Hayashi projections are orthogonal, mutually disjoint, and resolve the active Markov support. [11] No further closing-matrix assumption is needed for this specialization.

The remaining projector passage is now formalized as well. For every physical slice of the i-th normal representative, $P_s A_i P_t = 0$ unless $s = t = i$. For the matching label,

$$P_i A_i P_i = A_i.$$

The zero-weight Markov summands retained by the formal decomposition annihilate all these physical slices, so this conclusion does not require replacing the active-support resolution by an ambient resolution of the identity. Acting sitewise on every nonempty periodic chain then gives

$$P_s^{\otimes N} \sigma^{(N)}(K) P_s^{\otimes N} = n_s \sigma^{(N)}(\mathcal{K}_s), \qquad N \geq 1,$$

with the common copy weight absorbed into $\mathcal{K}_s$. [12]

The literal identity resolution in equation Pis is recovered in the ambient physical space by choosing a BNT label s_0 from a copy with unit-modulus weight. Assign every zero-weight Markov summand to s_0, while retaining the unique source label on each positive-weight summand. The completed projections

$$\widehat{P}_s = \sum_{k:\widehat{j}_k = s} Q_k$$

then satisfy

$$\sum_s \widehat{P}_s = \mathbb{1}, \qquad \widehat{P}_s \widehat{P}_t = 0 \quad (s \neq t).$$

[10] The relevant declarations are `MPSTensor.SectorDecomposition.normalizedThreeSiteClosingMatrix`, `MPOTensor.reducedBlockState_reindex_isThreeSiteFamilyClosure_of_sameMPV`, and `MPOTensor.reducedBlockState_four_threeSiteFamilyClosure_nonzero_closing`.

[11] The relevant declaration is `MPOTensor.exists_bntSectorProjectors_four_of_sameMPVPos_isSAL`.

[12] The relevant declarations are `MPOTensor.bntSectorProjection_mul_physicalSlice_mul_eq_zero`, `MPOTensor.sum_bntSectorProjection_mul_physicalSlice_mul_self`, `MPOTensor.changePhysicalBasis_bntSectorProjection_basis`, `MPOTensor.sitewise_bntSectorProjection_mul_mpo_mul_self`, and `MPOTensor.exists_bntProjectorSelection_positiveLength_of_sameMPVPos_isSAL`.

Since a zero-weight Q_k annihilates every physical slice, this reassignment preserves, whenever $s \neq t$ or $i \neq s$,

$$\widehat{P}_s K_i^{\beta\alpha} \widehat{P}_t = 0.$$

Thus the retained zero-weight summands account exactly for the difference between the active-support resolution and the source's equation *Pis*. The construction and its SAL specialization are `MPOTensor.completedBntSectorProjection` and `MPOTensor.exists_bntSeparatingProjectors_of_sameMPVPos_isSAL_of_weight_copy_independent`.

The source standing context is recovered by `MPOTensor.exists_bntSeparatingProjectors_of_horizontalCF_isSourceZCL_isSAL`. It applies the ZCL equation to the horizontal canonical-form gauge identity, uses simplicity to exclude a zero physical-trace transfer, and thereby derives the equality of the copy weights proved at lines 1646–1665. It then invokes the copy-independent specialization above. Consequently the final theorem assumes only the simple biCF, ZCL, and SAL context of Appendix C.2, rather than assuming copy independence separately.

The empty-word value $N = 0$ is excluded because its closed virtual trace records the bond dimension and is not a physical chain. Positivity and ZCL of every selected absorbed BNT representative are now formalized. Positivity follows by compressing the positive full-chain operator and dividing by the positive integer n_s. ZCL follows by restricting the full transfer equation to the s-th canonical block and using nonnilpotence to exclude the zero transfer. [13] The strict normalization needed in the sector entropy identity is now formalized: an MPDO tensor with source ZCL has positive trace at every nonempty chain length, since its normalized physical-trace transfer is a nonzero idempotent. Applied to the absorbed BNT sectors above, this proves $p_j^{(N)} > 0$ rather than silently inferring it from the weaker displayed inequality $p_j \geq 0$. [14] The weighted orthogonal-direct-sum entropy formula used in Appendix C.2, lines 1760–1770, is also formalized as `vonNeumannEntropy_blockDiagonal_smul`, together with the form for pairwise-annihilating support operators `vonNeumannEntropy_eq_sum_of_pairwise_annihilating_supports`. The remaining source step after the sector-compression identity is now formalized: the same sector probabilities occur in all four marginals, the Shannon terms cancel, and strict positivity together with strong subadditivity forces equality in every sector. Thus every absorbed BNT representative satisfies SAL under the biCF hypothesis imposed at the start of Case II in the source. [15] This completes the entropy argument

[13] The relevant declarations are `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isMPDO_of_projectorSelection`, `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isMPDO_of_sameMPVPos_isSAL`, and `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isSourceZCL`.

