

Safe-Box Entropy and the Sublinear Collar Bound

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Verdict: scope-restriction (resolved).

1 The source propositions

Propositions 9.5 and 9.6 of the two-dimensional area-law manuscript of September 24, 2026 [Ope26] (08-scanner.tex, lines 681–944, at revision adc7f1241b42e322a6451854ab7e4b4c146bf78a) establish the small-exponent entropy bound on safe rectangles and deduce from it the sublinear mutual-information bound across the collar of a separated template.

Fix a local Hilbert-space dimension $q \geq 1$, a finite interaction range $R \in \mathbb{N}$, an interaction norm bound $J \geq 0$, and a spectral gap $\Delta > 0$. Consider a finite lattice domain $\Lambda \subset \mathbb{Z}^2$, a local Hamiltonian on Λ with those parameters and ground energy e_0 , a unit ground vector Ω , and a subset $A \subseteq \Lambda$ with cut boundary ∂A in the induced graph on Λ . Write Z for the set of endpoints of these cut edges, and interpret distance to the cut as distance to Z , with distance to the empty set equal to infinity. For any region $Y \subseteq \Lambda$, write $S_\Omega(Y)$ for the von Neumann entropy of the reduced density matrix of Ω on Y .

Fix a safety parameter $D_0 > 2R + 10$. A lattice rectangle $Q \subset \mathbb{Z}^2$ with nonempty coordinate intervals has size $s = \text{size}(Q)$ equal to the larger of its two side lengths in lattice sites. The rectangle Q is D_0 -safe for the cut A if the ℓ_∞ distance $\text{dist}_\infty(Q, \partial A)$ exceeds $D_0 s$. For each integer scale $r \geq 1$, let $F_{\text{box}}(r)$ denote the supremum of $S_\Omega(A \cap Q)$ over all finite domains Λ , gapped ground vectors Ω , cuts $A \subseteq \Lambda$, and D_0 -safe rectangles Q of size at most r . Proposition 9.5 of Ref. [Ope26] (08-scanner.tex, lines 681–805) states that for the target exponent $e_* = 2 \times 10^{-6}$, there exists a constant $C_{\text{box}} > 0$, depending only on q , R , J , and Δ , such that for every $D_0 > 2R + 10$ and every integer $r \geq 1$,

$$F_{\text{box}}(r) \leq C_{\text{box}} r^{1+e_*}. \quad (1)$$

The paper proves (1) from an initial safe-box exponent $0 < e_0 < 1$ by iterating the one-step exponent map $e \mapsto (1 - \ell_0)e$, where $\ell_0 = (1 - e_0)/200000$, finitely many times until the exponent is at most e_* .

Proposition 9.6 of Ref. [Ope26] (08-scanner.tex, lines 808–944) applies (1) to a template $T \subset \mathbb{Z}^2$ at scanner scale $n \geq 2$ and template scale s_0 . With the

collar exponent $\ell = 10^{-5}$ and collar depth $L = \lfloor n^{1-\ell} \rfloor$, let T_L denote the ℓ_∞ neighborhood of T of radius L , and define the core region $X = A \cap T$ and the collar truncation set $S_0 = A \cap T_L$. For a template T satisfying $\text{dist}_\infty(T, \partial A) > 4D_0s_0$ and $L \leq s_0$, Proposition 9.6 bounds the mutual information $I_\Omega(X : \Lambda \setminus S_0)$ between X and $\Lambda \setminus S_0$ by

$$2I_\Omega(X : \Lambda \setminus S_0) \leq Cn^{1-1/100000}, \quad (2)$$

for a constant $C > 0$ and scale threshold $N \geq 2$ chosen after the fixed system and template parameters and before the finite domain Λ , Hamiltonian, ground vector Ω , cut A , and template T .

