

The Finite Periodic One-Species PVBS Gap

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Verdict: scope-restriction (open).

1 Source and scope

Bachmann and Nachtergaele, arXiv:1112.4097, Section II after equation (8), give the one-particle dispersion $\epsilon_i(k) = 1 - 2\lambda_i \cos(k + \theta_{i0})/(1 + \lambda_i^2)$. They then conjecture that the exact thermodynamic gap for the multi-species model is the minimum of $(1 - \lambda_i)^2/(1 + \lambda_i^2)$ over species. The result proved here is the finite periodic one-species, zero-phase case, with positive hopping $\mu \neq 1$. It is not a formal proof of that general thermodynamic conjecture.

2 Normalization and the finite ring

The local parent interaction is exactly

$$h_\mu = |11\rangle\langle 11| + \frac{(|01\rangle - \mu|10\rangle)(\langle 01| - \mu\langle 10|)}{1 + \mu^2}. \quad (1)$$

On a ring of $N \geq 2$ sites, sum this interaction over the N ordered windows $(i, i + 1)$ modulo N . For $N = 2$, this includes both orientations; a convention with only one undirected bond is not covered by the claimed constant. The local formula is identified with the repository's actual orthogonal-projector parent interaction, rather than defining a replacement Hamiltonian.

Let $\mathcal{N}$ count occupied sites. The exact periodic identity is

$$H_\mu = a(\mu)H_1 + c(\mu)\mathcal{N}, \quad a(\mu) = \frac{2\mu}{1 + \mu^2}, \quad c(\mu) = \frac{(\mu - 1)^2}{1 + \mu^2}. \quad (2)$$

Indeed, the hopping coefficients agree, while the difference in diagonal terms is $[(\mu^2 - \mu)n_L + (1 - \mu)n_R]/(1 + \mu^2)$ at each bond. Periodic summation identifies the two occupation sums.¹

¹Formal declarations: `MPSTensor.parentInteraction_pvbs_two_apply`, `MPSTensor.parentHamiltonianES_pvbs_decomposition`, and `MPSTensor.parentHamiltonianES_pvbs_gap_exact`.

3 Lower bound and attainment

For $\mu \geq 0$, the critical Hamiltonian H_1 and the coefficient $a(\mu)$ are nonnegative. Every configuration except the vacuum has at least one particle. Thus for a vector with zero vacuum amplitude,

$$\langle v, H_\mu v \rangle \geq c(\mu) \|v\|^2, \quad \|H_\mu v\| \geq c(\mu) \|v\|. \quad (3)$$

In particular this holds on the orthogonal complement of the actual kernel, since the vacuum is a zero-energy vector.

The uniform one-particle vector $W_N = \sum_j |0 \cdots 1_j \cdots 0\rangle$ is annihilated by every critical local term and has particle number one. Hence $H_\mu W_N = c(\mu) W_N$. For positive $\mu \neq 1$, its eigenvalue is positive, and symmetry makes W_N orthogonal to the kernel. This nonzero vector attains the bound, so the exact finite-ring gap is $c(\mu)$.

At $\mu = 1$, the same vector joins the ground space. The vanishing of $c(1)$ does not say that a fixed finite ring has zero gap above its enlarged kernel. The separate Fourier-magnon argument below supplies nonzero low-energy excitations as N grows.

4 Critical thermodynamic limit

On the entire one-particle sector at $\mu = 1$, the double-occupation projector vanishes and the local interaction is the singlet exchange term. The actual parent therefore acts by the normalized cycle Laplacian on the particle amplitudes. The first Fourier character gives a nonzero vector ψ_N with

$$H_1 \psi_N = \eta_N \psi_N, \quad \eta_N = 1 - \cos(2\pi/N) > 0 \quad (N \geq 2). \quad (4)$$

Since the Hamiltonian is symmetric and $\eta_N > 0$, this vector is orthogonal to its entire kernel. As $\eta_N \rightarrow 0$, any proposed positive norm-gap bound on all sufficiently large rings is contradicted by this family. This is the finite-periodic-volume criterion of thermodynamic gaplessness, without constructing an infinite-volume operator or claiming that any fixed finite ring has zero gap above its full kernel.²

5 Remaining scope

The coefficient and norm-gap theorems prove the positive-hopping one-species finite-ring assertion. The general multi-species thermodynamic conjecture, nonzero phases, and open-chain spectral-gap estimates remain outside this statement. The open verdict tracks that broader source scope, not a missing hypothesis in the finite one-species result.

²Formal declarations: `MPSTensor.parentHamiltonianES_pvbs_critical_fourierMagnon` and `MPSTensor.not_exists_eventual_uniform_gap_pvbs_critical`.