[14] The general declarations are `MPOTensor.trace_mpo_ne_zero_of_isSourceZCL` and `MPOTensor.trace_mpo_pos_of_isMPDO_isSourceZCL`; their specialization to the absorbed BNT representative is `MPOTensor.trace_mpo_commonWeightAbsorbedBasisMPOTensor_pos`.

[15] The principal declarations are `MPOTensor.blockEntropy_eq_sum_bntSectorProbability`, `MPOTensor.blockEntropy_add_blockEntropy_eq_bntSector`, and `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isSAL_of_sameMPVPos`.

of [CPGSV16, Appendix C.2, lines 1748–1781] with the source's stated biCF hypothesis. There is a local endpoint typo in this passage. Line 1771 writes $m < \lfloor N/2 \rfloor$, but Definition 4.6 gives the equality chain through $I_{\lfloor N/2 \rfloor}$, and the conclusion at lines 1781–1783 compares $m = L$ with $m = L + 1$. When $L + 1 = \lfloor N/2 \rfloor$, that comparison requires the endpoint $m = \lfloor N/2 \rfloor$. The formal entropy equivalence therefore uses $1 \leq m \leq \lfloor N/2 \rfloor$.

The neighboring operators are now assembled from these sector tensors. The operator

$$\eta_{k,h} = \sum_{\gamma} (r_k)_\gamma \otimes (l_h)_\gamma$$

acts on the neighboring bond space, the bond contraction of two neighboring factorized slices exhibits $\eta_{k,h}$ between the outer sector tensors, and its trace is the pairing $T_{k,h} = \mathrm{tr}(\eta_{k,h}) = (r_k|l_h)$ of the closed sector tensors obtained by tracing the physical legs.[16] The source derives positivity of the $\eta_{k,h}$ from the all-length projected chain identity at lines 1446–1450, choosing the sector tensors so that each factor is positive semidefinite. That all-length algebraic identity is now formalized, but it does not by itself choose coherent positive representatives for all neighboring pairs. Positive semidefiniteness of the chosen representatives therefore still enters the construction of the explicit η-data as a hypothesis.

The earlier summary figure `MPDO_etahk=RhLk.png` at source line 1391 draws r_h next to l_k while naming the operator $\eta_{k,h}$. The later definition `etarl`, the identity $T_{k,h} = (r_k|l_h)$, and the products $\eta_{k,l} \otimes \eta_{l,h}$ in Proposition `3to5` consistently fix the intended convention as $\eta_{k,h} = r_k l_h$. The formalization follows this later consistent convention; the earlier figure is an index-swap typo.

The Hayashi decomposition of the concrete three-site marginal is now formalized for every $N \geq 4$. Independent bijections on the first two tensor factors preserve equality in strong subadditivity: each marginal is carried to the corresponding reindexed marginal, and the four entropies are unchanged. The canonical bijection between a singleton block and one physical-site index therefore carries the SAL equality from flattened block coordinates to the tripartite coordinates of the $(N-1)$-site state. Applying the forward Hayashi characterization gives a decomposition before the last $N - 4$ sites of the third region are traced out. Partial trace preserves the same middle-site direct sum, and the contiguous-block reduction identity gives

$$\sigma_3^{(N)}(\mathcal{K}) = \mathrm{tr}_{4,\ldots,N-1}\left[\sigma_{N-1}^{(N)}(\mathcal{K})\right].$$

This proves Lemma `Lsigma3` at every admissible chain length, with no claim that the decomposition is common to different values of N. [17]

[16]The formal declarations are `MPOTensor.sectorEta`, `MPOTensor.physicalSlice_neighboring_contraction`, `MPOTensor.closedSectorL`, `MPOTensor.closedSectorR`, `MPOTensor.trace_sectorEta`, and `MPOTensor.ExplicitEtaOperators.ofSectorTensors`.