2 The formal statement and hypothesis reduction

The conditional template mutual-information theorem derives (2) from the analytic estimate (1). The latter is now proved independently, so it is not an additional hypothesis of the physical collar theorem. For the conditional argument, fix $q \geq 1$, $R \in \mathbb{N}$, $J \geq 0$, $\Delta > 0$, $D_0 \in \mathbb{N}$ with $D_0 \geq 1$, and a template constant $C_{\text{tpl}} \geq 24$. Assuming (1) at $e_* = 2 \times 10^{-6}$, there exist $C > 0$ and $N \geq 2$ such that every finite instance with $n \geq N$, $\text{dist}_\infty(T, \partial A) > 4D_0s_0$, and $L = \lfloor n^{1-1/100000} \rfloor \leq s_0$ satisfies

$$2I_\Omega(X : \Lambda \setminus S_0) \leq Cn^{62497/62500} \leq Cn^{1-1/100000}. \quad (3)$$

All intermediate hypotheses of the scanner chain are eliminated in five steps leading to (3).¹

1. Post-replica collar mutual information from the sharp comparison.

Fix the final scanner exponents $\mu = 1/4$, $\ell = 10^{-5}$, $\omega = 1/20000$, $\kappa = 10^{-6}$, and $\nu = 1/1000$, and define the integer scales

$$\begin{aligned} L &= \lfloor n^{1-\ell} \rfloor, & m &= \lfloor n^\mu \rfloor, & D &= \lceil n^\kappa \rceil, \\ K &= L \operatorname{div}(8m), & r_0 &= \lceil C_{\text{tr}}(\log n)^2 \rceil, & a &= \frac{W}{K}, \end{aligned} \quad (4)$$

where m is the band spacing, D is the separation scale, K is the band count, r_0 is the truncation radius with coefficient $C_{\text{tr}} > 0$, $W \in [1, n^\omega]$ is the total band weight, and a is the per-band weight (`08-scanner.tex`, lines 32–58 and 839–865). Let Ω_t be the unit ground vector of the Hamiltonian truncated to S_0 , with ground energy $e_{\text{tr}} \leq \min(n^{-1000}, g/4)$, normalized gap $g/2$ where $g = \Delta / \max\{1, J(\|f_{1,\Delta/2}\|_1 + 1)\}$, and trace distance $d_{\text{tr}}(\Omega_t, \Omega) \leq \sqrt{2 \min(n^{-1000}, g/4)/g}$. Let E be a typical spectral window of the

¹The conditional template mutual-information theorem is `TNLean.PEPS.AreaLaw.Scan.exists_template_final_mutualInformation_of_boxEntropy`, whose tracking issues [issue 8756](#) and [issue 9334](#) were resolved by [pull request 9463](#).

reduced state of Ω_t on X with dimension $d = |E|$, spectral mass $z > 0$, and auxiliary Schmidt entropy S' satisfying $|S' - S_{\Omega_t}(X)| \leq c$ for $c = n^{3/5} - \log z$.

Assume the target and positive-depth prefix entropy bounds on the original ground vector Ω at exponent $e_{\text{box}} = e_*$:

$$S_{\Omega}(X) \leq C_e n^{1+e_*}, \quad S_{\Omega}(Q_j) \leq C_e (nL^{e_*} + n), \quad (5)$$

for every positive-depth prefix Q_j and a constant $C_e \geq 0$. Regional continuity transfers (5) to Ω_t with shell bound $B_{\text{sh}} = C_e(nL^{e_*} + n)n^{-247}$. At replica count k , averaging over the nm charge rounds and the parameter interval $[n^{-\nu}, 2n^{-\nu}]$ selects a charge round s_k and parameter $p_k \in [n^{-\nu}, 2n^{-\nu}]$ (a rescaling by two of the paper's interval $[n^{-\nu}/2, n^{-\nu}]$ in `08-scanner.tex`, lines 380–384 and 880–883). Because $e_* - \mu + \nu < -8\nu$ (with margin 120499/500000) and $(\omega - (1 - \ell - \mu))/4 + \nu < -8\nu$ (with margin 35697/200000), the normalized charge entropy defect $\mathcal{Q}_k(s_k, p_k)/(KnD)$ at the selected point is at most $n^{-8\nu} = n^{-1/125}$ for large k . (An intermediate formal lemma recorded the weaker bound $n^{-\nu}$, which is too weak after taking the eighth root in the energy estimate; the paper's bound $\delta_n \leq n^{-0.18}$ in `08-scanner.tex`, lines 880–883, is stronger than $n^{-8\nu}$ and needs no modification.) Substituting $(\mathcal{Q}_k/(KnD))^{1/8} \leq n^{-\nu}$ and $p_k \leq 2n^{-\nu}$ into the selected-point energy cap with exponent $2\omega + \ell + 2\kappa - \nu = -111/125000 < -1/2000$ gives the normalized energy defect bound

$$0 \leq \delta_k \leq n^{-1/2000} + \frac{\text{rem}_k}{g/2}, \quad (6)$$

where $\text{rem}_k \rightarrow 0$ as $k \rightarrow \infty$ (`08-scanner.tex`, lines 457–533 and 839–911).²

