CPGSV17 Blocked-Basis Chi Families and Uniform BNT Labels

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Verdict: scope-restriction (resolved).

1 Source statement

Cirac–Pérez-García–Schuch–Verstraete state in [CPGSV16, Theorem 4.14(ii)] that there is a set of diagonal matrices

$$\chi_{\alpha,\beta,\gamma}$$

with positive entries such that, for every length L, the operators $O_L(M_\alpha)$ linearly span an algebra whose structure coefficients satisfy

$$c^{(L)}_{\alpha,\beta,\gamma} = \mathrm{tr}(\chi^L_{\alpha,\beta,\gamma}).$$

The important point for the present note is that the labels α, β, γ are BNT labels, not basis labels depending on L. Thus the same diagonal matrix $\chi_{\alpha,\beta,\gamma}$ is used for all positive lengths; only the exponent changes.

Here $O_L(M_\alpha)$ denotes the length-L operator associated to the normal tensor labelled by α in the basis of normal tensors. The phrase "BNT labels" refers to these labels α, β, γ. The support algebra $\mathcal{A}_n$ below is the formal support algebra chosen for the blocked length n in the Lean development.

2 Formalized blocked-basis analogue

The current blocked coefficient construction in `TNLean/MPS/MPDO/AlgebraStructure.lean` chooses bases of the support algebras $\mathcal{A}_n$. Its multiplication coefficients have indices

$$i, j \in \mathrm{Basis}(\mathcal{A}_n), \qquad k \in \mathrm{Basis}(\mathcal{A}_{2n}),$$

because multiplication is represented as

$$\mathcal{A}_n \times \mathcal{A}_n \longrightarrow \mathcal{A}_{2n}.$$

The declarations `MPOTensor.AlgebraStructureData.BlockedStructureChiFamily` and `MPOTensor.AlgebraStructureData.HasBlockedStructureChiTracePowerForm` therefore express a positive-length, length-dependent blocked-basis analogue:

$$c^{(n)}_{i,j,k} = \sum_r \chi^n_{n,i,j,k,r} = \mathrm{tr}(\chi^n_{n,i,j,k}).$$

Here n is the source blocking length of the two factors in $\mathcal{A}_n$, while the coefficient k is taken in the basis of $\mathcal{A}_{2n}$. Both the family and its trace-power predicate are indexed only by $0 < n$, so there is no length-zero component and no degenerate constraint $c^{(0)}_{i,j,k} = \dim(\chi_{0,i,j,k})$, which is not part of the intended positive-chain statement.

3 Gap

Verdict: scope restriction.

This blocked-basis analogue is not the uniform BNT-label statement of Theorem 4.14(ii). It allows the diagonal matrix to depend on n, because the available indices themselves depend on the blocked bases $\mathcal{A}_n$ and $\mathcal{A}_{2n}$. The source theorem instead provides one matrix for each triple of BNT labels and uses that same matrix for every length L.

There is a second scope restriction in the operation being compared. In the paper, the coefficients $c^{(L)}_{\alpha,\beta,\gamma}$ belong to the same-length operator algebra:

$$O_L(M_\alpha)O_L(M_\beta) = \sum_\gamma c^{(L)}_{\alpha,\beta,\gamma} O_L(M_\gamma).$$

The current Lean coefficient family comes from the blocked support-algebra product

$$\mathcal{A}_n \times \mathcal{A}_n \longrightarrow \mathcal{A}_{2n},$$

so the two input factors have source length n, while the result is expanded in a basis at length $2n$. The exponent n in

$$c^{(n)}_{i,j,k} = \mathrm{tr}(\chi^n_{n,i,j,k})$$

is therefore the source length of the two factors, not the codomain length. This convention is only a blocked-basis analogue; it is not a replacement for the same-length multiplication statement in Theorem 4.14(ii).

This note isolates the chi-uniformity and same-length coefficient comparison part of Theorem 4.14(ii). The idempotent condition

$$m_\gamma = \sum_{\alpha,\beta} c^{(1)}_{\alpha,\beta,\gamma} m_\alpha m_\beta$$

from paper lines 981–985 is a separate condition in the same theorem; it is not represented by the blocked-basis trace-power predicate.

Consequently, `MPOTensor.AlgebraStructureData.HasBlockedStructureChiTracePowerForm` should not be cited as a formalization of Theorem 4.14(ii). It is a local blocked-basis coefficient statement, useful only after the uniform BNT-label statement has been separated from the chosen blocked bases.

4 Explicit vertical closure for the rescaling-stable example R

Verdict: example-specific explicit vertical closure; the general blocked-basis scope restriction and the corrected status of printed Theorem 4.14 remain.

Let E_{ij} denote the 2×2 matrix unit and set

$$A^0 = \frac{4}{5}E_{00}, \qquad A^1 = \frac{4}{5}E_{11}, \qquad A^2 = \frac{3}{5}E_{01}, \qquad A^3 = \frac{3}{5}E_{10}.$$

The horizontal MPO tensor R is defined by

$$R_{ab}^{pq} = \frac{25}{32}\left(A^a \otimes \overline{A^b}\right)_{pq}.$$

Its normalized vertical component is the 16-letter tensor $M_0^{(a,b)} = A^a \otimes \overline{A^b}$, and $O_L(M_0)$ denotes the length-L closed operator of that component. Finally, write

$$W = \begin{pmatrix} 16/25 & 9/25 \\ 9/25 & 16/25 \end{pmatrix}, \qquad W_L = W^{\otimes L}.$$

Thus W^L below is an ordinary matrix power, whereas W_L is a Kronecker power.

