

Tensor Network Theory:

A formalization blueprint

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Last update: August 26, 2026

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Chapter 1

Introduction

The central result is the Fundamental Theorem for translation-invariant matrix product vectors. The basic object is a family of matrices $A = \{A^i\}_{i=0}^{d-1}$, which assigns to each chain length N the vector $|V^{(N)}(A)\rangle$ with coefficients indexed by $\sigma = (i_1, \dots, i_N)$,

$$V^{(N)}(A)_\sigma = \text{tr}(A^{i_1} \dots A^{i_N}). \quad (1.1)$$

The main argument, for translation-invariant tensors, follows the canonical-form and basis-of-normal-tensors argument of [CPGSV16, CPGSV21], with the single-block injective case anchored in the earlier representation theorem of [PGVWC07].

Notation

$\mathbb{C}, \mathbb{N}, \mathbb{Z}$	complex numbers, natural numbers, integers
$M_D(\mathbb{C}), \text{GL}_D(\mathbb{C})$	complex matrices and invertible complex matrices
$\text{tr}(X), \text{span}_{\mathbb{C}}(S)$	trace and complex linear span
$A = \{A^i\}_{i=0}^{d-1}$	a translation-invariant MPS tensor
$w = (i_1, \dots, i_L), A^w$	a word and the product $A^{i_1} \dots A^{i_L}$
$ V^{(N)}(A)\rangle$	the length- N matrix product vector
$V^{(N)}(A)_\sigma$	the coefficient of $ V^{(N)}(A)\rangle$ at σ
$O_{AB}(N)$	the bilinear overlap $\sum_\sigma V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}$
$\mathcal{E}_A(X)$	the transfer map $\sum_i A^i X (A^i)^\dagger$
$A^{[L]}$	the length- L blocked tensor

In canonical form one writes

$$A^i = \bigoplus_{j=1}^g \mu_j A_j^i, \quad (1.2)$$

$$B^i = \bigoplus_{k=1}^h \nu_k B_k^i. \quad (1.3)$$

The blocks in equations 1.2 and 1.3 are normal tensors. If $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every positive chain length N , then the bases of normal tensors have the same size. After relabelling the blocks of B , each matched pair satisfies

$$B_j^i = \zeta_j X_j A_j^i X_j^{-1}, \quad (1.4)$$

for some invertible matrix X_j and nonzero scalar ζ_j , and the equal case compares the coefficient power sums attached to the relabelled bases. The block gauges in equation 1.4 and the weight comparison then combine into a single gauge conjugacy of the two weighted block-diagonal tensors in equations 1.2 and 1.3. For a single injective block this reduces to the familiar conclusion $B^i = X A^i X^{-1}$ for all physical indices i , the single-block specialization of equation 1.4.

Chapters 2–10 carry out this argument. Chapter 2 fixes the matrix product vector notation, word evaluations, transfer maps, gauge equivalence, injectivity, blocking, and site-periodic tensor families. Chapter 3 proves the single-block theorem by trace pairings, linear extension, simplicity of matrix algebras, and Skolem–Noether. Chapters 4, 5, and 6 develop the transfer-map, Perron–Frobenius, and overlap-decay tools used to control normal blocks. The general theory of Schwarz inequalities and multiplicative domains is developed in the companion quantum-channel volume [LTC26, Chapter “Schwarz Inequalities and Multiplicative Domains”]. Chapters 7 and 8 supply the blocking, primitivity, irreducibility, and canonical-form reductions. Chapter 8 includes the invariant-projection splitting. Chapter 9 supplies bases of normal tensors and the power-sum comparison.

Chapter 11 uses the Fundamental Theorem to obtain virtual symmetry gauges, projective bond representations, and string-order criteria; its Kraus-mixing step uses the rectangular Kraus-freedom theorem from the companion quantum-channel volume [LTC26, “Rectangular Kraus freedom”], not any additional input for the Fundamental Theorem.

Beyond Chapter 11, the later chapters develop parent-Hamiltonian consequences, correlation estimates, tensor-network examples, matrix product operators and density operators, renormalization fixed points, and periodic tensor families. They also retain the matrix-product-specific normal-form and asymptotic results needed for these applications. General quantum-channel representations, positive-map examples, dynamical semigroups, operator convexity, and entropy are developed in the companion quantum-channel volume [LTC26, Chapters “Quantum Channels and Positive Maps,” “Channel Representations and Normal Forms,” “Positive but Not Completely Positive Maps,” “Operator Convexity and Jensen Inequalities,” “Quantum Dynamical Semigroups,” and “Quantum Entropy”].

Chapter 2

Matrix Product Vectors

Fix a physical dimension d and a bond dimension D . This chapter studies the tensors that generate matrix product vector (MPV) families. All tensors here are translation-invariant with periodic boundary conditions (PBC).

2.1 Basic definitions

Definition 2.1.1 (MPS tensor). [Lean](#) [✓ Stmt](#) A (translation-invariant, PBC) *MPS tensor* with physical dimension d and bond dimension D is a collection of matrices $\{A^i\}_{i=0}^{d-1}$, where $A^i \in M_D(\mathbb{C})$, indexed by a physical index $i \in \{0, \dots, d-1\}$. Such a tensor defines an MPV family. Diagrammatically,

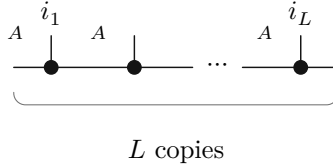


The black node denotes the tensor A , the horizontal legs are virtual, and the upper leg is the physical index i .

Definition 2.1.2 (Word evaluation). [Lean](#) [✓ Stmt](#) Given a word $w = (i_1, \dots, i_L) \in \{0, \dots, d-1\}^L$, the *word evaluation* is the matrix product

$$A^w := A^{i_1} A^{i_2} \dots A^{i_L} \in M_D(\mathbb{C}). \quad (2.1)$$

The empty word evaluates to the identity, $A^\emptyset = \mathbb{1}_D$. In tensor-network notation,



in which each black node denotes the same local tensor A , the virtual legs remain open, and the physical legs are labelled by the word $(i_1, \dots, i_L)$.

Lemma 2.1.3 (Word evaluation along a listed configuration). [Lean](#) [✓ Stmt](#) For a configuration $\sigma = (\sigma_0, \dots, \sigma_{N-1})$, let $w_\sigma = [\sigma_0, \dots, \sigma_{N-1}]$. Then $A^{w_\sigma} = A^{\sigma_0} A^{\sigma_1} \dots A^{\sigma_{N-1}}$.

Proof. ✓ Proof This follows by induction on N from the recursive definition of word evaluation. $\square$

Lemma 2.1.4 (Multiplicativity of word evaluation). Lean ✓ Stmt For any words w_1, w_2 , $A^{w_1 \cdot w_2} = A^{w_1} A^{w_2}$.

Proof. ✓ Proof The claim follows from the definition of A^w . $\square$

Lemma 2.1.5 (Centre of $M_D(\mathbb{C})$). Lean ✓ Stmt If a matrix $Z \in M_D(\mathbb{C})$ commutes with every element of $M_D(\mathbb{C})$, then there exists $\lambda \in \mathbb{C}$ such that $Z = \lambda \mathbb{1}_D$.

Proof. ✓ Proof Since Z commutes with every matrix unit, all off-diagonal entries of Z vanish and all diagonal entries are equal. $\square$

Lemma 2.1.6 (Rectangular amplified commutant). Lean ✓ Stmt Let $(A_i)_{i \in I}$ span $M_D(\mathbb{C})$, and let $C : \mathbb{C}^{m_2} \otimes \mathbb{C}^D \rightarrow \mathbb{C}^{m_1} \otimes \mathbb{C}^D$ satisfy, for every $i \in I$,

$$(\mathbb{1}_{m_1} \otimes A_i)C = C(\mathbb{1}_{m_2} \otimes A_i).$$

Then there is a matrix $F : \mathbb{C}^{m_2} \rightarrow \mathbb{C}^{m_1}$ such that $C = F \otimes \mathbb{1}_D$.

Proof. ✓ Proof Each (a, b) multiplicity block of C commutes with every A_i . Since the A_i span the full matrix algebra, Lemma 2.1.5 makes this block a scalar multiple of the identity. These scalars are the entries of F . $\square$

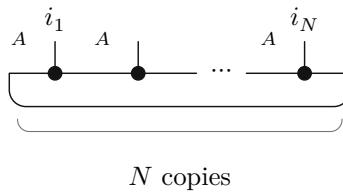
Definition 2.1.7 (Matrix product vector). Lean ✓ Stmt The *matrix product vector* (MPV) of a tensor A at system size N is the vector

$$|V^{(N)}(A)\rangle = \sum_{i_1, \dots, i_N=0}^{d-1} \text{tr}(A^{i_1} A^{i_2} \dots A^{i_N}) |i_1, \dots, i_N\rangle \in (\mathbb{C}^d)^{\otimes N}. \quad (2.2)$$

Equivalently, for a configuration $\sigma = (i_1, \dots, i_N) \in \{0, \dots, d-1\}^N$, we write

$$V^{(N)}(A)_\sigma := \text{tr}(A^{i_1} A^{i_2} \dots A^{i_N}). \quad (2.3)$$

The coefficient function $\sigma \mapsto V^{(N)}(A)_\sigma$ gives the components of the vector in (2.2); the displayed ket is the corresponding vector in $(\mathbb{C}^d)^{\otimes N}$. For a general word w , we also write $c_w(A) := \text{tr}(A^w)$. The *MPV family* generated by A is the collection $\mathcal{V}(A) = \{|V^{(N)}(A)\rangle\}_{N \geq 1}$. The coefficient $V^{(N)}(A)_\sigma$ is the periodic contraction

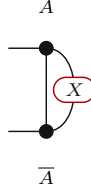


of N copies of the local tensor A , with the outer virtual legs closed by the trace.

Definition 2.1.8 (Transfer map). Lean ✓ Stmt The *transfer map* associated to a tensor A is the linear map $\mathcal{E}_A : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ defined by

$$\mathcal{E}_A(X) = \sum_{i=0}^{d-1} A^i X (A^i)^\dagger. \quad (2.4)$$

Diagrammatically, the transfer map is the double-layer contraction



in which the upper node denotes A , the lower node denotes $A^\dagger$, and the physical index is summed over between them.

Remark 2.1.9. ✓ Stmt The transfer map is automatically positive: if $X \geq 0$ then $\mathcal{E}_A(X) \geq 0$, because each summand $A^i X (A^i)^\dagger$ is positive semidefinite. The abstract framework is developed in the companion quantum-channel volume [LTC26, “Positive map”].

2.2 Site-periodic tensor families

Definition 2.2.1 (Site-periodic tensor family). Lean ✓ Stmt For a fixed period m , a *site-periodic tensor family* is a family of local tensors indexed by $\{0, \dots, m-1\}$, with local physical dimension d and bond dimension D .

Definition 2.2.2 (Periodic same-state and gauge-equivalence relations). Lean Lean ✓ Stmt For site-periodic tensors A, B at fixed period m :

- equality of the induced site-periodic states;
- cyclic gauge equivalence.

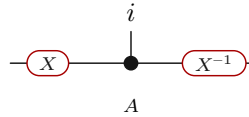
Lemma 2.2.3 (Equivalence structures for periodic relations). Lean Lean ✓ Stmt Both periodic relations are equivalence relations (reflexive, symmetric, transitive).

2.3 Gauge equivalence and same MPV

Definition 2.3.1 (Gauge equivalence). Lean ✓ Stmt Two tensors A and B of the same bond dimension D are *gauge equivalent* if there exists an invertible matrix $X \in \text{GL}_D(\mathbb{C})$ such that, for every $i \in \{0, \dots, d-1\}$,

$$B^i = X A^i X^{-1}. \quad (2.5)$$

In tensor-network notation, (2.5) is represented by



where the black node denotes the tensor A , the upper leg is the physical index i , and the two red side nodes denote the gauge matrices acting on the virtual legs.

Definition 2.3.2 (Same MPV). Lean ✓ Stmt Two tensors A and B (of the same bond dimension) generate the same MPV family if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every system size N .

Definition 2.3.3 (Same MPV - different bond dimensions). [Lean](#) [✓ Stmt](#) For tensors A and B of possibly different bond dimensions D_1 and D_2 , we write $\mathcal{V}(A) = \mathcal{V}(B)$ if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every word length N , including the empty word. This algebraic relation is used for exact direct-sum identities.

Definition 2.3.4 (Positive-length same MPV - different bond dimensions). [Lean](#) [✓ Stmt](#) For tensors A and B of possibly different bond dimensions, we write $\mathcal{V}^+(A) = \mathcal{V}^+(B)$ if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every $N > 0$.

Remark 2.3.5. [✓ Stmt](#) Definition 2.3.2 is the same-bond-dimension specialization of Definition 2.3.3. We keep both because later results use both the same-dimension and different-dimension forms.

Definition 2.3.6 (Nonzero proportional MPV). [Lean](#) [✓ Stmt](#) Two tensors A and B (of possibly different bond dimensions) generate nonzero proportional MPV families if for every $N > 0$ there is a nonzero scalar $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. This is the projective reading of the proportionality hypothesis in [CPGSV16, Theorem 2.10].

Definition 2.3.7 (Eventually nonzero proportional MPV). [Lean](#) [✓ Stmt](#) Two tensors A and B are eventually nonzero proportional if, for all sufficiently large N , there is a nonzero scalar $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. This auxiliary form is used with the eventual linear-independence statement Lemma E.1.10.

Lemma 2.3.8 (Elementary nonzero proportionality rules). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Nonzero proportionality implies eventual nonzero proportionality, is symmetric, and equality of MPV families is the special case with scalar 1, at every positive length.*

Proof. [✓ Proof](#) The eventual form follows by viewing an all-length statement as an eventual one. Symmetry follows by inverting the scalar c_N at each length, and equality is the case $c_N = 1$. $\square$

Lemma 2.3.9 (Coefficient isolation after complementary matching). [Lean](#) [✓ Stmt](#) *Let R be a ring, let V be an R -module, and let I and K be finite index sets. Let $T \subseteq I$, and let $\{u_i\}_{i \in I}$ and $\{v_k\}_{k \in K}$ be vectors in V . Suppose that, for every $i \notin T$, there are a scalar α_i and an index $\phi(i) \in K$ such that $u_i = \alpha_i v_{\phi(i)}$. If*

$$\sum_{i \in I} a_i u_i = \sum_{k \in K} b_k v_k$$

and the coproduct-indexed family $w: T \amalg K \rightarrow V$, defined by $w(\text{inl}(i)) = u_i$ and $w(\text{inr}(k)) = v_k$, is linearly independent, then $a_i = 0$ for every $i \in T$.

Proof. [✓ Proof](#) Substitute $u_i = \alpha_i v_{\phi(i)}$ for $i \notin T$ and collect the resulting coefficients over the fibres of ϕ . The given equality is then a linear relation in the combined family. Linear independence makes every coefficient vanish, in particular each a_i with $i \in T$. $\square$

Lemma 2.3.10 (Scaling of word evaluation). [Lean](#) [✓ Stmt](#) *Let ζA denote the tensor with matrices $\{\zeta A^i\}_{i=0}^{d-1}$. Then, for any word w , $(\zeta A)^w = \zeta^{|w|} A^w$, where $|w|$ is the length of w .*

Proof. [✓ Proof](#) Induction on w : the base case gives $\zeta^0 = 1$, and each step picks up one factor of ζ . $\square$

Lemma 2.3.11 (Scaling of MPV components). [Lean](#) [✓ Stmt](#) *Let ζA denote the tensor with matrices $\{\zeta A^i\}_{i=0}^{d-1}$. Then, for every N and every configuration $\sigma = (i_1, \dots, i_N)$, $V^{(N)}(\zeta A)_\sigma = \zeta^N V^{(N)}(A)_\sigma$.*

Proof. ✓ Proof Apply Lemma 2.3.10 to the word $w = (i_1, \dots, i_N)$ of length N , then take the trace; the scalar ζ^N pulls out by linearity. $\square$

Definition 2.3.12 (Gauge-phase equivalence). Lean Lean ✓ Stmt Two tensors A and B are *gauge-phase equivalent* if there exist $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \in \mathbb{C} \setminus \{0\}$ such that, for every physical index i ,

$$B^i = \zeta X A^i X^{-1}. \quad (2.6)$$

Proof. ✓ Proof Gauge equivalence is the special case $\zeta = 1$. $\square$

Definition 2.3.13 (Unit gauge-phase equivalence). Lean ✓ Stmt Two tensors are *unit gauge-phase equivalent* when the gauge-phase equation holds with $|\zeta| = 1$.

Lemma 2.3.14 (Unit gauge phase implies gauge phase). Lean ✓ Stmt *Unit gauge-phase equivalence implies gauge-phase equivalence.*

Proof. ✓ Proof A unit-modulus scalar is nonzero. $\square$

Lemma 2.3.15 (Gauge-phase equivalence across equal bond dimensions). Lean Lean ✓ Stmt *Gauge-phase equivalence is symmetric after identifying equal bond dimensions. Moreover, suppose that A , B , and C have identified bond dimensions, that X_1 and X_2 are invertible, that $\zeta_1, \zeta_2 \neq 0$, and that, for every physical index i ,*

$$\begin{aligned} B^i &= \zeta_1 X_1 A^i X_1^{-1}, \\ C^i &= \zeta_2 X_2 A^i X_2^{-1}. \end{aligned}$$

Then

$$C^i = (\zeta_2 \zeta_1^{-1}) (X_2 X_1^{-1}) B^i (X_1 X_2^{-1}), \quad (2.7)$$

so B and C are gauge-phase equivalent after the corresponding identification of their bond dimensions.

Proof. ✓ Proof Symmetry follows by replacing (X, ζ) with (X^{-1}, ζ^{-1}) . Formula (2.7) follows by substituting the first gauge formula into the second. $\square$

Lemma 2.3.16 (Gauge covariance of word evaluation). Lean ✓ Stmt *If $B^i = X A^i X^{-1}$ for all i , then $B^w = X A^w X^{-1}$ for every word w .*

Proof. ✓ Proof Induction on the word length, using $XX^{-1} = \mathbb{1}$. $\square$

Theorem 2.3.17 (Gauge invariance of MPVs). Lean ✓ Stmt *If A and B are gauge equivalent, then they generate the same MPV family.*

Proof. ✓ Proof Let $B^i = X A^i X^{-1}$. By Lemma 2.3.16, $B^w = X A^w X^{-1}$ for every word w . Then $\text{tr}(B^w) = \text{tr}(X A^w X^{-1}) = \text{tr}(A^w)$ by cyclicity of the trace. $\square$

Theorem 2.3.18 (Gauge-phase scaling of MPVs). Lean ✓ Stmt *If $B^i = \zeta X A^i X^{-1}$ for every physical index i , then for every system size N and configuration σ one has $V^{(N)}(B)_\sigma = \zeta^N V^{(N)}(A)_\sigma$.*

Proof. ✓ Proof Combine gauge covariance of word evaluation with scalar scaling, then take traces. $\square$

2.4 Injectivity and normality

Definition 2.4.1 (Injectivity). [Lean](#) [✓ Stmt](#) A tensor A is *injective* if its matrices span the full matrix algebra: $\text{span}_{\mathbb{C}}\{A^i : i = 0, \dots, d-1\} = M_D(\mathbb{C})$.

The inverse-tensor characterization of injectivity says that A is injective exactly when there is a tensor A^{-1} such that

$$\sum_i (A^i)_{\alpha,\beta} (A^{-1,i})_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \quad (2.8)$$

Thus (2.8) identifies the two pairs of virtual indices when A^{-1} is contracted with A through their common physical index. In shorthand, $A^{-1}A = \mathbb{1}$. This is the identity used in [CPGSV16, Section 2.3, lines 324–331].

Definition 2.4.2 (L -block injectivity). [Lean](#) [✓ Stmt](#) A tensor A is *L -block injective* if

$$\text{span}_{\mathbb{C}}\{A^{i_1} \dots A^{i_L} : (i_1, \dots, i_L) \in \{0, \dots, d-1\}^L\} = M_D(\mathbb{C}). \quad (2.9)$$

An injective tensor is 1-block injective.

Lemma 2.4.3 (One-site injectivity gives one-block injectivity). [Lean](#) [✓ Stmt](#) If the one-site matrices $\{A^i\}_{i=0}^{d-1}$ span $M_D(\mathbb{C})$, then the length-one word products span $M_D(\mathbb{C})$, so A is 1-block injective.

Proof. [✓ Proof](#) The length-one words are exactly the letters $i \in \{0, \dots, d-1\}$, and evaluating a length-one word gives the matrix A^i . Thus the length-one word span is the same span as in one-site injectivity. $\square$

Definition 2.4.4 (Normal tensor). [Lean](#) [✓ Stmt](#) A tensor A is *normal* [CPGSV21] if there exists a positive integer L_0 such that A is L_0 -block injective.

Lemma 2.4.5 (Gauge invariance of block injectivity). [Lean](#) [✓ Stmt](#) If A and B are gauge equivalent and A is L -block injective, then B is L -block injective.

Proof. [✓ Proof](#) Conjugation by an invertible matrix maps the span of the length- L words of A onto the span of the length- L words of B . Since conjugation is an automorphism of the full matrix algebra, fullness of the first span implies fullness of the second. $\square$

Theorem 2.4.6 (Gauge transport of normality). [Lean](#) [✓ Stmt](#) If A and B are gauge equivalent and A is normal, then B is normal.

Proof. [✓ Proof](#) Apply Lemma 2.4.5 at a blocking length for which A is block-injective. $\square$

Remark 2.4.7. [✓ Stmt](#) The algebraic characterization of normality in Definition 2.4.4 — eventual full matrix span, equivalently eventual block injectivity — is equivalent to the spectral characterization used in [CPGSV21, Definition IV.1]: after TP normalization, the transfer map $\mathcal{E}_A$ is a primitive channel, equivalently irreducible with trivial peripheral spectrum. This equivalence is proved in [SPGWC10, Proposition 3]. In what follows, the directions of the equivalence are supplied by Lemma 7.7.2.1, Theorem 7.7.4.2 for primitive normalized tensors with positive-definite fixed point, the Burnside route through Theorem 7.8.2 for the aperiodic algebraic criterion, and Theorem 8.9.2.1. In particular, every subsequent use of “normal” here refers to this single notion; Definition 8.9.2 employs the spectral formulation, but it describes the same class of tensors.

2.5 Canonical form

Definition 2.5.1 (Block-injective canonical form). [Lean](#) [✓ Stmt](#) Here, a *block-injective canonical form* for a tensor consists of:

1. a number of blocks $r \geq 1$,
2. bond dimensions $D_1, \dots, D_r$,
3. injective tensors $\{A_k^i\}_{i=0}^{d-1}$ with $A_k^i \in M_{D_k}(\mathbb{C})$ for each block $k \in \{1, \dots, r\}$,
4. scaling factors $\mu_1, \dots, \mu_r \in \mathbb{C}$.

The associated tensor is the block-diagonal tensor

$$A^i = \bigoplus_{k=1}^r \mu_k A_k^i. \quad (2.10)$$

This block-injective formulation of the canonical form is used in [\[PGVWC07\]](#); see also [\[CPGSV21\]](#) for the normal canonical form.

Remark 2.5.2. [✓ Stmt](#) Here we use the block-injective canonical form: each block is injective in the sense of Definition 2.4.1. This is stronger than the NT-block canonical form of [\[CPGSV21\]](#), where the blocks are only normal in the spectral sense. Chapter 8 treats the normal canonical form needed for the blocking-based multi-block theory, while the periodic irreducible-form extension is kept separate. This section isolates the stronger block-injective form used in the injective and block-injective arguments.

Definition 2.5.3 (Block projection). [Lean](#) [✓ Stmt](#) For a finite direct sum $\bigoplus_k \mathbb{C}^{D_k}$, the block projection P_j is the matrix that is the identity on the j -th summand and zero on all other summands.

Definition 2.5.4 (Block-diagonal matrix). [Lean](#) [✓ Stmt](#) A matrix on $\bigoplus_k \mathbb{C}^{D_k}$ is block diagonal if it is a direct sum of square matrices, one on each summand.

Lemma 2.5.5 (Off-block characterization). [Lean](#) [✓ Stmt](#) A matrix on $\bigoplus_k \mathbb{C}^{D_k}$ is block diagonal if and only if every entry from one summand to a distinct summand is zero.

Proof. [✓ Proof](#) The forward direction follows from the definition of a direct sum of matrices. Conversely, if all off-block entries vanish, the diagonal blocks reconstruct the original matrix. $\square$

Lemma 2.5.6 (Commutation with block projections). [Lean](#) [✓ Stmt](#) If a matrix on $\bigoplus_k \mathbb{C}^{D_k}$ commutes with every block projection P_j , then it is block diagonal.