[17]The formal declarations are `isSSAEquality_submatrix_prodEquiv` and `MPOTensor.exists_etaStructure_reducedBlockState_three_of_isSAL`.

The neighboring operators and their all-length algebraic identity are now formalized. The inverse-map tensors also determine a source-minimal physical-sector factorization. The nonzero dimensions of its left and right factors are derived from the trace-one Hayashi density matrices, rather than assumed. For an injective tensor satisfying SAL, the four-site Hayashi decomposition and a nonzero entry of the normalized tail therefore produce this raw factorization without ZCL or positivity hypotheses. For one such factorization, every complete nonempty cyclic neighboring product is positive semidefinite as a diagonal sector compression of the corresponding finite-chain operator. [18] The positive choice of the individual $\eta_{k,h}$ and primitivity of the active sector trace matrix are now obtained from this factorization. Positivity of a complete cyclic product alone would not supply coherent positive representatives for all neighboring pairs: recurrence, injective spanning, and the directed-cut argument control the choices simultaneously across the sector graph.

The positivity half of the projected chain identity is now formalized: for an MPDO, the chain of the physically conjugated tensor is a congruence of the original chain, so every cyclic tensor product of neighboring operators is a diagonal sector block of a positive operator,

$$0 \leq \left[Q_{k_1} \otimes \cdots \otimes Q_{k_N}\right] \tilde{\sigma} \left[Q_{k_1} \otimes \cdots \otimes Q_{k_N}\right] = \bigotimes_{n=1}^{N} \eta_{k_n,k_{n+1}}.$$

The length-one and length-two specializations give $\eta_{k,k} \geq 0$ and $\eta_{k,h} \otimes \eta_{h,k} \geq 0$ unconditionally. A rescaling lemma for the factors of a nonzero positive semidefinite Kronecker product then yields, for every pair with $\eta_{k,h} \neq 0$ and $\eta_{h,k} \neq 0$, a scalar $c \neq 0$ with $c\,\eta_{k,h} \geq 0$ and $c^{-1}\eta_{h,k} \geq 0$. Under the additional hypothesis of *symmetric support* ($\eta_{k,h} = 0$ forces $\eta_{h,k} = 0$), choosing one scalar per unordered pair produces a positive semidefinite family which agrees with the raw family on the diagonal and preserves every two-site block $\eta_{k,h} \otimes \eta_{h,k}$.[19]

The pairwise construction has two limitations. Its symmetric-support hypothesis is not stated in the source, and it preserves only the chain blocks of length at most two. The source instead rescales the sector tensors and therefore

[18] The formal declarations are `MPOTensor.inverseMapPhysicalSectorFactorization`, `MPOTensor.inverseMapPhysicalSectorFactorization_neighboringOperator_eq_sectorEta`, `MPOTensor.nonempty_physicalSectorFactorization_of_isSAL`, `MPOTensor.exists_physicalSectorFactorization_with_cyclicNeighboringProduct_posSemidef_of_isSAL`, `MPOTensor.etaOfSectorTensors`, `MPOTensor.sum_cyclic_sectorTensor_eq_prod_etaOfSectorTensors`, `MPOTensor.trace_sector_word_eq_prod_etaOfSectorTensors`, `MPOTensor.mpo_submatrix_sector_eq_cyclicEtaTensorProduct`, and `MPOTensor.mpo_reindex_sectorChainEquiv_eq_blockDiagonal_cyclicEta`. For the source-minimal physical-sector factorization, the corresponding direct statements are `MPOTensor.PhysicalSectorFactorization.mpo_submatrix_sector_eq_cyclicNeighboringProduct` and `MPOTensor.PhysicalSectorFactorization.mpo_reindex_sectorChainEquiv_eq_blockDiagonal`.

[19] The formal declarations are `Matrix.exists_smul_posSemidef_of_kronecker_posSemidef`, `MPOTensor.mpo_conjugatePhysical_eq`, `MPOTensor.cyclicEtaTensorProduct_posSemidef`, `MPOTensor.sectorEta_self_posSemidef`, `MPOTensor.sectorEta_kronecker_posSemidef`, `MPOTensor.exists_smul_posSemidef_sectorEta`, and `MPOTensor.exists_explicitEtaOperators_sectorEta`.

preserves every length simultaneously. These limitations are overcome below by reparameterizing zero-weight sectors, proving recurrence of the remaining support, and applying a vertex rephasing across the whole sector graph.