The explicit module `TNLean/MPS/MPDO/RescalingStableExplicitVerticalBNT.lean` defines the one-label clause `MPOTensor.RescalingStableLengthDependentRFP.R_oneLabelBNTAlgebraTensorClause`. There is one BNT label. Its normalized vertical component has 16 physical letters, bond dimension 4, and

$$M_0^{(a,b)} = A^a \otimes \overline{A^b}.$$

The multiplicity matrix and the distinct chi matrix are

$$\mu_0 = \begin{pmatrix} 25/32 \end{pmatrix}, \qquad \chi_{0,0,0} = \operatorname{diag}(1, 7/25).$$

The vertical reconstruction is

$$\operatorname{vertical}(R) = \frac{25}{32}M_0.$$

For every positive length L, the component operator satisfies

$$O_L(M_0)^2 = \left(1 + \left(\frac{7}{25}\right)^L\right) O_L(M_0) = \operatorname{tr}(\chi_{0,0,0}^L) O_L(M_0).$$

At length one, $m_0 = \mathrm{tr}(\mu_0) = 25/32$ and $c^{(1)}_{0,0,0} = \mathrm{tr}(\chi_{0,0,0}) = 32/25$, so the idempotent identity is

$$\frac{25}{32} = \frac{32}{25}\left(\frac{25}{32}\right)^2.$$

Thus the separately defined coefficient family `MPOTensor.RescalingStableLengthDependentRFP.oneLabelCoeffs` and matrix `MPOTensor.RescalingStableLengthDependentRFP.oneLabelChi` are now the actual coefficient and diagonal data of this explicit vertical clause.

The contraction orientation is essential. Let $\mathrm{rot}(\mathrm{vertical}(R))$ denote the MPO obtained by separating the 16-valued physical index of the vertical tensor into two four-valued indices. Then

$$\mathrm{mpo}(\mathrm{rot}(\mathrm{vertical}(R)), L) = \left(\frac{25}{32}\right)^L O_L(M_0).$$

This rotated vertical contraction is not the original horizontal tensor R, even though the physical and bond dimensions both equal 4. At length one, the original horizontal operator $\mathrm{mpo}(R, 1)$ has rank two, whereas $O_1(M_0)$ is a pure-state operator of rank one. They are therefore not proportional, and one must not write

$$\mathrm{mpo}(R, L) = \left(\frac{25}{32}\right)^L O_L(M_0).$$

The earlier horizontal countercalculation remains the regression boundary. The ordinary matrix power W^L is distinct from the L-fold Kronecker power W_L in the factorization of the original horizontal operator. If that operator satisfied the false scalar product law, then already at $L = 1$ one would have

$$\frac{25}{32} W^2 = \frac{32}{25} W.$$

The $(0, 0)$ entries are instead $337/800$ and $512/625$.

This closure is specific to R. It does not identify the blocked support algebra bases with uniform BNT labels in general, prove that every blocked-basis χ family is length independent, or attach arbitrary uniform rescalings of χ to tensor presentations. The formal algebra characterization of Theorem 4.14(i)$\Leftrightarrow$(ii) is complete, while the printed theorem remains partial because clause (iii) requires the documented active-support correction.

5 Relation to the algebra-to-fusion converse

The line ranges must be kept distinct. Lines 1830–1922 prove Proposition 4.13, the vertical canonical-form statement; they do not extract $\chi_{\alpha,\beta,\gamma}$. The construction of χ in the forward direction (i) $\Rightarrow$ (ii) occurs at lines 2020–2042. The converse (ii) $\Rightarrow$ (i) begins at line 2046 and takes the positive χ-coefficient formula and the idempotent identity as hypotheses.

Let $E = E_1$ be the one-site transfer map. The present algebra predicate gives the fixed-point-space identity

$$\mathrm{Fix}\big((E^n)^\dagger\big) = \mathrm{Fix}(E^\dagger), \qquad n > 0.$$

This rules out root-of-unity peripheral eigenvalues of $E^\dagger$, but it does not imply the transfer-retract idempotence equation $E^2 = E$.

This failure is formalized by `MPOTensor.exists_hasBlockedAdjointFixedPointAlgebraTower_not_isZCL`. The counterexample is the phase-flip channel with Kraus operators $K_0 = \frac{3}{5}I$ and $K_1 = \frac{4}{5}Z$. It is trace-preserving, fixes the positive-definite matrix I, and has off-diagonal eigenvalue $-\frac{7}{25}$. Thus every positive power has the same adjoint fixed-point algebra, while the transfer map is not idempotent. Consequently, the requested implication from the present support-algebra predicate is false even under the additional trace-preserving and faithful-fixed-point hypotheses.

The converse direction (ii) $\Rightarrow$ (i) in the proof of Theorem 4.14 uses a stronger coefficient argument. In Appendix C.4, lines 2046–2085, the authors compare the two BNT expansions coming from the one-site and two-site decompositions. The source identity `eq:algebra-appendix` is stated at lines 1933–1936 and used in this converse at lines 2048–2050:

$$\sum_\gamma \mathrm{tr}(\nu_\gamma^L) O_L(A_\gamma) = \sum_{\alpha,\beta,\gamma} \mathrm{tr}\big((\mu_\alpha \otimes \mu_\beta \otimes \chi_{\alpha,\beta,\gamma})^L\big) O_L(M_\gamma).$$

Since both A_γ and M_γ are BNT, the comparison first gives, up to relabelling (line 2053),

$$A_\gamma = e^{i\theta_\gamma} Z_\gamma M_\gamma Z_\gamma^{-1}.$$

Proposition 4.13 then removes the phase and identifies Z_γ with a unitary U_γ, as in line 2057, so

$$A_\gamma = U_\gamma M_\gamma U_\gamma^{-1}$$

with U_γ unitary. Since, for all sufficiently large L, the operators $O_L(A_\gamma) = O_L(M_\gamma)$ are linearly independent, and all coefficients are positive, the simple-matrix lemma (lines 1155–1163, used at line 2058) gives the multiset identity

$$\{\nu_{\gamma,k}\}_k = \{\mu_{\alpha,k_1}\mu_{\beta,k_2}\chi_{\alpha,\beta,\gamma,k_3}\}_{\alpha,\beta,k_1,k_2,k_3}.$$