Proof. [✓ Proof](#) Compare the (i, j) block of the identity $XP_j = P_jX$ for $i \neq j$. The right multiplication by P_j keeps that block, while the left multiplication by P_j kills it. Hence every off-block entry is zero, and Lemma 2.5.5 applies. $\square$

Definition 2.5.7 (Block-diagonal tensor from blocks). [Lean](#) [✓ Stmt](#) Given r blocks with bond dimensions $(D_k)_{k=1}^r$, per-block tensors $\{A_k^i\}$, and scaling factors $(\mu_k)_{k=1}^r$, the associated *block-diagonal tensor* is the tensor of total bond dimension $D = \sum_{k=1}^r D_k$ defined by

$$A^i = \bigoplus_{k=1}^r \mu_k A_k^i.$$

Lemma 2.5.8 (Word evaluation of a block-diagonal tensor). [Lean](#) [✓ Stmt](#) For a word w , the word evaluation of a block-diagonal tensor remains block diagonal:

$$A^w = \bigoplus_k \mu_k^{|w|} A_k^w. \quad (2.11)$$

Proof. [✓ Proof](#) Induct on the word and multiply block-diagonal matrices block by block; every letter contributes one scalar factor μ_k on the k -th block. $\square$

Theorem 2.5.9 (MPV decomposition). [Lean](#) [✓ Stmt](#) The MPV of a block-diagonal tensor decomposes as

$$|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(A_k)\rangle. \quad (2.12)$$

Equivalently, for each configuration σ one has

$$V^{(N)}(A)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(A_k)_\sigma.$$

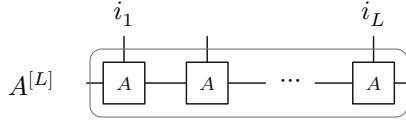
Proof. [✓ Proof](#) By Lemma 2.5.8, the full word evaluation is block diagonal with blocks $\mu_k^N A_k^w$ for a word w of length N . Taking the trace of a block-diagonal matrix gives the sum of the block traces: $\text{tr}(A^w) = \sum_k \mu_k^N \text{tr}(A_k^w)$. $\square$

2.6 Blocking

Definition 2.6.1 (Blocked tensor). [Lean](#) [✓ Stmt](#) Given a tensor A and a blocking length L , the L -blocked tensor is the tensor with physical index set $\{0, \dots, d-1\}^L$ (hence physical dimension d^L) and the same bond dimension D , whose matrices are indexed by words $(i_1, \dots, i_L) \in \{0, \dots, d-1\}^L$:

$$(A^{[L]})^{(i_1, \dots, i_L)} = A^{i_1} A^{i_2} \dots A^{i_L}. \quad (2.13)$$

Blocking coarse-grains L neighbouring sites into one tensor: the inner virtual bonds are contracted, the L physical legs merge into a single composite index, and the two outer bonds remain as the bond of $A^{[L]}$.



Lemma 2.6.2 (Blocked word evaluation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $w = (J_1, \dots, J_m)$ is a word in the blocked alphabet, where each J_k is an L -tuple in $\{0, \dots, d-1\}^L$, then concatenating the block words gives a word $\tilde{w}$ of length mL in the original alphabet and $(A^{[L]})^w = A^{\tilde{w}}$.

Proof. [✓ Proof](#) Induct on the blocked word w and split off the first block word. Lemma 2.1.4 turns concatenation of blocks into multiplication of the corresponding evaluations. $\square$

Lemma 2.6.3 (Blocking preserves MPV coefficients). [Lean](#) [✓ Stmt](#) For each basis state $\sigma = (\sigma_1, \dots, \sigma_N)$ of the blocked system, where each σ_k is an L -tuple in $\{0, \dots, d-1\}^L$, there exists a basis state $\tilde{\sigma} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_{NL})$ of the original system such that $V^{(N)}(A^{[L]})_\sigma = V^{(NL)}(A)_{\tilde{\sigma}}$.

Proof. ✓ Proof Concatenate the block words and apply the blocked word evaluation lemma (Lemma 2.6.2). $\square$

Theorem 2.6.4 (Blocking preserves same MPV). Lean ✓ Stmt *If A and B generate the same MPV family, then for any blocking length L , the blocked tensors $A^{[L]}$ and $B^{[L]}$ also generate the same MPV family.*

Proof. ✓ Proof By Lemma 2.6.3, each MPV coefficient of the blocked tensor $A^{[L]}$ equals a coefficient of A at the corresponding concatenated configuration. The same holds for $B^{[L]}$ (with the same concatenated configuration). Since A and B agree on all configurations, so do $A^{[L]}$ and $B^{[L]}$. $\square$

2.7 MPV overlap

Definition 2.7.1 (MPV as Hilbert-space vector). Lean ✓ Stmt We regard $|V^{(N)}(A)\rangle$ as an element of $\mathbb{C}^{d^N}$ indexed by configurations $\sigma \in \{0, \dots, d-1\}^N$, with σ -component $V^{(N)}(A)_\sigma$ as in Definition 2.1.7. This allows us to use the standard Euclidean inner product on $\mathbb{C}^{d^N}$.

Definition 2.7.2 (MPV inner product). Lean ✓ Stmt The *MPV inner product* between two tensors A and B at system size N is

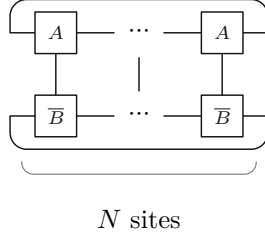
$$\langle V^{(N)}(A) | V^{(N)}(B) \rangle := \sum_{\sigma \in \{0, \dots, d-1\}^N} \overline{V^{(N)}(A)_\sigma} V^{(N)}(B)_\sigma, \quad (2.14)$$

i.e. the Euclidean inner product on $\mathbb{C}^{d^N}$ (conjugate-linear in the first argument).

Definition 2.7.3 (MPV overlap). Lean ✓ Stmt The *MPV overlap* between two tensors A (of bond dimension D_1) and B (of bond dimension D_2) at system size N is

$$O_{AB}(N) := \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}. \quad (2.15)$$

Diagrammatically, the overlap contracts the MPV ring of A against the conjugate ring of B along their shared physical indices, both virtual rings closed by the trace:



This pairing is linear in the first MPV coefficient and conjugate-linear in the second. By Lemma 2.7.4, it is the complex conjugate of the Hilbert-space inner product from Definition 2.7.2.

Lemma 2.7.4 (Overlap equals conjugate inner product). Lean ✓ Stmt $O_{AB}(N) = \overline{\langle V^{(N)}(A) | V^{(N)}(B) \rangle}$.

Proof. ✓ Proof The overlap sums $V_\sigma \overline{W_\sigma}$ while the inner product sums $\overline{V_\sigma} W_\sigma$. Complex conjugation swaps these. $\square$

Lemma 2.7.5 (Overlap determined by positive-length MPV). Lean ✓ Stmt If $V^{(N)}(A)_\sigma = V^{(N)}(A')_\sigma$ and $V^{(N)}(B)_\sigma = V^{(N)}(B')_\sigma$ for all $N > 0$ and all σ , then $O_{AB}(N) = O_{A'B'}(N)$ for all positive N .

Proof. ✓ Proof Direct pointwise rewriting: if the MPV coefficients agree for all positive chain lengths, the overlap sums are identical. □

Chapter 3

The Single-Block Fundamental Theorem

An injective MPS tensor that generates the same matrix-product vectors as a second tensor of equal bond dimension is conjugate to it [CPGSV21, Corollary IV.5]: there is an invertible X with $B^i = XA^iX^{-1}$ for every physical index i . The argument is purely algebraic. Equality of the matrix-product vectors gives $\text{tr}(A^w) = \text{tr}(B^w)$ for every word w ; the length-two and length-three traces produce a linear map T with $T(A^i) = B^i$ that is multiplicative and nonzero; a nonzero multiplicative endomorphism of the simple algebra $M_D(\mathbb{C})$ is a unital automorphism; and by Skolem–Noether every automorphism of $M_D(\mathbb{C})$ is conjugation by an invertible X . Evaluating at A^i closes the proof. The same conjugation, one normal block at a time, is the central step of the multi-block theorem in Chapter 10.

3.1 The multiplicative linear extension

Nondegeneracy of the trace pairing on $M_D(\mathbb{C})$ is a generic algebra fact, proved in the companion quantum-channel volume [LTC26, “Nondegeneracy of the trace pairing”].

Lemma 3.1.1 (Same matrix-product vectors give trace agreement). Lean ✓ Stmt *If A and B generate the same matrix-product vectors, then $\text{tr}(A^w) = \text{tr}(B^w)$ for every word w .*

Proof. ✓ Proof By (2.3), the configuration $\sigma = w$ gives $V^{(|w|)}(A)_w = \text{tr}(A^w)$. Equality of the matrix-product vectors gives $V^{(N)}(A) = V^{(N)}(B)$ for all N , so $\text{tr}(A^w) = \text{tr}(B^w)$. □

The trace conditions constrain every product of the matrices A^i . To bring the algebra structure of $M_D(\mathbb{C})$ to bear, record how a matrix pairs against the family $\{A^i\}$ under the trace.

Definition 3.1.2 (Trace pairing map). Lean ✓ Stmt For a tensor A , the trace pairing map $\Phi_A : M_D(\mathbb{C}) \rightarrow \mathbb{C}^d$ is defined by

$$\Phi_A(M)_i = \text{tr}(MA^i). \tag{3.1}$$

Theorem 3.1.3 (The trace pairing map is injective for injective tensors). Lean Lean ✓ Stmt *If A is injective, then $\ker \Phi_A = \{0\}$.*

Proof. ✓ Proof If $\Phi_A(M) = 0$, then (3.1) gives $\text{tr}(MA^i) = 0$ for every i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so $\text{tr}(MN) = 0$ for all N , and $M = 0$ by the corresponding theorem in the companion quantum-channel volume [LTC26, “Nondegeneracy of the trace pairing”]. □

Lemma 3.1.4 (Injectivity transfers across a range inclusion). Lean ✓ Stmt Let $\Phi, \Psi : V \rightarrow W$ be $\mathbb{C}$ -linear maps with V finite-dimensional. If $\ker \Phi = \{0\}$ and $\text{range } \Phi \subseteq \text{range } \Psi$, then $\ker \Psi = \{0\}$.

Proof. ✓ Proof By rank-nullity, $\ker \Phi = \{0\}$ gives $\dim \text{range } \Phi = \dim V$, and the range inclusion gives

$$\dim V = \dim \text{range } \Phi \leq \dim \text{range } \Psi \leq \dim V.$$

so $\dim \text{range } \Psi = \dim V$ and $\ker \Psi = \{0\}$. □

Theorem 3.1.5 (The linear extension exists and is unique). Lean ✓ Stmt If A is injective and A, B generate the same matrix-product vectors, then there is a unique linear map $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ with $T(A^i) = B^i$ for all i .

Proof. ✓ Proof Length-two trace agreement gives $\Phi_A(A^i) = \Phi_B(B^i)$ for every i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so $\text{range } \Phi_A \subseteq \text{range } \Phi_B$. With $\ker \Phi_A = \{0\}$ (Theorem 3.1.3), Lemma 3.1.4 gives $\ker \Phi_B = \{0\}$. Let g be a left inverse of Φ_B and set $T = g \circ \Phi_A$; then

$$T(A^i) = g(\Phi_A(A^i)) = g(\Phi_B(B^i)) = B^i.$$

Uniqueness holds because the $\{A^i\}$ span $M_D(\mathbb{C})$. □

Theorem 3.1.6 (The linear extension is multiplicative). Lean ✓ Stmt Under the hypotheses of Theorem 3.1.5, the map T satisfies $T(MN) = T(M)T(N)$ for all $M, N \in M_D(\mathbb{C})$.

Proof. ✓ Proof Length-two trace agreement gives $\Phi_B(T(A^i)) = \Phi_B(B^i) = \Phi_A(A^i)$; since the $\{A^i\}$ span $M_D(\mathbb{C})$, this extends to $\Phi_B \circ T = \Phi_A$, so $\text{range } \Phi_A = \text{range}(\Phi_B \circ T) \subseteq \text{range } \Phi_B$. With $\ker \Phi_A = \{0\}$ (Theorem 3.1.3), Lemma 3.1.4 makes Φ_B injective. Length-three trace agreement gives, for all i, j , and k ,

$$\Phi_B(T(A^i A^j))_k = \Phi_A(A^i A^j)_k = \text{tr}(A^i A^j A^k) = \text{tr}(B^i B^j B^k) = \Phi_B(B^i B^j)_k.$$

Hence injectivity of Φ_B yields $T(A^i A^j) = B^i B^j$. Extend in two stages: for fixed j , the maps $M \mapsto T(MA^j)$ and $M \mapsto T(M)B^j$ agree on the spanning $\{A^i\}$, hence $T(MA^j) = T(M)B^j$ for all M ; then for fixed M , the maps $N \mapsto T(MN)$ and $N \mapsto T(M)T(N)$ agree on the spanning $\{A^j\}$, hence $T(MN) = T(M)T(N)$ for all N . □

3.2 Inner automorphism and the single-block theorem

The rigidity of the full matrix algebra now finishes the proof. A nonzero multiplicative map on $M_D(\mathbb{C})$ is forced to be bijective (simplicity), unit-preserving (surjectivity), and conjugation by a single invertible matrix (Skolem–Noether). The next four results establish these steps and the nonvanishing of T , which the final theorem combines into the gauge X .

Theorem 3.2.1 (A nonzero multiplicative endomorphism of $M_D(\mathbb{C})$ is bijective). Lean ✓ Stmt A nonzero multiplicative $\mathbb{C}$ -linear endomorphism of $M_D(\mathbb{C})$ is bijective.

Proof. ✓ Proof The kernel of a multiplicative linear map is a two-sided ideal. The algebra $M_D(\mathbb{C})$ is simple, so the kernel is $\{0\}$ or all of $M_D(\mathbb{C})$; the latter forces the map to be zero. Hence the map is injective, and surjective by finite-dimensionality. □

Lemma 3.2.2 (A surjective multiplicative map preserves the unit). [Lean](#) [✓ Stmt](#) *If $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ is a surjective multiplicative $\mathbb{C}$ -linear map, then $T(\mathbb{1}) = \mathbb{1}$, so T is a $\mathbb{C}$ -algebra homomorphism.*

Proof. [✓ Proof](#) Choose X with $T(X) = \mathbb{1}$. Then

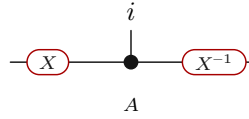
$$T(\mathbb{1}) = T(X)T(\mathbb{1}) = T(X\mathbb{1}) = T(X) = \mathbb{1}.$$

□

Lemma 3.2.3 (An injective tensor forces a nonzero partner). [Lean](#) [✓ Stmt](#) *Let $D \geq 1$. If A is injective and A, B generate the same matrix-product vectors, then the B^i are not all zero.*

Proof. [✓ Proof](#) If $B^i = 0$ for every i , then Lemma 3.1.1 on length-one words gives $\text{tr}(A^i) = 0$ for all i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so the trace functional vanishes identically, contradicting $\text{tr}(\mathbb{1}) = D \neq 0$. □

Theorem 3.2.4 (Single-block Fundamental Theorem). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor and B a tensor of the same bond dimension. If A and B generate the same matrix-product vectors, then there is $X \in \text{GL}_D(\mathbb{C})$ with $B^i = XA^iX^{-1}$ for all i . In tensor-network notation the gauge conjugates the local tensor on its virtual legs, with the physical index i on the upper leg:*



Proof. [✓ Proof](#) Theorem 3.1.5 gives the unique linear T with $T(A^i) = B^i$, and Theorem 3.1.6 makes it multiplicative. It is nonzero: otherwise $B^i = 0$ for all i , against Lemma 3.2.3. By Theorem 3.2.1, T is bijective, and by Lemma 3.2.2 it fixes $\mathbb{1}$, so T is a $\mathbb{C}$ -algebra automorphism. Theorem [LTC26, “Skolem–Noether”] gives $X \in \text{GL}_D(\mathbb{C})$ with $T(M) = XM X^{-1}$; evaluating at $M = A^i$ gives $B^i = XA^iX^{-1}$. □

Chapter 4

Transfer Maps and Quantum-Channel Interfaces

This chapter records the transfer-map interfaces used by the MPS development. The underlying theory is developed in the companion quantum-channel volume [LTC26, “Positive map” and “Quantum channel”].

4.1 Irreducibility

Theorem 4.1.1 (Positive-definite transport by a coisometry). [Lean](#) [✓ Stmt](#) *Let $A > 0$ be a square complex matrix and let V be a possibly rectangular matrix satisfying $VV^\dagger = \mathbb{1}$. Then*

$$VAV^\dagger > 0. \quad (4.1)$$

Proof. [✓ Proof](#) For a row vector x , one has

$$x(VAV^\dagger)x^\dagger = (xV)A(xV)^\dagger. \quad (4.2)$$

Moreover, $xV = 0$ implies

$$x = x(VV^\dagger) = 0. \quad (4.3)$$

Thus every nonzero x gives $xV \neq 0$, and positive definiteness of A makes the displayed quadratic form strictly positive. $\square$

Theorem 4.1.2 (Injectivity implies irreducibility). [Lean](#) [✓ Stmt](#) *If A is an injective MPS tensor, then its transfer map $\mathcal{E}_A$ is irreducible.*

Proof. [✓ Proof](#) If P is an invariant projection for $\mathcal{E}_A$, then $(\mathbb{1} - P)A^iP = 0$ for all i . Since the $\{A^i\}$ span $M_D(\mathbb{C})$, $(\mathbb{1} - P)MP = 0$ for all M , so $P = 0$ or $P = \mathbb{1}$. $\square$

4.2 Transfer maps

Theorem 4.2.1 (Transfer map is CP). [Lean](#) [✓ Stmt](#) *The transfer map $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ is completely positive.*

Proof. ✓ Proof It is already written in the Kraus form (established in the companion quantum-channel volume [LTC26, “Completely positive map”]), with Kraus operators $\{A^i\}$. $\square$

Theorem 4.2.2 (Transfer matrix of an MPS transfer map). Lean ✓ Stmt *For an MPS tensor $A = \{A^i\}_{i=0}^{d-1}$, the matrix representing $\mathcal{E}_A$ under vectorization is*

$$K_A = \sum_{i=0}^{d-1} \overline{A^i} \otimes A^i.$$

Proof. ✓ Proof Apply the standard vectorization identity $\text{vec}(AXB) = (B^\top \otimes A) \text{vec}(X)$ to each summand $A^i X (A^i)^\dagger$ and sum over the physical index. $\square$

Lemma 4.2.3 (Trace adjointness of a transfer map). Lean ✓ Stmt *The trace-adjoint identity of Theorem [LTC26, “Trace adjointness of a Kraus map”] holds when the Kraus family is the family of matrices of an MPS tensor and the two Kraus maps are written as transfer maps.*

Proof. ✓ Proof

Identify the two Kraus maps with the corresponding transfer maps. $\square$

Lemma 4.2.4 (Transfer map under scalar multiplication). Lean ✓ Stmt *Let A be an MPS tensor and let $w \in \mathbb{C}$. Then, for every bond matrix X ,*

$$\mathcal{E}_{wA}(X) = w\bar{w} \mathcal{E}_A(X).$$

Proof. ✓ Proof Expand the transfer map and pull the scalar w and its conjugate out of each summand. $\square$

Lemma 4.2.5 (Left-canonical transfer maps preserve trace). Lean ✓ Stmt *If A is left canonical, then, for every bond matrix X ,*

$$\text{tr}(\mathcal{E}_A(X)) = \text{tr}(X).$$

Proof. ✓ Proof Cyclicity of trace gives

$$\text{tr}(\mathcal{E}_A(X)) = \text{tr}\left(\left(\sum_i (A^i)^\dagger A^i\right) X\right) = \text{tr}(X),$$

since left canonicity makes the sum in parentheses equal to the identity. $\square$

Theorem 4.2.6 (Transfer map is a channel). Lean ✓ Stmt *If $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, then the transfer map $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ is a quantum channel (CPTP).*

Proof. ✓ Proof Complete positivity is Theorem 4.2.1. Trace preservation follows by cyclicity and the normalization hypothesis:

$$\text{tr}(\mathcal{E}_A(X)) = \sum_i \text{tr}(A^i X (A^i)^\dagger) = \text{tr}\left(\left(\sum_i (A^i)^\dagger A^i\right) X\right) = \text{tr}(X).$$

$\square$

Chapter 5

Perron–Frobenius Theory for Channels and Transfer Maps

This chapter establishes elementary fixed-point existence for quantum channels and the Perron–Frobenius theory: existence, positive definiteness, and uniqueness of fixed points for injective and irreducible transfer maps, canonical gauge constructions, and the spectral-radius identification of Perron eigenvalues. The ergodicity of irreducible channels and the CP-map irreducibility criterion modeled on Wolf’s spectral characterization are developed in the companion quantum-channel volume [LTC26, “Cesàro convergence for irreducible channels” and “Irreducibility iff the restricted CP spectral properties hold”]. The conceptual source is Wolf’s treatment of irreducible positive maps [Wol12, Chapter 6, especially Theorems 6.2–6.5 and Corollary 6.3] and [EHK78].

5.1 Positive definiteness

Theorem 5.1.1 (Transfer-map PSD fixed point — irreducible version). Lean ✓ Stmt *Let A be an MPS tensor whose transfer map $\mathcal{E}_A$ is irreducible. If $\rho \geq 0$, $\rho \neq 0$, and $\mathcal{E}_A(\rho) = \rho$, then ρ is positive definite.*

Proof. ✓ Proof The transfer map is completely positive, so Theorem [LTC26, “PSD fixed point is positive definite — irreducible version”] applies. □

Theorem 5.1.2 (PSD fixed point is positive definite). Lean ✓ Stmt *Let A be an injective MPS tensor and let $\rho \geq 0$, $\rho \neq 0$, be a fixed point of $\mathcal{E}_A$. Then ρ is positive definite.*

Proof. ✓ Proof Injectivity implies irreducibility of the transfer map. Apply Theorem 5.1.1. □

5.2 Uniqueness

Theorem 5.2.1 (Irreducible fixed-point uniqueness). Lean ✓ Stmt *Let A be an MPS tensor whose transfer map is irreducible. If $\rho, \sigma \geq 0$ are fixed points of $\mathcal{E}_A$ and $\rho \neq 0$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.*

Proof. ✓ Proof The transfer map is completely positive, so Theorem [LTC26, “Irreducible fixed-point uniqueness — CP version”] applies. □

Theorem 5.2.2 (Uniqueness of PSD fixed point). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor. If $\rho, \sigma \geq 0$ are nonzero fixed points of $\mathcal{E}_A$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.*

Proof. [✓ Proof](#) Injectivity implies irreducibility of the transfer map. Apply Theorem 5.2.1. $\square$

5.3 Existence and the Perron–Frobenius theorem

Theorem 5.3.1 (Existence of PSD fixed point). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, so that $\mathcal{E}_A$ is trace-preserving. Then there exists $\rho \geq 0$, $\rho \neq 0$, with $\mathcal{E}_A(\rho) = \rho$.*

Proof. [✓ Proof](#) Under the normalization $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, the transfer map is a quantum channel (Theorem 4.2.6). The Cesàro fixed-point theorem (the corresponding theorem in the companion quantum-channel volume [LTC26, “Cesàro fixed-point theorem”]) then provides the fixed point; injectivity is not needed for this step. $\square$

Theorem 5.3.2 (Quantum Perron–Frobenius). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an injective MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then the transfer map $\mathcal{E}_A$ has a unique positive semidefinite fixed point ρ up to scaling, and ρ is positive definite.*

Proof. [✓ Proof](#) Combine Theorems 5.3.1, 5.1.2, and 5.2.2. $\square$

Theorem 5.3.3 (Quantum Perron–Frobenius for irreducible transfer maps). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an MPS tensor such that its transfer map is irreducible and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, so that $\mathcal{E}_A$ is trace-preserving. Then $\mathcal{E}_A$ has a unique positive definite fixed point, up to scalar multiple.*

Proof. [✓ Proof](#) Existence is the channel fixed-point theorem (Theorem 5.3.1). Irreducibility upgrades the fixed point to positive definite, and Theorem 5.2.1 gives uniqueness up to scalar multiple. $\square$

5.4 Right- and left-canonical gauges

Definition 5.4.1 (Left-canonical normalization). [Lean](#) [✓ Stmt](#) *An MPS tensor $A = (A^i)_i$ is left-canonical when*

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}.$$

Equivalently, its Kraus map is trace-preserving.

Theorem 5.4.2 (Left-canonical (trace-preserving) gauge). [Lean](#) [Lean](#) [✓ Stmt](#) *Let σ be a fixed point of the adjoint transfer map $\sum_i (A^i)^\dagger \sigma A^i = \sigma$, and let S be invertible with $S^\dagger S = \sigma$. Define $A'^i = S A^i S^{-1}$. Then $\sum_i (A'^i)^\dagger A'^i = \mathbb{1}$, i.e. the gauged Kraus map is trace-preserving. This is the left-canonical normalization.*

Proof. [✓ Proof](#) Conjugating each summand gives $(S A^i S^{-1})^\dagger (S A^i S^{-1}) = (S^\dagger)^{-1} (A^i)^\dagger \sigma A^i S^{-1}$. The adjoint fixed-point substitution leaves $(S^\dagger)^{-1} \sigma S^{-1}$, and $\sigma = S^\dagger S$ cancels the outer factors. See Section A.1 for the full calculation. $\square$

Theorem 5.4.3 (Square-root trace-preserving gauge from an adjoint fixed point). [Lean](#) [✓ Stmt](#)
 Let σ be positive definite and satisfy $\sum_i (A^i)^\dagger \sigma A^i = \sigma$. Define

$$B^i = \sigma^{1/2} A^i \sigma^{-1/2}. \quad (5.1)$$

Then B is trace-preserving: $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

Proof. [✓ Proof](#)

This is the specialization $K_i = A^i$ of Theorem [\[LTC26\]](#), “Trace-preserving gauge from an adjoint fixed point, Kraus form”. Each conjugated summand is $(B^i)^\dagger B^i = \sigma^{-1/2} (A^i)^\dagger \sigma A^i \sigma^{-1/2}$. Substituting $\sum_i (A^i)^\dagger \sigma A^i = \sigma$ and cancelling the outer inverse square roots gives $\mathbb{1}$. The complete calculation appears in Section [A.1](#). $\square$

Theorem 5.4.4 (Square-root trace-preserving gauge from an adjoint eigenvector). [Lean](#) [✓ Stmt](#)
 Let σ be positive definite and satisfy $\sum_i (A^i)^\dagger \sigma A^i = r\sigma$ for $r > 0$. Define

$$B^i = \sigma^{1/2} (r^{-1/2} A^i) \sigma^{-1/2}. \quad (5.2)$$

Then B is trace-preserving: $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

Proof. [✓ Proof](#) Rescale A by $r^{-1/2}$. The adjoint transfer map is quadratic in the tensor, so the eigenvector equation becomes a fixed-point equation. Theorem [5.4.3](#) then gives the result. $\square$

Remark 5.4.5. [✓ Stmt](#) The right-canonical gauge of Theorem [\[LTC26\]](#), “Right-canonical (unital) gauge” and the left-canonical gauge of Theorem [5.4.2](#) are generally different similarity transforms: one is built from a fixed point of $\mathcal{E}_A$, the other from a fixed point of the adjoint transfer map. This section does not claim that a single gauge makes the transfer map simultaneously unital and trace-preserving.