²The moment-margin declarations are `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_truncated_target_actualSharpUpper` in `TNLean/PEPS/AreaLaw/Scan/SourceTargetSharpUpper` and `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_truncated_target_sharp_moments` in `TNLean/PEPS/AreaLaw/Scan/SourceTargetSharpMoments`; the endpoint budget and entropy-transfer declarations carrying (5) are `TNLean.PEPS.AreaLaw.Scan.CollarScan.eventually_actualScan_suppliedPrevector_endpoint_budget` in `TNLean/PEPS/AreaLaw/Scan/ActualEndpointRoughBudget`, `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_source_actualScan_suppliedPrevector_endpoint_budget` in `TNLean/PEPS/AreaLaw/Scan/SourceEndpointRoughBudget`, `TNLean.PEPS.AreaLaw.Scan.eventually_abs_regionalEntropy_sub_le_of_truncation` in `TNLean/PEPS/AreaLaw/Scan/TargetEntropyTransfer`, `TNLean.PEPS.AreaLaw.Scan.roughBudget_le_source_power` and `TNLean.PEPS.AreaLaw.Scan.selectedDensity_budget_le_principal` in `TNLean/PEPS/AreaLaw/Scan/SourceRoughBudgetPower`, and `TNLean.PEPS.AreaLaw.Scan.exists_final_selectedDensity_le` in `TNLean/PEPS/AreaLaw/Scan/SelectedDensityPower`; the density and energy exponent-margin declarations are `TNLean.PEPS.AreaLaw.Scan.ScannerExponents.eventually_principalDensity_le_rpow`, `TNLean.PEPS.AreaLaw.Scan.ScannerExponents.exists_principalDensity_le_rpow`, `TNLean.PEPS.AreaLaw.Scan.final_principalDensity_margins`, `TNLean.PEPS.AreaLaw.Scan.exists_final_principalDensity_le_rpow`, `TNLean.PEPS.AreaLaw.Scan.final_eightNuDensity_margins`, `TNLean.PEPS.AreaLaw.Scan.exists_final_principalDensity_le_eightNu`, and `TNLean.PEPS.AreaLaw.Scan.exists_final_selectedDensity_le_eightNu` in `TNLean/PEPS/AreaLaw/Scan/StrongSelectedDensityPower`, together with `TNLean.PEPS.AreaLaw.Scan.finalSelectedEnergyCap_le`, `TNLean.PEPS.AreaLaw.Scan.final_selectedEnergy_margins`, `TNLean.PEPS.AreaLaw.Scan.eventually_final_selectedEnergy_le_rpow`, `TNLean.PEPS.AreaLaw.Scan.exists_final_selectedEnergy_le_rpow`, and `TNLean.PEPS.AreaLaw.Scan.exists_final_selectedEnergy_add_remainder_le_rpow`; the composed

Subtracting the sharp upper and lower norm rates at (s_k, p_k) , substituting (6), bounding the terminal pre-charge entropies on Ω_t by $2S_{\Omega_t}(X) + 2B_{\text{sh}} + 6nD \log q$, transferring mutual information from Ω_t to Ω with the continuity term $C_{\text{MI}} = 16\sqrt{d_{\text{tr}}(\Omega_t, \Omega)} \log(e \dim \mathcal{H}_X)$, and taking $k \rightarrow \infty$ at fixed n , $W \in [1, n^\omega]$, and smoothing parameter $\tau \in (0, 1/2]$ yields

$$\begin{aligned} 2I_\Omega(X : \Lambda \setminus S_0) &\leq \frac{4S_{\Omega_t}(X) + 4c - 2 \log z}{W} \\ &\quad + \left(\tau + \frac{n^{-1/2000}}{\tau} \right) (4S_{\Omega_t}(X) + 2c + 2B_{\text{sh}} + 6nD \log q) \\ &\quad + \left(3 + \frac{1}{a} \right) h(\tau) + 37C_0Ba - \frac{\log z}{a} + 2c + C_{\text{MI}}, \end{aligned} \quad (7)$$

where $h(\tau)$ is the binary entropy and C_0B is the quadratic marginal moment coefficient with $B = C_B n D (\log n)^{12}$ (08-scanner.tex, lines 868–911 and 936–941).³