Summing this identity gives

$$\mathrm{tr}(\nu_\gamma) = \sum_{\alpha,\beta} m_\alpha m_\beta c^{(1)}_{\alpha,\beta,\gamma} = m_\gamma.$$

Only after this trace equality is obtained do the normalizing maps in [CPGSV16, lines 2065–2085] have the form

$$\widetilde{R}_2\Big(\bigoplus_\gamma X_\gamma\Big) = \bigoplus_\gamma \frac{\nu_\gamma}{m_\gamma} \otimes X_\gamma, \qquad \widetilde{R}_1\Big(\bigoplus_\gamma X_\gamma\Big) = \bigoplus_\gamma \frac{\mu_\gamma}{m_\gamma} \otimes X_\gamma.$$

These are the maps used in the retained-sector composites $T_0 = \widetilde{R}_2 \widetilde{T} R_1$ and $S_0 = \widetilde{R}_1 \widetilde{S} R_2$. The first denominator is normalized by $\mathrm{tr}(\nu_\gamma) = m_\gamma$, proved above; the second by the defining equality $m_\gamma = \mathrm{tr}(\mu_\gamma)$ for the original block coefficient. When a vertical canonical form has a discarded zero sector, these composites are trace preserving only on the retained physical supports. Their completion to maps on the full physical matrix algebras is described in `cpgsv17_vertical_isometry_zero_sector.tex`.

The present algebra predicate `MPOTensor.HasBlockedAdjointFixedPointAlgebraTower` does not include these BNT-label coefficients, the positive $\chi_{\alpha,\beta,\gamma}$, or the length-one idempotent trace identity. It proves the fixed-point-space equation above, not the displayed BNT coefficient comparison. Thus a source-faithful construction must first obtain the BNT-label theorem witness from the MPDO tensor, use the trace identity to construct the retained-sector composites, and complete them on the discarded physical complements. The one-site transfer-retract criterion `MPOTensor.transferRetractData_one_iff_isZCL` concerns the separate transfer-map idempotence predicate and cannot supply these physical maps.

The scalar normalization step is now isolated independently of the still missing unconditional channel construction. The structure `MPOTensor.BNTMultiplicitySpectrumComparison` records the one-site traces and the sectorwise multiset equality obtained from the vertical BNT comparison at lines 2048–2058. Given this comparison, the BNT algebra clause proves

$$\mathrm{tr}(\nu_\gamma) = m_\gamma$$

through `MPOTensor.BNTAlgebraClause.twoMultiplicityTrace_eq_traceScalar`. Thus the idempotent law now discharges the precise scalar obligation at lines 2058–2064; it does not construct the comparison from the MPDO tensor and does not yet define the maps T and S.

The power-sum part of the preceding comparison is also isolated. The constructor `MPOTensor.BNTMultiplicitySpectrumComparison.ofEventuallyEqualPowerSums` proves that positivity and eventual sectorwise equality of the power sums imply the required multiset equality. It combines the geometric-sequence extrapolation already used in the BNT Fundamental Theorem with the unequal-cardinality power-sum identity, and therefore proves the power-sum step used at lines 2053–2058.

The tensor-level BNT comparison at lines 2048–2058 is now supplied by the literal two-site spectrum comparison and exact sector-gauge construction documented in `cpsv16_two_site_sector_unitary_gauge_gap.tex`. The predicate `MPOTensor.BNTAlgebraTensorClause` now chooses the one-site vertical canonical decomposition and identifies the operators and trace scalars in the abstract BNT algebra clause with $O_L(M_\alpha)$ and $m_\alpha = \sum_q \omega_{\alpha,q}$, respectively. It does not itself contain a second vertical canonical decomposition. Instead, the mixed-prefix argument constructs the required relabelling and simultaneous unitary identifications from the tensor-attached clause. The resulting positive multiset equality supplies the spectrum comparison without an additional mathematical hypothesis; the completion is used by `MPOTensor.BNTAlgebraTensorClause.isRFPViaTS`.

6 BNT-label coefficient objects

The same-length coefficient family $c^{(L)}_{\alpha,\beta,\gamma}$, indexed by fixed BNT labels, is represented separately from the blocked-basis coefficients.[1] When the Appendix C.4 construction has already produced a diagonal $\chi_{\alpha,\beta,\gamma}$-family, there is also a canonical coefficient family

$$c^{(L)}_{\alpha,\beta,\gamma} = \sum_r \chi^L_{\alpha,\beta,\gamma,r}.$$

This is recorded separately so that the χ-family can serve as the single source of those coefficients in the source-side special case.[2] A same-length BNT product predicate records the algebra identity

$$O_L(M_\alpha)O_L(M_\beta) = \sum_\gamma c^{(L)}_{\alpha,\beta,\gamma} O_L(M_\gamma)$$

for an abstract family of length-L BNT operators.[3] The idempotent condition

$$m_\gamma = \sum_{\alpha,\beta} c^{(1)}_{\alpha,\beta,\gamma} m_\alpha m_\beta$$

is also represented as a predicate on the trace scalars $m_\alpha = \mathrm{tr}(\mu_\alpha)$.[4] A positive BNT-label χ-witness consists of a length-independent diagonal family $\chi_{\alpha,\beta,\gamma}$, positivity of every diagonal entry, and the positive-length trace-power identity.[5] If the coefficients are chosen canonically from the same χ-family, then positivity of the diagonal entries alone gives such a witness.[6] A blocked-basis comparison predicate records that the chosen blocked multiplication coefficients are the pullback of the source-length BNT-label coefficients along a source label map from $\mathcal{A}_n$ and a target label map from $\mathcal{A}_{2n}$. The label-assignment datum is separated from the coefficient equality, so the Appendix C.3 construction target can be stated without also assuming the comparison formula.[7] This predicate inherits the second scope restriction above: its left hand side is expanded in a basis of $\mathcal{A}_{2n}$, while the source theorem's BNT product law is same-length. It therefore records the formal blocked-basis comparison predicate and is not itself the

[1] `MPOTensor.BNTLabelCoefficientFamily`.

