

Fibonacci String-Net: Entries of the Two Operator Blocks

2026-09-24

Verdict: local-correction (resolved).

1 What the source prints

Appendix D.1.1 of [BMW⁺17] gives the categorical data of the Fibonacci theory: labels $1, \tau$ with $N_{11}^1 = N_{\tau 1}^\tau = N_{1\tau}^\tau = N_{\tau\tau}^1 = N_{\tau\tau}^\tau = 1$, quantum dimensions $d_1 = 1$, $d_\tau = \phi = (1 + \sqrt{5})/2$, the F-symbols

$$[F_d^{abc}]_e^f = \delta_{abe}\delta_{cde}\delta_{adf}\delta_{bcf} F_{def}^{abc}, \quad [F_\tau^{\tau\tau\tau}] = \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix}, \quad (1)$$

all other admissible components being 1, and the G-symbols $G_{def}^{abc} = F_{def}^{abc}/(v_e v_f)$ with $v_i = \sqrt{d_i}$. The operator tensor itself is given only as a diagram. The text then states that, after the zero rows and columns are removed, the projector matrix product operator has bond dimension 5 and consists of two blocks B_1, B_τ of dimensions 2 and 3 that satisfy the Fibonacci fusion rules, with weights $w_1 = 1/(1 + \phi^2)$ and $w_\tau = \phi/(1 + \phi^2)$.¹ No numeric entry of B_1 or B_τ is printed.

2 The entries used

Write a physical letter as a pair (x', x) of an outgoing and an incoming label and let the bond index at site j run over the admissible letters $(x'_j, x_j) \in \{(1, \tau), (\tau, 1), (\tau, \tau)\}$. The τ block used in the formalization is

$$(B_\tau^{(x', x)})_{(x'_j, x_j), (x'_{j+1}, x_{j+1})} = \delta_{(x'_j, x_j), (x', x)} [F_{x'_{j+1}}^{\tau x \tau}]_{x'}^{x_{j+1}}, \quad (2)$$

with the bare F-symbols of (1) and without the closed-loop factors v_a . The trivial block B_1 is the admissibility projector of bond dimension 2, carrying the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ on its diagonal letters. The placement (2) was fixed by an

¹Source traceability: `References/1511.08090/AnyonsPEPS.tex`, lines 1240–1269.

exhaustive search over the assignments of the six F-symbol slots to the available labels, keeping those for which the fusion rules hold at lengths 2, 3, 4.²

3 What is checked

Every entry of the τ block equals the corresponding F-symbol of (1); the blocks have the recorded dimensions 2 and 3; their periodic operators satisfy $O_1^2 = O_1$, $O_1 O_\tau = O_\tau O_1 = O_\tau$ and $O_\tau^2 = O_1 + O_\tau$ at every positive length; and the trace of the bond-five tensor $B_1 \oplus B_\tau$ against $\Delta = w_1 I_2 \oplus w_\tau I_3$ with the printed weights is $P_L = w_1 O_1 + w_\tau O_\tau$, which is idempotent at every positive length.³ Since the source prints no numeric entries, what is recorded is a choice of explicit blocks with the printed block dimensions, weights and fusion rules, not an error in the source.

4 Relation to the source's tensor

The source draws one site of its operator as the crossing of the horizontal operator line f with a vertical edge line e , with corner plaquettes b (upper left), a (upper right), c (lower left) and d (lower right), and gives it the entry

$$S_f[(b, e), (c, e); (b, c), (a, d)] = G_{def}^{abc} \sqrt{v_a v_b v_c v_d}, \quad G_{def}^{abc} = \frac{F_{def}^{abc}}{v_e v_f}, \quad v_i = \sqrt{d_i}, \quad (3)$$

where a physical letter records the left plaquette and the edge label, the edge label passes through the site unchanged, and the bond letters are the admissible pairs of plaquettes above and below the operator line; the upper-right plaquette of a site is the upper-left plaquette of the next one.⁴ The Fibonacci G-symbols have the tetrahedral symmetry $G_{def}^{abc} = G_{abd}^{fce}$. On the configurations whose edge labels are all τ this moves the G-symbol of (3) to the F-symbol slots of (2), and the remaining factors are $\sqrt{v_a v_b v_c v_d} / (v_b v_d) = h(a, d) / h(b, c)$ with $h(u, l) = (v_u / v_l)^{1/2}$. Hence the diagonal bond matrix h conjugates every letter of the source's tensor on edge label τ into the corresponding letter of B_1 or B_τ ,

$$h S_f^{(x', x)} h^{-1} = B_f^{(x', x)}, \quad (4)$$

the entries of S_f involving $\phi^{\pm 1/4}$ while those of B_f lie in $\mathbb{Z}[\phi^{-1/2}]$. The similarity cancels in the trace, so on edge label τ the periodic operators of the source's

²Verification record: `Notes/OpenProblemsTN/checks/p5_more_examples_data.md`, §3.1. It is a record of the search, not a source.

³Results: `FibonacciCompression.fibTauGolden_eq_fSymbol`, `FibonacciCompression.i`, `sMPOFusionAlgebra_fibBlock`, `FibonacciCompression.fibProjector_eq_trace_fibDelta`, `FibonacciCompression.fibProjector_mul_self`.

