

Simultaneous Left Inverse of Injective Operator Blocks

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Verdict: false-source (resolved).

1 Source statement

To extract the F -symbols, Bultinck et al. apply a left inverse of the labelled block tensors to both sides of an identity between sums over the final label d [BMW⁺17]. They write that injectivity of the single-block tensors implies that the matrix $(B_d^{ij})_{\alpha\beta}$, with rows indexed by the physical pair ij and columns by the bond pair $\alpha\beta$, has a left inverse B_d^+ with

$$B_d^+ B_{d'} = \delta_{dd'} \mathbb{K}_{\chi_d} \otimes \mathbb{K}_{\chi_d}. \quad (1)$$

¹ The case $d = d'$ is injectivity of B_d . The case $d \neq d'$ asks that one left inverse for B_d annihilates every other block.

2 Counterexample

Relation (1) for all labels together says that the one-site map

$$\Phi : \mathbb{C}^{p^2} \longrightarrow \bigoplus_d M_{\chi_d}, \quad v \longmapsto \left(\sum_{i,j} v_{ij} B_d^{ij} \right)_d, \quad (2)$$

is surjective. The row of (B_d^+) indexed by (α, β) is a preimage of the matrix unit $E_{\alpha\beta}$ in the summand d . Surjectivity forces

$$p^2 \geq \sum_d \chi_d^2. \quad (3)$$

Injectivity of each block separately does not imply it. Take physical dimension $p = 2$. Let B_1 have bond dimension $\chi_1 = 2$ and let its four letters B_1^{ij} form a basis of M_2 . Let B_2 have bond dimension $\chi_2 = 1$ with some nonzero letter. Both

¹Source traceability: `References/1511.08090/AnyonsPEPS.tex`, line 269, within the derivation at lines 252–277.

blocks are injective. They are not gauge equivalent, since their bond dimensions differ. The map Φ goes from $\mathbb{C}^4$ to $M_2 \oplus M_1 \cong \mathbb{C}^5$, so its rank is at most $4 < 5$. Hence no simultaneous left inverse exists. The data are nondegenerate: every block is injective and every weight is nonzero.

3 Missing hypothesis

The step needs joint linear independence of the labelled blocks at one site, that is, surjectivity of (2). Injectivity of each block together with pairwise non-equivalence does not supply it.

4 Elimination after blocking

Blocking removes the hypothesis. Let the blocks $B_1, \dots, B_m$, read as tensors on the doubled physical alphabet, be normal, of positive bond dimension, and pairwise not gauge equivalent up to a nonzero scalar: there is no invertible X and no $\zeta \neq 0$ with $B_{d'}^{ij} = \zeta X B_d^{ij} X^{-1}$ for all ij when $d \neq d'$. Then there is $L > 0$ such that the tuples $(B_1^w, \dots, B_m^w)$, for the words w of length L , span $\bigoplus_d M_{\chi_d}$. This is the block-injectivity argument for a basis of normal tensors [CPGSV16], which uses only normality and pairwise non-equivalence up to a scalar.² Hence the map (2) of the blocked tensors $B_d^{[L]}$ is surjective, and (1) holds at one blocked site.

The scalar in the non-equivalence cannot be omitted. For $B_2 = -B_1$ the words of length L satisfy $B_2^w = (-1)^L B_1^w$, so the tuples span only a copy of M_{χ_1} at every length, although B_1 and B_2 are in general not gauge equivalent.

The rest of the extraction survives blocking with the same fusion tensors. If the letters of the product $O_a O_b$ are, in suitable coordinates, block diagonal with blocks B_c repeated N_{ab}^c times, then so are all words of these letters, with the same coordinates and the blocks replaced by the corresponding words of B_c . The split compressions of the pairwise products, and so the fusion tensors, the reconstruction identity, and both zipper identities, therefore hold for the blocked tensors without change. The complete zipper data of the source thus exist for the blocked tensors at one common positive length, as in the common blocking of the source.^{3 4}

5 Remaining restriction

The one-site statement (1) remains false in general, by the counterexample above, and it is not formalized. What is formalized is its blocked form, under

²Source traceability: `Papers/1606.00608/MPD0-22-12-17-2.tex`, lines 317–345.

³Source traceability: `References/1511.08090/AnyonsPEPS.tex`, lines 427–431.

⁴Formal statements: `MPSTensor.exists_positive_wordTupleSpanTop_of_isNormal`, `MPOTensor.exists_blockTensor_isMPOBlockLeftInverse`, `MPOTensor.CompleteZipperFusionFamily.ofCompressionBlocked`, and `MPOTensor.CompleteZipperFusionFamily.ofStarBlocked`.

pairwise non-equivalence up to a nonzero scalar. The unblocked construction from split compressions keeps the simultaneous left inverse as an input.⁵

References

- [BMW⁺17] Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Anyons and matrix product operator algebras. *Annals of Physics*, 378:183–233, 2017.
- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.

⁵Formal definition: `MPOTensor.CompleteZipperFusionFamily.ofCompression`.