[2] `MPOTensor.BNTLabelCoefficientFamily.ofChi`, `MPOTensor.BNTLabelCoefficientFamily.ofChi_coeff`, `MPOTensor.BNTLabelCoefficientFamily.ofChi_coeff_eq_trace_matrix_pow`, `MPOTensor.BNTLabelCoefficientFamily.ofChi_hasPositiveLengthChiTracePowerForm`.

[3] `MPOTensor.BNTLabelOperatorFamily.HasSameLengthProductForm`.

[4] `MPOTensor.BNTLabelTraceScalarFamily.HasIdempotentCoefficientForm`.

[5] `MPOTensor.PositiveBNTLabelChiTracePowerForm`.

[6] `MPOTensor.PositiveBNTLabelChiTracePowerForm.ofChi`.

[7] `MPOTensor.BNTBlockedBasisLabelAssignment`, `MPOTensor.BNTBlockedBasisLabelAssignment.blockedLabel`, `MPOTensor.BNTBlockedBasisCoefficientComparison`.

same-length product statement of Theorem 4.14(ii). The generic declarations in this section do not construct the tensor-attached objects by themselves. That construction is now supplied by `MPOTensor.BNTAlgebraTensorClause` and the reflected-target argument in `TNLean/MPS/MPDO/BNTAlgebraTensorClauseReflectedTarget.lean`. Once both the same-length BNT product form and a positive BNT-label χ-witness are available, the same-length product expansion with coefficients written as $\mathrm{tr}(\chi^L_{\alpha,\beta,\gamma})$ is formalized as a direct corollary.[8] Once both the comparison predicate and a positive BNT-label χ-witness are available, the coefficient-level blocked trace-power formula is formalized as a direct corollary.[9] The same hypotheses also give a positive blocked-basis χ-witness by pulling back the uniform BNT-label χ-family along the comparison maps.[10] Likewise, once the idempotent coefficient condition and the positive BNT-label χ-witness are available, the length-one idempotent identity is formalized with the coefficients rewritten as traces of the corresponding χ-matrices.[11]

The BNT-label objects above are now also gathered into one theorem-data record: fixed coefficients, same-length operators and product law, trace scalars and idempotent condition, positive BNT-label χ-witness, and blocked-basis comparison.[12] This record is not an additional mathematical assumption in the source theorem; it is the single formal object collecting the source data supplied by the Appendix C.3–C.4 construction from an MPDO tensor. There is also a construction for the source-side special case in which the coefficient family is the one canonically determined by the same χ-family; it takes the product law, idempotent law, and blocked-basis comparison for those canonical coefficients.[13] From such data, the source predicates themselves and the corresponding equation-level consequences are available: the same-length product law, the idempotent scalar law, the BNT-label χ-positivity and trace-power predicates, the coefficient trace-power formula, positivity of the diagonal χ-entries, the same-length chi-trace product law, the chi-trace idempotent law, the blocked-basis coefficient trace-power formula, the blocked-basis coefficient comparison, the positive blocked χ-witness, the pulled-back blocked trace-power predicate, and the pulled-back blocked-basis χ-family with positive entries and

[8] `MPOTensor.BNTLabelOperatorFamily.HasSameLengthProductForm.eq_sum_chi_trace_pow`, `MPOTensor.BNTLabelOperatorFamily.HasSameLengthProductForm.eq_sum_ofChi_trace_pow`.

[9] `MPOTensor.BNTBlockedBasisCoefficientComparison.blocked_coeff_eq_trace_pow`, `MPOTensor.BNTBlockedBasisCoefficientComparison.blocked_coeff_eq_ofChi_trace_pow`, `MPOTensor.BNTBlockedBasisCoefficientComparison.blocked_coeff_eq_pulledBlockedChi_trace_pow`.

[10] `MPOTensor.BNTBlockedBasisCoefficientComparison.pulledBlockedChiFamily_toDiagonal_of_pos`, `MPOTensor.BNTBlockedBasisCoefficientComparison.pulledBlockedChi_tracePowerCoeff_of_pos`, `MPOTensor.BNTBlockedBasisCoefficientComparison.pulledBlockedChi_dim_of_pos`, `MPOTensor.BNTBlockedBasisCoefficientComparison.pulledBlockedChi_trace_matrix_pow_of_pos`, `MPOTensor.BNTBlockedBasisCoefficientComparison.toPositiveBlockedStructureChiTracePowerForm`.

[11] `MPOTensor.BNTLabelTraceScalarFamily.HasIdempotentCoefficientForm.eq_sum_chi_trace`, `MPOTensor.BNTLabelTraceScalarFamily.HasIdempotentCoefficientForm.eq_sum_ofChi_trace`.

[12] `MPOTensor.BNTLabelTheoremData`.