5.5 Perron–Frobenius eigenvector existence

This section establishes the existence of a positive definite eigenvector for the adjoint transfer map of an irreducible MPS tensor, and uses it to construct a TP-gauge normalization. The PSD-eigenvector step corresponds to [\[Wol12, Theorem 6.5\]](#) (general positive maps) and [\[EHK78\]](#); the upgrade to positive definiteness uses the irreducible-map theory of [\[Wol12, Theorem 6.3\]](#). The TP-gauge application follows [\[CPGSV16, Appendix A\]](#).

Theorem 5.5.1 (Irreducibility of the transfer map of an irreducible tensor). [Lean](#) [✓ Stmt](#) If an MPS tensor A is irreducible, then its transfer map $\mathcal{E}_A$ is irreducible.

Proof. [✓ Proof](#) If P is a nontrivial invariant projection for $\mathcal{E}_A$, then

$$(\mathbb{1} - P)A^i P = 0$$

for every i , which is exactly a nontrivial invariant projection for the family $\{A^i\}$, contradicting irreducibility of A . $\square$

Theorem 5.5.2 (Positive-definite adjoint eigenvector for irreducible tensors). [Lean](#) [✓ Stmt](#) Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive definite matrix σ and a positive real $r > 0$ such that $\mathcal{E}_A^\dagger(\sigma) = r\sigma$.

This combines the eigenvector-existence part of [\[Wol12, Theorem 6.5\]](#) with the irreducible upgrade to positive definiteness ([\[Wol12, Theorem 6.3\(2\)–\(3\)\]](#)); the application to the adjoint transfer map follows [\[CPGSV16, Appendix A\]](#).

Proof. ✓ Proof By Theorem 5.5.1, $\mathcal{E}_A$ is irreducible, and since $\mathcal{K}_A = \mathcal{E}_A$ (the corresponding theorem in the companion quantum-channel volume [LTC26, “Kraus-map form of the transfer map”]), its Kraus map $\mathcal{K}_A$ is irreducible too (the corresponding theorem in the companion quantum-channel volume [LTC26, “Irreducibility in Kraus-map notation”]). Applying Theorem [LTC26, “Positive-definite adjoint eigenvector for irreducible Kraus families”] to $K_i = A^i$ gives a positive definite σ and a positive real $r > 0$ with $\sum_i (A^i)^\dagger \sigma A^i = r\sigma$, i.e. $\mathcal{E}_A^\dagger(\sigma) = r\sigma$. $\square$

Theorem 5.5.3 (TP-gauge existence for irreducible tensors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive real r , a positive definite matrix σ , and a tensor B gauge-equivalent to $r^{-1/2}A$ such that*

$$B^i = \sigma^{1/2} (r^{-1/2} A^i) \sigma^{-1/2}, \quad (5.3)$$

and $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

This is the “spectral rescaling + TP gauge” step of [CPGSV16, Appendix A]. The similarity is generally non-unitary.

Proof. ✓ Proof By Theorem 5.5.2, obtain σ positive definite and $r > 0$ with $\mathcal{E}_A^\dagger(\sigma) = r\sigma$. Set $c = r^{-1/2}$ and $A'^i = cA^i$. Then $\mathcal{E}_{A'}^\dagger(\sigma) = \sigma$. By the square-root trace-preserving eigenvector gauge construction (Theorem 5.4.4), the tensor defined by (5.3) satisfies $\sum_i (B^i)^\dagger B^i = \mathbb{1}$. Lemma A.1.1 gives the required gauge equivalence to $r^{-1/2}A$. $\square$

Theorem 5.5.4 (Unital-gauge existence for irreducible tensors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive real r , a positive definite matrix ρ , and a tensor B gauge-equivalent to $r^{-1/2}A$ such that*

$$B^i = r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}, \quad (5.4)$$

and $\sum_i B^i (B^i)^\dagger = \mathbb{1}$.

This is the Perron–Frobenius unital-gauge orientation used in [PGVWC07, Theorem 4, lines 765–770], stated for one irreducible nonzero block.

Proof. ✓ Proof Apply Theorem 5.5.2 to the conjugate-transposed Kraus family. Irreducibility transfers to that family, and the nonzero Kraus hypothesis is preserved by taking adjoints. Translating the resulting adjoint eigenvector equation back gives $\mathcal{E}_A(\rho) = r\rho$ with ρ positive definite and $r > 0$. The unital gauge theorem then gives the representative (5.4), and the same square-root similarity gives gauge equivalence to $r^{-1/2}A$. $\square$

Chapter 6

Peripheral Channel Structure and Transfer-Operator Gaps

This chapter studies two spectral phenomena for finite-dimensional quantum channels and MPS transfer maps, following [PGVWC07] and [Wol12].

The first is *peripheral* structure: for an irreducible unital Schwarz map with a faithful adjoint fixed point, the eigenvalues on the unit circle form a finite cyclic group (the corresponding theorem in the companion quantum-channel volume [LTC26, “Peripheral eigenvalues: cyclic structure for Kraus maps”]), and under trace preservation the order of this group divides the bond dimension D (Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”]). Removing this period by unitary conjugation and cyclic corner projections produces the periodicity-free cyclic-sector decomposition (Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”]) used by the canonical-form reduction of Chapter 8.

The second is the *mixed transfer operator* F_{AB} of two MPS tensors A and B , and its spectral radius $\rho_{\text{spec}}(F_{AB})$. We show that $\rho_{\text{spec}}(F_{AB}) \leq 1$ under trace-preserving normalization (Theorem 6.3.1), that equality forces A and B to be gauge-phase equivalent (Theorem 6.3.4), and that mismatched bond dimensions always force a strict gap (Theorem 6.4.2). Since the MPV overlap $O_{AB}(N)$ equals the operator trace $\text{Tr}(F_{AB}^N)$ (Theorem 6.2.1), these gaps translate directly into the overlap-decay theorems (Theorem 6.5.5 and its rectangular and irreducible-trace-preserving variants) used throughout the fundamental theorem. The complementary case — a single tensor whose transfer map has a trivial peripheral spectrum away from the fixed-point projection — gives the rank-one Perron limit $\text{Tr}(\mathcal{E}_A^n) \rightarrow 1$ (Theorem 6.10.1) and hence primitive self-overlap convergence $O_{AA}(N) \rightarrow 1$ (Theorem 6.10.2).

Throughout, $\rho_{\text{spec}}(\cdot)$ denotes the spectral radius of a linear endomorphism of a finite-dimensional matrix space: $\rho_{\text{spec}}(F_{AB}) < 1$ for a mixed transfer operator and $\rho_{\text{spec}}(\mathcal{E}_A - P) < 1$ for a transfer map after removing its fixed-point projection P . The unqualified phrase *spectral gap* is reserved for the energy gap of a many-body Hamiltonian, discussed later in the full blueprint.

The algebraic mixed-transfer identities in the first two sections do not use normalization. Starting with the spectral-radius estimates, we assume the trace-preserving normalization

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}. \quad (6.1)$$

Right-canonical (unital) and left-canonical (trace-preserving) gauges are generally different similarity transforms of the same tensor, recorded in Remark 6.3.5; both are used below, depending on which fixed point — of the transfer map or of its adjoint — drives the gauge.

The proofs of the technical mixed-transfer identities, Frobenius-norm estimates, gauge-rigidity intertwiners, spectral-radius decay estimates, the rank-one Perron projection, and the cyclic-decomposition machinery removing peripheral periodicity are collected in Appendix B.

6.1 Mixed transfer operator

Definition 6.1.1 (Mixed transfer operator). Lean ✓ Stmt The *mixed transfer operator* (or *cross transfer operator*) for two MPS tensors A and B of the same bond dimension D is the linear map $F_{AB} : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ defined by

$$F_{AB}(X) := \sum_{i=0}^{d-1} A^i X(B^i)^\dagger. \quad (6.2)$$

Definition 6.1.2 (Rectangular mixed transfer operator). Lean ✓ Stmt For MPS tensors A of bond dimension D_1 and B of bond dimension D_2 , possibly with $D_1 \neq D_2$, the *rectangular mixed transfer operator* is the map $F_{AB}^{\text{rect}} : M_{D_1 \times D_2}(\mathbb{C}) \rightarrow M_{D_1 \times D_2}(\mathbb{C})$ defined by

$$F_{AB}^{\text{rect}}(X) := \sum_{i=0}^{d-1} A^i X(B^i)^\dagger. \quad (6.3)$$

The self-transfer identity $F_{AA} = \mathcal{E}_A$, the iterated word expansion of F_{AB} , and the trace-expansion lemma over matrix units are proved in Section B.1.

6.2 MPV overlap as transfer trace

Recall from (2.15) that $O_{AB}(N) = \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}$. This section rewrites that scalar as the operator trace of the mixed transfer power.

Lemma 6.2.1 (Overlap equals transfer trace). Lean ✓ Stmt For tensors A and B of bond dimension $D \geq 1$,

$$O_{AB}(N) = \text{Tr}(F_{AB}^N). \quad (6.4)$$

Proof. ✓ Proof Section B.1 expands the operator trace over matrix units (Lemma [LTC26, “Trace expansion over matrix units”]) and the transfer power by words (Lemma B.1.2); reassembling the sums over matrix-unit indices gives

$$\text{Tr}(F_{AB}^N) = \sum_{\sigma} \text{tr}(A^\sigma) \overline{\text{tr}(B^\sigma)} = \sum_{\sigma} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma} = O_{AB}(N).$$

The rectangular case, Lemma 6.2.2, is the same argument over the rectangular matrix-unit basis. $\square$

Lemma 6.2.2 (Rectangular overlap as transfer trace). Lean ✓ Stmt For tensors A of bond dimension $D_1 \geq 1$ and B of bond dimension $D_2 \geq 1$,

$$O_{AB}(N) = \text{Tr}((F_{AB}^{\text{rect}})^N). \quad (6.5)$$

Proof. ✓ Proof The detailed word-index calculation is the same as the proof of Theorem 6.2.1, over the rectangular matrix-unit basis; see Section B.1. $\square$

6.3 Eigenvalue bound and transfer-operator gap

The rectangular eigenvalue bound is the MPS formulation of the generic bound for trace-preserving mixed Kraus maps. Under the identification of the rectangular mixed transfer with the mixed Kraus map, the two tensor normalizations are precisely the trace-preserving conditions. For the rigidity theorem, one gauges A and B separately using positive fixed points of their transfer maps to obtain unital Kraus families, and then applies the Kadison–Schwarz equality argument ([Wol12, Theorem 5.3]) to a block embedding of a mixed-transfer eigenvector.

Theorem 6.3.1 (Eigenvalue bound). Lean ✓ Stmt *Let A and B be normalized MPS tensors. Every eigenvalue μ of F_{AB} satisfies $|\mu| \leq 1$.*

Proof. ✓ Proof This is the equal-bond-dimension case $D_1 = D_2 = D$ of Theorem 6.3.2. The argument uses only the trace-preserving normalization (6.1) and the general eigenvalue bound for trace-preserving Kraus maps; it does not appeal to Perron–Frobenius theory. $\square$

Theorem 6.3.2 (Rectangular eigenvalue bound). Lean ✓ Stmt *Let A and B be normalized MPS tensors of bond dimensions $D_1 \geq 1$ and $D_2 \geq 1$. Every eigenvalue μ of F_{AB}^{rect} satisfies $|\mu| \leq 1$.*

Proof. ✓ Proof This is the MPS formulation of the corresponding theorem in the companion quantum-channel volume [LTC26, “Eigenvalue bound for a trace-preserving mixed Kraus map”]: the map F_{AB}^{rect} is the mixed Kraus map of the two matrix families, and the normalizations of A and B are exactly its two trace-preserving hypotheses. $\square$

Lemma 6.3.3 (Spectral radius bound). Lean ✓ Stmt *For normalized tensors, $\rho_{\text{spec}}(F_{AB}) \leq 1$.*

Proof. ✓ Proof The spectral radius is the maximum of $|\mu|$ over the eigenvalues, and each satisfies $|\mu| \leq 1$ by Theorem 6.3.1. $\square$

Theorem 6.3.4 (Spectral radius ≥ 1 implies gauge-phase equivalence). Lean ✓ Stmt *Let A and B be injective normalized MPS tensors. If $\rho_{\text{spec}}(F_{AB}) \geq 1$, then A and B are gauge-phase equivalent.*

Proof. ✓ Proof Since $\rho_{\text{spec}}(F_{AB}) \leq 1$ by Lemma 6.3.3, the hypothesis gives an eigenvalue μ with $|\mu| = 1$ and a nonzero eigenvector X satisfying $F_{AB}(X) = \mu X$. The proof proceeds in five steps, elaborated in Section B.4.

Step 1 (Separate gauging). Choose positive definite fixed points ρ_A and ρ_B of $\mathcal{E}_A$ and $\mathcal{E}_B$ by Theorem 5.3.2. Gauge the tensors separately to unital Kraus families by Theorem [LTC26, “Right-canonical (unital) gauge”]: $A'^i = \rho_A^{-1/2} A^i \rho_A^{1/2}$ and similarly for B . Set $X' = \rho_A^{-1/2} X \rho_B^{-1/2}$.

Step 2 (Block Kraus family and off-diagonal embedding). Form

$$C^i := \begin{pmatrix} A'^i & 0 \\ 0 & B'^i \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & X' \\ 0 & 0 \end{pmatrix}.$$

The matrices $\{C^i\}$ form a unital Kraus family on $M_{2D}(\mathbb{C})$, and $E_C(Y) = \mu Y$.

Step 3 (Kadison–Schwarz equality and Kraus relations). The positive definite fixed points of the adjoint maps give a faithful weighted trace for the block map. Applying the Kadison–Schwarz inequality to Y and pairing its positive semidefinite gap with this fixed point shows that the gap vanishes. The Kraus relation of the corresponding theorem in the companion quantum-channel volume [LTC26, “KS equality implies Kraus commutation”] then gives $Y(C^i)^\dagger = \mu(C^i)^\dagger Y$. Applying the same argument to the complementary product gives $C^i Y = \mu Y C^i$.

Step 4 (Intertwining identity and invertibility). Expanding the block relations from Step 3 gives

$$X'(B'^i)^\dagger = \mu(A'^i)^\dagger X', \quad A'^i X' = \mu X' B'^i.$$

The corresponding multiplicative identities make $X'^\dagger X'$ and $X' X'^\dagger$ nonzero positive semidefinite fixed points of the appropriate transfer maps. Injectivity upgrades them to positive definite matrices, so X' is invertible.

Step 5 (Skolem–Noether conclusion). The intertwining identity and invertibility of X' give an algebra isomorphism between the full matrix algebras generated by the two families. By Theorem [LTC26, “Skolem–Noether”], this isomorphism is inner. Undoing the separate gauges gives $B^i = \bar{\mu} X A^i X^{-1}$ for every i , which is the gauge-phase relation (2.6).

The gauging is applied separately to $\mathcal{E}_A$ and $\mathcal{E}_B$; the mixed transfer map F_{AB} need not be simultaneously unital and trace-preserving. $\square$

Remark 6.3.5 (Two gauge conventions). ✓ Stmt The right-canonical gauge used here, $A'^i = \rho^{-1/2} A^i \rho^{1/2}$, makes $\{A'^i\}$ a unital Kraus family: $\sum_i A'^i (A'^i)^\dagger = \mathbb{1}$. The convention $\tilde{A}^i = \rho^{1/2} A^i \rho^{-1/2}$ makes the family trace-preserving: $\sum_i (\tilde{A}^i)^\dagger \tilde{A}^i = \mathbb{1}$. Both conventions appear in practice: the unital form is used in the transfer-operator gap proof above, while the trace-preserving form is used in the canonical-form reduction of Chapter 8.

Theorem 6.3.6 (Strict transfer-operator gap). Lean ✓ Stmt *Let A and B be injective normalized MPS tensors that are not gauge-phase equivalent. Then $\rho_{\text{spec}}(F_{AB}) < 1$.*

Proof. ✓ Proof Lemma 6.3.3 gives $\rho_{\text{spec}}(F_{AB}) \leq 1$. Equality would imply gauge-phase equivalence by Theorem 6.3.4, contradicting the hypothesis. $\square$

6.4 Transfer-operator gap — rectangular case

Lemma 6.4.1 (Rectangular intertwining forces equal bond dimension). Lean ✓ Stmt *Let A and B be injective tensors of bond dimensions D_1 and D_2 . If there are $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$ and $\mu \in \mathbb{C}$ such that, for every physical index i ,*

$$X(B^i)^\dagger = \mu(A^i)^\dagger X, \quad A^i X = \mu X B^i, \quad (6.6)$$

then $D_1 = D_2$.

Proof. ✓ Proof The first relation in (6.6) shows that $\ker(X)$ is invariant under every $(B^i)^\dagger$: if $Xv = 0$, then $X(B^i)^\dagger v = \mu(A^i)^\dagger Xv = 0$. Since the $\{B^i\}$ span the full matrix algebra, so do the $\{(B^i)^\dagger\}$ (conjugate transposition is a conjugate-linear bijection and therefore preserves complex linear spans), so $\ker(X)$ is invariant under every matrix; injectivity of the resulting map forces $\ker(X) = 0$, giving $D_2 \leq D_1$. Taking adjoints of the second relation gives $X^\dagger (A^i)^\dagger = \bar{\mu} (B^i)^\dagger X^\dagger$, so the same argument applied to $X^\dagger$ shows $\ker(X^\dagger)$ is invariant under every $(A^i)^\dagger$ and hence trivial, giving $D_1 \leq D_2$. $\square$

Theorem 6.4.2 (Rectangular transfer-operator gap). Lean ✓ Stmt *Let A be an injective normalized tensor of bond dimension D_1 and B an injective normalized tensor of bond dimension D_2 , with $D_1 \neq D_2$. Then $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$.*

Proof. ✓ Proof Theorem 6.3.2 gives $|\mu| \leq 1$ for every eigenvalue of F_{AB}^{rect} . If its spectral radius were 1, there would be a modulus-one eigenvector $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$.

Gauge A and B to unital families A' and B' and transport X to a nonzero rectangular intertwiner X' satisfying (B.2). Lemma 6.4.1 would then give $D_1 = D_2$, contradicting the hypothesis. $\square$

Theorem 6.4.3 (Rectangular transfer powers below unit spectral radius). Lean ✓ Stmt *Let A and B have bond dimensions D_1 and D_2 , respectively. If $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$, then, for every $X \in M_{D_1 \times D_2}(\mathbb{C})$,*

$$(F_{AB}^{\text{rect}})^n(X) \rightarrow 0.$$

This is the decay mechanism for the off-diagonal transfer operators in [CPGSV16, Appendix B, equations (B.3)–(B.4) and the paragraph following equation (B.6)].

Proof. ✓ Proof

The Gelfand formula implies convergence of $(F_{AB}^{\text{rect}})^n$ to zero in operator norm. Evaluation at a fixed rectangular matrix is continuous, and therefore preserves this limit. $\square$

Theorem 6.4.4 (Rectangular overlap decay). Lean ✓ Stmt *If $D_1 \neq D_2$ and both tensors are injective and normalized, then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. ✓ Proof The gap $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$ from Theorem 6.4.2 and Lemma [LTC26, “Powers vanish below unit spectral radius”] imply operator-norm convergence $(F_{AB}^{\text{rect}})^N \rightarrow 0$, so its operator trace tends to zero. The identity (6.5) therefore gives $O_{AB}(N) \rightarrow 0$. $\square$

6.5 MPV overlap decay

We first record two spectral-radius identities for complex normed algebras. They will also control the scalar weights of normal fixed-point blocks in Chapter 18.

Theorem 6.5.1 (Spectral radius under nonzero scaling). Lean ✓ Stmt *Let a belong to a complex Banach algebra and let $c \in \mathbb{C} \setminus \{0\}$. Then*

$$\rho_{\text{spec}}(ca) = |c| \rho_{\text{spec}}(a).$$

Proof. ✓ Proof Multiplication by c sends the spectrum of a onto the spectrum of ca . Taking the supremum of the moduli of the spectral values gives the formula. $\square$

Theorem 6.5.2 (Spectral radius of a nonzero idempotent). Lean ✓ Stmt *Let a be a nonzero element of a complex normed algebra. If $a^2 = a$, then*

$$\rho_{\text{spec}}(a) = 1.$$

Proof. ✓ Proof The identity $a^2 = a$ gives $a(a - 1) = 0$. Since $a \neq 0$, the element $a - 1$ is not invertible, so 1 belongs to the spectrum of a . Every spectral value λ satisfies $\lambda^2 = \lambda$, hence the spectrum is contained in $\{0, 1\}$. $\square$

The Banach-algebra fact that powers below unit spectral radius vanish is proved in Section B.5.

Lemma 6.5.3 (An idempotent below unit spectral radius vanishes). Lean ✓ Stmt *Let a be an element of a complex Banach algebra with $a^2 = a$ and $\rho_{\text{spec}}(a) < 1$. Then $a = 0$.*

Proof. ✓ Proof

Lemma [LTC26, “Powers vanish below unit spectral radius”] gives $a^n \rightarrow 0$. Idempotence gives $a^{n+1} = a$ for every n , so uniqueness of limits yields $a = 0$. $\square$

Theorem 6.5.4 (Transfer powers converge to zero). Lean ✓ Stmt *Let A and B be injective normalized tensors that are not gauge-phase equivalent. Then $F_{AB}^n(X) \rightarrow 0$ for every $X \in M_D(\mathbb{C})$.*

Proof. ✓ Proof The gap $\rho_{\text{spec}}(F_{AB}) < 1$ from Theorem 6.3.6 and the equal-bond-dimension case of Theorem 6.4.3 give $F_{AB}^n(X) \rightarrow 0$ for every X . $\square$

Theorem 6.5.5 (Overlap decay). Lean ✓ Stmt *Let A and B be injective normalized tensors of the same bond dimension that are not gauge-phase equivalent. Then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. ✓ Proof By (6.4) and the trace-expansion formula of [LTC26, “Trace expansion over matrix units”],

$$O_{AB}(N) = \text{Tr}(F_{AB}^N) = \sum_{p,q} (F_{AB}^N(E_{pq}))_{pq}.$$

Theorem 6.5.4 makes every summand tend to zero. Since the sum is finite, $O_{AB}(N) \rightarrow 0$. $\square$

The block-separation consequences of this gap — cross-correlation decay between distinct blocks and the persistence of self-correlation at a fixed point — used in the canonical-form separation of Chapter 8, are recorded in Section B.5.

6.6 Transfer-operator gap under irreducible-TP hypotheses

The preceding transfer-operator gap results require *injectivity* of both tensors. The following variants weaken this to *irreducibility* together with the trace-preserving normalization (6.1), following Cirac et al. [CPGSV16] and the irreducibility theory of [Wol12, Section 6.2]. The argument replaces the Kraus-commutation step with a Cauchy–Schwarz rigidity argument for the Perron–Frobenius fixed point.

Theorem 6.6.1 (Gauge-equivalence from irreducible-TP spectral radius). Lean Lean ✓ Stmt *Let A and B be irreducible trace-preserving normalized MPS tensors of bond dimension D . If $\rho_{\text{spec}}(F_{AB}) \geq 1$, then A and B are gauge-phase equivalent.*

Proof. ✓ Proof Choose a modulus-one eigenvector as in Theorem 6.3.4. The proof follows its five steps, with one change. Separate gauging, the block Kraus family, and the Kadison–Schwarz equality proceed as before. The equality makes the relevant positive products of the transported intertwiner nonzero positive semidefinite fixed points. Under irreducibility, Theorem 5.1.1 makes these fixed points positive definite, so the transported intertwiner is invertible. Lemma B.4.1 supplies (B.2), and Lemma B.4.2 then gives gauge-phase equivalence. $\square$

Theorem 6.6.2 (Strict transfer-operator gap (irreducible-TP)). Lean ✓ Stmt *Let A and B be irreducible trace-preserving normalized MPS tensors of the same bond dimension that are not gauge-phase equivalent. Then $\rho_{\text{spec}}(F_{AB}) < 1$.*

Proof. ✓ Proof Lemma 6.3.3 gives $\rho_{\text{spec}}(F_{AB}) \leq 1$, while the contrapositive of Theorem 6.6.1 excludes equality. $\square$

Theorem 6.6.3 (Overlap decay (irreducible-TP)). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving normalized tensors of the same bond dimension that are not gauge-phase equivalent. Then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) Theorem 6.6.2 gives $\rho_{\text{spec}}(F_{AB}) < 1$, so $F_{AB}^N \rightarrow 0$. The overlap–trace identity (6.4) gives $O_{AB}(N) \rightarrow 0$. $\square$

Theorem 6.6.4 (Rectangular transfer-operator gap (irreducible-TP)). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving normalized tensors of bond dimensions D_1 and D_2 , respectively, with $D_1 \neq D_2$. Then $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$.*

Proof. [✓ Proof](#) Suppose instead that $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) = 1$, and choose a modulus-one eigenvector $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$. Gauge A and B separately to unital families A' and B' and transport X to $X' \neq 0$. The matrix $X'X'^\dagger$ is a nonzero positive semidefinite fixed point of the appropriate transfer map on $M_{D_1}(\mathbb{C})$, so Theorem 5.1.1 makes it positive definite. Thus X' has full row rank and $D_1 \leq D_2$. Similarly, $X'^\dagger X'$ is positive definite, so X' has full column rank and $D_2 \leq D_1$. Hence $D_1 = D_2$, a contradiction. $\square$

Theorem 6.6.5 (Rectangular overlap decay (irreducible-TP)). [Lean](#) [✓ Stmt](#) *If $D_1 \neq D_2$ and both tensors are irreducible and trace-preserving normalized, then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) Theorem 6.6.4 gives $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$, so $(F_{AB}^{\text{rect}})^N \rightarrow 0$. The rectangular overlap–trace identity (6.5) gives $O_{AB}(N) \rightarrow 0$. $\square$

6.7 Overlap rigidity for normal blocks

The BNT matching argument ultimately turns a non-decaying mixed overlap into a gauge-phase match. The following two results isolate that spectral step.