⁴Source traceability: `References/1511.08090/AnyonsPEPS.tex`, lines 1044–1050 and 1257–1268, figure file `StringNetMPO_RHS.pdf`.

blocks equal those of B_1 and B_τ at every length.⁵

Edge label 1 (formerly a scope restriction, now resolved). The source’s fusion statement concerns the full alphabet of four letters per site (plaquette and edge label). When an edge carries the label 1, admissibility of G_{d1f}^{abc} forces $a = b$ and $c = d$, so the two plaquettes on either side of the edge coincide and the right bond letter equals the left one. Moreover $G_{c1f}^{bbc} v_b v_c = 1$ whenever $N_{bc}^f > 0$. Hence the letter of S_f at $((b, 1), (c, 1))$ is the matrix unit E_{pp} with $p = (b, c)$ when (b, c) is a bond letter of the block f , and zero otherwise. The similarity h leaves these letters unchanged, so (4) holds on the full alphabet for the blocks B_1, B_τ extended by these matrix units at edge label 1. The four compression data of the products $B_a \otimes B_b$ extend to the sixteen letters of the extended products with the same gauges: the conjugated edge-1 letters are block upper triangular, with the edge-1 letters of the target blocks on the diagonal and further entries only in the columns of the zero slots. The remainder no longer vanishes, but it is nilpotent and the trace identity of the compression still holds. Hence the source’s blocks satisfy the Fibonacci fusion rules on the full alphabet at every positive length, and no restriction to edge label τ remains.⁶

The review [CPGSV21] writes the operator tensor with plaquettes A (upper left), B (upper right), C (lower left), D (lower right), and entry $(1/\sqrt{d_A d_D}) (F_B^{aC\alpha})_A^D$.⁷ At $\alpha = \tau$ this is the bare block (2) multiplied by $1/\sqrt{d_{x'} d_D}$, with D the lower-right plaquette. The bond gauge $g(u, l) = d_l^{-1/2}$ turns this factor into the letterwise scalar $(d_{x'} d_x)^{-1/2}$, and the periodic operator is the congruence

$$O_f^{\text{rev}} = W^{-1/2} O_f W^{-1/2}, \quad W = \text{diag} \prod_k d_{x_k}, \quad (5)$$

not a similarity of O_f . Read in the orthonormal basis of plaquette configurations, these operators do not satisfy the fusion rules: at one site $O_1^{\text{rev}} = \text{diag}(0, 1/\phi)$, whose square has the entry $1/\phi^2 \neq 1/\phi$. The operators $O_f^{\text{rev}} W = W^{-1/2} O_f W^{1/2}$ are similar to O_f and do satisfy them.⁸

Local fix (review normalization). The review states the fusion rules $O_a O_b = \sum_c N_{ab}^c O_c$ for the operators built from the tensor 2F , and identifies 2F with the operator tensor.⁹ The Fibonacci appendix attaches the prefactor $1/\sqrt{d_A d_D}$ to

⁵Results: `FibonacciCompression.fibStringNetEdgeTau_conj`, `FibonacciCompression.mpo_fibStringNetTensor_edgeTau`.

⁶Results: `FibonacciCompression.fibStringNetTensor_conj`, `FibonacciCompression.mpo_fibStringNetTensor`, `FibonacciCompression.isMPOFusionAlgebra_fibBlockFull`, `FibonacciCompression.isMPOFusionAlgebra_fibStringNetTensor`.

⁷Review traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2613–2625, figure file `fig3_mpo.pdf`.

⁸Results: `FibonacciCompression.mpo_fibReviewTensor`, `FibonacciCompression.not_isMPOFusionAlgebra_fibReviewTensor`, `FibonacciCompression.isMPOFusionAlgebra_fibReviewWeightedTensor`.

⁹Review traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 1309 and 1341.

that tensor. Read literally in the orthonormal basis of plaquette configurations, the resulting operators (5) contradict the review’s own fusion statement, as shown above. The review is therefore internally inconsistent at this point. Either the prefactor is a normalization slip, or the review’s operators stand for the operators $O_f^{\text{ev}}W$, composed with the weight W on plaquette configurations. The formalization records both readings: the literal one fails the fusion rules, and the weighted one satisfies them.

The review also prints the selection rule for $(F_d^{abc})_e^f$ as “ $N_{ab}^c > 0$, $N_{cd}^e > 0$, $N_{ad}^f > 0$, $N_{bcf} > 0$ ”.¹⁰ This rule is garbled: the first condition names the label c where the fusion channel e belongs, and the last one is misprinted. The formalized review tensor uses instead the source’s F-symbols (1), with the selection rule $\delta_{abe}\delta_{cde}\delta_{adf}\delta_{bcf}$.

5 The empty chain

The source asks for $P_L^2 = P_L$ at every length; the exclusion of the empty chain $L = 0$, where P_0 is a scalar different from 0 and 1, is recorded separately in `docs/paper-gaps/bmwshv17_fibonacci_projector_positive_length.tex`.

References

- [BMW⁺17] Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Anyons and matrix product operator algebras. *Annals of Physics*, 378:183–233, 2017.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

¹⁰Review traceability: `Papers/2011.12127/TN-Review-main.tex`, line 2623.