2. Final-scale error absorption and transport smallness. Specializing (7) to $W = n^{1/20000}$ and $\tau = n^{-1/4000}$, the transport smallness condition $al_q(r_0) \leq c_t$, where $\ell_q(r_0) = 1 + (2r_0 + 1)^2 \log q$, holds for all sufficiently large n because $\ell_q(r_0) = O((\log n)^4)$ and $a \leq C_a n^{-0.74994}$. From $X \subseteq S_0$ and $|S_0| \leq (C_T + 1)n^2$, the target dimension factor obeys $\log(e \dim \mathcal{H}_X) \leq 1 + ((C_T + 1) \log q)n^2$, so $C_{\text{MI}} = O(n^{-248})$. Under (5), the seven error terms in (7) have exponents

$$\begin{aligned} &0.999952, \quad 0.999752, \quad 0.74969, \quad 0.250061, \\ &-99.25006, \quad 0.6, \quad -248, \end{aligned} \quad (8)$$

whose maximum in (8) is $62497/62500 = 0.999952 = 1 - 1/100000 - 19/500000$ (08-scanner.tex, lines 839–942). Absorbing the logarithmic factors into that margin yields (3) conditionally on (5), the scale inequalities $2r_0 \leq D$, $1 \leq D$, and $4D \leq m$, and the scanner geometry.⁴

selected-point comparison, density, and energy bounds are `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_selected_actualCharge_sharp_comparison`, `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_selected_actualCharge_sharp_density` in [TNLean/PEPS/AreaLaw/Scan/SourceSelectedSharpDensity](#), and `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_selected_actualCharge_sharp_energy` in [TNLean/PEPS/AreaLaw/Scan/SourceSelectedSharpEnergy](#).

³The mutual-information comparison and replica limit declarations are `TNLean.PEPS.AreaLaw.Scan.two_information_le_sharpInformationBound` and `TNLean.PEPS.AreaLaw.Scan.information_le_of_eventually_sharpInformationBound` in [TNLean/PEPS/AreaLaw/Scan/SharpInformationScalar](#), and `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_actual_mutualInformation_bound` in [TNLean/PEPS/AreaLaw/Scan/SourceActualInformation](#).

⁴The final-scale scalar absorption and transport smallness declarations are `TNLean.PEPS.AreaLaw.Scan.finalSharpError`, `TNLean.PEPS.AreaLaw.Scan.exists_finalSharp_source_error_le`, and `TNLean.PEPS.AreaLaw.Scan.exists_finalSharp_transport_smallness`; the target dimension bound is `TNLean.PEPS.AreaLaw.Scan.log_exp_configuration_card_le_of_card` in [TNLean/PEPS/AreaLaw/Scan/PhysicalInformationDimension](#); the conditional final-scale mutual-information theorem is `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_finalScale_mutualInformation` in [TNLean/PEPS/AreaLaw/Scan/SourceFinalInformation](#).

3. Numerical admissibility of the rounded scales. At the rounded definitions (4), the inequalities $2r_0 \leq D$, $1 \leq D$, and $4D \leq m$ hold for all sufficiently large n because $r_0 = O((\log n)^2)$, $D = \lceil n^{10^{-6}} \rceil$, and $m = \lfloor n^{1/4} \rfloor$. This removes the three numerical scale premises, leaving only the scanner geometry and the entropy bounds (5).⁵

4. Target and prefix entropies from the safe-box bound. For any separated template T with $\text{dist}_\infty(T, \partial A) > 4D_0s_0$ and $L \leq s_0 \leq n$, the dyadic safe-box partitions of T and of each positive-depth shell cover $X = A \cap T$ and every prefix Q_j by D_0 -safe rectangles (08-scanner.tex, lines 574–590, 641–667, and 828–844). Subadditivity of entropy then derives both bounds in (5) from the single safe-box bound (1) at $e_* = 2 \times 10^{-6}$.⁶

5. Template scanner geometry. Constructing the scanner on the separated template T with $C_{\text{tpl}} \geq 24$ and $D_0 \geq 1$ supplies the induced domain graph, support anchors, quadratic volume bound $|S_0| \leq (C_T + 1)n^2$, layer cardinalities, anchor multiplicity, and boundary clearance, reducing all hypotheses of (3) to (1).