Lemma 6.7.1 (Mixed-transfer spectral radius from unit-modulus overlap). [Lean](#) [✓ Stmt](#) *Let A and B have the same bond dimension. If $\|\langle V^{(N)}(A) | V^{(N)}(B) \rangle\| \rightarrow 1$, then the mixed transfer map has spectral radius at least 1.*

Proof. [✓ Proof](#) If the mixed-transfer spectral radius were less than 1, the iterated mixed transfer map would tend to zero in operator norm, hence so would its trace. By (6.4), the overlap would tend to zero, contradicting the hypothesis. $\square$

Theorem 6.7.2 (Gauge-phase equivalence from unit-modulus overlap). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible MPS tensors of the same bond dimension with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ and $\sum_i (B^i)^\dagger B^i = \mathbb{1}$. If $\|\langle V^{(N)}(A) | V^{(N)}(B) \rangle\| \rightarrow 1$, then A and B are gauge-phase equivalent.*

Proof. [✓ Proof](#) By Lemma 6.7.1, the mixed-transfer spectral radius is at least 1, and Theorem 6.6.1 then gives gauge-phase equivalence. $\square$

6.8 Peripheral spectral refinements

Every result in this section concerning peripheral eigenvalues, cyclic projections, or peripheral eigenspaces is stated for a finite Kraus map and therefore assumes complete positivity. These results do not supply the abstract Schwarz-map case of Wolf Theorem 6.6; the restriction and its elimination plan are documented in the scope-restriction note [con26a].

6.8.1 Cyclic decomposition of irreducible finite Kraus maps

The construction — normalizing a peripheral eigenvalue to a unitary eigenvector, forming its discrete Fourier (cyclic) spectral projections, and the supporting corner-algebra apparatus (restriction to an invariant corner, corner rank, and the compression isometry $PM_D(\mathbb{C})P \cong M_n(\mathbb{C})$) — is given in Section B.7.

Lemma 6.8.1.1 (Cyclic shift of the Kraus operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $P_0, \dots, P_{m-1}$ be orthogonal projections summing to the identity and cyclically permuted by a Kraus map, $\sum_v K_v P_{k+1} K_v^\dagger = P_k$. Then the projections are mutually orthogonal, none of them vanishes, and each Kraus operator intertwines consecutive projections:*

$$K_v P_{k+1} = P_k K_v. \quad (6.7)$$

Proof. [✓ Proof](#)

Compressing $\sum_v K_v P_{k+1} K_v^\dagger = P_k$ by $\mathbb{1} - P_k$ exhibits a vanishing sum of positive semidefinite matrices, so each summand vanishes: $(\mathbb{1} - P_k) K_v P_{k+1} = 0$. Mutual orthogonality follows by compressing the resolution of the identity by a fixed P_k , another vanishing sum of positive semidefinite matrices. Expanding $K_v P_{k+1}$ and $P_k K_v$ through the resolution of the identity, the off-diagonal compressions vanish by orthogonality, leaving (6.7). If some projection vanished, the cyclic action would propagate the vanishing around the cycle, contradicting the resolution of the identity. $\square$

Theorem 6.8.1.2 (Cyclic shift along a Kraus word). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Lemma 6.8.1.1, a product of ℓ Kraus operators moves each cyclic projection back by ℓ steps. In particular, every product of m Kraus operators commutes with every cyclic projection.*

Proof. [✓ Proof](#) Induct on the length of the word using (6.7). $\square$

The primitivity and irreducibility of the restricted sector dynamics, the invariance of each cyclic corner under the period, and the closure of the peripheral eigenvalues under products and inverses are proved in Section B.7.

6.9 Self-overlap and the period-one criterion

Theorem 6.9.1 (Self-overlap along peripheral-period multiples). [Lean](#) [✓ Stmt](#) *Let $D > 0$ and $m > 0$. Let A be a left-canonical irreducible MPS tensor with bond dimension D , and suppose that the peripheral eigenvalues of its transfer map are exactly $\{z \in \mathbb{C} : z^m = 1\}$. Then $O_{AA}(mn) \rightarrow m$.*

Proof. [✓ Proof](#) Apply Theorem 8.11.1 to the period- m blocking $A^{[m]}$. It gives m nonzero cyclic-sector tensors $C_0, \dots, C_{m-1}$, each left-canonical, whose unit-weight direct sum has the same MPV family as $A^{[m]}$. The cyclic-sector relations make every C_u primitive and irreducible. Distinct sectors are not gauge-phase equivalent. Hence the primitive self-overlap limit and the equal- and unequal-dimension overlap-decay theorems give

$$O_{C_u C_v}(n) \rightarrow \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}$$

Expanding the overlap of the unit-weight direct sum and taking the finite sum of these limits yields

$$O_{A^{[m]}A^{[m]}}(n) = \sum_{u,v \in \mathbb{Z}_m} O_{C_u C_v}(n) \rightarrow m.$$

Finally, blocking identifies $O_{A^{[m]}A^{[m]}}(n) = O_{AA}(mn)$, which proves the claim. $\square$

Theorem 6.9.2 (Primitivity and normality from normalized self-overlap). [Lean](#) [✓ Stmt](#) *Let A be an irreducible left-canonical tensor of positive bond dimension. If $O_{AA}(N) \rightarrow 1$, then the transfer map of A is primitive and A is normal.*

Proof. [✓ Proof](#) Let m be the peripheral period. Along its multiples, $O_{AA}(mn) \rightarrow m$. The assumed full limit gives $O_{AA}(mn) \rightarrow 1$, hence $m = 1$. The transfer map is therefore primitive, and irreducibility and left-canonicity give normality. $\square$

Theorem 6.9.3 (Normality from normalized self-overlap). [Lean](#) [✓ Stmt](#) *Let A be an irreducible left-canonical tensor of positive bond dimension. If $O_{AA}(N) \rightarrow 1$, then A is normal.*

Proof. [✓ Proof](#) This is the normality conclusion of Theorem 6.9.2. $\square$

6.10 Primitive overlap convergence (complementary transfer-map gap form)

The theorems in this section are stated directly in terms of the complementary map $E - P$. This is the analytic form needed for the later overlap argument, corresponding to the primitive-case specialization of [Wol12, Theorem 6.7] where T_ϕ is a rank-one projection. For primitive normalized transfer maps, the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”] supplies this complementary transfer-map input under explicit hypotheses; the rank-one Perron projection $P = P_\rho$, its trace, and the concrete irreducible instantiation of the complementary spectral-radius gap are given in Section B.6.

Theorem 6.10.1 (Complementary transfer-map gap implies trace convergence to 1). [Lean](#) [✓ Stmt](#) *Let E be trace-preserving with fixed point ρ , where $\text{tr}(\rho) \neq 0$, and let P be the fixed-point projection of the corresponding definition in the companion quantum-channel volume [LTC26, “Fixed-point projection”]. If $\rho_{\text{spec}}(E - P) < 1$, then $\text{Tr}(E^n) \rightarrow 1$ as $n \rightarrow \infty$.*

Proof. [✓ Proof](#)

By the power decomposition (established in the companion quantum-channel volume [LTC26, “Power decomposition”]), $E^n = P + (E - P)^n$ for $n \geq 1$. Since $\rho_{\text{spec}}(E - P) < 1$, the Gelfand formula gives $(E - P)^n \rightarrow 0$, so $\text{Tr}(E^n) \rightarrow \text{Tr}(P)$. Lemma [LTC26, “Trace of the fixed-point projection”] (Section B.6) gives $\text{Tr}(P) = 1$. $\square$

Theorem 6.10.2 (Self-overlap convergence from a complementary transfer-map gap). [Lean](#) [✓ Stmt](#) *Let A be a normalized MPS tensor with a nonzero PSD fixed point ρ of the transfer map. If $\rho_{\text{spec}}(\mathcal{E}_A - P) < 1$, where P is the fixed-point projection, then $O_{AA}(N) \rightarrow 1$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) By the overlap–trace identity (6.4) with $A = B$, one has $O_{AA}(N) = \text{Tr}(\mathcal{E}_A^N)$. By Theorem 6.10.1, $\text{Tr}(\mathcal{E}_A^N) \rightarrow 1$. $\square$

Chapter 7

Wielandt Bound

This chapter proves the quantum Wielandt bounds of [SPGWC10]. For a primitive normalized tensor, it compares the uniform vector-spreading index $q(\mathcal{E}_A)$ with the first exact word length $\iota(A)$ satisfying $S_{\iota(A)}(A) = M_D(\mathbb{C})$ and proves $\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2$. The cumulative bound $T_{D^2}(A) = M_D(\mathbb{C})$ is an intermediate step in the proof of Lemma 1.

7.1 Cumulative span

The exact-word identities used in this section are proved in Section C.1. The stabilization and spectral linear algebra behind the cumulative estimates are collected in Section C.2.

Definition 7.1.1 (Word span). [Lean](#) [✓ Stmt](#) The *word span* at length n is the subspace

$$S_n(A) = \text{span}_{\mathbb{C}}\{A^w : |w| = n\} \subseteq M_D(\mathbb{C}).$$

This is [SPGWC10, Section II].

Definition 7.1.2 (Cumulative span). [Lean](#) [✓ Stmt](#) The *cumulative span* at level n is $T_n(A) = S_0(A) + S_1(A) + \dots + S_n(A)$.

Definition 7.1.3 (Fixed-length vector span of an MPS tensor). [Lean](#) [✓ Stmt](#)

For an MPS tensor A , its fixed-length vector span is the fixed-length vector span of the underlying finite matrix family.

7.1.1 Paper primitivity and indices

For the quantitative bounds, write $d' = \dim S_1(A)$ for the number of linearly independent Kraus operators. The source definitions and Proposition 3 of [SPGWC10] distinguish uniform vector spreading, eventual exact-word spanning, and spectral strong irreducibility.

Definition 7.1.1.1 (Kraus rank). [Lean](#) [✓ Stmt](#) The *Kraus rank* is $\text{kr}(A) = \dim S_1(A)$.

Definition 7.1.1.2 (Eventually full Kraus rank). [Lean](#) [✓ Stmt](#) The Kraus family has *eventually full Kraus rank* if there is a positive integer i such that $S_i(A) = M_D(\mathbb{C})$.

Definition 7.1.1.3 (Paper primitivity). [Lean](#) [✓ Stmt](#)

The transfer map is *primitive in the sense of [SPGWC10]* if there is a positive integer q such that

$$H_q(A, \varphi) = \mathbb{C}^D \quad (7.1)$$

for every nonzero $\varphi \in \mathbb{C}^D$. The same q works for all nonzero vectors.

Definition 7.1.1.4 (Full-Kraus-rank index). [Lean](#) [✓ Stmt](#) The *full-Kraus-rank index* is

$$\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}.$$

If the set is empty, its value is defined to be zero.

Definition 7.1.1.5 (Primitivity index). [Lean](#) [✓ Stmt](#) The *primitivity index* is

$$q(\mathcal{E}_A) = \min\{q \geq 1 : H_q(A, \varphi) = \mathbb{C}^D \text{ for every } \varphi \neq 0\}.$$

If the set is empty, its value is defined to be zero.

Definition 7.1.1.6 (Strengthened strong irreducibility). [Lean](#) [✓ Stmt](#) We call the tensor *strongly irreducible* here if there is a matrix $\rho > 0$ such that

$$\begin{aligned} \mathcal{E}_A(\rho) &= \rho, \\ \text{spec}_{\text{per}}(\mathcal{E}_A) &= \{1\}, \end{aligned}$$

and $\mathcal{E}_A$ is irreducible. The condition in [SPGWC10, Proposition 3(c)] requires that 1 be the only peripheral eigenvalue and that its eigenspace be one-dimensional, generated by a positive-definite fixed point. The definition above strengthens this condition by also requiring irreducibility explicitly.

The next seven results use the following notation. For nonnegative integers d and D , let $K = \{K^i\}_{i=0}^{d-1}$ be a finite family of $D \times D$ matrices, and let $\mathcal{E}_K$ be its Kraus map.

Theorem 7.1.1.7 (Powers of a transfer-map eigenmatrix). [Lean](#) [✓ Stmt](#) If $\mathcal{E}_A(X) = \mu X$, then $\mathcal{E}_A^n(X) = \mu^n X$ for every $n \geq 0$.

Proof. [✓ Proof](#) This is the corresponding theorem in the companion quantum-channel volume [LTC26, “Powers on an eigenspace”] for the eigenspace of μ . $\square$

Theorem 7.1.1.8 (Hermitian power-fixed point from a peripheral eigenvector). [Lean](#) [✓ Stmt](#) Let A be a normalized MPS tensor, and suppose $\mathcal{E}_A(X) = \mu X$ with $X \neq 0$, $\mu \neq 1$, and $\mu^p = 1$. Then there is a nonzero Hermitian matrix H such that

$$\begin{aligned} \text{tr}(H) &= 0, \\ \mathcal{E}_A^p(H) &= H. \end{aligned}$$

Moreover, H is not positive semidefinite.

Proof. [✓ Proof](#)

Apply Theorem [LTC26, “Hermitian power-fixed point from a finite-order eigenmatrix”] to the transfer channel. The resulting matrix is nonzero, Hermitian, trace zero, and fixed by $\mathcal{E}_A^p$. A nonzero positive-semidefinite matrix has positive trace, so this matrix is not positive semidefinite. $\square$

Theorem 7.1.1.9 (Uniqueness of power-fixed positive matrices). [Lean](#) [✓ Stmt](#) Suppose there is a length $q \geq 0$ such that $H_q(A, \varphi) = \mathbb{C}^D$ for every nonzero $\varphi \in \mathbb{C}^D$. Let $p > 0$. If ρ and σ are nonzero positive-semidefinite fixed points of $\mathcal{E}_A^p$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.

Proof. [✓ Proof](#)

This is Theorem [LTC26](#), “Uniqueness of power-fixed positive matrices for a Kraus map”] for the Kraus family formed by the matrices of the tensor. $\square$

Theorem 7.1.1.10 (Kraus-span equivalence). [Lean](#) [✓ Stmt](#) Under normalization, primitivity in Wolf’s sense is equivalent to eventual full Kraus rank.

Theorem 7.1.1.11 (Combined primitivity equivalence). [Lean](#) [✓ Stmt](#) Under normalization, primitivity in Wolf’s sense is equivalent to the conjunction of eventual full Kraus rank, normality, and strong irreducibility.

Remark 7.1.1.12 (Eventually full Kraus rank and normality). [Lean](#) [✓ Stmt](#) Eventual full Kraus rank is equivalent to normality. The remaining pairwise primitivity equivalences are stated in Theorem [7.1.1.13](#).

Theorem 7.1.1.13 (Strengthened form of the primitivity equivalence). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $D > 0$ and suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. The following are equivalent:

1. there is a positive q for which $H_q(A, \varphi) = \mathbb{C}^D$ for every $\varphi \neq 0$;
2. there is a positive i for which $S_i(A) = M_D(\mathbb{C})$;
3. $\mathcal{E}_A$ is strongly irreducible.

Proposition 3 of [\[SPGWC10\]](#) states the same equivalence with item (3) replaced by the literal source condition: 1 is the only peripheral eigenvalue, and its one-dimensional eigenspace is generated by a positive-definite fixed point. Here item (3) includes the additional irreducibility condition.

Proof. [✓ Proof](#)

If $S_i(A) = M_D(\mathbb{C})$ and $\varphi \neq 0$, then $S_i(A)\varphi = \mathbb{C}^D$, proving (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (3), regard the matrices of A as a finite Kraus family. Theorems [\[LTC26](#), “Positive-definite fixed point from vector spreading”] and [\[LTC26](#), “Primitivity from vector spreading”] give a positive-definite fixed point and show that 1 is the only peripheral eigenvalue. Theorem [\[LTC26](#), “Irreducibility from vector spreading”] gives irreducibility. Together these facts say that $\mathcal{E}_A$ is strongly irreducible.

For (3) $\Rightarrow$ (2), write $\mathcal{E}_A^m = P_\rho + (\mathcal{E}_A - P_\rho)^m$. The complementary powers tend to zero. If $S_m(A) \neq M_D(\mathbb{C})$ for every m , choose a nonzero matrix B orthogonal to $S_m(A)$. The associated trace pairing vanishes on the left, while positive definiteness gives a lower bound $c\|B\|^2$ for the P_ρ term and the complementary term is $o(\|B\|^2)$. For large m this is a contradiction. $\square$

Theorem 7.1.1.14 (Primitive-channel specialization of the index bound). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem [7.1.1.13](#), paper primitivity implies

$$q(\mathcal{E}_A) \leq \iota(A).$$

Proposition 1 of [\[SPGWC10\]](#) states this inequality for every quantum channel; the present theorem records its specialization under normalization and paper primitivity.

Proof. [✓ Proof](#) By Theorem [7.1.1.13](#), $\iota(A)$ is a positive length with $S_{\iota(A)}(A) = M_D(\mathbb{C})$. Therefore $H_{\iota(A)}(A, \varphi) = \mathbb{C}^D$ for every $\varphi \neq 0$, so the defining minimum for $q(\mathcal{E}_A)$ is at most $\iota(A)$. $\square$

Theorem 7.1.1.15 (Unique fixed points of full rank). [Lean](#) [✓ Stmt](#) Let $D > 0$ and suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. If there is a positive n for which $S_n(A) = M_D(\mathbb{C})$, then there is a unique positive definite density matrix $\rho \in M_D(\mathbb{C})$ such that every fixed point of $\mathcal{E}_A$ is a scalar multiple of ρ .

This is [Wol12, Theorem 6.15].

Proof. [✓ Proof](#) By Theorem 7.1.1.13, the span hypothesis makes $\mathcal{E}_A$ strongly irreducible, giving a positive-definite fixed point ρ_0 together with peripheral primitivity (unique peripheral eigenvalue 1) and irreducibility of $\mathcal{E}_A$. Peripheral primitivity and irreducibility of $\mathcal{E}_A$ together give the complementary transfer-map gap at ρ_0 (Lemma C.5.5): the restriction of $\mathcal{E}_A$ off the fixed-point projection at ρ_0 has spectral radius strictly less than one. This gap forces every fixed point of $\mathcal{E}_A$ to be a scalar multiple of ρ_0 ; rescaling ρ_0 by the inverse of its (positive) trace gives the unique density-matrix representative ρ .

The source proves this instead by passing to $\mathcal{E}_A^n$, showing every $\mathcal{E}_A^n(|\psi\rangle\langle\psi|)$ has full rank from the span hypothesis, and ruling out a fixed-point space of dimension larger than one via Corollary 6.5 of [Wol12]. Both routes use only Wolf’s own Chapter 6 machinery; the present proof reuses the already-formalized Proposition 3 equivalence. $\square$

Lemma 7.1.1.16 (Monotonicity of cumulative span). [Lean](#) [✓ Stmt](#) For every n , one has $T_n(A) \leq T_{n+1}(A)$.

Proof. [✓ Proof](#) The span T_{n+1} includes all words of length at most $n+1$ and therefore contains all words of length at most n . $\square$

Lemma 7.1.1.17 (Dimension bound). [Lean](#) [✓ Stmt](#) For every n , one has $\dim T_n(A) \leq D^2$.

Proof. [✓ Proof](#) Since $T_n(A) \subseteq M_D(\mathbb{C})$ and $\dim M_D(\mathbb{C}) = D^2$, the result follows. $\square$

Theorem 7.1.1.18 (Cumulative span stabilisation). [Lean](#) [✓ Stmt](#) If $T_n(A) = T_{n+1}(A)$, then $T_m(A) = T_n(A)$ for every $m \geq n$.

Proof. [✓ Proof](#) If $T_n = T_{n+1}$, then $S_{n+1} \subseteq T_n$. Left-multiplying any element of S_{n+1} by A^i gives an element of S_{n+2} , so $S_{n+2} \subseteq T_{n+1} = T_n$. By induction, $S_m \subseteq T_n$ for every $m > n$, hence $T_m = T_n$. $\square$

Theorem 7.1.1.19 (Dimension growth). [Lean](#) [✓ Stmt](#) If $T_n(A) \neq T_{n+1}(A)$, then $\dim T_{n+1}(A) > \dim T_n(A)$.

Proof. [✓ Proof](#) The inclusion $T_n \leq T_{n+1}$ from Lemma 7.1.1.16, together with $T_n \neq T_{n+1}$, gives strict subspace inclusion and hence strict dimension growth. $\square$

Lemma 7.1.1.20 (Word span characterises block injectivity). [Lean](#) [✓ Stmt](#) One has $S_L(A) = M_D(\mathbb{C})$ if and only if A is L -block injective.

Proof. [✓ Proof](#) Both conditions assert that the span of all products of length L equals $M_D(\mathbb{C})$. $\square$

Theorem 7.1.1.21 (Positive-length propagation of block injectivity). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every n ,

$$S_{n+1}(A) = \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_n(A).$$

Consequently, if $L > 0$ and A is L -block injective, then A is m -block injective for every $m \geq L$. This auxiliary length-shift statement combines the block-injectivity discussion in [CPGSV16, Section II] with [CPGSV16, Appendix C.3, Lemma L]; it is not stated there as a separate lemma.

Proof. ✓ Proof Every word of length $n + 1$ factors into its first letter and a word of length n , which proves the displayed equality by bilinearity. Write $L = n + 1$. Since $S_L(A) = M_D(\mathbb{C})$, one has $S_n(A) \leq S_L(A)$. Hence

$$M_D(\mathbb{C}) = S_L(A) = \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_n(A) \leq \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_L(A) = S_{L+1}(A).$$

Iterating this implication gives $S_m(A) = M_D(\mathbb{C})$ for every $m \geq L$. □

7.2 Nonzero trace product

Theorem 7.2.1 (Cumulative span reaches $M_D(\mathbb{C})$). Lean ✓ Stmt *Let $D > 0$. If A is normal, then $T_{D^2}(A) = M_D(\mathbb{C})$. This cumulative estimate supports the proof of [SPGWC10, Lemma 1]; it is not a conclusion of source Theorem 1.*

Proof. ✓ Proof This is Theorem C.2.4. □

Theorem 7.2.2 (Cumulative span reaches $M_D(\mathbb{C})$ — sharp bound). Lean ✓ Stmt *Let $D > 0$. If A is normal, then $T_{D^2-d'+1}(A) = M_D(\mathbb{C})$, where $d' = \dim S_1(A) = \text{kr}(A)$. This is the cumulative-span estimate used to prove the sharp trace conclusion in [SPGWC10, Lemma 1].*

Proof. ✓ Proof If $T_n(A) \neq T_{n+1}(A)$, then the dimension strictly increases. Since $S_1(A) \subseteq T_1(A)$ and $\dim S_1(A) = d'$, one has $\dim T_1(A) \geq d'$. After at most $D^2 - d'$ additional strict-growth steps, the dimension reaches $D^2 = \dim M_D(\mathbb{C})$. Stabilisation before reaching $M_D(\mathbb{C})$ contradicts normality. Hence $T_{D^2-d'+1}(A) = M_D(\mathbb{C})$. □

Theorem 7.2.3 (Eigenvector extraction for an MPS tensor). Lean ✓ Stmt *The conclusion of Theorem [LTC26, “Eigenvector extraction from a full cumulative span”] holds for the matrix family of an MPS tensor whenever its cumulative span at level N is the full matrix algebra.*

Proof. ✓ Proof

Apply the finite-family theorem to the matrices of the tensor. □

Theorem 7.2.4 (Sharp nonzero trace product for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. Then there is a word w with*

$$1 \leq |w| \leq D^2 - \text{kr}(A) + 1, \\ \text{tr}(A^w) \neq 0.$$

This is the sharp trace conclusion of [SPGWC10, Lemma 1].