[13] `MPOTensor.BNTLabelTheoremData.ofChi`.

trace-power formula are immediate consequences.[14]

The same data can be stated in the existential form closest to the paper: one object records the BNT-label type, the same-length operator spaces, their algebraic structure, and the corresponding theorem data.[15] The existential nonempty statement can be unpacked directly into a concrete witness carrying the source predicates, the corresponding same-length and idempotent equations written with $c^{(L)}_{\alpha,\beta,\gamma}$, and the same two equations with coefficients written as χ-traces; the blocked-basis χ-family is also identified as the pullback of the BNT-label χ-family along the comparison maps. The existential statement has the analogous canonical construction from a positive χ-family, again leaving the product, idempotent, and blocked-comparison statements as inputs rather than reconstructing them from an MPDO tensor.[16] Such a witness immediately gives the source predicates and their consequences: the coefficient trace-power formula, the same-length product law, the idempotent scalar law, positivity of the diagonal χ-entries, the blocked-basis coefficient comparison, the same-length chi-trace product law, the chi-trace idempotent law, the blocked-basis coefficient trace-power formula, the positive blocked-basis χ-witness, and the pulled-back

[14] MPOTensor.BNTLabelTheoremData.same_length_product_form, MPOTensor.BNTLabelTheoremData.idempotent_coefficient_form, MPOTensor.BNTLabelTheoremData.positive_chi_pos_entries, MPOTensor.BNTLabelTheoremData.positive_chi_trace_power, MPOTensor.BNTLabelTheoremData.labelAssignment, MPOTensor.BNTLabelTheoremData.sourceLabel, MPOTensor.BNTLabelTheoremData.targetLabel, MPOTensor.BNTLabelTheoremData.same_length_product_eq_sum, MPOTensor.BNTLabelTheoremData.idempotent_eq_sum, MPOTensor.BNTLabelTheoremData.coeff_eq_trace_pow, MPOTensor.BNTLabelTheoremData.coeff_eq_ofChi_coeff, MPOTensor.BNTLabelTheoremData.same_length_product_form_ofChi, MPOTensor.BNTLabelTheoremData.idempotent_coefficient_form_ofChi, MPOTensor.BNTLabelTheoremData.same_length_product_eq_sum_ofChi, MPOTensor.BNTLabelTheoremData.idempotent_eq_sum_ofChi, MPOTensor.BNTLabelTheoremData.chi_entry_pos, MPOTensor.BNTLabelTheoremData.blocked_coeff_eq, MPOTensor.BNTLabelTheoremData.blockedComparison_ofChi, MPOTensor.BNTLabelTheoremData.blocked_coeff_eq_ofChi, MPOTensor.BNTLabelTheoremData.same_length_product_eq_sum_chi_trace_pow, MPOTensor.BNTLabelTheoremData.idempotent_eq_sum_chi_trace, MPOTensor.BNTLabelTheoremData.blocked_coeff_eq_trace_pow, MPOTensor.BNTLabelTheoremData.toPositiveBlockedStructureChiTracePowerForm, MPOTensor.BNTLabelTheoremData.positiveBlockedChi, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_toDiagonal_of_pos, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_tracePowerCoeff_of_pos, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_dim_of_pos, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_trace_matrix_pow_of_pos, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_tracePower, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_posEntries, MPOTensor.BNTLabelTheoremData.positiveBlockedChi_entry_pos, MPOTensor.BNTLabelTheoremData.blocked_coeff_eq_positiveBlockedChi_trace_pow.

[15] MPOTensor.BNTLabelTheoremWitness, MPOTensor.HasBNTLabelTheoremWitness, MPOTensor.BNTLabelTheoremWitness.toTheoremData, MPOTensor.BNTLabelTheoremWitness.coeffs, MPOTensor.BNTLabelTheoremWitness.operators, MPOTensor.BNTLabelTheoremWitness.traceScalars, MPOTensor.BNTLabelTheoremWitness.positiveChi, MPOTensor.BNTLabelTheoremWitness.blockedComparison, MPOTensor.BNTLabelTheoremWitness.same_length_product_form, MPOTensor.BNTLabelTheoremWitness.idempotent_coefficient_form, MPOTensor.BNTLabelTheoremWitness.positive_chi_pos_entries, MPOTensor.BNTLabelTheoremWitness.positive_chi_trace_power, MPOTensor.BNTLabelTheoremWitness.labelAssignment, MPOTensor.BNTLabelTheoremWitness.sourceLabel, MPOTensor.BNTLabelTheoremWitness.targetLabel, MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi.

[16] MPOTensor.BNTLabelTheoremWitness.ofChi.

blocked trace-power predicate and χ-family for the chosen algebra-structure data.[17]

The witness must not be constructed from an arbitrary MPDO tensor or from the present support-algebra predicate. Such a construction would imply the false converse exhibited above. In Theorem 4.14(ii), the BNT-label coefficients, their positive trace-power representation, and the length-one idempotent identity are the hypothesis of the converse, not consequences of the mere existence of stabilized adjoint fixed-point algebras.

7 Length-independent zipper specialization

For a fusion-isometry family already satisfying Theorem 4.14(iii), suppose that its positive trace-power coefficients are independent of the positive chain length. The observation at line 1010 then gives

$$\chi_{\alpha,\beta,\gamma} = 1_{r_{\alpha,\beta,\gamma}}.$$

Substitution in the weighted fusion identity gives both zipper equations and the completeness equation

$$(M_\alpha M_\beta)^{ij} = U^\dagger_{\alpha,\beta} \left(\bigoplus_\gamma 1_{r_{\alpha,\beta,\gamma}} \otimes M_\gamma^{ij} \right) U_{\alpha,\beta}.$$

These statements are formalized by `MPOTensor.BNTFusionIsometryFamily.fusionIsometry_mul_mulTensor_of_lengthIndependent`, `MPOTensor.BN`

