

Ising String-Net Operators: the Normalization of the σ Tensor

2026-09-24, updated 2026-09-27

Verdict: local-correction (resolved).

1 The source statement

Appendix D.2, “Ising string-net”, of Ref. [BMW⁺17] lists the labels $1, \sigma, \psi$ of the Ising category, the fusion rules with the single nontrivial rule $\sigma \times \sigma = 1 + \psi$, the quantum dimensions $d_1 = d_\psi = 1$, $d_\sigma = \sqrt{2}$, and the nontrivial F -symbols $F_{\sigma ab}^{\sigma\sigma\sigma} = \pm 1/\sqrt{2}$ and $F_{\sigma\sigma\sigma}^{\psi\sigma\psi} = F_{\psi\sigma\sigma}^{\sigma\psi\sigma} = -1$. It then says that the matrix product operator tensors are built from the G -symbols $G_{def}^{abc} = F_{def}^{abc}/(v_e v_f)$, $v_i = \sqrt{d_i}$, “similarly as for the Fibonacci model”. For the Fibonacci model the source states that the operator blocks satisfy the fusion rules; for the Ising model it prints only the fusion rules of the category.¹ The Ising analogue of the Fibonacci statement is, for the three blocks O_1, O_ψ, O_σ on a ring of N sites, the algebra

$$O_\psi^2 = O_1, \quad O_\sigma O_\psi = O_\psi O_\sigma = O_\sigma, \quad O_\sigma^2 = O_1 + O_\psi, \quad (1)$$

with O_1 the unit on the admissible closed paths. The same abstract algebra is printed in Ref. [AMF16] for a different realization, the defect operators of the Ising lattice model acting on spin and dual-spin configurations.²

2 The chosen normalization

The formal tensors have physical index the fusion-tree label (x', ρ, x) with $x \in x' \otimes \rho$ (ten labels) and bond index the pair of outer labels, and their letters are the F -symbols $[F_y^{ax'\rho}]_{y'x}$ themselves, without the factors $v_e v_f$ of the G -symbols. In addition the σ tensor is stored as $\sqrt{2}A_\sigma$, so that its entries are $0, \pm 1, \pm\sqrt{2}$ rather than $0, \pm 1/\sqrt{2}, \pm 1$, and every entry of the three tensors lies in $\mathbb{Z}[\sqrt{2}]$. The factor $\sqrt{2}$ is a per-site scalar, so the periodic operator of the stored tensor

¹Source traceability: `References/1511.08090/AnyonsPEPS.tex`, lines 1257–1268 and 1305–1323.

²Source traceability: `References/1601.07185/source/Ising-Defects.tex`, lines 1051–1055.

is $2^{N/2}$ times that of $(\sqrt{2})^{-1}(\sqrt{2}A_\sigma)$, and the rule $O_\sigma^2 = O_1 + O_\psi$ for the latter is equivalent to

$$O_N(\sqrt{2}A_\sigma)^2 = 2^N(O_N(A_1) + O_N(A_\psi)). \quad (2)$$

Both forms are proved.

The omission of the factors $v_e v_f$ does not change the periodic operators; this is proved in the next section. The identification of the formal letters with the F -symbols was first recorded in the verification data `Notes/OpenProblemsTN/checks/asym_ising_action_data.md`; it is now a formal statement.

3 Relation to the source's tensor

The source's operator tensor (the Fibonacci figure, reused for Ising) is the crossing of the operator line f with a vertical edge line e , with corner plaquettes b (upper left), a (upper right), c (lower left) and d (lower right), and entry $G_{def}^{abc}\sqrt{v_a v_b v_c v_d}$.³ On the fusion-tree labels of the formal tensors the upper label is (b, e, a) , the lower label is (c, e, d) , and the bond letters are the admissible pairs (b, c) and (a, d) of plaquettes above and below the operator line. With the F -symbols and selection rule $\delta_{abe}\delta_{cde}\delta_{adf}\delta_{bcf}$ of the source, the letters of A_1 , A_ψ and $A_\sigma = (\sqrt{2})^{-1}\hat{A}_\sigma$ are the F -symbols F_{abd}^{fce} , and the Ising G -symbols have the tetrahedral symmetry $G_{def}^{abc} = G_{abd}^{fce}$. Hence the nonzero entries of the source's tensor are those of the formal block multiplied by $\sqrt{v_a v_b v_c v_d}/(v_b v_d) = g(a, d)/g(b, c)$, with $g(u, l) = (v_u/v_l)^{1/2}$, and the diagonal bond matrix g conjugates every letter of the source's tensor into the letter of A_1 , A_ψ or A_σ :

$$g S_f^{kh} g^{-1} = A_f^{kh}. \quad (3)$$

The similarity cancels in the trace, so the periodic operators of the source's tensors equal $O_N(A_1)$, $O_N(A_\psi)$ and $O_N(A_\sigma) = 2^{-N/2}O_N(\hat{A}_\sigma)$ at every length, and the algebra (1) holds for the source's tensors themselves.⁴

4 Formal boundary

The closed-path kernels of O_1 and O_ψ , the eight products with at most one factor σ , and both forms of (2) are proved at every positive length for the v -free tensors, and transfer to the source's G -symbol tensors by (3). What remains is the choice of storing $\hat{A}_\sigma = \sqrt{2}A_\sigma$ so that every entry lies in $\mathbb{Z}[\sqrt{2}]$, a per-site scalar recorded above; nothing is open.

³Source traceability: `References/1511.08090/AnyonsPEPS.tex`, lines 1044–1050 and 1262–1266, figure file `StringNetMPO_RHS.pdf`.

⁴Results: `IsingTwist.isingGSymbol_tetrahedral`, `IsingTwist.isingStringNetTensor_conj`, `IsingTwist.mpo_isingStringNetTensor`, `IsingTwist.mpo_isingStringNetTensor_sigma`, `IsingTwist.isMPOFusionAlgebra_isingStringNetTensor`.

References

- [AMF16] David Aasen, Roger S. K. Mong, and Paul Fendley. Topological defects on the lattice: I. The Ising model. *Journal of Physics A: Mathematical and Theoretical*, 49:354001, 2016.
- [BMW⁺17] Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Anyons and matrix product operator algebras. *Annals of Physics*, 378:183–233, 2017.