Proof. ✓ Proof Theorem 7.1.1.13 gives eventual full Kraus rank, hence normality. Put $b = D^2 - \text{kr}(A) + 1$ and let V_n be the span of A^w with $1 \leq |w| \leq n$. Then $S_1(A) \subseteq V_1$, so $\dim V_1 \geq \text{kr}(A)$. If $V_n = V_{n+1}$ for some $n \geq 1$, then V_n is closed under left multiplication by every A^i . It therefore contains every positive-length word product; normality forces $V_n = M_D(\mathbb{C})$. Thus, until the full matrix algebra is reached, the dimensions increase strictly, and the bound on $\dim M_D(\mathbb{C})$ gives $V_b = M_D(\mathbb{C})$. Express $\mathbb{1}$ as a linear combination of the positive-length words defining V_b . Since $\text{tr}(\mathbb{1}) = D \neq 0$, at least one has nonzero trace. □

Theorem 7.2.5 (Sharp nonzero trace product). Lean ✓ Stmt *Let $D > 0$. If A is normal, then there exists a word w of length at most $D^2 - \text{kr}(A) + 1$ such that $\text{tr}(A^w) \neq 0$. This normality variant has the same sharp length bound as [SPGWC10, Lemma 1]; the source hypotheses are stated in Theorem 7.2.4.*

Proof. ✓ Proof By Theorem 7.2.2, the identity $\mathbb{1}$ lies in $T_{D^2-d'+1}(A)$ and is therefore a linear combination of word products of length at most $D^2 - d' + 1$. At least one such product has nonzero trace. $\square$

Theorem 7.2.6 (Sharp nonzero trace product — positive length). Lean ✓ Stmt *If A is normal and $D \geq 1$, then there exists a word w of positive length $1 \leq |w| \leq D^2 - \text{kr}(A) + 1$ such that $\text{tr}(A^w) \neq 0$. This strengthens Theorem 7.2.5 by requiring $|w| \geq 1$, a feature needed for the blocking argument in the general Wielandt bound below.*

Proof. ✓ Proof Define the *positive-level* cumulative span $V_n = \text{span}_{\mathbb{C}}\{A^w : 1 \leq |w| \leq n\}$. The same stabilization-or-strict-growth dichotomy shows $V_{D^2-d'+1} = M_D(\mathbb{C})$: if V_n stabilises and is not $M_D(\mathbb{C})$, then V_n is closed under left multiplication by each A^i , so $S_N(A) \subseteq V_n \neq M_D(\mathbb{C})$ for every $N \geq 1$, contradicting normality. Since $V_{D^2-d'+1} = M_D(\mathbb{C})$, the identity $\mathbb{1}$ lies in $V_{D^2-d'+1}$ and is therefore a linear combination of word products of positive length. At least one such product has nonzero trace because $\text{tr}(\mathbb{1}) = D \neq 0$ for $D \geq 1$. $\square$

7.3 Eigenvector spreading

Definition 7.3.1 (Cumulative vector span). Lean ✓ Stmt Given an MPS tensor A and a vector $\varphi \in \mathbb{C}^D$, the *cumulative vector span* is

$$K_n(A, \varphi) = \text{span}_{\mathbb{C}}\{A^w \varphi : |w| \leq n\} \subseteq \mathbb{C}^D.$$

This is [SPGWC10, proof of Lemma 2(a)].

Theorem 7.3.2 (Eigenvector spreading). Lean ✓ Stmt *Let A be a normal MPS tensor and let $A^{i_0} \varphi = \mu \varphi$, where $\mu \neq 0$ and $\varphi \neq 0$. Then $K_{D-1}(A, \varphi) = \mathbb{C}^D$. This cumulative form supplies the dimension-growth step in [SPGWC10, Lemma 2(a)].*

Proof. ✓ Proof The vector span K_n is monotone and $\dim K_n \leq D$. If $K_n = K_{n+1}$ for some $n < D - 1$, the span stabilises by Theorem C.2.7. But $T_{D^2}(A) = M_D(\mathbb{C})$ by Theorem 7.2.1, so Lemma C.2.8 gives $K_{D^2}(A, \varphi) = \mathbb{C}^D$, contradicting early stabilisation. Hence $\dim K_n$ grows by at least 1 at each step, reaching D by step $D - 1$. $\square$

Theorem 7.3.3 (Fixed-length eigenvector spreading for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. If $\varphi \neq 0$, $\mu \neq 0$, and $A^{i_0} \varphi = \mu \varphi$, then $H_{D-1}(A, \varphi) = \mathbb{C}^D$. This is [SPGWC10, Lemma 2(a)].*

Proof. ✓ Proof Proposition 3, in the form of Theorem 7.1.1.13, gives eventual full Kraus rank and hence normality. The cumulative spaces $K_n(A, \varphi)$ cannot stabilise below $\mathbb{C}^D$, because every matrix eventually lies in an exact word span. Their dimensions therefore reach D by $n = D - 1$, giving $K_{D-1}(A, \varphi) = \mathbb{C}^D$. Finally, the eigenvector relation pads every shorter word to length $D - 1$ by a nonzero power of μ , so $K_{D-1}(A, \varphi) = H_{D-1}(A, \varphi)$. $\square$

7.4 Quantum Wielandt bounds

Section C.3 proves the one-step augmentation facts used in cases (2) and (3). The blocking identities used in the general case are proved in Section C.4.

Remark 7.4.1 (Explicit blocking-length estimates). ✓ Stmt Theorem 7.2.1 gives the D^2 cumulative-span bound. The full-rank index bound $\iota(A) \leq (D^2 - d' + 1)D^2$ ([SPGWC10, Theorem 1, case (1)]) is proved in Theorem 7.4.8 below. The left-canonical normal blocked-injectivity estimate D^4 is proved below in Theorem 7.5.5. A sharper $3D^5$ canonical-form block-separation estimate, developed in the full blueprint, combines this D^4 estimate with the direct-sum argument of [PGVWC07].

Theorem 7.4.2 (Wielandt case (2), one-step invertible element). Lean ✓ Stmt *Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If $S_1(A)$ contains an invertible matrix X , then*

$$S_{D^2 - \text{kr}(A) + 1}(A) = M_D(\mathbb{C}). \quad (7.2)$$

In the terminology of [SPGWC10], this is the case where the first application space $S_1(A)$ contains an invertible operator. This is the exact word-span form of [SPGWC10, Theorem 1, case (2)].

Proof. ✓ Proof Adjoin X as an extra first Kraus operator, obtaining $\tilde{A}$. Theorem 7.1.1.13 gives normality of A , hence normality of $\tilde{A}$; the Kraus rank and all exact word spans are unchanged. The first Kraus operator of $\tilde{A}$ is invertible, so Theorem C.3.5 gives $S_{D^2 - \text{kr}(\tilde{A}) + 1}(\tilde{A}) = M_D(\mathbb{C})$. Transfer this equality back to A . □

Corollary 7.4.3 (Wielandt case (2), index bound). Lean ✓ Stmt *Under the hypotheses of Theorem 7.4.2, one has $\iota(A) \leq D^2 - \text{kr}(A) + 1$, where $\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}$.*

Proof. ✓ Proof The word-span equality (7.2) gives an admissible value in the defining minimum for $\iota(A)$. □

Lemma 7.4.4 (Rank-one extraction from a non-invertible eigenvector). Lean Lean ✓ Stmt *Let A be a normal MPS tensor with bond dimension $D \geq 1$. Suppose that a Kraus operator A^{i_0} is non-invertible and that there exist a nonzero vector $\varphi \in \mathbb{C}^D$ and a nonzero scalar μ such that $A^{i_0}\varphi = \mu\varphi$. Then, for every $\psi \in \mathbb{C}^D$, $|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A)$.*

Proof. ✓ Proof Let $\mathcal{L}(M) = A^{i_0}M$, and choose r so that $\mathcal{L}^r$ annihilates the generalized zero-eigenspace of $\mathcal{L}$; equivalently, $\text{range}(\mathcal{L}^r)$ is its stabilized nonzero-spectral part. The dimension-growth argument gives a level n_0 for which

$$(A^{i_0})^r S_{n_0}(A) = \{(A^{i_0})^r M : M \in M_D(\mathbb{C})\}.$$

Since $A^{i_0}\varphi = \mu\varphi$ and $\mu \neq 0$, iterating gives $(A^{i_0})^r\varphi = \mu^r\varphi$ and hence $\varphi = \mu^{-r}(A^{i_0})^r\varphi \in \text{range}((A^{i_0})^r)$. Therefore, for every $\psi \in \mathbb{C}^D$, $|\varphi\rangle\langle\psi| \in S_{r+n_0}(A)$. The corresponding Fitting-index estimate gives $r + n_0 \leq D^2 - D + 1$. Since

$$A^{i_0}|\varphi\rangle\langle\psi| = |A^{i_0}\varphi\rangle\langle\psi| = \mu|\varphi\rangle\langle\psi|,$$

one has $|\varphi\rangle\langle\psi| = \mu^{-1}A^{i_0}|\varphi\rangle\langle\psi|$. Thus $|\varphi\rangle\langle\psi| \in S_k(A)$ implies $|\varphi\rangle\langle\psi| \in S_{k+1}(A)$, because left multiplication by A^{i_0} maps $S_k(A)$ into $S_{k+1}(A)$. Iterating from $k = r + n_0$ and using $r + n_0 \leq D^2 - D + 1$ gives $|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A)$. □

Theorem 7.4.5 (Sharp rank-one extraction for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. Suppose A^{i_0} is not invertible and $A^{i_0}\varphi = \mu\varphi$ with $\varphi \neq 0$ and $\mu \neq 0$. Then, for every $\psi \in \mathbb{C}^D$,*

$$|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A).$$

This is the quantitative rank-one conclusion of [SPGWC10, Lemma 2(b)].

Proof. ✓ Proof Theorem 7.1.1.13 first gives normality. Apply the Fitting decomposition to left multiplication $\mathcal{L}(M) = A^{i_0}M$. Choose r so that $\mathcal{L}^r$ kills its generalized zero-eigenspace. Dimension growth then gives n_0 with

$$(A^{i_0})^r S_{n_0}(A) = \{(A^{i_0})^r M : M \in M_D(\mathbb{C})\},$$

$$r + n_0 \leq D^2 - D + 1.$$

Since $(A^{i_0})^r \varphi = \mu^r \varphi$ and $\mu \neq 0$, the vector φ lies in the range of $(A^{i_0})^r$. Hence $|\varphi\rangle\langle\psi| \in S_{r+n_0}(A)$ for every ψ . The identity $|\varphi\rangle\langle\psi| = \mu^{-1} A^{i_0} |\varphi\rangle\langle\psi|$ pads this element to the exact length $D^2 - D + 1$. $\square$

Theorem 7.4.6 (Wielandt case (3), one-step non-invertible element). Lean ✓ Stmt *Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If $S_1(A)$ contains a non-invertible matrix X with a nonzero eigenvalue, then $S_{D^2}(A) = M_D(\mathbb{C})$. More explicitly, it is enough to have $X\varphi = \mu\varphi$ with $\varphi \neq 0$ and $\mu \neq 0$. In the terminology of [SPGWC10], this is the case where the first application space $S_1(A)$ contains a non-invertible operator with a nonzero eigenvalue. This is the exact word-span form of [SPGWC10, Theorem 1, case (3)].*

Proof. ✓ Proof Theorem 7.1.1.13 gives normality. Adjoin X as an extra first Kraus operator. The augmented tensor remains normal and has the same exact word spans as A . The eigenvector relation for X is now a one-generator eigenvector relation. The eigenvector-spreading argument gives full fixed-length vector span by length $D-1$, and Lemma 7.4.4 gives the rank-one operators $|\varphi\rangle\langle\psi|$ at length $D^2 - D + 1$. Multiplying these two fixed-length spans gives S_{D^2} for the augmented tensor, hence for A . $\square$

Corollary 7.4.7 (Wielandt case (3), index bound). Lean ✓ Stmt *Under the hypotheses of Theorem 7.4.6, one has $\iota(A) \leq D^2$.*

Proof. ✓ Proof The equality $S_{D^2}(A) = M_D(\mathbb{C})$ gives an admissible value in the defining minimum for $\iota(A)$. $\square$

Theorem 7.4.8 (Wielandt's inequality — general bound). Lean ✓ Stmt *Let A be a normalized MPS tensor with $D > 0$, $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, that is primitive in the sense of Definition 7.1.1.3. Then*

$$\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2,$$

where $\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}$ is the full-Kraus-rank index. This is [SPGWC10, Theorem 1, case (1)].

Proof. ✓ Proof Theorem 7.1.1.13 gives normality. By Theorem 7.2.4, there exists a *positive-length* word w with $|w| \leq D^2 - d' + 1$ and $\text{tr}(A^w) \neq 0$, where $d' = \text{kr}(A)$. The matrix A^w has nonzero trace, hence a nonzero eigenvalue μ and eigenvector φ by Theorem [LTC26, “Nonzero trace implies eigenvector”]. Set $n = |w|$ and let $B = A^{[n]}$ be the blocked tensor. Then B is normal by Theorem C.4.1, and the encoding of w as a single block index gives $B^{i_0} = A^w$.

1. If A^w is invertible, then $S_{D^2}(B) = M_D(\mathbb{C})$, whence $S_{D^2n}(A) = M_D(\mathbb{C})$.
2. If A^w is non-invertible, then the eigenvector φ spans a proper subspace, and the rank-one extraction argument gives $S_{D^2}(B) = M_D(\mathbb{C})$, whence $S_{D^2n}(A) = M_D(\mathbb{C})$.

In the invertible case, the invertible Kraus operator B^{i_0} allows direct dimension growth, giving $S_{D^2-k_B+1}(B) = M_D(\mathbb{C})$, where $k_B = \text{kr}(B)$. In the non-invertible case, the bound $S_{D^2}(B) = M_D(\mathbb{C})$ follows from tracking indices through the rank-one extraction argument of [SPGWC10, Lemma 2(b)]; the separate blocked support construction in Theorem C.4.6 does not supply this explicit bound. In both cases, $\iota(A) \leq D^2 n \leq D^2(D^2 - d' + 1)$. $\square$

Remark 7.4.9 (Quantitative and qualitative Wielandt bounds). ✓ Stmt The quantitative bound $\iota(A) \leq (D^2 - d' + 1)D^2$ is cited from [SPGWC10, Theorem 1, case (1)]. The later Fundamental Theorem needs only the qualitative conclusion $S_N(A) = M_D(\mathbb{C})$ for some N . The blocked construction in Theorem C.4.6 supplies this conclusion directly from normality; the explicit index bound is not needed there.

7.5 Fixed-length matrix spanning

The preceding sections establish the cumulative-spanning part of the Lemma 2(a) argument from [SPGWC10]: eigenvector spreading gives a cumulative vector span $K_{D-1}(A, \varphi) = \mathbb{C}^D$. The next stage is to pass to a span at one fixed length, namely to find n_0 such that $S_{n_0}(A) = M_D(\mathbb{C})$ (i.e. word products of *exactly* one length span all matrices). The product-span facts used here are proved in Section C.1; the blocked fixed-length construction is in Section C.4.

Lemma 7.5.1 (Eigenvector padding). Lean ✓ Stmt Suppose $A^{i_0}\varphi = \mu\varphi$ with $\mu \neq 0$. Then $K_n(A, \varphi) = H_n(A, \varphi)$ for every n . In particular, if $K_n(A, \varphi) = \mathbb{C}^D$, then $H_n(A, \varphi) = \mathbb{C}^D$.

Proof. ✓ Proof Any word w with $|w| \leq n$ can be padded to exact length n by appending $k = n - |w|$ copies of the index i_0 . The eigenvector relation gives $A^{w i_0^k}\varphi = \mu^k A^w\varphi$, so the padded vector is a nonzero scalar multiple of the original. Hence every generator of K_n lies in H_n . The reverse inclusion $H_n \leq K_n$ is immediate. $\square$

Theorem 7.5.2 (Fixed-length vector spanning implies matrix spanning). Lean ✓ Stmt Assume:

1. $H_n(A, \varphi) = \mathbb{C}^D$ (length- n word products applied to φ span all of $\mathbb{C}^D$), and
2. for each standard basis vector e_j , the rank-one operator $|\varphi\rangle\langle e_j|$ lies in $S_m(A)$.

Then $S_{n+m}(A) = M_D(\mathbb{C})$.

Proof. ✓ Proof Fix a matrix unit E_{ij} . By hypothesis (1), there exists $M_i \in S_n(A)$ with $M_i\varphi = e_i$. By hypothesis (2), $|\varphi\rangle\langle e_j| \in S_m(A)$. Their product satisfies

$$M_i|\varphi\rangle\langle e_j| = |M_i\varphi\rangle\langle e_j| = |e_i\rangle\langle e_j| = E_{ij}.$$

Lemma C.1.1 therefore gives $E_{ij} \in S_{n+m}(A)$. Since the matrix units span $M_D(\mathbb{C})$, $S_{n+m}(A) = M_D(\mathbb{C})$. $\square$

Lemma 7.5.3 (Block injectivity and blocked-tensor injectivity). Lean ✓ Stmt A tensor A is L -block injective if and only if its L -blocked tensor $A^{[L]}$ is injective.

Proof. ✓ Proof The blocked physical alphabet is a reindexing of words of length L . Decoding blocked indices therefore identifies the two spanning families, and hence the two injectivity conditions are equivalent. $\square$

Lemma 7.5.4 (Blocking transfer). Lean ✓ Stmt If the blocked tensor $A^{[L]}$ satisfies $S_n(A^{[L]}) = M_D(\mathbb{C})$, then $S_{nL}(A) = M_D(\mathbb{C})$. In particular, A is L -block injective if and only if $A^{[L]}$ is one-site injective.

Proof. ✓ Proof By the blocked-tensor definition (2.13), each length- n word in the blocked alphabet $\{0, \dots, d^L - 1\}$ decodes to a length- nL word in the original alphabet $\{0, \dots, d - 1\}$, so $S_n(A^{[L]}) \subseteq S_{nL}(A)$. For $n = 1$, the d^L blocked letters are indexed by the length- L words of A , giving $S_1(A^{[L]}) = S_L(A)$, which proves the stated equivalence. $\square$

Theorem 7.5.5 (Bounded injective blocking for left-canonical normal tensors). Lean ✓ Stmt *Let A be a left-canonical normal MPS tensor with bond dimension $D > 0$. Then there is a positive blocking length $L \leq D^4$ such that the blocked tensor $A^{[L]}$ is one-site injective.*

This is the blocked-injectivity form of the quantum Wielandt input used in the Fundamental Theorem for translation-invariant tensors. It corresponds to the statement in [CPGSV16, Section II] that every normal tensor becomes injective after at most D^4 blockings, using the index estimate of [SPGWC10, Theorem 1]. The left-canonical/trace-preserving hypothesis is included because this is the canonical-form context in which the blocked-injectivity input is used.

Proof. ✓ Proof Normality is eventual full Kraus rank, so Theorem 7.1.1.13 gives paper primitivity. Theorem 7.4.8 then yields

$$\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2.$$

Trace preservation gives a nonzero Kraus operator, hence $\text{kr}(A) \geq 1$ and the right-hand side is at most D^4 . Set $L = \iota(A)$. By definition, $L > 0$ and $S_L(A) = M_D(\mathbb{C})$. Lemma 7.5.4 identifies the latter equality with one-site injectivity of $A^{[L]}$. $\square$

Theorem 7.5.6 (Exact quantum-Wielandt blocking length). Lean ✓ Stmt *Let A be a left-canonical normal MPS tensor with bond dimension $D > 0$. Then $S_{D^4}(A) = M_D(\mathbb{C})$; equivalently, A is D^4 -block injective.*

Proof. ✓ Proof Choose L with $0 < L \leq D^4$ and $S_L(A) = M_D(\mathbb{C})$, and write $D^4 = L + k$. Induction on k , using positive-length propagation at each step, gives $S_{L+k}(A) = M_D(\mathbb{C})$. $\square$

Theorem 7.5.7 (Common exact block-injectivity length). Lean ✓ Stmt *Let $(A_j)_{j=1}^g$ be a finite family of positive-bond-dimension, left-canonical normal tensors. There is a positive integer L such that every A_j is L -block injective.*

Proof. ✓ Proof Apply Theorem 7.5.5 to each block. The product of the finitely many positive lengths is a common positive multiple, and exact block injectivity persists under positive multiples. $\square$

Corollary 7.5.8 (Uniform injective blocking for separated normal-canonical families). Lean ✓ Stmt *Let $(A_x)_{x \in X}$ be a finite separated normal-canonical family whose blocks have positive virtual bond dimension. Then there is a positive integer L such that every blocked tensor $A_x^{[L]}$ is one-site injective. The same conclusion holds simultaneously for two finite separated normal-canonical families whose blocks have positive virtual bond dimension.*

Here the family is a basis of normal-canonical representatives in the sense of Definition 8.9.1.2; the blocks are distinct and the weight moduli are non-increasing but need not be strictly ordered. The representative family is assumed to have been selected; this statement does not reconstruct the full CPSV sector decomposition with repeated copies inside one sector. This scope restriction is recorded in [con26g].

Proof. ✓ Proof Apply Theorem 7.5.5 to each block. Taking the product of the finitely many resulting positive blocking lengths gives a common positive multiple, and injectivity persists under positive further blocking. $\square$

Remark 7.5.9 (Quantitative and blocked rank-one arguments). ✓ Stmt The quantitative rank-one statement of [SPGWC10, Lemma 2(b)] is Theorem 7.4.5. Theorem 7.5.2 turns that rank-one conclusion and fixed-length vector spreading into matrix spanning. Section C.4 also proves a separate blocked construction, Theorem C.4.6, which first extracts a rank-one operator in a blocked tensor and then applies the fixed-length matrix-spanning argument; its fixed-length conclusion is a consequence of its normality hypothesis, not the quantitative statement of source Lemma 2(b).

Remark 7.5.10 (Use of the quantum Wielandt theorem in the Fundamental Theorem). The cumulative estimate $T_{D^2}(A) = M_D(\mathbb{C})$ in Theorem 7.2.1 supports the proof of Lemma 1; it is not a conclusion of source Theorem 1. The source theorem instead bounds the first positive N for which $S_N(A) = M_D(\mathbb{C})$. This exact-length statement matches injectivity of Γ_N and passes through blocking by Lemma 7.5.4.

The blocked normality and rank-one constructions used below are proved in Section C.4.

7.6 Primitive MPS tensors

Definition 7.6.1 (Primitive MPS tensor). Lean ✓ Stmt Assume $D > 0$. A tensor A together with a matrix $\rho \in M_D(\mathbb{C})$ is *primitive* if

$$\begin{aligned} \sum_i (A^i)^\dagger A^i &= \mathbb{1}, \\ \rho &\geq 0, \quad \rho \neq 0, \quad \mathcal{E}_A(\rho) = \rho, \\ \rho_{\text{spec}}(\mathcal{E}_A - P) &< 1, \end{aligned}$$

where P is the fixed-point projection associated to ρ . When the choice of ρ is irrelevant, we simply say that A is a primitive MPS tensor. This condition combines a complementary spectral gap with a nonzero positive semidefinite fixed point. It is not the paper definition in Definition 7.1.1.3, which is the uniform spreading condition (7.1). With the additional hypothesis $\rho > 0$, the complementary-gap condition implies strong irreducibility and hence paper primitivity by Theorem 7.1.1.13.

7.7 Primitivity and normality

Throughout this section, primitivity means the condition of Definition 7.6.1: the tensor is in TP gauge, it has a nonzero PSD fixed point, and the complementary map has spectral radius < 1 .

7.7.1 From primitivity to normality

The primitivity condition of Definition 7.6.1 combines TP normalization, a nonzero PSD fixed point, and a complementary transfer-map gap. It is connected to normality through the following results.

Remark 7.7.1.1 (Complementary-gap and paper primitivity). ✓ Stmt Definition 7.6.1 assumes TP normalization, a nonzero positive semidefinite fixed point ρ , and $\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1$. Without $\rho > 0$, these hypotheses do not imply irreducibility or normality; for example, a fixed point supported on $\text{diag}(1, 0)$ need not see the complementary invariant subspace.

If $\rho > 0$, the transfer map is irreducible and the complementary gap gives peripheral spectrum $\{1\}$. Thus the tensor is strongly irreducible in the sense of Definition 7.1.1.6. Proposition 3 then gives uniform spreading and eventual full Kraus rank. This implication compares the two notions; it does not identify Definition 7.6.1 with the paper definition.

7.7.2 Primitive implies irreducible

Lemma 7.7.2.1 (Primitive tensors with positive-definite fixed point are irreducible). [Lean](#)

[✓ Stmt](#) If A is a primitive MPS tensor with PSD fixed point ρ and ρ is positive definite, then A is an irreducible tensor. This is part of [SPGWC10, Proposition 3] (see also [CPGSV16, Section 2.3]).

Proof. [✓ Proof](#) By contrapositive. Assume A has an invariant projection P ($P \neq 0$, $P \neq 1$, $(1 - P)A^i P = 0$ for all i). Set $\sigma = P\rho P$.

1. *Invariance of σ under $\mathcal{E}_A^n$:* the projection relation $(1 - P)A^i P = 0$ implies $(1 - P)\mathcal{E}_A^n(\sigma) = 0$ for all n (by induction on n).
2. *Convergence:* By Lemma C.5.6, $\mathcal{E}_A^n(\sigma) \rightarrow \frac{\text{tr}(\sigma)}{\text{tr}(\rho)}\rho$. Hence $(1 - P)\frac{\text{tr}(\sigma)}{\text{tr}(\rho)}\rho = 0$.
3. *Nonzero scalar:* Since $P \neq 0$ and ρ is positive definite, $\text{tr}(P\rho P) > 0$, so $\frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \neq 0$. Therefore $(1 - P)\rho = 0$.
4. *Contradiction:* Since ρ is positive definite, it is a unit in $M_D(\mathbb{C})$. From $(1 - P)\rho = 0$ we get $1 - P = 0$, i.e. $P = 1$, contradicting $P \neq 1$.

□

7.7.3 From irreducibility to full spanning via Burnside's theorem

The irreducibility condition (Definition 8.1.1.1) is formulated in terms of invariant *projections*. For Burnside-style arguments it is more natural to work with invariant *subspaces* of $\mathbb{C}^D$ under a finite matrix family.

Definition 7.7.3.1 (Algebra span). [Lean](#) [✓ Stmt](#) For a finite matrix family A , let $\text{alg}(A)$ be the unital $\mathbb{C}$ -subalgebra of $M_D(\mathbb{C})$ generated by the matrices $\{A^i\}_{i=0}^{d-1}$: $\text{alg}(A) = \mathbb{C}\langle A^i : i = 0, \dots, d-1 \rangle \subseteq M_D(\mathbb{C})$.