[17] `MPOTensor.BNTLabelTheoremWitness.same_length_product_form`, `MPOTensor.BNTLabelTheoremWitness.idempotent_coefficient_form`, `MPOTensor.BNTLabelTheoremWitness.positive_chi_pos_entries`, `MPOTensor.BNTLabelTheoremWitness.positive_chi_trace_power`, `MPOTensor.BNTLabelTheoremWitness.sourceLabel`, `MPOTensor.BNTLabelTheoremWitness.targetLabel`, `MPOTensor.BNTLabelTheoremWitness.coeff_eq_trace_pow`, `MPOTensor.BNTLabelTheoremWitness.coeff_eq_ofChi_coeff`, `MPOTensor.BNTLabelTheoremWitness.same_length_product_form_ofChi`, `MPOTensor.BNTLabelTheoremWitness.idempotent_coefficient_form_ofChi`, `MPOTensor.BNTLabelTheoremWitness.same_length_product_eq_sum_ofChi`, `MPOTensor.BNTLabelTheoremWitness.idempotent_eq_sum_ofChi`, `MPOTensor.BNTLabelTheoremWitness.same_length_product_eq_sum`, `MPOTensor.BNTLabelTheoremWitness.idempotent_eq_sum`, `MPOTensor.BNTLabelTheoremWitness.chi_entry_pos`, `MPOTensor.BNTLabelTheoremWitness.blocked_coeff_eq`, `MPOTensor.BNTLabelTheoremWitness.blockedComparison_ofChi`, `MPOTensor.BNTLabelTheoremWitness.blocked_coeff_eq_ofChi`, `MPOTensor.BNTLabelTheoremWitness.same_length_product_eq_sum_chi_trace_pow`, `MPOTensor.BNTLabelTheoremWitness.idempotent_eq_sum_chi_trace`, `MPOTensor.BNTLabelTheoremWitness.blocked_coeff_eq_trace_pow`, `MPOTensor.BNTLabelTheoremWitness.toPositiveBlockedStructureChiTracePowerForm`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_toDiagonal_of_pos`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_tracePowerCoeff_of_pos`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_dim_of_pos`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_trace_matrix_pow_of_pos`, `MPOTensor.HasBNTLabelTheoremWitness.positive_blocked_chi_witness`, `MPOTensor.HasBNTLabelTheoremWitness.exists_blocked_chi_trace_power_form`, `MPOTensor.HasBNTLabelTheoremWitness.exists_blocked_coeff_eq_trace_pow`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_tracePower`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_posEntries`, `MPOTensor.BNTLabelTheoremWitness.positiveBlockedChi_entry_pos`, `MPOTensor.BNTLabelTheoremWitness.blocked_coeff_eq_positiveBlockedChi_trace_pow`.

TFusionIsometryFamily.mulTensor_mul_fusionIsometry_conjTranspose_of_lengthIndependent, and MPOTensor.BNTFusionIsometryFamily.mulTensor_eq_conjTranspose_mul_unweightedDirectSum_mul_of_lengthIndependent. They are the identity-weight specialization of the equations labelled zippercondition2 and inversegaugeone in the "Fusion tensors" section of the arXiv:1511.08090 source file AnyonsPEPS.tex.

This result has an explicit scope restriction: it assumes the special length-independent coefficient system discussed at lines 995–1010, rather than proving that every renormalization fixed point has such coefficients. For this specialization, the canonical bond reassociation and the two iterated fusion isometries now define the comparison of the full left- and right-associated direct sums. The formalized letterwise identity states that this comparison intertwines the two unweighted direct sums. The matrix is oriented from the right-associated sum to the left-associated sum, hence oppositely to the printed F-matrix convention in equation Fmove; it is not identified with that F-matrix.

The two products of the full comparison with its adjoint are now identified exactly:

$$CC^\dagger = U^{\mathrm{L}}(U^{\mathrm{L}})^\dagger, \qquad C^\dagger C = U^{\mathrm{R}}(U^{\mathrm{R}})^\dagger.$$

These are the range projections of the two iterated fusion isometries. They are formalized by MPOTensor.BNTFusionIsometryFamily.tripleFusionComparison_mul_conjTranspose and MPOTensor.BNTFusionIsometryFamily.conjTranspose_mul_tripleFusionComparison. The present fusion assumptions give $(U^{\mathrm{L}})^\dagger U^{\mathrm{L}} = 1$ and $(U^{\mathrm{R}})^\dagger U^{\mathrm{R}} = 1$. Without further information these identities make the full comparison only a partial isometry between the two ranges. The invertible total fusion matrix of arXiv:1511.08090, lines 187–191, supplies the corresponding completeness in the source argument.

The simultaneous inverse relation at line 269 of the arXiv:1511.08090 source,

$$B_d^+ B_{d'} = \delta_{d,d'} 1,$$

separates all final-label blocks at once. Its exact finite-word consequence is now an explicit hypothesis in MPOTensor.BNTFusionIsometryFamily.HasFinalLabelSelectorWords. Under this hypothesis at a positive word length, the selector polynomial also supplies the missing completeness. Every nonempty word in the unweighted direct sum lies in the range of the corresponding fusion isometry. Summing the selector polynomials over the final label gives the identity on that direct sum. Consequently

$$U^{\mathrm{L}}(U^{\mathrm{L}})^\dagger = 1, \qquad U^{\mathrm{R}}(U^{\mathrm{R}})^\dagger = 1, \qquad CC^\dagger = C^\dagger C = 1.$$

These statements are formalized by MPOTensor.BNTFusionIsometryFamily.fusionIsometry_mul_conjTranspose_eq_one_of_lengthIndependent_of_selectorWords, MPOTensor.BNTFusionIsometryFamily.leftFusionIsometry_mul_conjTranspose_eq_one_of_lengthIndependent_of_selectorWords, MPOTensor.BNTFusionIsometryFamily.rightFusionIsometry_mul_conjTranspose_eq_one_of_lengthIndependent_of_selectorWords, `MPOTensor.BNTFusionIsometryFamily.tripleFusionComparison_mul_conjTranspose_eq_one_of_lengthIndependent_of_selectorWords`, and `MPOTensor.BNTFusionIsometryFamily.conjTranspose_mul_tripleFusionComparison_eq_one_of_lengthIndependent_of_selectorWords`. The positive-length restriction is necessary because the telescoping identity uses at least one conjugated letter.