The obstruction is already present when every neighboring bond space is one-dimensional. Take four sectors $0, 1, 2, 3$ and set

$$\eta_{k,k} = 1, \qquad \eta_{0,1} = \eta_{0,2} = \eta_{1,3} = 1, \qquad \eta_{2,3} = i,$$

with all other off-diagonal entries equal to zero. Every directed cyclic product is either zero or one, and is therefore positive semidefinite. The sector rephasing which preserves each same-sector product has the form

$$\eta_{k,h} \longmapsto z_k^{-1} z_h \eta_{k,h}, \qquad |z_k| = 1.$$

Positivity on the edges $0 \to 1$, $0 \to 2$, and $1 \to 3$ forces $z_1 = z_0$, $z_2 = z_0$, and $z_3 = z_1$, respectively. The transformed edge $2 \to 3$ then still has phase i. Hence no coherent choice of sector phases makes all four nonzero neighboring operators positive.

A sufficient condition for the graph-theoretic rephasing step is recurrence of the nonzero support: every nonzero directed edge lies on a nonzero directed cycle. Under this hypothesis, positivity of all cyclic tensor products would first show that each edge is a phase times a positive operator, then force trivial phase around every cycle, and finally permit a coherent choice of vertex phases. The source does not state recurrence as a hypothesis. It is derived below from injectivity after zero-weight sectors are reparameterized. This derivation does not use the later primitivity argument, and hence avoids circularity.

For the present formal notion of a Hayashi decomposition, recurrence is false even for the concrete inverse-map neighboring operators. The reason is that the structure permits zero sector weights. Consider the scalar-bond tensor on a two-dimensional physical space given by

$$K^{00} = 1, \qquad K^{01} = K^{10} = K^{11} = 0.$$

It is injective and generates the rank-one projector onto the constant-zero word at every length. Decompose the middle physical space into its two one-dimensional coordinate sectors, with weights $p_0 = 1$ and $p_1 = 0$, and choose the normalized left and right states in both sectors to be supported on the zero coordinate. For the scalar tail $R = 1$, the inverse-map sector tensors satisfy

$$r_0 = r_1 = 1, \qquad l_0 = 1, \qquad l_1 = 0, \qquad \eta_{k,h} = p_h.$$

Thus $1 \to 0$ is a nonzero support edge, whereas no edge enters sector 1. This edge is not recurrent. Hence no theorem deriving recurrence may quantify over an arbitrary decomposition satisfying only $p_k \geq 0$.

The source-faithful repair retains the full physical direct sum, since a zero-weight summand can have nonzero left and right factor dimensions. In the chosen inverse-map factorization, $p_k = 0$ already implies $l_k = 0$. Replacing r_k by zero on precisely those sectors therefore leaves every product $l_k \otimes r_k$

unchanged and kills the spurious outgoing edges of the zero-weight sectors. Incoming edges were already zero because their target left tensor vanishes. This reparameterization and its neighboring-operator formula are now formalized by `MPOTensor.PhysicalSectorFactorization.zeroRightTensorOnInactive`, `MPOTensor.sectorTensorL_eq_zero_of_weight_eq_zero`, `MPOTensor.zeroWeightReparameterizedInverseMapPhysicalSectorFactorization`, and `MPOTensor.zeroWeightReparameterizedInverseMapPhysicalSectorFactorization_neighboringOperator`. Every neighboring operator incident to an inactive sector therefore vanishes, while active–active operators are unchanged. On the active support, recurrence follows from an algebraic triangle-closing argument once the virtual matrices furnished by all sector entries span the full matrix algebra and every active sector furnishes a nonzero matrix. This argument is now formalized: a nonzero edge $k \to h$ can be closed through one further sector j, giving $h \to j \to k$, by nondegeneracy and cyclicity of the matrix trace pairing.[20] Injectivity now supplies the spanning hypothesis for every physical-sector factorization.[21] Positive-weight endpoint nonvanishing is now proved directly from the three-site closure and the Hayashi state decomposition.[22] Consequently the zero-weight reparameterized inverse-map support is recurrent, by injective spanning and triangle closure.[23]

3.1 Recurrence of the sector support graph: definitions and the per-cycle rescaling

The directed graph on sector indices with an edge $k \to h$ whenever $\eta_{k,h} \neq 0$, reachability along its nonzero edges, and recurrence of this graph (every nonzero edge lies on a nonzero directed cycle) are now formalized without further hypothesis.[24] The edge is on η, not on the trace matrix T: this matches the phase-mismatch mechanism of the four-sector counterexample above, which is already visible at the level of the individual neighboring operators.