Definition 7.7.3.2 (Invariant submodule). [Lean](#) [✓ Stmt](#) A subspace $W \leq \mathbb{C}^D$ is A -invariant if $A^i W \subseteq W$ for every i .

Definition 7.7.3.3 (Irreducible action). [Lean](#) [✓ Stmt](#) The matrices in A act *irreducibly* on $\mathbb{C}^D$ if every A -invariant subspace is trivial, i.e. equal to 0 or to $\mathbb{C}^D$.

Theorem 7.7.3.4 (Irreducible tensor implies irreducible action). [Lean](#) [✓ Stmt](#) If A is irreducible as a tensor (Definition 8.1.1.1), then the Kraus operators act irreducibly on $\mathbb{C}^D$ (Definition 7.7.3.3).

Proof. [✓ Proof](#) Let $W \leq \mathbb{C}^D$ be an A -invariant subspace. In finite dimension the orthogonal projection P onto W exists. Invariance of W implies $(1 - P)A^i P = 0$ for all i . Thus P is a nontrivial invariant projection unless $W = 0$ or $W = \mathbb{C}^D$, contradicting irreducibility of the tensor. □

Theorem 7.7.3.5 (Burnside's theorem for matrix algebras). [Lean](#) [✓ Stmt](#) Assume $D > 0$. If A acts irreducibly on $\mathbb{C}^D$, then the generated unital subalgebra is the full matrix algebra: $\text{alg}(A) = M_D(\mathbb{C})$. This is the complex finite-dimensional case of Burnside's theorem (equivalently, Jacobson's density theorem); see [Jac09].

Proof. ✓ Proof Identify $\text{alg}(A)$ with its image under the algebra equivalence $M_D(\mathbb{C}) \simeq_{\mathbb{C}} \text{End}_{\mathbb{C}}(\mathbb{C}^D)$. The irreducible-action hypothesis makes $\mathbb{C}^D$ a simple module for this algebra. Over the algebraically closed field $\mathbb{C}$, Schur's lemma identifies the endomorphisms commuting with the action with scalars, and Jacobson density then forces the action map to be surjective onto all of $\text{End}_{\mathbb{C}}(\mathbb{C}^D)$. $\square$

Remark 7.7.3.6. ✓ Stmt This Burnside/Jacobson-density step is a substantial external algebraic ingredient in the chapter. We use it here as the finite-dimensional complex Burnside theorem, namely the statement that an irreducible matrix algebra action already generates the full endomorphism algebra.

7.7.4 From primitivity to strong irreducibility and normality

The results in this subsection remove the one-step padding assumption from the normality conclusions. They connect primitivity in Definition 7.6.1, formulated via a complementary transfer-map gap, directly to normality (eventually full Kraus rank), using the convergence of the transfer-map iterates to the projection onto the fixed-point subspace.

Lemma 7.7.4.1 (Primitive tensors with positive-definite fixed point are strongly irreducible).

Lean ✓ Stmt *If A is a primitive MPS tensor with positive-definite fixed point ρ , then it satisfies the strengthened condition in Definition 7.1.1.6: $\mathcal{E}_A$ is irreducible, the fixed point ρ is positive definite, and the peripheral spectrum of $\mathcal{E}_A$ is $\{1\}$. The source clause is [SPGWC10, Proposition 3(c)]; the definition used here records irreducibility separately.*

Proof. ✓ Proof Lemma C.5.7 gives irreducibility of $\mathcal{E}_A$. The complementary-gap hypothesis gives peripheral primitivity directly: the only eigenvalue of modulus one is 1. Together with the positive-definite fixed point ρ , these are exactly the three conditions in Definition 7.1.1.6. The complementary-gap consequences are proved in Section C.5. $\square$

Theorem 7.7.4.2 (Primitive MPS with positive-definite fixed point is normal). Lean ✓ Stmt *If A is a primitive MPS tensor with positive-definite fixed point ρ , then A is normal (has eventually full Kraus rank). No identity-in-the-one-step-span condition is needed.*

This connects the complementary-gap condition in Definition 7.6.1 directly to normality.

Proof. ✓ Proof By Lemma 7.7.4.1, A is strongly irreducible. Theorem 7.1.1.13 identifies strong irreducibility with eventual full Kraus rank, hence with normality. $\square$

7.8 One-step padding and exact word spans

Cumulative spanning, $T_N(A) = M_D(\mathbb{C})$, is weaker than exact-length spanning $S_L(A) = M_D(\mathbb{C})$. If $\mathbb{1} \in S_1(A)$, shorter words can be padded to a common length. Section C.6 proves this conversion and records a counterexample without the one-step padding condition.

Theorem 7.8.1 (Homogeneous word-span propagation under normalization). Lean Lean Lean

✓ Stmt *Suppose*

$$\sum_i A^i (A^i)^\dagger = \mathbb{1}, \tag{7.3}$$

and $S_L(A) = M_D(\mathbb{C})$. Then $S_m(A) = M_D(\mathbb{C})$ for every $m \geq L$. Equivalently, L -block injectivity propagates to every larger homogeneous length.

Proof. ✓ **Proof** If $S_n(A) = M_D(\mathbb{C})$, then for every $X \in M_D(\mathbb{C})$, substituting (7.3) gives

$$X = \left(\sum_i A^i (A^i)^\dagger \right) X = \sum_i A^i ((A^i)^\dagger X) \in S_1(A) S_n(A) \subseteq S_{n+1}(A).$$

Indeed, $(A^i)^\dagger X \in S_n(A)$ because $S_n(A) = M_D(\mathbb{C})$. Thus $S_{n+1}(A) = M_D(\mathbb{C})$. Iterating this implication gives the result for every $m \geq L$. □

Theorem 7.8.2 (Algebra span plus one-step padding implies normality). Lean ✓ Stmt *If $\text{alg}(A) = M_D(\mathbb{C})$ and $\mathbb{1} \in S_1(A)$, then A is normal.*

Proof. ✓ **Proof** Since $\text{alg}(A) = M_D(\mathbb{C})$, the ascending chain condition for finite-dimensional subspaces gives an N such that $T_N(A) = M_D(\mathbb{C})$. Apply Theorem C.6.5. □

Remark 7.8.3 (Normal canonical form avoids one-step padding). The identity-in-the-one-step-span results in Section C.6 are needed only when cumulative spanning is upgraded to exact-length spanning. The normal-canonical theory used later in Chapters 8–9 does not pass through this step: it works directly with irreducible TP-normalized blocks whose blocked transfer maps are primitive and, once the corresponding weighted block data are available, places them in normal canonical form in the sense of [CPGSV16]. There are analogous cumulative-span lemmas for blocked irreducible tensors and eigenvector-spreading arguments, but they are not needed for the implication stated below.

7.9 Block injectivity from TP/primitive/irreducible blocks

Theorem 7.9.1 (TP-primitive irreducible tensor is injective after blocking). Lean ✓ Stmt *Let A be a left-canonical MPS tensor with $D > 0$. Suppose that A is tensor-irreducible and that its transfer map is primitive. Then there exists a positive blocking length L such that $A^{[L]}$ is one-site injective.*

Proof. ✓ **Proof** If $D = 1$, trace preservation gives a nonzero one-site Kraus matrix, and every nonzero scalar matrix spans $M_1(\mathbb{C})$. Hence A is one-site injective, so Lemma 2.4.3 gives the positive witness $L = 1$. Suppose now that $D \geq 2$. Theorem 8.11.1.1 gives a block-injectivity witness L . If $L = 0$, Corollary C.1.3 would force $D = 1$, a contradiction. Thus $L > 0$, and the blocked-chain equivalence identifies this L -block injectivity with one-site injectivity of the blocked tensor $A^{[L]}$. □

Remark 7.9.2. Blocking preserves normality by Theorem C.4.1, proved in Section C.4.

Chapter 8

Canonical Form Reduction

This is the first of three chapters that together prove the Fundamental Theorem of matrix product states. Here an arbitrary translation-invariant MPS tensor is reduced to a weighted family of primitive blocks; Chapter 9 constructs the basis of normal tensors carried by those blocks and compares their coefficients, and Chapter 10 proves the theorem itself. The reduction follows [PGVWC07], [CPGSV16], [CPGSV21], and [Wol12, Chapter 6]. Its goal is to expose the primitive blocks inside the tensor. The transfer operator of an arbitrary tensor can be reducible and periodic. The reduction first splits invariant subspaces. For each irreducible trace-preserving block, it then powers the transfer map, extracts the adjoint-fixed cyclic projections, compresses to their corners, and proves that each corner is primitive and tensor-irreducible. The blocks are returned in a fixed gauge. The periodic decomposition theorem of [PGVWC07, Theorem 5] gives p translated p -periodic state summands when $p \mid N$ and a zero state otherwise. We state this fixed-length decomposition below before using its blocked cyclic-sector counterpart in Theorem 8.3.1.1.

Definition 8.1 (Periodic projector-component chain). [Lean] [Lean] [Lean] [✓ Stmt] Let $A = (A^i)_{i=0}^{d-1}$ have bond dimension D , let $(Q_u)_{u \in \mathbb{Z}/p\mathbb{Z}}$ be cyclic projections, and fix a starting sector u . For each $0 \leq j < N$, the corresponding chain of length N has local matrices

$$B_{u,j}^i = Q_{u+j} A^i Q_{u+j+1}, \quad (8.1)$$

where sector indices are read modulo p . It has the same bond dimension D as A and repeats with period p .

Theorem 8.2 (Boundary traces and component-chain coefficients). [Lean] [Lean] [✓ Stmt] Suppose that the orthogonal projections satisfy $\sum_u Q_u = \mathbb{1}$ and $Q_u A^i = A^i Q_{u+1}$. For a word $\sigma = (i_0, \dots, i_{N-1})$, define the boundary product

$$F_u(\sigma) = Q_u A^{i_0} Q_{u+1} A^{i_1} \dots Q_{u+N-1} A^{i_{N-1}} Q_{u+N}. \quad (8.2)$$

At every length $N \geq 0$,

$$V^{(N)}(A)_\sigma = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \text{tr} F_u(\sigma). \quad (8.3)$$

If $N > 0$, then $\text{tr} F_u(\sigma)$ is the coefficient of the explicit component chain from Definition 8.1. At $N = 0$, the boundary product is $F_u(\emptyset) = Q_u$, so

$$\text{tr}(\mathbb{1}) = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \text{tr}(Q_u). \quad (8.4)$$

This is not the sum of the ordinary empty component-chain coefficients: an empty chain evaluates to $\mathbb{1}$ and does not retain its starting projection.

Proof. ✓ **Proof** The cyclic relation and orthogonality give $F_u(\sigma) = Q_u A^{i_0} \dots A^{i_{N-1}}$. Hence

$$\mathrm{tr}(A^{i_0} \dots A^{i_{N-1}}) = \mathrm{tr} \left(\left(\sum_u Q_u \right) A^{i_0} \dots A^{i_{N-1}} \right) = \sum_u \mathrm{tr} F_u(\sigma). \quad (8.5)$$

For $N > 0$, multiplying the local matrices in (8.1) gives $F_u(\sigma)$, proving the coefficient identity. For $N = 0$, the same trace calculation reduces to (8.4). $\square$

Theorem 8.3 (Normalized periodic state-vector decomposition). Lean Lean Lean ✓ **Stmt** This is a normalized form of the state-vector conclusion in [PGVWC07, Theorem 5]. Let A be an irreducible left-canonical tensor whose peripheral transfer spectrum is the set of p -th roots of unity. If $p \mid N$, there are orthogonal cyclic projections $(Q_u)_{u \in \mathbb{Z}/p\mathbb{Z}}$ such that $\sum_u Q_u = \mathbb{1}$, $Q_u A^i = A^i Q_{u+1}$, and $Q_{u+N} = Q_u$. For every ring length $N \geq 0$, its coefficients satisfy the boundary-aware decomposition

$$V^{(N)}(A)_\sigma = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \mathrm{tr} F_u(\sigma). \quad (8.6)$$

For $N > 0$, this becomes the source's decomposition into the p explicit component chains:

$$|V^{(N)}(A)\rangle = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} |V^{(N)}(B_u)\rangle. \quad (8.7)$$

Each B_u is a p -periodic MPS of bond dimension D with local matrices given by (8.1). At $N = 0$, (8.6) instead uses the boundary traces $\mathrm{tr}(Q_u)$ from (8.4); it does not identify them with ordinary empty-chain coefficients. If $p \nmid N$, then $|V^{(N)}(A)\rangle = 0$.

Proof. ✓ **Proof** Choose the cyclic projections supplied by periodicity. If $p \mid N$, then $Q_{u+N} = Q_u$, and Theorem 8.2 gives (8.6) at every length. For $N > 0$, the explicit component-chain decomposition follows by inserting $\sum_u Q_u = \mathbb{1}$ and multiplying the local corner matrices; this proves (8.7). The separate coefficient identity in Theorem 8.2 identifies its summands with the boundary traces. If $p \nmid N$, then $Q_{u+N} \neq Q_u$; cyclicity of the trace and orthogonality give $\mathrm{tr} F_u(\sigma) = 0$ for every u and σ . $\square$

Two normalizations appear. The unital orientation $\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}$ is used in [PGVWC07, Theorem 4]; the trace-preserving (left-canonical) orientation $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$ is used in the TP-gauge and after-blocking reductions below. They exchange under the conjugate-transposed Kraus family $K_i := (A^i)^\dagger$, whose transfer map is the Frobenius adjoint of $\mathcal{E}_A$.

8.1 Invariant-subspace decomposition and irreducible blocks

The block decomposition rests on one principle: a positive semidefinite fixed point of $\mathcal{E}_A$ that is not full rank has a proper invariant support, which block-diagonalizes the Kraus operators without changing the MPV family. Iterating the resulting splitting decomposes any tensor into irreducible blocks.

Lemma 8.1.1 (Lower-triangular Kraus condition implies corner invariance). Lean ✓ **Stmt** Let A be an MPS tensor and let P be an orthogonal projection. If $(\mathbb{1} - P)A^i P = 0$ for every physical index i , then $P\mathcal{E}_A(PXP)P = \mathcal{E}_A(PXP)$ for every $X \in M_D(\mathbb{C})$.

Proof. ✓ Proof Expand the transfer map using (2.4) and use $(\mathbb{1} - P)A^iP = 0$ to reduce each Kraus term to the P -corner. $\square$

Theorem 8.1.2 (Invariant projection gives a two-block decomposition). Lean ✓ Stmt Suppose P is an orthogonal projection satisfying $(\mathbb{1} - P)A^iP = 0$ for every physical index i . Then there exist n, m with $n + m = D$ and MPS tensors A_1, A_2 of bond dimensions n and m such that A and the two-block tensor $A_1 \oplus A_2$ generate the same MPV family in the sense of Definition 2.3.3.

Proof. ✓ Proof Diagonalize P by a unitary U . Since $P^2 = P$, the diagonal form of P has only 0- and 1-eigenvalues, so the basis splits as $n + m = D$. Conjugating A by U preserves the MPV family by Theorem D.1.1. In this basis the relation $(\mathbb{1} - P)A^iP = 0$ makes each letter upper block triangular. Write the conjugated letters as

$$B^i = \begin{pmatrix} A_1^i & * \\ 0 & A_2^i \end{pmatrix}.$$

Since $B^{i_1} \dots B^{i_N}$ is again upper block triangular, its trace satisfies

$$\text{tr}(B^{i_1} \dots B^{i_N}) = \text{tr}(A_1^{i_1} \dots A_1^{i_N}) + \text{tr}(A_2^{i_1} \dots A_2^{i_N}). \quad (8.8)$$

Hence the block-diagonal tensor $A_1 \oplus A_2$ has the same MPV coefficients as the conjugated tensor. $\square$

Definition 8.1.3 (Invariant projection condition). Lean ✓ Stmt An MPS tensor A has an *invariant projection* if there exists an orthogonal projection P with $P \neq 0$, $P \neq \mathbb{1}$, and $(\mathbb{1} - P)A^iP = 0$ for all i . A tensor is *irreducible* if it has no nontrivial invariant projection (Definition 8.1.1.1).

Theorem 8.1.4 (PSD fixed point gives invariant projection). Lean ✓ Stmt Let $\rho \geq 0$ be a fixed point of $\mathcal{E}_A$, and let P be its support projection. Then $(\mathbb{1} - P)A^iP = 0$ for all i : the range of P is invariant under every A^i .

Proof. ✓ Proof From $\mathcal{E}_A(\rho) = \rho$ and positivity of each summand $A^i\rho(A^i)^\dagger$, one gets $(\mathbb{1} - P)A^i\rho(A^i)^\dagger(\mathbb{1} - P) = 0$ for each i . Since ρ is supported on the range of P , this forces $(\mathbb{1} - P)A^iP = 0$.

This is the finite-dimensional case of [Wol12, Proposition 6.10]: the support projection of any stationary state is sub-harmonic. The equivalence between irreducibility and absence of nontrivial invariant projections is [Wol12, Theorem 6.2]; see also [CPGSV21, Section IV.A.1]. $\square$

Theorem 8.1.5 (Two-block decomposition). Lean ✓ Stmt If $\mathcal{E}_A$ has a nonzero PSD fixed point ρ that is not positive definite, then there exist MPS tensors A_1, A_2 of smaller bond dimensions with $\mathcal{V}(A) = \mathcal{V}(A_1 \oplus A_2)$.

Proof. ✓ Proof The support projection P of ρ is neither 0 nor $\mathbb{1}$, since $\rho \neq 0$ and ρ is not positive definite. Theorem 8.1.4 gives $(\mathbb{1} - P)A^iP = 0$, so a unitary change of basis diagonalizing P puts A^i in upper-triangular block form with diagonal blocks B_{11}^i, B_{22}^i . Theorem D.1.1 preserves the MPV under the conjugation, and the trace identity (8.8) shows that deleting the off-diagonal blocks preserves all MPV coefficients. $\square$

8.1.1 Iterated reduction to irreducible blocks

Iterating the two-block splitting on bond dimension decomposes any tensor into irreducible blocks; the remaining lemmas record the spectral facts about irreducible unital blocks and the algebra-spanning input used later.

Definition 8.1.1.1 (Irreducible tensor). [Lean](#) [✓ Stmt](#) An MPS tensor A is *irreducible* if it has no nontrivial invariant projection.

Theorem 8.1.1.2 (Irreducible block decomposition). [Lean](#) [✓ Stmt](#) Every MPS tensor A admits a block-diagonal tensor $\bigoplus_{k=1}^r A_k$ with each A_k irreducible, $\mathcal{V}(A) = \mathcal{V}(\bigoplus_k A_k)$, and $\sum_{k=1}^r D_k = D$.

Proof. [✓ Proof](#) By strong induction on D . If A is irreducible, take $r = 1$. Otherwise a nontrivial invariant projection P gives the upper-triangular splitting of Theorem 8.1.5; deleting the off-diagonal part preserves the MPV family and yields two blocks of strictly smaller bond dimension. Apply the induction hypothesis to each block.

Both [PGVWC07, Theorem 4] and [CPGSV21, Section IV.A.1] obtain the block decomposition by the same invariant-projection splitting. $\square$

8.2 Nonzero blocks and the trace-preserving gauge

In an irreducible block decomposition some blocks may have all-zero Kraus operators. Such a block contributes nothing to a matrix product vector at any positive length and is therefore discarded. The structural direct-sum identity implies that the retained block dimensions satisfy $\sum_k D_k \leq D$ without using an empty-word coefficient. Each retained block is then placed in the trace-preserving gauge.

Theorem 8.2.1 (Blockwise TP-gauge normalization of irreducible blocks). [Lean](#) [✓ Stmt](#) Suppose A generates the same MPV family as $\bigoplus_{k=1}^r B_k$, where each B_k is irreducible with at least one nonzero Kraus operator. Then there are nonzero weights $(\mu_k)_{k=1}^r$ and blocks $(C_k)_{k=1}^r$ such that, for every N and every σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k C_k\right)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(C_k)_\sigma, \quad (8.9)$$

each C_k is irreducible and left-canonical, $\sum_i (C_k^i)^\dagger C_k^i = \mathbb{1}$, with positive bond dimension. This is the blockwise form used in the arbitrary-tensor trace-preserving gauge below.

Proof. [✓ Proof](#) The Perron–Frobenius TP-gauge construction is applied to each irreducible block using the gauge (5.3). Gauge equivalence preserves irreducibility and the MPV family, and the positive scaling factors are absorbed into the weights μ_k . $\square$

8.2.1 Trace-preserving gauge for arbitrary tensors

Theorem 8.2.1.1 (TP-gauge reduction for arbitrary tensors). [Lean](#) [✓ Stmt](#) For any MPS tensor A , there exist nontrivial blocks $(B_k)_{k=1}^r$ with nonzero weights $(\mu_k)_{k=1}^r$, each block irreducible, left-canonical ($\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$), and of positive bond dimension, such that, for every $N \geq 1$ and every σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma, \quad (8.10)$$

and $\sum_{k=1}^r D_k \leq D$.

Proof. ✓ Proof Theorem D.1.3 supplies the nonzero irreducible blocks $(C_k)_{k=1}^r$, with $V^{(N)}(A)_\sigma = V^{(N)}(\bigoplus_{k=1}^r C_k)_\sigma$ for every $N \geq 1$ and every σ . Applying the Perron–Frobenius TP gauge (5.3) to each C_k gives a left-canonical block B_k with weight μ_k , and

$$V^{(N)}\left(\bigoplus_{k=1}^r C_k\right)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma.$$

The gauge preserves every block dimension, so the structural bound $\sum_k D_k \leq D$ is unchanged. $\square$

8.3 Blocking, period removal, and primitive blocks

Blocking by an integer p carries MPV equalities, weights, transfer maps, and identifications of the blocked physical alphabet. For an irreducible trace-preserving block, powering first collapses the peripheral eigenvalue set to $\{1\}$; the powered map can nevertheless split into several cyclic corners. The reduction compresses to those corners and proves primitivity there. A later least-common-multiple step places a finite family of the resulting sectors over one blocked alphabet.

8.3.1 Period removal by cyclic sectors

The period-removing blocking of [CPGSV16, Section 2.3] has four distinct steps. Power the transfer map by its period, obtain the adjoint-fixed cyclic projections, compress to their corners, and prove that each corner restriction is irreducible. Only then does the peripheral set $\{1\}$ imply primitivity of the sector maps. The cyclic projections come from [Wol12, Theorem 6.6].

Theorem 8.3.1.1 (Period removal by blocking). Lean ✓ Stmt *Let A be a left-canonical irreducible tensor. Then there is a period-removing length $m \geq 1$ such that $A^{[m]}$ admits:*

1. *left-canonical sector tensors $(C_k)_{k=1}^m$ of nonzero bond dimension,*
2. *a unit-weight direct-sum MPV decomposition $A^{[m]} \sim \bigoplus_{k=1}^m C_k$,*
3. *orthogonal projections $(P_k)_{k=1}^m$ summing to $\mathbb{1}$ with $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$,*
4. *commutation relations $P_k(A^{[m]})^i = (A^{[m]})^i P_k$ for every blocked letter i ,*
5. *the sector trace formula $f_{C_k}(\sigma) = \text{tr}(P_k(A^{[m]})^{\sigma_1} \dots (A^{[m]})^{\sigma_N})$,*
6. *linear isomorphisms onto the compression corners $\varphi_k : M_{\dim C_k}(\mathbb{C}) \xrightarrow{\sim} P_k M_D(\mathbb{C}) P_k$,*
7. *transfer-intertwining identities $\varphi_k(\mathcal{E}_{C_k^\dagger}(X)) = \mathcal{E}_{(P_k A^{[m]})^\dagger}(\varphi_k(X))$,*
8. *multiplicativity and adjoint compatibility $\varphi_k(XY) = \varphi_k(X)\varphi_k(Y)$ and $\varphi_k(X^\dagger) = \varphi_k(X)^\dagger$.*

This is the cyclic-sector part of the period-removal step of [CPGSV16, Section 2.3]; the projections P_k come from the peripheral spectrum of the adjoint transfer map ([Wol12, Theorem 6.6]).

Proof. ✓ Proof The peripheral cyclic-structure theorem identifies the period and its primitive root. The cyclic decomposition of the adjoint transfer map gives projections (P_k) with $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$. Iterating the cyclic relation m times gives $(\mathcal{E}_{A^\dagger})^m(P_k) = P_k$, and the blocked-transfer identity $\mathcal{E}_{(A^{[m]})^\dagger} = (\mathcal{E}_{A^\dagger})^m$ identifies these with the adjoint transfer map of $A^{[m]}$. Apply the block decomposition from adjoint-fixed projections. $\square$

Theorem 8.3.1.2 (Unconditional primitive blocked sectors). [Lean](#) [✓ Stmt](#) Let A be a left-canonical irreducible tensor whose adjoint transfer map has peripheral spectrum $\{\gamma^j : 0 \leq j < m\}$ for a primitive m th root of unity γ , with blocked cyclic-sector decomposition $(C_u)_{u=1}^m, (P_u)_{u=1}^m$, and linear isomorphisms $\varphi_u : M_{\dim C_u}(\mathbb{C}) \xrightarrow{\sim} P_u M_D(\mathbb{C}) P_u$. Assume explicitly that the P_u are orthogonal projections summing to $\mathbb{1}$ and satisfying $\mathcal{E}_{A^\dagger}(P_{u+1}) = P_u$, that each sector has nonzero bond dimension, and that the compression maps satisfy

$$\begin{aligned}\varphi_u(\mathcal{E}_{C_u^\dagger}(X)) &= \mathcal{E}_{(P_u A^{[m]})^\dagger}(\varphi_u(X)), \\ \varphi_u(XY) &= \varphi_u(X)\varphi_u(Y), \\ \varphi_u(X^\dagger) &= \varphi_u(X)^\dagger.\end{aligned}\tag{8.11}$$

Here $P_u A^{[m]}$ denotes the blocked tensor with letters $P_u(A^{[m]})^\alpha$. Then every sector tensor satisfies $\sigma_\partial(\mathcal{E}_{C_u}) = \{1\}$ and is tensor-irreducible.