Under the same selector hypothesis, `MPOTensor.BNTFusionIsometryFamily.tripleFusionComparison_finalSector_submatrix_eq_zero` proves that the full comparison has zero corner between two distinct final labels. The proof takes one rectangular bond-space corner of the letterwise intertwining identity, extends it to words, and applies the selector polynomial.

Under the additional assumption that the selected final tensor is injective at the present blocking, `MPOTensor.BNTFusionIsometryFamily.exists_tripleFusionComparison_finalSector_eq_kronecker_one_of_separation` combines this separation with the amplified-commutant argument. For every selected final label ε, it produces a rectangular multiplicity matrix F_ε for which the reindexed diagonal corner is

$$C^{\varepsilon,\varepsilon}_{\alpha,\beta,\gamma} = F_\varepsilon \otimes 1_{D_\varepsilon},$$

and retains the vanishing of $C^{\varepsilon,\varepsilon'}_{\alpha,\beta,\gamma}$ when $\varepsilon' \neq \varepsilon$. This is the bond-factor conclusion appearing at line 277 of the arXiv:1511.08090 source, but the comparison used here is oriented oppositely to the printed convention.

This remains a scope-restricted result when only the abstract fusion-isometry family is supplied. Once its labelled tensors are linked to the source BNT witness, however, the simultaneous final-label selectors are derived rather than assumed, as described below. One-letter injectivity at a chosen present blocking remains separate. The factorization theorem alone permits a rectangular matrix F_ε.

There is now a separate numerical squareness result. Under length independence and eventual linear independence, `MPOTensor.BNTFusionIsometryFamily.card_leftFinalMultiplicity_eq_card_rightFinalMultiplicity_of_lengthIndependent` specializes the coefficient associativity relation to

$$\sum_\delta r^\delta_{\alpha,\beta} r^\varepsilon_{\delta,\gamma} = \sum_\delta r^\delta_{\beta,\gamma} r^\varepsilon_{\alpha,\delta}.$$

This is exactly the equality of fixed-final multiplicity dimensions at lines 238–241 of the arXiv:1511.08090 source. It proves that the matrix F_ε has square shape when both the factorization theorem and the additional eventual-independence hypothesis apply. It does not prove that F_ε is invertible by itself.

Under the positive-length selector hypothesis above, the range-completeness obstruction from lines 181–191 has now been removed. The two-sided unitarity of the full comparison, together with the vanishing of its off-diagonal final-label corners, implies mathematically that every fixed-final bond-space corner is two-sided unitary. The two identities are now formalized by `MPOTensor.BNTFusionIsometryFamily.tripleFusionComparison_finalSector_mul_conjTrans pose_eq_one and MPOTensor.BNTFusionIsometryFamily.conjTranspose_mul_tripleFusionComparison_finalSector_eq_one.

Combining these corner identities with $C^{\varepsilon,\varepsilon} = F_\varepsilon \otimes 1_{D_\varepsilon}$ gives the two unitary identities for F_ε by cancelling the identity factor. This is formalized by MPOTensor.BNTFusionIsometryFamily.exists_tripleFusionComparison_finalSector_eq_kronecker_one_and_unitary under the explicit assumption $0 < D_\varepsilon$. The dimensional qualification is necessary because injectivity is vacuous when $D_\varepsilon = 0$. Thus the fixed-final invertibility obstruction has been removed under the simultaneous-selector, present-blocking injectivity, and positive-bond-dimension hypotheses.

For an arbitrary coefficient-form BNT, the positive dimensions, a common positive blocking length, and the simultaneous one-letter span of the blocked representatives are now derived by MPSTensor.IsCPSVBasisOfNormalTensors.exists_blocked_wordTupleSpanTop_one. In particular, that theorem supplies a span containing every tuple that is the identity on one labelled block and zero on the others. One-letter injectivity and selector coefficients can therefore be extracted after the same common blocking, without adding either property to the definition of a BNT.

The pre-blocking common word span is exposed by MPSTensor.IsCPSVBasisOfNormalTensors.exists_positive_wordTupleSpanTop. Its selector coefficients bridge directly to the fusion-family formulation. Consequently MPOTensor.BNTFusionIsometryFamily.fusionIsometry_mul_conjTranspose_eq_one_of_bnt_of_lengthIndependent proves

$$U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = I$$

from the BNT witness and the length-independent fusion data, with no selector or coisometry hypothesis. The selector length supplied by the BNT argument is positive; the empty-chain case is not used. This coisometry proof is a short derived argument from the cited source ingredients, not a proof printed verbatim in either paper.

For the positive-diagonal fusion family, under the explicit hypothesis that identifies the labelled doubled-index tensors with a BNT witness, transport through the same positive block length is now formalized by MPOTensor.BNTFusionIsometryFamily.exists_positive_block_with_injective_fusion. The theorem derives the common one-letter span and injectivity of every blocked labelled tensor, and proves the blocked conjugation identity, both zipper orientations, and literal reconstruction. Its multiplicity is exactly the dimension of the specialized chi space. The blocking length remains positive. The empty chain is neither needed nor used: although its conjugation formula would reduce to coisometry, the equality above is instead derived from positive-length BNT selectors.

The simultaneous block-letter inverse is extracted from the same one-letter product-algebra span by MPSTensor.WordTupleSpanTop.exists_mpo_block_left_inverse. Reindexing the product bond space as a pair, the adjoint of $U_{\alpha,\beta}$ is the synthesis map and $U_{\alpha,\beta}$ is the analysis map. These maps and all the derived blocked identities are assembled by `MPOTensor.BNTFusionIsometryFamily.exists_blocked_completeZipperFusionFamily`. Consequently the existing printed F-matrix and pentagon theorem apply to the length-independent source BNT fusion family without an additional injectivity, selector, inverse, or coisometry hypothesis.

The labelled fusion-isometry family and its identification with the BNT of the vertical canonical form are now obtained from the RFP tensor by `MPOTensor.HasBNTFusionTensorClause.of_isRFPViaTS`. This closes the corrected active-support form of the Appendix C.3–C.4 implication. It is logically separate from the complete-zipper construction below.