3 The physical safe-box bootstrap

The physical safe-box estimate is proved first for integer clearance. For fixed q, R, J, Δ , a single $C_b > 0$ works for every integer $d > 2R + 10$ and every positive integer scale.⁷ Its composition with the conditional template theorem gives the original-state information bound at integer clearance, without assuming an entropy estimate.

In the bootstrap argument (08-scanner.tex, lines 681–805), given an initial safe-box exponent $0 < e_0 < 1$, the bootstrap fixes $g_0 = (1 - e_0)/2$, $\mu = 1 - g_0$, $\nu = g_0/1000$, $\ell = \nu/100$, $\kappa = \min\{\ell, e_*/4\}$, and total band weight $W = 4$, with $a = 4/K$ and K of order $n^{g_0 - \ell}$. The sharp comparator cannot be used at these scales because its marginal-moment condition $a\sqrt{B} \leq c_0$ with B of order $n^{1+\kappa}$ would require a negative exponent in

$$\frac{1 + \kappa}{2} - g_0 + \ell = \frac{e_0}{2} + \frac{\kappa}{2} + \ell > 0. \quad (9)$$

Because the exponent in (9) is strictly positive, the bootstrap instead uses the rough lower norm comparison at $W = 4$, $K \geq 8$, and $a = 4/K$, which requires

⁵The rounded-scale mutual-information theorem is `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_sourceTarget_finalScale_mutualInformation_of_rounded_scales`.

⁶The template entropy reductions are `TNLean.PEPS.AreaLaw.Scan.exists_template_source_entropy_bounds` and `TNLean.PEPS.AreaLaw.Scan.exists_template_final_source_entropy_bounds_of_boxEntropy` in [TNLean/PEPS/AreaLaw/Scan/TemplateSourceEntropy](#).

⁷The theorem is `TNLean.PEPS.AreaLaw.Scan.exists_boxEntropy_le_boxError`.

no marginal-moment bound and gives, for every $0 < \tau \leq 1/2$,

$$4S' - 6 \left(\tau \log d + \frac{h(\tau)}{a} \right) - 3 \frac{\delta_k}{\tau} \max\{0, 2S'\} - R_k \leq -\frac{1}{k} \log \mathcal{N}_k, \quad (10)$$

where $\mathcal{N}_k$ is the squared filtered norm and $R_k \rightarrow 0$ as $k \rightarrow \infty$ (`07-comparators.tex`, lines 585–651; `08-scanner.tex`, lines 755–769). On each D_0 -safe rectangle Q of size s , the scanner is specialized to $T = Q \cap \mathbb{Z}^2$ and $n = 24s$, where the weak clearance $\text{dist}_\infty(Q, \partial A) > D_0 s$ implies $2L + 10r_0 \leq D_0 s$ and $(D_0 + 1)L \leq D_0 s$ for large s (`08-scanner.tex`, lines 693–755).⁸ Combining (10) with the rough upper bound at $W = 4$ establishes the one-step improvement $e \mapsto (1 - \ell_0)e$ for $e_* < e \leq e_0$, and finite iteration yields the integer-clearance safe-box estimate.

4 Arbitrary real clearance

The source allows every real $D_0 > 2R + 10$, whereas the safe-box bootstrap uses integer clearance. Put $d = 2R + 10$. The integer estimate at $d + 1$ can be transferred to d without changing the domain, Hamiltonian, or ground vector. Indeed, partition a d -safe rectangle of size $s \geq 2$ into at most four rectangles of size at most $\lceil s/2 \rceil$. Since $d \geq 10$, one has $(d + 1)\lceil s/2 \rceil \leq ds$, so each child is $(d + 1)$ -safe. Subadditivity and the volume bound for the remaining small sizes give

$$F_{\text{box},d}(r) \leq C_w r^{1+e_*}, \quad C_w = 4C_b + 4 \log q + 1. \quad (11)$$

Here C_b is the physical coefficient at clearance $d + 1$; neither it nor C_w depends on the subsequently chosen real D_0 .