Proof. [✓ Proof](#) Theorem 8.4.3 gives irreducibility of $(\mathcal{E}_A^\dagger)^m$ on each corner P_u . The compression equivalences in (8.11) intertwine the adjoint transfer map of C_u with the corresponding corner restriction of $(\mathcal{E}_A^\dagger)^m$. The peripheral spectrum of the blocked adjoint map is

$$\sigma_\partial((\mathcal{E}_A^\dagger)^m) = \{(\gamma^j)^m : 0 \leq j < m\} = \{1\},\tag{8.12}$$

because γ is a primitive m th root of unity. Hence each nonzero irreducible corner restriction is primitive. Primitivity and irreducibility are preserved across φ_u ; then use Theorem D.4.4 to pass from the adjoint blocked map to the primal transfer map of C_u . $\square$

Theorem 8.3.1.3 (Period removal with primitive irreducible sectors). [Lean](#) [✓ Stmt](#) Let A be a left-canonical irreducible tensor. Then there is a period $m \geq 1$ and a unit-weight decomposition of $A^{[m]}$ into sector blocks $(C_k)_{k=1}^m$, each of which is left-canonical, has primitive transfer map, is tensor-irreducible, and has nonzero bond dimension. Equivalently, the cyclic-sector decomposition of $A^{[m]}$ already satisfies the trace-preservation, primitivity, and irreducibility hypotheses needed for the common-blocking step (Lemma D.5.5).

Proof. [✓ Proof](#) Left-canonicity makes $K_i := (A^i)^\dagger$ unital, irreducibility passes to the adjoint map, and the quantum Perron–Frobenius theorem provides a positive-definite fixed point; the cyclic peripheral-spectrum theorem and Theorem 8.3.1.1 then produce the cyclic-sector decomposition. Keep the projections and compression equivalences, and apply Theorem 8.3.1.2 to the sector blocks. $\square$

Theorem 8.3.1.4 (Common blocking length for cyclic sectors). [Lean](#) [✓ Stmt](#) If two tensors generate the same MPV family at every positive length, then the cyclic-sector decompositions on the two sides can be expressed at one common positive blocking length p . For every $N \geq 1$ and every blocked σ , each blocked tensor equals its powered nonzero part:

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_j (\mu_j^A)^p (A_j)^{[p]}\right)_\sigma,\tag{8.13}$$

$$V^{(N)}(B^{[p]})_\sigma = V^{(N)}\left(\bigoplus_k (\mu_k^B)^p (B_k)^{[p]}\right)_\sigma.\tag{8.14}$$

The two powered nonzero parts agree:

$$V^{(N)}\left(\bigoplus_j (\mu_j^A)^p (A_j)^{[p]}\right)_\sigma = V^{(N)}\left(\bigoplus_k (\mu_k^B)^p (B_k)^{[p]}\right)_\sigma.\tag{8.15}$$

Proof. ✓ Proof Start from Theorem D.5.4. With p_A, p_B the least common multiples of the period-removal lengths on the two sides, use $p = \text{lcm}(p_A, p_B)$. The prescribed-common-multiple lemma constructs both one-sided sector families at this length, and the positive-length blocking lemmas carry the one-sided tensor/nonzero-part equalities over to the common alphabet, giving (8.13) and (8.14). The two-sided blocking lemma applied to (D.16) gives (8.15). $\square$

8.4 Cyclic-sector irreducibility

The powered map is analyzed corner by corner before the resulting sector families are placed over a common blocked alphabet. The compression and common-alphabet identities are proved in Section D.5.

Theorem 8.4.1 (Fixed-point upgrade on an invariant corner). Lean ✓ Stmt *Let A have positive bond dimension, suppose that $\mathcal{E}_A$ is irreducible, and assume $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Let Q be an orthogonal projection. If $(\mathcal{E}_{A^\dagger})^m$ preserves the corner $QM_D(\mathbb{C})Q$, then*

$$(\mathcal{E}_{A^\dagger})^m(Q) = Q.$$

Proof. ✓ Proof Choose a positive-definite fixed point ρ of $\mathcal{E}_A$. The weighted trace $X \mapsto \text{tr}(\rho X)$ is invariant under $\mathcal{E}_{A^\dagger}$ and its powers. Corner preservation gives

$$Q - (\mathcal{E}_{A^\dagger})^m(Q) = Q(\mathcal{E}_{A^\dagger})^m(\mathbb{1} - Q)Q \geq 0.$$

The invariant weighted trace of this positive semidefinite difference is zero. Since $\rho > 0$, the difference vanishes. $\square$

Theorem 8.4.2 (Fixed-point algebra rigidity on cyclic sectors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with positive bond dimension and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Let $(P_k)_{k \in \mathbb{Z}/m\mathbb{Z}}$ be orthogonal projections such that*

$$\sum_k P_k = \mathbb{1}, \quad \mathcal{E}_{A^\dagger}(P_{k+1}) = P_k.$$

If $X, Y \in P_k M_D(\mathbb{C}) P_k$ are fixed by $(\mathcal{E}_{A^\dagger})^m$, then

$$\mathcal{E}_{A^\dagger}(XY) = \mathcal{E}_{A^\dagger}(X)\mathcal{E}_{A^\dagger}(Y).$$

Proof. ✓ Proof Irreducibility passes to the adjoint transfer map. The cyclic relation gives equality in both Kadison–Schwarz inequalities:

$$\begin{aligned} \mathcal{E}_{A^\dagger}(P_k^\dagger P_k) &= \mathcal{E}_{A^\dagger}(P_k)^\dagger \mathcal{E}_{A^\dagger}(P_k), \\ \mathcal{E}_{A^\dagger}(P_k P_k^\dagger) &= \mathcal{E}_{A^\dagger}(P_k) \mathcal{E}_{A^\dagger}(P_k)^\dagger. \end{aligned}$$

Hence every P_k lies in the multiplicative domain, so for every Z ,

$$\begin{aligned} \mathcal{E}_{A^\dagger}(P_k Z) &= \mathcal{E}_{A^\dagger}(P_k) \mathcal{E}_{A^\dagger}(Z), \\ \mathcal{E}_{A^\dagger}(Z P_k) &= \mathcal{E}_{A^\dagger}(Z) \mathcal{E}_{A^\dagger}(P_k). \end{aligned}$$

If $X \in P_k M_D(\mathbb{C}) P_k$ is fixed by $(\mathcal{E}_{A^\dagger})^m$, then its orbit sum

$$R_X = \sum_{\ell=0}^{m-1} (\mathcal{E}_{A^\dagger})^\ell(X)$$

is fixed by $\mathcal{E}_{A^\dagger}$. By Theorem 8.8.1, $R_X = c\mathbb{1}$. Orthogonality of the cyclic sectors and the multiplicative-domain identities give $X = P_k R_X P_k = c P_k$. Similarly, $Y = c' P_k$. Since $\mathcal{E}_{A^\dagger}(P_k) = P_{k-1}$, both sides of the required identity equal $cc' P_{k-1}$. $\square$

Lemma 8.4.3 (Sector irreducibility for the adjoint blocked map). [Lean](#) [✓ Stmt](#) Let A be a left-canonical irreducible MPS tensor with $D > 0$, let m be the period of its cyclic-sector decomposition, and let $(P_u)_{u=1}^m$ be the cyclic projections of the adjoint transfer map with $\mathcal{E}_{A^\dagger}(P_{u+1}) = P_u$ and $\sum_u P_u = \mathbb{1}$. Then the restriction of $(\mathcal{E}_{A^\dagger})^m$ to each corner $P_u M_D(\mathbb{C}) P_u$ is irreducible.

Definition 8.4.4 (Primitive irreducible cyclic-sector decomposition). [Lean](#) [✓ Stmt](#) A has primitive irreducible cyclic sectors of period $m \geq 1$ if there is a family $(C_k)_{k=1}^m$ with $A^{[m]} \sim \bigoplus_{k=1}^m C_k$ with unit weights, each C_k left-canonical, with primitive transfer map, tensor-irreducible, and of positive bond dimension.

8.5 Translation-invariant canonical form

Definition 8.5.1 (Unnormalized positive-length PGVWC07 witness). [Lean](#) [✓ Stmt](#) An unnormalized positive-length PGVWC07 witness for a tensor A consists of finitely many blocks $(A_k)_{k=1}^r$ of positive bond dimensions and positive real weights $(a_k)_{k=1}^r$. Each block is unital, has only scalar transfer-map fixed points, and has a diagonal positive-definite fixed point Λ_k for the adjoint transfer map. Moreover,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r a_k A_k\right)_\sigma \quad (N > 0),$$

and the retained bond dimensions satisfy $\sum_k \dim(A_k) \leq D$.

Theorem 8.5.2 (Construction of the positive-length PGVWC07 witness). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Every MPS tensor has an unnormalized positive-length PGVWC07 witness. More precisely, the construction has four stages. First, a nonzero irreducible block remains irreducible under nonzero scalar rescaling and can be put in a unital gauge with scalar fixed-point space and a diagonal positive-definite dual fixed point. Second, this construction is applied to each member of a finite nonzero irreducible block family. Third, the nonzero irreducible decomposition of an arbitrary tensor supplies that family and the total bond-dimension bound. Finally, these blocks, weights, and fixed points form a witness in the sense of Definition 8.5.1.

Proof. [✓ Proof](#) Follow [PGVWC07, Theorem 4]. The nonzero irreducible decomposition is obtained by strict dimension induction. On a nonzero current block, Perron–Frobenius gives a positive semidefinite eigenvector at a positive spectral radius. If it is singular, positivity makes its support invariant: $(\mathbb{1} - P)A^i P = 0$. After diagonalizing P , every letter is upper block triangular and the finite-ring trace splits as

$$\mathrm{tr}(A^{i_1} \dots A^{i_N}) = \mathrm{tr}(A_1^{i_1} \dots A_1^{i_N}) + \mathrm{tr}(A_2^{i_1} \dots A_2^{i_N}).$$

Discard all-zero blocks and repeat on the two strictly smaller diagonal blocks. This gives the finite nonzero irreducible family of Theorem D.1.3.

For each retained irreducible block, divide by the square root of its positive spectral radius and conjugate by the square root of its positive-definite Perron eigenvector. Theorem 5.5.4 gives the unital identity

$$\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}.$$

Nonzero scalar rescaling preserves irreducibility. The proof passes from tensor irreducibility to irreducibility of the matrix action, carries invariant subspaces through the gauge similarity, and

returns to tensor irreducibility using Theorem 7.7.3.4 and Lemma C.2.2. Hence Theorem 8.8.1 makes every fixed point scalar. Theorem 8.10.2 then diagonalizes a positive-definite dual fixed point by a unitary without changing unitality.

Apply this single-block construction to every member of the finite family. The square roots of the block spectral radii become positive weights, gauge invariance preserves every positive-length MPV coefficient, and the block dimensions are unchanged. Combining this blockwise result with the initial decomposition yields the arbitrary-input unnormalized form and its bound $\sum_k \dim(A_k) \leq D$. Together these conclusions give the structured witness. $\square$

Theorem 8.5.3 (Finite normalization of a nonempty PGVWC07 witness). [Lean](#) [Lean](#) [✓ Stmt](#) *Let W be an unnormalized positive-length witness with positive weights a_k . If some positive-length MPV coefficient of the original tensor is nonzero, then W has at least one block. For any nonempty witness, define*

$$s := \max_{1 \leq k \leq r} a_k, \\ \nu_k := \frac{a_k}{s}.$$

Then $s > 0$, every ν_k is positive, $|\nu_k| \leq 1$, and $|\nu_k| = 1$ for at least one k . For every $N > 0$,

$$V^{(N)}(A)_\sigma = s^N V^{(N)}\left(\bigoplus_k \nu_k A_k\right)_\sigma. \quad (8.16)$$

Proof. [✓ Proof](#) If the witness had no blocks, its weighted block tensor would have zero MPV coefficients at every positive length, contrary to the assumed nonzero coefficient. Thus $r > 0$. Since the finite family (a_k) is nonempty and every $a_k > 0$, its maximum s is positive. The definition $\nu_k = a_k/s$ gives $0 < \nu_k \leq 1$, and any maximizing index has $\nu_k = 1$. Finally, $a_k = s\nu_k$ for every k , so homogeneity of a length- N matrix product vector gives (8.16). $\square$

Theorem 8.5.4 (Normalized exact form for a nonzero MPV family). [Lean](#) [✓ Stmt](#) *Suppose that $V^{(N)}(A)_\sigma \neq 0$ for some $N > 0$ and some word σ . Then there are a scalar $s > 0$ and a nonempty finite family of positive-bond-dimension blocks $(A_k)_{k=1}^r$ with positive weights $(\mu_k)_{k=1}^r$ such that $s^{-1}A$ and $\bigoplus_k \mu_k A_k$ have identical MPV coefficients at every positive length. The weights satisfy $|\mu_k| \leq 1$, with equality for at least one block, and*

$$\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}, \\ \sum_i (A_k^i)^\dagger \Lambda_k A_k^i = \Lambda_k \quad (8.17)$$

for a diagonal positive-definite matrix Λ_k . Every fixed point of $\mathcal{E}_{A_k}$ is a scalar multiple of $\mathbb{1}$, and $\sum_k \dim(A_k) \leq D$.

Proof. [✓ Proof](#) Choose the structured witness from Theorem 8.5.2. The assumed nonzero coefficient makes its block family nonempty, so Theorem 8.5.3 supplies $s > 0$ and the normalized positive weights. Equation (8.16) and homogeneity give

$$V^{(N)}(s^{-1}A)_\sigma = s^{-N} V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_k \mu_k A_k\right)_\sigma.$$

The witness satisfies all required block conditions and the bond-dimension bound. $\square$

Theorem 8.5.5 (Zero/nonzero alternative for the normalized exact form). Lean ✓ Stmt For every MPS tensor A , one of the following alternatives holds:

1. $V^{(N)}(A)_\sigma = 0$ for every $N > 0$ and every word σ ;
2. there are a scalar $s > 0$ and a nonempty normalized block family satisfying all conclusions of Theorem 8.5.4.

Proof. ✓ Proof Split according to whether there exist $N > 0$ and σ with $V^{(N)}(A)_\sigma \neq 0$. In that case apply Theorem 8.5.4. If no such pair exists, every positive-length coefficient vanishes. $\square$

Theorem 8.5.6 (Translation-invariant canonical form). Lean ✓ Stmt Let A be a translation-invariant MPS representation of bond dimension D . There is a positive scalar s and a finite family of blocks A_k , possibly empty, such that the globally rescaled tensor $s^{-1}A$ has the same MPV coefficients on every positive-length ring as $\bigoplus_k \mu_k A_k^i$. The weights are positive and satisfy $1 \geq |\mu_k|$; if the family is nonempty, then some weight has norm 1. For each block,

$$\begin{aligned} \sum_i A_k^i (A_k^i)^\dagger &= \mathbb{1}, \\ \sum_i (A_k^i)^\dagger \Lambda_k A_k^i &= \Lambda_k. \end{aligned} \tag{8.18}$$

Here Λ_k is diagonal, positive, and full-rank, and each block has positive bond dimension. Moreover, the only fixed points of $\mathcal{E}_{A_k}(X) = \sum_i A_k^i X (A_k^i)^\dagger$ are the scalar multiples of $\mathbb{1}$: for every X , the identity $\mathcal{E}_{A_k}(X) = X$ implies that there exists $c \in \mathbb{C}$ such that $X = c\mathbb{1}$. The total bond dimension of the decomposition is at most D .

This is the positive-length form of [PGVWC07, Theorem 4], with the global spectral-radius normalization step made explicit.

Proof. ✓ Proof Apply Theorem 8.5.5. If every positive-length coefficient vanishes, take $s = 1$ and the empty retained family; the block conditions and bond-dimension bound are vacuous, and the MPV identity is the zero identity. In the nonzero branch, use the nonempty normalized family supplied by the second alternative. $\square$

Remark 8.5.7 (CPSV blocked canonical form (source statement)). The blocked canonical-form reduction of [CPGSV16, Section 2.3] and [CPGSV21, Section IV.A] states that an arbitrary translation-invariant MPS tensor becomes, after blocking, a weighted family of normal tensors, up to a summand that vanishes on every positive-length ring. Four statements must be distinguished. Theorem 8.5.6 gives the unconditional PGVWC07 normalized exact form before period removal. Theorem 8.11.1.2 gives the prepared trace-preserving primitive-block reduction. For a family with a nonzero positive-length coefficient, Theorem E.3.7 normalizes the weights and Theorem 9.6.3.14 gives the normalized BNT form, including its per-site scalar. If every positive-length coefficient vanishes, there is no nonempty family from which to choose a unit-modulus weight; this exceptional zero family is represented by the empty retained family instead.

8.6 Normal tensors and canonical forms

This section records the CPSV definitions of *normal tensor* and *canonical form* from [CPGSV16]. The basis of normal tensors is stated in Chapter 9. Write

$$\sigma_\partial(\mathcal{E}) = \{\lambda \in \sigma(\mathcal{E}) : |\lambda| = r(\mathcal{E})\} \tag{8.19}$$

for the peripheral spectrum of a transfer map $\mathcal{E}$, where $r(\mathcal{E})$ is its spectral radius. The stronger conditions used later (Definition 8.9.2 and the canonical-form conditions of Chapter 9) add left-canonical normalization and spectral primitivity. The weight moduli are taken non-increasing, not strictly decreasing; equal-modulus and repeated-copy sectors are retained and compared by the coefficient argument of Chapter 9.

Definition 8.6.1 (CPSV normal tensor (NT)). [Lean](#) [✓ Stmt](#) A tensor A of physical dimension d and bond dimension D is a *normal tensor* if, after the spectral-radius normalization of [CPGSV16], (i) A admits no nontrivial invariant orthogonal projection and (ii) the associated CPM $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ has spectral radius $r(\mathcal{E}_A) = 1$ and peripheral spectrum $\sigma_\partial(\mathcal{E}_A) = \{1\}$.

Remark 8.6.2. Condition (i) cannot be dropped. A reducible tensor may have peripheral spectrum $\{1\}$ while still admitting a nontrivial invariant projection, so condition (ii) alone does not characterize normality.

Theorem 8.6.3 (Algebraically normal left-canonical tensors are CPSV normal). [Lean](#) [✓ Stmt](#) Let $D > 0$. If A is algebraically normal and left-canonical, then

$$A \text{ is a CPSV normal tensor.} \quad (8.20)$$

Proof. [✓ Proof](#) Algebraic normality together with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ makes the transfer map strongly irreducible. It consequently has a positive-definite fixed point, no nontrivial invariant projection, and peripheral spectrum $\{1\}$. The fixed-point eigenvalue and left-canonical normalization give $r(\mathcal{E}_A) = 1$. Thus no scalar rescaling remains, and the two conditions in Definition 8.6.1 hold for A itself. $\square$

Theorem 8.6.4 (Normality and left-canonical normalization under entrywise conjugation). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a tensor or matrix, an overline denotes entrywise conjugation. If A and B are gauge equivalent through X , then $\overline{A}$ and $\overline{B}$ are gauge equivalent through $\overline{X}$. If A is left-canonical, then $\overline{A}$ is left-canonical. If A is algebraically normal, then $\overline{A}$ is algebraically normal. Independently, if A is a CPSV normal tensor, then $\overline{A}$ is a CPSV normal tensor.

Proof. [✓ Proof](#) Entrywise conjugating $B^i = X A^i X^{-1}$ gives $\overline{B^i} = \overline{X} \overline{A^i} (\overline{X})^{-1}$. Conjugating the left-canonical identity gives

$$\sum_i (\overline{A^i})^\dagger \overline{A^i} = \overline{\sum_i (A^i)^\dagger A^i} = \overline{\mathbb{1}} = \mathbb{1}. \quad (8.21)$$

Entrywise conjugation maps the span of the length- N products of A onto the span of those of $\overline{A}$, so it preserves positive-length block injectivity and hence algebraic normality. For the normal-tensor assertion, the transfer map of $\overline{A}$ is the anti-linear conjugate $\sigma \circ \mathcal{E}_A \circ \sigma$ by (8.22). By the corresponding theorem in the companion quantum-channel volume [LTC26, “Entrywise conjugation transports peripheral eigenvalues”] this conjugation preserves the spectral radius and primitivity, and by Theorem 8.6.6 entrywise conjugation preserves irreducibility. Hence both clauses of Definition 8.6.1 hold for $\overline{A}$. $\square$

Theorem 8.6.5 (Transfer maps and positive diagonal matrices under entrywise conjugation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let σ denote entrywise conjugation. For every tensor A , as operators on the matrix space,

$$\mathcal{E}_{\overline{A}} = \sigma \circ \mathcal{E}_A \circ \sigma; \quad (8.22)$$

evaluating at $\overline{X}$ gives, for every matrix X ,

$$\mathcal{E}_{\overline{A}}(\overline{X}) = \overline{\mathcal{E}_A(X)}. \quad (8.23)$$

If Λ is positive definite and diagonal, then

$$\overline{\Lambda} = \Lambda. \quad (8.24)$$

Proof. ✓ Proof Entrywise conjugation commutes with finite sums, matrix products, and conjugate transposition, which gives (8.22) on every matrix, and (8.23) follows since σ is an involution. The diagonal entries of a positive-definite complex matrix are real, while all off-diagonal entries of a diagonal matrix vanish. This proves (8.24). $\square$

Theorem 8.6.6 (Irreducibility under entrywise conjugation). Lean ✓ Stmt *If A is irreducible, then $\overline{A}$ is irreducible.*

Proof. ✓ Proof Suppose P is a nontrivial invariant orthogonal projection for $\overline{A}$, so $(1-P)\overline{A}^i P = 0$ for every i . Conjugating entrywise gives $(1-\overline{P})A^i \overline{P} = 0$, and $\overline{P}$ is again a Hermitian idempotent distinct from 0 and 1. Hence $\overline{P}$ is a nontrivial invariant orthogonal projection for A , contradicting the irreducibility of A . $\square$

Definition 8.6.7 (CPSV canonical form). Lean Lean Lean Lean Lean ✓ Stmt Let A have bond dimension D . It is in the canonical form of [CPGSV16, Section 2.3] if there are positive integers $D_1, \dots, D_r$, normal tensors A_k of bond dimension D_k , nonzero weights $\mu_k \in \mathbb{C}$, and, writing $S = \sum_k D_k$, a matrix $U \in \mathbb{C}^{S \times D}$ such that $S \leq D$ and, for every physical index i ,

$$\begin{aligned} UU^\dagger &= \mathbb{1}_S, \\ A^i &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U. \end{aligned} \quad (\text{CF})$$

Thus the complement of the retained direct sum consists literally of zero bond-space coordinates. In particular, $r = 0$ is possible only when $A = 0$.

The convention in line 246 of [CPGSV16] is a separate normalization of these data, not part of canonical-form membership:

$$\begin{aligned} |\mu_k| &\leq 1 \quad \text{for every } k, \\ A \neq 0 &\implies \exists k, |\mu_k| = 1. \end{aligned} \quad (8.25)$$

Every weight μ_k is nonzero. This is the reading of [CPGSV16, Section 2.3] adopted here: the remark in line 219 there places the degenerate case in the block dimensions ($D_k = 0$), and lines 224–225 choose each μ_k so that the transfer map of $\mu_k A_k$ has spectral radius one, which a vanishing weight cannot satisfy. The reading loses nothing, since if $\mu_{k_0} = 0$ then $\mu_{k_0} A_{k_0}^i = 0$ is the zero matrix of size D_{k_0} and

$$\bigoplus_{k=1}^r \mu_k A_k^i = \bigoplus_{k \neq k_0} \mu_k A_k^i \oplus 0_{D_{k_0}}, \quad (8.26)$$

so A^i is the same coisometric image of the direct sum over the remaining indices, with the D_{k_0} coordinates of the omitted summand moved to the zero complement. No ordering or separation condition is imposed.

The basis-of-normal-tensors refinement is Definition 9.1.1; it is not part of canonical form itself.

Theorem 8.6.8 (Physical relabeling preserves CPSV normal and canonical form). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $e : \{0, \dots, d' - 1\} \rightarrow \{0, \dots, d - 1\}$ be a bijection and set

$$(e^*A)^j = A^{e(j)}. \quad (8.27)$$

The transfer maps of A and e^*A agree. In particular, physical relabeling preserves left-canonical normalization and CPSV normality. If A has canonical-form data

$$A^i = U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U,$$

then e^*A has the same weights and coisometry and the relabeled normal blocks:

$$(e^*A)^j = U^\dagger \left(\bigoplus_{k=1}^r \mu_k (e^*A_k)^j \right) U. \quad (8.28)$$

Proof. [✓ Proof](#) A bijection of the physical alphabet only permutes the summands in $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$. Hence the transfer map, its spectral radius, its peripheral spectrum, and its invariant orthogonal projections are unchanged. Formula (8.28) is obtained from Formula (CF) by replacing i with $e(j)$. $\square$

Theorem 8.6.9 (Transfer maps under a bond-space gauge). [Lean](#) [✓ Stmt](#) If $B^i = X A^i X^{-1}$ for every physical index i , then there is an invertible matrix C such that

$$\mathcal{E}_B(Y) = C^{-1} \mathcal{E}_A(CY C^\dagger) (C^\dagger)^{-1}.$$

Proof. [✓ Proof](#) Expand both transfer maps and take $C = X^{-1}$. $\square$

Theorem 8.6.10 (Gauge invariance of transfer quasi-idempotence). [Lean](#) [✓ Stmt](#) Let A and B be gauge-equivalent MPS tensors and let $c \in \mathbb{C}$. Then

$$\mathcal{E}_B^2 = c \mathcal{E}_B \iff \mathcal{E}_A^2 = c \mathcal{E}_A.$$

Proof. [✓ Proof](#) Theorem 8.6.9 identifies the two transfer maps by an algebra automorphism. Such an automorphism preserves composition and scalar multiplication, and therefore preserves the displayed equation. $\square$

Theorem 8.6.11 (Spectral radius under congruence similarity). [Lean](#) [✓ Stmt](#) For an invertible matrix C , the linear maps

$$E \quad \text{and} \quad Y \mapsto C^{-1} E (CY C^\dagger) (C^\dagger)^{-1}$$

have the same spectral radius.