Reachability defined by reflexive transitive closure is equivalent to the existence of a finite directed walk. Consequently, recurrence of the sector support is exactly the return-walk hypothesis in the closed-walk coboundary theorem.[25]

Every nonempty closed directed walk is also represented by a finite cyclic family of its source vertices. Consecutive vertices are connected by the corre-

[20] The formal declarations are `MPOTensor.PhysicalSectorFactorization.sectorVirtualMatrix`, `MPOTensor.PhysicalSectorFactorization.exists_two_edge_return_of_neighboringOperator_ne_zero`, and `MPOTensor.PhysicalSectorFactorization.neighboringOperator_reaches_of_ne_zero`.

[21] This is `MPOTensor.PhysicalSectorFactorization.sectorVirtualMatrixFamily_span_eq_top`.

[22] The formal declaration is `MPOTensor.exists_active_sectorVirtualMatrix_ne_zero`.

[23] This is `MPOTensor.zeroWeightReparameterizedInverseMapPhysicalSectorFactorization_isRecurrentSupport`.

[24] The formal declarations are `MPOTensor.IsSectorEdge`, `MPOTensor.SectorReaches`, and `MPOTensor.IsRecurrentSupport`, in `TNLean/MPS/MPDO/SectorRecurrence.lean`.

[25] The formal declarations are `TNLean.Algebra.DirectedWalk.reaches_iff_reflTransGen`, `MPOTensor.sectorReaches_iff_directedWalkReaches`, and `MPOTensor.isRecurrentSupport_iff_directedWalkReturns`.

sponding edge, including the final return to the initial vertex, and the cyclic product of edge weights equals the walk weight.[26]

The finite-factor rescaling used implicitly by the four-sector counterexample is now a general lemma about complex matrices, independent of the MPDO setting. If a finite Kronecker product $A_0 \otimes \cdots \otimes A_{N-1}$ of nonzero matrices is positive semidefinite, then there are nonzero scalars $c_0, \ldots, c_{N-1}$ with $\prod_n c_n = 1$ and every $c_n A_n \geq 0$; the proof peels off one factor at a time using the two-factor rescaling already available. A companion lemma shows that the rescaling scalar of a single nonzero matrix is unique up to a positive real factor: if $c_1 M$ and $c_2 M$ are both positive semidefinite for a nonzero M, then c_1/c_2 is a positive real number.[27]

Reindexing the cyclic tensor product $\Omega_k = \bigotimes_n \eta_{k_n,k_{n+1}}$ along the shared horizontal-bond indices identifies it, entrywise, with the finite Kronecker product of the consecutive neighboring operators around the cycle.[28] Combined with the finite-factor rescaling lemma and with positivity of Ω_k at every length (`MPOTensor.cyclicEtaTensorProduct_posSemidef`), this gives a coherent choice of phases around any *one fixed* nonzero cycle: nonzero scalars c_n with $\prod_n c_n = 1$ and every $c_n \eta_{k_n,k_{n+1}} \geq 0$, both for an abstract neighboring-operator family and for the concrete inverse-map-constructed `MPOTensor.sectorEta`.[29] This theorem concerns one fixed cycle: it does not by itself assert that the choices made on two different cycles agree on a shared edge, which is the remaining content of the coherent vertex-rephasing step.

3.2 The source's one-sided directed-cut obstruction

The source's local-orthogonality contradiction uses the decomposition $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ together with the one-sided product vanishing $\mathcal{A}_1\mathcal{A}_2 = 0$ at Appendix C.2, lines 1465–1470. It does not require $\mathcal{A}_2\mathcal{A}_1 = 0$. The underlying matrix-algebra statement is now formalized as `Matrix.submodule_sup_ne_top_of_mul_eq_zero`: two nonzero linear subspaces of a full complex matrix algebra cannot both span the algebra and have all products vanish in one order.