Every rectangle satisfying the source's strict distance inequality at $D_0 > d$ is d -safe. Thus (11) gives $S_\Omega(A \cap Q) \leq C_w \text{size}(Q)^{1+e_*}$ for every such rectangle. This pointwise statement is equivalent to (1): for rectangles of size at most r , monotonicity of the positive power bounds every entropy by $C_w r^{1+e_*}$; conversely, take $r = \text{size}(Q) \geq 1$. The formal theorem states the pointwise

⁸The bootstrap scale numerics, weak-rectangle scanner geometry, and rough lower comparison are `TNLean.PEPS.AreaLaw.Scan.BootstrapParameters.scannerExponents` in [TNLean/PEPS/AreaLaw/Scan/BootstrapScannerExponents](#), `TNLean.PEPS.AreaLaw.Scan.ScannerExponents.exists_fixedWeightFour_transport_smallness` in [TNLean/PEPS/AreaLaw/Scan/FixedWeightTransportSmallness](#), `TNLean.PEPS.AreaLaw.Scan.ScannerExponents.exists_rounded_admissibility` in [TNLean/PEPS/AreaLaw/Scan/ScannerRoundedAdmissibility](#), `TNLean.PEPS.AreaLaw.Scan.ScannerExponents.exists_rounded_clearance_threshold` in [TNLean/PEPS/AreaLaw/Scan/RoundedClearance](#), `TNLean.PEPS.AreaLaw.Scan.rectangleBootstrapScanner` and `TNLean.PEPS.AreaLaw.Scan.exists_rectangleBootstrapScanner_geometry` in [TNLean/PEPS/AreaLaw/Scan/RectangleBootstrapGeometry](#), `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_actualScan_rough_lower_norm` in [TNLean/PEPS/AreaLaw/Scan/ActualRoughLower](#), `TNLean.PEPS.AreaLaw.Scan.CollarScan.exists_actualScan_supportedPrevector_rough_lower` in [TNLean/PEPS/AreaLaw/Scan/ActualRoughLowerPrevector](#), and `TNLean.PEPS.AreaLaw.Scan.tendsto_roughLowerRemainder_of_label_dimension` in [TNLean/PEPS/AreaLaw/Scan/ActualRoughLowerRemainder](#).

bound and introduces neither a second notion of safety nor another entropy supremum.⁹

For Proposition 9.6, apply the conditional template information theorem at the integer clearance d with (11). It supplies $C_m > 0$ and a common threshold before every finite physical instance. A template separated at $4D_0s_0$ is also separated at $4ds_0$. Consequently the information bound applies to the same original vector and the unchanged regions $X = A \cap T$ and $S_0 = A \cap T_L$. Dividing its coefficient by two gives $I_\Omega(X : \Lambda \setminus S_0) \leq (C_m/2)n^{1-1/100000}$. The cut and volume bounds then supply all clauses of the sublinear collar bound with the single coefficient $\max(C_m/2, 10)$.¹⁰ This holds for every real $D_0 > 2R + 10$ and $C_{\text{tpl}} \geq 24$, with no assumed safe-box estimate or mutual-information inequality.

This resolves the safe-box and collar restrictions described here. The two-family repair construction of Proposition 11.2 and its use in Theorem 1.1 are proved separately.¹¹

References

- [Ope26] OpenAI. A two-dimensional area law from a global spectral gap, 2026. September 24, 2026 manuscript.

⁹The clearance comparison is `TNLean.PEPS.AreaLaw.boxEntropy_power_le_of_succ_clearance`; the real-clearance conclusion is `TNLean.PEPS.AreaLaw.exists_regionalEntropy_safe_rect_le_boxError`.

¹⁰The theorem is `TNLean.PEPS.AreaLaw.sublinearCollarBound_real`.

¹¹The theorems are `TNLean.PEPS.AreaLaw.twoFamilyDecomposition` and `TNLean.PEPS.AreaLaw.uniformAreaLaw`.