Independently of this positive-diagonal specialization, the source-faithful complete zipper fusion structure is now formalized by `MPOTensor.CompleteZipperFusionFamily`. Its data are the integer fusion multiplicities, independently specified synthesis and analysis tensors, their biorthogonality, the product-letter reconstruction equation `inversegaugeone`, the two zipper equations, injectivity of the labelled blocks, and the simultaneous one-letter inverse from lines 269–277 of `AnyonsPEPS.tex`. In particular, the product-MPO projector discussed at lines 181–191 is not identified with the identity on the ambient product bond space. Applying `inversegaugeone` to the two associative decompositions at lines 238–247 reconstructs every triple-product letter through either fusion tree. Contracting these equal reconstructions with the simultaneous inverse derives equality of the two triple-product support projectors. These assumptions suffice to construct the printed-orientation matrix `MPOTensor.CompleteZipperFusionFamily.printedFMatrix`, whose rows are right-tree multiplicities and whose columns are left-tree multiplicities. The exact equation `Fmove` and both inverse identities are proved by `MPOTensor.CompleteZipperFusionFamily.rightTripleSynthesis_mul_printedFMatrix`, `MPOTensor.CompleteZipperFusionFamily.printedFMatrix_mul_inversePrintedFMatrix`, and `MPOTensor.CompleteZipperFusionFamily.inversePrintedFMatrix_mul_printedFMatrix`. Thus no length-independent coefficient system, selector words, or additional present-blocking injectivity hypothesis is imposed on this source-labelled result.

This complete zipper fusion structure does not close the gap for the positive-diagonal family: constructing its complete zipper data from the assumptions of arXiv:1606.00608 remains open. Within the complete zipper family itself, the two-sided printed F-move and the genuine fourfold, entrywise pentagon equation of `AnyonsPEPS.tex`, lines 279–283, are now formalized.

The five categorical fourfold multiplicity spaces, their fusion maps, and the five lifted printed-F edges are now recorded by `MPOTensor.CompleteZipperFusionFamily.FourfoldLeftAssocMultiplicity` through `MPOTensor.CompleteZipperFusionFamily.twoEdgePrintedFMatrix`. These spaces use the integer fusion multiplicities N_{ab}^{c} of the complete zipper family. They are not identified with the positive-diagonal weighted coordinates of the restricted family. The five elementary F-moves identify the three-edge and two-edge composites with the same expansion in the right-associated fourfold fusion map. Its explicit analysis map cancels that expansion, proving the forward equality `MPOTensor.`

`CompleteZipperFusionFamily.threeEdgePrintedFMatrix_eq_twoEdgePrintedFMatrix`. The inverse edge matrices then give both `MPOTensor.CompleteZipperFusionFamily.threeEdgeInversePrintedFMatrix_eq_twoEdgeInversePrintedFMatrix` and the literal indexed equation (33), `MPOTensor.CompleteZipperFusionFamily.inversePrintedFMatrix_pentagon`.

Local fixes in the fusion-tensor display. The prose at line 161 of `AnyonsPEPS.tex` displays the arrow for $X^c_{ab,\mu}$ in the direction from the product bond space to the c bond space. The equations at lines 163–200 and the printed F-move at lines 248–251 require the reverse direction, $X^c_{ab,\mu} : \mathbb{C}^{D_c} \to \mathbb{C}^{D_a} \otimes \mathbb{C}^{D_b}$, with $X^{c+}_{ab,\mu}$ as its left inverse. The formalization uses this equation-compatible orientation. The same display writes the Kronecker factor in $X^{d+}_{ab,\nu} X^c_{ab,\mu}$ as δ_{de}, although no label e occurs in the relation. Its typed form is δ_{dc} (equivalently δ_{cd}), which is the index relation used in the formalization.

Local fix in the pentagon index orientation. Equation `Fmove` at lines 248–251 places the right-tree indices above and the left-tree indices below: $(F^{abc}_d)^{f\lambda\sigma}_{e\mu\nu}$. In contrast, every factor in equation `pentagoneq` at lines 280–283 places the left-tree indices above and the right-tree indices below. With the multiplicity types fixed by the fusion trees at line 279, the latter entries are entries of the inverse F-matrix. The forward edge matrices in the formalization follow `Fmove`; equivalently, the literal upper/lower placement in `pentagoneq` is read using `MPOTensor.CompleteZipperFusionFamily.inversePrintedFMatrix`. Both the forward identity and this inverse, literally indexed form are stated by the two matrix equalities and the entrywise theorem above.

8 Completion status for Theorem 4.14

The tensor-attached BNT route described above is complete. The mixed one-site/two-site prefix argument gives the unitary relabelling and positive multiset identity for the $\nu_{\gamma,k}$. The retained-sector composites $T_0 = \widetilde{R}_2 \widetilde{T} R_1$ and $S_0 = \widetilde{R}_1 \widetilde{S} R_2$, together with the zero-sector measure-and-prepare completion, give the two trace-preserving completely positive maps of Definition 4.1. This implication is `MPOTensor.BNTAlgebraTensorClause.isRFPViaTS`.

Under the literal CPSV canonical-form and MPDO hypotheses, the two directions are combined in `MPOTensor.isRFPViaTS_iff_hasBNTAlgebraTensorClause`. The corresponding equivalence `MPOTensor.isRFPViaTS_iff_hasBNTFusionTensorClause` uses the documented active-support correction of Theorem 4.14(iii), not its unrestricted printed form. These results do not identify the positive-diagonal family with a complete zipper fusion family and do not supply the stronger categorical F-matrix data discussed above. The one-site criterion for `MPOTensor.TransferRetractData` also remains separate: it is equivalent to doubled-index transfer-map idempotence rather than Definition 4.1.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.