The zero-weight reparameterization is now formalized: it retains every physical summand, sets the right sector tensor to zero when its Hayashi weight vanishes, and thereby isolates every zero-weight sector without changing the physical slices. The concrete inverse-map construction gives a short proof of recurrence. Positive-weight sectors contain nonzero virtual matrices, injectivity makes the full sector family span $M_D(\mathbb{C})$, and the trace-pairing argument closes

[26] The formal declarations are `TNLean.Algebra.DirectedWalk.closedConsVertices`, `TNLean.Algebra.DirectedWalk.closedConsVertices_edge`, and `TNLean.Algebra.DirectedWalk.weight_closedCons_eq_fin_prod`.

[27] The formal declarations are `Matrix.exists_pi_smul_posSemidef_of_finKronecker_posSemidef` and `Matrix.exists_pos_real_smul_eq_of_smul_posSemidef`, in `TNLean/Algebra/KroneckerFactorPositivity.lean`.

[28] The formal declaration is `MPOTensor.cyclicEtaTensorProduct_submatrix_eq_finKronecker`.

[29] The formal declarations are `MPOTensor.exists_pi_smul_posSemidef_of_cyclicEtaTensorProduct_posSemidef` and `MPOTensor.exists_pi_smul_posSemidef_of_sectorEta_cycle`.

every nonzero edge through a third sector. Thus recurrence is a conclusion, not an additional hypothesis.

For the stronger one-component assertion used in trace-matrix primitivity, let S be a nonempty proper forward-closed set of positive-weight sectors and let C be its complement. The source argument obtains $\mathcal{A}_S\mathcal{A}_C = 0$ from the absence of edges from S to C; this directed-cut implication is formalized by `MPOTensor.PhysicalSectorFactorization.oneSiteSectorSpace_mul_eq_zero`. The one-site matrices in this directed-cut space are now identified with the sector virtual matrices used in the spanning theorem. On the nonzero-weight subtype, the spaces belonging to S and C are both nonzero, their sum is the full matrix algebra, and their products in the order $\mathcal{A}_S\mathcal{A}_C$ vanish. The one-sided product obstruction therefore proves that the nonzero neighboring support is strongly connected; no absence of edges from C to S is needed. Positivity identifies this support with the positive entries of the restricted real trace matrix, so the latter is irreducible. The relevant declarations are `MPOTensor.PhysicalSectorFactorization.oneSiteSectorMatrix_eq_sectorVirtualMatrix`, `MPOTensor.PhysicalSectorFactorization.activeSector_reflTransGen_neighboringOperator_ne_zero`, and `MPOTensor.PhysicalSectorFactorization.activeSectorTraceMatrix_isIrreducible`.

The periodicity step is treated differently from the source. At lines 1457–1462 the source passes to a power T^n, invokes blocking to remove periodic components, and then reduces to primitive diagonal components. The present argument does not formalize that reduction. Instead, nondegeneracy and cyclicity of the trace pairing give a closed support walk of length two through every nonzero-weight sector. Triangle closure, applied to an outgoing edge, gives a closed support walk of length three through the same sector. These two coprime return lengths are an auxiliary replacement for the source's blocking step. The length-two result and positivity of the corresponding diagonal entry of T^2 are formalized by `MPOTensor.PhysicalSectorFactorization.exists_active_two_cycle` and `MPOTensor.PhysicalSectorFactorization.activeSectorTraceMatrix_pow_two_diag_pos`. The length-three return is `MPOTensor.PhysicalSectorFactorization.activeSectorTraceMatrix_pow_three_diag_pos`. The elementary theorem `Matrix.IsIrreducible.isPrimitive_of_pow_two_diagonal_pos_of_pow_three_diagonal_pos` combines irreducibility with these two coprime returns to exhibit one common exponent N for which every entry of T^N is positive. Thus `MPOTensor.PhysicalSectorFactorization.activeSectorTraceMatrix_isPrimitive` completes the active-sector primitivity argument. This replacement of the source's blocking step is a local fix, not an additional hypothesis.

3.3 Coherent vertex rephasing under recurrence

Granting recurrence, propagating the per-cycle phase choice above into a single vertex-indexed rescaling requires a coboundary argument, for which the uniqueness lemma above supplies the key input. Fix a nonzero edge $k \to h$: by uniqueness, *every* valid rescaling scalar of the single matrix $\eta_{k,h}$ (whichever cycle through the edge was used to produce it) agrees with every other one up to a positive real factor. Consequently, for any nonzero cycle through several edges, the finite-factor rescaling scalars obtained from that one cycle are, edge by edge, the unique per-edge scalars up to a positive real factor, so the product of the per-edge scalars around *any* nonzero cycle is a positive real number (apply the finite-factor lemma to that cycle and compare via uniqueness to the per-edge scalars). This is the statement that the phase assignment has trivial holonomy, which is the standard sufficient condition for a cocycle on a graph to be a coboundary: fixing a base sector in a recurrent component and defining its vertex phase by accumulating the per-edge scalars along any nonzero directed path from the base sector is then independent of the path chosen, because any two paths between the same pair of sectors close into a nonzero cycle (using recurrence to supply the return path) whose holonomy is trivial.

The graph-theoretic coboundary step is now formalized: for a recurrent directed relation, any group-valued edge assignment with trivial product around every closed directed walk is a vertex coboundary. [30] The proof chooses one representative of each reachability component and defines the vertex weight by multiplication along a path from that representative; a common return path proves independence of the chosen path.

The general MPDO-specific bridge is formalized under a recurrence hypothesis. For every nonzero neighboring operator, a directed return path closes the edge into a nonzero cycle. Positive rescaling of that cyclic tensor product therefore supplies a unit phase that makes the edge operator positive semidefinite. Positive rescaling of an arbitrary closed walk, together with uniqueness of the unit phase, shows that these chosen edge phases have product one around the walk. The coboundary theorem then gives unit vertex phases z_k with edge phase $z_k^{-1} z_h$. The declarations `MPOTensor.exists_vertexPhase_smul_posSemidef` and `MPOTensor.exists_rephased_inverseMapPhysicalSectorFactorization` carry out this argument, including the dependent sector bond spaces and the reciprocal rephasing of the physical-sector tensors. The latter leaves every complete cyclic neighboring product unchanged by `MPOTensor.PhysicalSectorFactorization.rephase_cyclicNeighboringProduct`. Composing the positive rephased factorization with the conditional bond construction gives `MPOTensor.nonempty_etaLocalStructureData_of_recurrentSupport`.

This general result is explicitly scope-restricted: recurrence is an additional hypothesis not present in Lemma C.4. For the concrete inverse-map factorization, that restriction is now eliminated. The declarations `MPOTensor.zeroWeightReparameterizedInverseMapPhysicalSectorFactorization_isRecurrentSupport`, `MPOTensor.exists_positive_inverseMapPhysicalSectorFactorization`, and `MPOTensor.exists_positive_physicalSectorFactorization_of_isSAL` retain and reparameterize the zero-weight sectors, derive recurrence from the source hypotheses, and produce the coherently positive factorization.

[30]The formal declaration is `TNLean.Algebra.DirectedWalk.exists_vertex_of_closedWalk_weight_eq_one`.

The canonical family `MPOTensor.ExplicitEtaOperators.ofHayashiMarkov` does not supply this missing result. Its operators are partial traces of normalized sector states, and hence its trace matrix is identically one. In the source inverse-map construction the trace matrix

$$T_{k,h} = \mathrm{tr}(\eta_{k,h})$$

is a generally nonconstant nonnegative matrix whose positive-weight restriction is asserted to be primitive. Thus the two families cannot be identified without an additional theorem; in general they are not equal.

The translation-invariant bond $B_{n,n+1}$ is now defined in the original physical coordinates and proved positive. Adjacent translates commute on every periodic chain of length at least three, including the pair crossing the periodic boundary. The exceptional chain of length two is also covered: the translates beginning at sites zero and one read the sites in the opposite cyclic orders. On the (k, h) sector block they act, respectively, as

$$I_{L_k \otimes R_h} \otimes \eta_{k,h}, \qquad \eta^{\mathrm{op}}_{h,k} \otimes I_{R_k \otimes L_h},$$

and therefore commute on complementary tensor factors. Proposition C.8 does not discuss this crossed finite-size ordering separately, although it is an instance of its translated-bond conclusion.[31]

Translates with disjoint cyclic two-site supports commute by locality, without any further physical hypothesis.[32] Thus the adjacent, crossed two-site, and disjoint local commutators are all formalized. Their finite case distinction is also assembled into pairwise commutativity of all translates on every periodic chain of length $N \geq 2$. [33]

The cyclic sector identity is now identified with the ordered bond product and transported through the local unitary, giving the exact finite-chain realization

$$\sigma^{(N)}(K) = c_N \prod_{n=1}^{N} B_{n,n+1}$$

with $c_N = 1$ for the canonical bond determined by the physical-sector factorization.[34] Together with positivity of the neighboring operators, this supplies the mathematical data needed to construct `MPOTensor.EtaLocalStructureData`. The conditional construction is `MPOTensor.PhysicalSectorFactorization.etaLocalStructureData`; its normalization scalar is one. The raw inverse-map physical-sector factorization and its coherently positive normalization are now

[31]The formal declarations are `MPOTensor.PhysicalSectorFactorization.physicalBond_adjacent_comm` and `MPOTensor.PhysicalSectorFactorization.physicalBond_two_zero_one_comm`.

[32]The formal declaration is `MPOTensor.PhysicalSectorFactorization.physicalBond_disjoint_comm`.

[33]The formal declaration is `MPOTensor.PhysicalSectorFactorization.physicalBond_pairwise_comm`.

[34]The formal declaration is `MPOTensor.PhysicalSectorFactorization.mpo_eq_product_physicalBond`.

constructed for an injective tensor satisfying SAL. ZCL is not a hypothesis of this construction.

The source Proposition `3to4` assumes SAL, not ZCL. ZCL enters only after this commuting product form has been obtained, in the later step that derives the GSNNCH–ZCL and RFP conclusions. Accordingly, adding ZCL to a conditional theorem would not be relevant to the coherent positive choice.

The coherent positive choice and the active-sector trace-matrix primitivity in Lemma `propSN` are now complete. The latter is `MPOTensor.exists_positive_physicalSectorFactorization_activeSectorTraceMatrix_isPrimitive_of_isSAL`. The positive commuting bond, its exact all-length product realization, and the resulting eta-local structure of Proposition `3to4` are available from a coherently positive physical-sector factorization. Their composition with the SAL theorem is `MPOTensor.nonempty_etaLocalStructureData_of_isSAL`, so the η-local conclusion of Proposition C.8 is now formalized from its source hypotheses.

There is a necessary distinction between the physical sector decomposition and the effective index set of the trace matrix. The former must retain every summand of the middle Hilbert space, including a summand with $p_k = 0$. After the zero-weight reparameterization above, however, $\eta_{k,h} = \eta_{h,k} = 0$ for every h whenever $p_k = 0$. Thus the trace matrix on the full ambient sector set has a zero row and a zero column and is not primitive. The primitive matrix in Lemma `propSN` must therefore be read on the effective sector set $\{k : p_k > 0\}$, as is customary when zero probability terms in a direct-sum state decomposition are omitted. The Lean definition `PhysicalSectorFactorization.activeSectorTraceMatrix` restricts the trace matrix to this subtype while leaving the physical sector equivalence, and hence the full Hilbert space, unchanged.

4 Classification

This note now records a closed coherent-positive-choice and primitivity gap in the SAL-to-positive-bond construction of `MPOTensor.EtaLocalStructureData`. Zero-weight sectors are reparameterized without changing the physical decomposition, recurrence is derived from injectivity, and the inverse-map sector tensors are coherently rephased so that every neighboring operator is positive semidefinite. The earlier scope-restricted recurrence theorem remains useful as the general graph-theoretic component, but recurrence is a conclusion for the reparameterized inverse-map factorization. The trace matrix restricted to positive-weight sectors is primitive. The unrestricted ambient trace matrix is not primitive when a zero-weight sector is present.

For the conditional bond construction, all translates commute pairwise at every length $N \geq 2$. The finite-chain MPO is the ordered product of those bonds, first in cyclic sector coordinates and then in the original physical coordinates. If the neighboring operators are positive semidefinite, these results provide the positive commuting bond and its exact realization with scalar one, and they determine the eta-local structure data. The raw physical-sector factorization already follows for an injective tensor satisfying SAL, and the coherent positive choice is now obtained by normalizing the inactive sectors and deriving recurrence from injectivity. The active-sector trace-matrix primitivity step is also complete. The projector form of the local-orthogonality contraction remains a separate source presentation, but it is no longer an obstruction to Lemma `propSN` because the directed-cut product identity is proved directly for the sector virtual matrix spaces.

References

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