Proof. [✓ Proof](#) The two maps are conjugate by the invertible linear transformation $Y \mapsto CY C^\dagger$, and therefore have the same spectrum. $\square$

Theorem 8.6.12 (Primitivity under congruence similarity). [Lean](#) [✓ Stmt](#) For an invertible matrix C , a linear map E is primitive if and only if

$$Y \mapsto C^{-1} E (CY C^\dagger) (C^\dagger)^{-1}$$

is primitive.

Proof. [✓ Proof](#) Congruence similarity is an algebra isomorphism on the endomorphism algebra and hence preserves the peripheral spectrum. $\square$

Theorem 8.6.13 (Gauge transport of CPSV normality). [Lean](#) [✓ Stmt](#) *Let X be invertible and suppose*

$$B^i = XA^iX^{-1}$$

for every physical index i . If B is a CPSV normal tensor, then A is a CPSV normal tensor.

Proof. [✓ Proof](#) With $S_X(Y) = XYX^\dagger$, the transfer maps satisfy

$$\mathcal{E}_B = S_X \circ \mathcal{E}_A \circ S_X^{-1}. \quad (8.29)$$

Similarity preserves the spectral radius and the peripheral spectrum. For irreducibility, start with an invariant orthogonal projection Q for one transfer map and form the support projection of the corresponding positive congruence $XQX^\dagger$ (using X^{-1} in the reverse direction). The intertwining in Formula (8.29) makes this support projection invariant for the other transfer map. Irreducibility forces its support to be zero or full, and invertibility of X then forces $Q = 0$ or $Q = 1$. Thus both conditions in Definition 8.6.1 are transported across the gauge. $\square$

Theorem 8.6.14 (Positive blocking preserves CPSV normal tensors). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be a CPSV normal tensor and let $L \geq 1$. Then the blocked tensor $A^{[L]}$ is a CPSV normal tensor.*

Proof. [✓ Proof](#) Choose a trace-preserving gauge $B^i = XA^iX^{-1}$. Positive blocking gives

$$(B^{[L]})^I = X(A^{[L]})^IX^{-1}. \quad (8.30)$$

The trace-preserving primitive and irreducible structure of B is preserved by positive blocking, so Theorem 8.6.3 makes $B^{[L]}$ CPSV normal. Gauge transport (Theorem 8.6.13) then gives CPSV normality of $A^{[L]}$. $\square$

Theorem 8.6.15 (Literal CPSV canonical form is preserved by positive blocking). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A have canonical-form data*

$$A^i = U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U, \quad UU^\dagger = \mathbb{1}.$$

For every $L \geq 1$, $A^{[L]}$ has canonical-form data with the same coisometry U , the blocked normal tensors $A_k^{[L]}$, and the weights μ_k^L :

$$(A^{[L]})^I = U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L (A_k^{[L]})^I \right) U. \quad (8.31)$$

Proof. [✓ Proof](#) Write $I = (i_1, \dots, i_L)$. Since $L \geq 1$, the word I is nonempty. Repeatedly inserting $UU^\dagger = \mathbb{1}$ into $A^{i_1} \dots A^{i_L}$ gives

$$\begin{aligned} A^{i_1} \dots A^{i_L} &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L A_k^{i_1} \dots A_k^{i_L} \right) U \\ &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L (A_k^{[L]})^I \right) U. \end{aligned}$$

The preceding theorem gives CPSV normality of every blocked block $A_k^{[L]}$, and the displayed identity is exactly the retained direct-sum reconstruction. $\square$

Theorem 8.6.16 (CPSV canonical-form representative after blocking). Lean ✓ Stmt For every tensor A , there are a positive integer p , a bond dimension D' , and a tensor B of physical dimension d^p and bond dimension D' such that $B \in \text{CF}$ and, for every $N \geq 1$,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}(B)_\sigma. \quad (8.32)$$

Here B is another tensor generating the same positive-length MPV family. The assertion $B \in \text{CF}$ is literal in the bond coordinates of B ; no assertion is made that $A^{[p]}$ itself is literally in canonical form in its original bond coordinates.

Proof. ✓ Proof Apply Theorem 8.11.1.3 to A and itself. It gives $p \geq 1$, weights μ_k , and left-canonical, primitive, tensor-irreducible blocks C_k such that, for every $N \geq 1$,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_k \mu_k C_k\right)_\sigma. \quad (8.33)$$

First, Theorem 8.11.1.1 makes each C_k algebraically normal. Second, Theorem 8.6.3 combines this conclusion with the left-canonical identity to make each C_k a CPSV normal tensor. Hence

$$B := \bigoplus_k \mu_k C_k \quad (8.34)$$

satisfies the literal block-diagonal equation (CF), so $B \in \text{CF}$. Equation (8.33) gives the second part of (8.32). $\square$

Theorem 8.6.17 (Closed-chain expansion of CPSV canonical form). Lean ✓ Stmt For canonical-form data as in (CF), every positive length N and every configuration σ satisfy

$$V^{(N)}(A)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(A_k)_\sigma. \quad (8.35)$$

Equivalently, $|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(A_k)\rangle$. This is the identity immediately following the canonical-form definition in [CPGSV16, Section 2.3].

Proof. ✓ Proof Induction on a nonempty physical word w of length N , using (CF) at each letter and $UU^\dagger = \mathbb{1}_S$ between adjacent letters, gives

$$A^w = U^\dagger \left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right) U. \quad (8.36)$$

Applying cyclicity of the trace and $UU^\dagger = \mathbb{1}_S$ to (8.36) gives

$$\text{tr}(A^w) = \text{tr} \left(\left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right) UU^\dagger \right) = \text{tr} \left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right). \quad (8.37)$$

Applying Theorem 2.5.9 to the right-hand side of (8.37) yields $\text{tr}(A^w) = \sum_{k=1}^r \mu_k^N \text{tr}(A_k^w)$, which is (8.35). $\square$

Theorem 8.6.18 (BNT characterization for CPSV canonical form). Lean ✓ Stmt Let A have canonical-form data as in (CF), and let $(B_j)_{j=1}^g$ have positive bond dimensions. The family $(B_j)_{j=1}^g$ is a basis of normal tensors for A if and only if

$$\begin{aligned} & (\forall j, B_j \text{ is normal}) \\ & \wedge (\forall k, \exists j, X_k, \phi_k, \forall i, A_k^i = e^{i\phi_k} X_k B_j^i X_k^{-1}) \\ & \wedge (\forall j \neq j', \nexists X, \phi, \forall i, B_j^i = e^{i\phi} X B_{j'}^i X^{-1}). \end{aligned} \quad (8.38)$$

Each displayed conjugacy also requires the two bond dimensions to agree, and every scalar $e^{i\phi}$ has modulus one. This is [CPGSV16, Proposition 2.7 and Appendix A], specialized to the literal canonical-form data.

Proof. ✓ Proof The canonical-form data supply normality of every A_k and the positive-length identity $|V^{(N)}(A)\rangle = \sum_k \mu_k^N |V^{(N)}(A_k)\rangle$. Substitution of these two facts into Theorem 9.1.4 gives (8.38). $\square$

The projector-closure result in Theorem 8.7.4 gives normal blocks and orthogonal ambient embeddings under the source hypotheses. Combining these embeddings into one coisometry gives Definition 8.6.7.

This source definition is distinct from the stronger normal-canonical-form conditions below. The renormalization fixed-point analysis uses a separate block-injective canonical-form criterion; it is not the CPSV canonical-form definition.

Remark 8.6.19 (Relation to the stronger conditions). A tensor A which is irreducible (Definition 8.1.1.1), has transfer map of spectral radius one, and has primitive transfer map is normal in the sense of Definition 8.6.1. The reverse implications from the stronger conditions of Definition 8.9.2 and of Chapter 9 require the fundamental-theorem step or the algebraic-to-spectral normality equivalence, established later.

8.6.1 Normalization conventions

Remark 8.6.1.1 (Scalar normalization conventions). The reductions use four scalar facts: scaling preserves injectivity; the transfer map under a scalar ζ rescales by $|\zeta|^2$; unit-modulus phase rescaling preserves $\sum_i (A^i)^\dagger A^i = \mathbb{1}$; and the MPV of a weighted block-diagonal tensor factorizes as $\mu_k^N V^{(N)}(A_k)_\sigma = \eta_k^N |\mu_k|^N V^{(N)}(A_k)_\sigma$, with $\eta_k := \mu_k / |\mu_k|$.

Left-canonical normalization is required before the blocks can be compared. Without it, the implication $\mathcal{V}(\bigoplus_k \mu_k A_k) = \mathcal{V}(\bigoplus_k \mu_k B_k) \Rightarrow \forall k, \mathcal{V}(A_k) = \mathcal{V}(B_k)$ already fails for two scalar blocks: for $\mu = (1, 2)$, $A = (1, 3/2)$, and $B = (3, 1/2)$, the two direct sums generate identical MPV families but the first blocks differ at $N = 1$. Left-canonical normalization removes this particular ambiguity; the sectors are then matched, up to a permutation, by the exact fixed-length linear independence of Chapter 9, which requires no ordering of the weight moduli.

8.7 Projector criterion for canonical decomposition

Definition 8.7.1 (Invariant-projector closure). Lean ✓ Stmt An MPS tensor A has *invariant-projector closure* if every orthogonal projection P satisfying $PA^i = PA^i P$ for all i also satisfies $A^i P = PA^i P$ for all i . This is the projector hypothesis in the sufficient condition for canonical form of [CPGSV16, Section 2].

Theorem 8.7.2 (Decomposition into irreducible corners under projector closure). [Lean] [✓ Stmt] Suppose A has invariant-projector closure. Then there are isometries $V_1, \dots, V_r$ with $V_k^\dagger V_k = \mathbb{1}$, pairwise orthogonal ranges, and $\sum_k V_k V_k^\dagger = \mathbb{1}$, such that $A^i V_k = V_k A_k^i$ for the corner tensors $A_k^i = V_k^\dagger A^i V_k$, every corner tensor A_k is irreducible, and $\mathcal{V}(A) = \mathcal{V}(\bigoplus_{k=1}^r A_k)$.

Proof. [✓ Proof] Proceed by strong induction on the bond dimension. If A is irreducible, the single corner $V = \mathbb{1}$ suffices. Otherwise, A has a nontrivial invariant projection P , which commutes with every A^i by Theorem D.2.1. Factoring $P = V_1 V_1^\dagger$ and $\mathbb{1} - P = V_2 V_2^\dagger$ through support isometries by Lemma D.2.3 splits A into two corner tensors of strictly smaller bond dimension. For $j = 1, 2$, commutation gives $A^i V_j = V_j (V_j^\dagger A^i V_j)$, and each corner inherits invariant-projector closure by Lemma D.2.4. The two recursively obtained families are combined. Ranges from different corners are orthogonal because

$$V_1^\dagger V_2 = V_1^\dagger P (\mathbb{1} - P) V_2 = 0.$$

Equality of matrix product vectors follows from Lemma D.2.6. □

Definition 8.7.3 (No nontrivial periodic vectors). [Lean] [✓ Stmt] An MPS tensor A has *no nontrivial p -periodic vectors* if, for every isometry V with $V^\dagger V = \mathbb{1}$ and $A^i V = V B^i$ for all i , with B irreducible (equivalently, for every restriction B of A to a minimal invariant subspace), and for every $\rho > 0$ and $r > 0$ with $\mathcal{E}_B(\rho) = r\rho$, every eigenvalue of $\mathcal{E}_B$ of modulus r equals r . The eigenvalue r is the spectral radius of $\mathcal{E}_B$ (Theorem [LTC26, “Spectral radius is the Perron eigenvalue”]), so the condition states that no block of the invariant-subspace decomposition of [CPGSV16, Section 2], rescaled to spectral radius one, has a peripheral eigenvalue $e^{2\pi i q/p} \neq 1$. Eigenvectors with such eigenvalues are the p -periodic vectors of [CPGSV16, Section 2].

Theorem 8.7.4 (Sufficient condition for canonical form). [Lean] [Lean] [✓ Stmt] Suppose A has invariant-projector closure and no nontrivial p -periodic vectors. Then there are normal tensors $A_1, \dots, A_r$ of positive bond dimensions $D_1, \dots, D_r$, with $\sum_k D_k \leq D$, and nonzero weights $\mu_1, \dots, \mu_r$ such that

$$\mathcal{V}^+(A) = \mathcal{V}^+\left(\bigoplus_{k=1}^r \mu_k A_k\right). \quad (8.39)$$

Moreover, there are isometries $V_k : \mathbb{C}^{D_k} \rightarrow \mathbb{C}^D$ with pairwise orthogonal ranges such that

$$\begin{aligned} A^i V_k &= V_k (\mu_k A_k^i), \\ A^i &= \sum_k V_k (\mu_k A_k^i) V_k^\dagger. \end{aligned} \quad (8.40)$$

This is the sufficient condition for canonical form of [CPGSV16, Section 2], with that paper’s convention $\sum_k D_k \leq D$ allowing zero blocks: a corner of A carrying the zero tensor cannot be rescaled to spectral radius one and is omitted because it vanishes at every positive length.

Proof. [✓ Proof] Decompose A into irreducible corners B_k along isometries V_k by Theorem 8.7.2; every matrix product vector coefficient of A is the sum of those of the corners. A corner whose matrices all vanish contributes nothing at positive length and is dropped. Every remaining corner has a Perron eigenpair $\mathcal{E}_{B_k}(\rho_k) = r_k \rho_k$ by Lemma D.3.1. The absence of p -periodic vectors, applied to the corner B_k , gives that every eigenvalue of $\mathcal{E}_{B_k}$ of modulus r_k equals r_k , so

$A_k := r_k^{-1/2} B_k$ is normal by Lemma D.3.2. Setting $\mu_k := r_k^{1/2}$ gives $\mu_k A_k = B_k$ at the level of matrices, so for every word w of positive length N the weight cancels:

$$\mu_k^N \operatorname{tr}(A_k^w) = (r_k^{1/2})^N (r_k^{-1/2})^N \operatorname{tr}(B_k^w) = \operatorname{tr}(B_k^w).$$

Summing over k gives (8.39), while the corner reconstruction gives (8.40). The retained dimensions satisfy $\sum_k D_k \leq \sum_k \operatorname{tr}(V_k V_k^\dagger) = \operatorname{tr}(\mathbb{1}) = D$. $\square$

Theorem 8.7.5 (Non-scalar fixed point gives a further split). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose A is unital, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If X is a Hermitian fixed point of $\mathcal{E}_A$ which is not a scalar multiple of $\mathbb{1}$, then there is a nonzero PSD fixed point ρ which is not positive definite, and consequently A admits a two-block MPV-equivalent decomposition with both block dimensions strictly smaller than D . This is the fixed-point shift and further decomposition step in [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) Let $\lambda_{\max}$ be the largest eigenvalue of X . Then $\rho := \lambda_{\max} \mathbb{1} - X$ is positive semidefinite, singular, and nonzero because X is not scalar. Unitality and the fixed-point equation for X give $\mathcal{E}_A(\rho) = \rho$. Apply Theorem 8.1.5. The non-scalar hypothesis is needed because scalar multiples of $\mathbb{1}$ are fixed by every unital map and yield no nontrivial support projection. $\square$

8.8 Fixed points and canonical gauges of irreducible blocks

Theorem 8.8.1 (Scalar fixed points in an irreducible unital block). [Lean](#) [✓ Stmt](#) If A is irreducible and unital, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and $\mathcal{E}_A(X) = X$, then X is a scalar multiple of $\mathbb{1}$. This is the final fixed-point assertion in the proof of [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) Tensor irreducibility gives irreducibility of the completely positive transfer map. By Wolf's Theorem 6.6 for irreducible unital Kraus maps, every fixed point is scalar. $\square$

Theorem 8.8.2 (Unital gauge from a full-rank Perron–Frobenius eigenvector). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let ρ be positive definite with $\mathcal{E}_A(\rho) = r\rho$ for some $r > 0$. For $B^i := r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}$,

$$\sum_i B^i (B^i)^\dagger = \mathbb{1}. \quad (8.41)$$

If $r = 1$, then the unscaled gauge $B^i := \rho^{-1/2} A^i \rho^{1/2}$ is unital and generates the same MPV family as A . This is the spectral-radius normalization and full-rank fixed-point gauge of [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) The positive square root of ρ is invertible. Substituting $B^i = r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}$ and using $\mathcal{E}_A(\rho) = r\rho$ gives

$$\sum_i B^i (B^i)^\dagger = r^{-1} \rho^{-1/2} \mathcal{E}_A(\rho) \rho^{-1/2} = \mathbb{1},$$

which is (8.41). For $r = 1$, the change from A to B is a similarity, so cyclicity of the trace gives equality of all MPV coefficients. $\square$

Definition 8.8.3 (CPSV canonical form II). [Lean](#) [Lean](#) [✓ Stmt](#) In canonical form II of [CPGSV16, Appendix A], there are positive retained dimensions D_k , normal tensors A_k , weights μ_k , and, for $S = \sum_k D_k \leq D$, a matrix $U \in \mathbb{C}^{S \times D}$ such that, for every physical index i ,

$$UU^\dagger = \mathbb{1}_S, \quad A^i = U^\dagger \left(\bigoplus_k \mu_k A_k^i \right) U. \quad (8.42)$$

Thus omitted bond-space coordinates are literally zero. Every retained block additionally satisfies

$$\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}_{D_k},$$

$$\mathcal{E}_{A_k}(\Lambda_k) = \Lambda_k,$$

where Λ_k is diagonal and positive definite. The two displayed equations are the only additional blockwise conditions; no ordering, injectivity, gauges, or basis-of-normal-tensors assumptions are included. The line-246 convention (8.25) remains a separate condition on the underlying canonical-form data.

Theorem 8.8.4 (Canonical form admits a canonical-form-II gauge). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)

Let A be a tensor in CPSV canonical form, with ambient bond dimension D . Then there is a tensor B of the same bond dimension and an $G \in \text{GL}(D, \mathbb{C})$ such that, for every physical index i ,

$$B^i = GA^iG^{-1}. \quad (8.43)$$

Moreover, B is in canonical form II and the total retained block dimension is unchanged. This is the nonsingular gauge assertion of [CPGSV16, Appendix A].

Proof. [✓ Proof](#) Write the retained part of the canonical-form decomposition as $C^i = \bigoplus_k \mu_k A_k^i$, and let $U \in \mathbb{C}^{S \times D}$ satisfy $UU^\dagger = \mathbb{1}_S$ and $A^i = U^\dagger C^i U$. For each normal block, first choose its pure trace-preserving Perron gauge and then diagonalize its positive definite fixed point by a unitary. This gives $X_k \in \text{GL}(D_k, \mathbb{C})$ and blocks B_k satisfying the two canonical-form-II identities. Put $X = \bigoplus_k X_k$ and extend it to the omitted zero coordinates by

$$G = U^\dagger XU + (\mathbb{1}_D - U^\dagger U). \quad (8.44)$$

Its inverse is

$$G^{-1} = U^\dagger X^{-1}U + (\mathbb{1}_D - U^\dagger U). \quad (8.45)$$

The identity $UU^\dagger = \mathbb{1}_S$ gives $GG^{-1} = G^{-1}G = \mathbb{1}_D$. It also gives

$$GU^\dagger = U^\dagger X, \quad (8.46)$$

and

$$UG^{-1} = X^{-1}U. \quad (8.47)$$

Therefore

$$GA^iG^{-1} = U^\dagger XC^iX^{-1}U = U^\dagger \left(\bigoplus_k \mu_k B_k^i \right) U, \quad (8.48)$$

which has the same ambient reconstruction and satisfies canonical form II. $\square$

Theorem 8.8.5 (Gauge equivalence under coisometric reconstruction). [Lean](#) [✓ Stmt](#) Let A and B have common bond dimension D , and let C and E have common retained bond dimension n . Suppose that $U_A, U_B \in \mathbb{C}^{n \times D}$ satisfy $U_A U_A^\dagger = U_B U_B^\dagger = \mathbb{1}_n$ and, for every physical index i ,

$$A^i = U_A^\dagger C^i U_A. \quad (8.49)$$

$$B^i = U_B^\dagger E^i U_B. \quad (8.50)$$

If C and E are gauge equivalent, then A and B are gauge equivalent.

Proof. ✓ Proof Choose $X \in \text{GL}(n, \mathbb{C})$ such that, for every physical index i , $E^i = XC^iX^{-1}$. Equality of the Gram matrices $U_A U_A^\dagger$ and $U_B U_B^\dagger$ gives a unitary $W \in \mathbb{C}^{D \times D}$ with

$$U_B^\dagger = W U_A^\dagger. \quad (8.51)$$

$$U_B = U_A W^\dagger. \quad (8.52)$$

Extend the retained gauge through U_A by

$$G = U_A^\dagger X U_A + (\mathbb{1}_D - U_A^\dagger U_A). \quad (8.53)$$

Its inverse is

$$G^{-1} = U_A^\dagger X^{-1} U_A + (\mathbb{1}_D - U_A^\dagger U_A). \quad (8.54)$$

The coisometry identity gives $GU_A^\dagger = U_A^\dagger X$ and $U_A G^{-1} = X^{-1} U_A$. Hence, with $Y = WG$,

$$B^i = W U_A^\dagger E^i U_A W^\dagger = W G A^i G^{-1} W^\dagger = Y A^i Y^{-1}. \quad (8.55)$$

□

Theorem 8.8.6 (Diagonal Perron normalization for an irreducible TP tensor). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ and $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that, for $B^i := U^\dagger A^i U$,*

$$\begin{aligned} \sum_i (B^i)^\dagger B^i &= \mathbb{1}, \\ \mathcal{E}_B(\Lambda) &= \Lambda. \end{aligned}$$

This proves the diagonal Perron normalization required in CFII. It does not prove that A is in the CPSV canonical form or that its block is normal; those require the corresponding primitivity and canonical-form hypotheses.

Proof. ✓ Proof The TP hypothesis makes $\mathcal{E}_A$ a channel with a nonzero PSD fixed point. Irreducibility makes $\mathcal{E}_A$ an irreducible CP map, and Theorem 5.1.1 upgrades the fixed point to positive definite. The spectral theorem diagonalizes it by a unitary conjugation, which preserves the TP normalization. □

8.9 Normal canonical form

This section defines the spectral *normal* canonical form used in the reduction. A separate block-injective canonical-form criterion is used in the renormalization fixed-point analysis. The supporting overlap consequences are collected in Section D.6 and Section 6.7.

Recall from Definition 2.7.3 and Lemma 6.2.1 that the bilinear overlap is

$$O_{AB}(N) = \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma} = \text{Tr}(F_{AB}^N). \quad (8.56)$$

This notation enters the self-overlap clause of the canonical-form conditions below.

Remark 8.9.1 (Normalization convention). [PGVWC07, Theorem 4] uses the unital gauge $\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}$; here the blocks are in the dual TP gauge $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$. A positive-definite fixed point of the adjoint transfer map provides the change of gauge between the two (Theorem 5.4.2).

Definition 8.9.2 (Normal canonical form conditions). [Lean](#) [✓ Stmt](#) Scaling factors $(\mu_k)_{k=1}^r$ and block tensors $(A_k)_{k=1}^r$ satisfy the *normal canonical form conditions* if:

1. each A_k is irreducible,
2. each A_k satisfies $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$,
3. each $\mathcal{E}_{A_k}$ is primitive (equivalently, $\sigma_\partial(\mathcal{E}_{A_k}) = \{1\}$),
4. $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_r|$,
5. all $\mu_k \neq 0$,
6. all bond dimensions are positive.

Here “normal” is the spectral sense of [CPGSV16], which by [SPGWC10, Proposition 3] is equivalent to the eventual block-injectivity formulation of Definition 2.4.4.

8.9.1 Separated one-copy normal canonical forms

The following specialization attaches BNT separation to the one-copy normal canonical form. Repeated representatives and their power-sum coefficients are instead recorded by the sector decomposition of Chapter 9.

Definition 8.9.1.1 (No gauge-phase-equivalent distinct blocks). [Lean](#) [✓ Stmt](#) A finite block family has no gauge-phase-equivalent distinct blocks when, for any two distinct indices with the same bond dimension, the corresponding blocks are not gauge-phase equivalent.

Definition 8.9.1.2 (Normal canonical form with BNT separation). [Lean](#) [✓ Stmt](#) This is a one-copy canonical-form condition: each listed block is one representative and repeated copies are not encoded as separate blocks. A family is in *normal canonical form with BNT separation* if it satisfies Definition 8.9.2 and distinct blocks of the same bond dimension are not gauge-phase equivalent. The weight moduli are non-increasing but need not be strictly decreasing, so equal-modulus blocks are allowed; the grouping into a BNT is governed by gauge-phase equivalence, not by distinctness of moduli. Repeated equal-modulus copies of one basis tensor are not separate blocks here; they appear as the weights $\mu_{j,q}$ in $c_N^{(j)} = \sum_q \mu_{j,q}^N$ in the sector decomposition below.

The off-diagonal decay needed for the BNT criterion follows from the separation condition together with the overlap-decay theorems of the preceding chapter.

Theorem 8.9.1.3 (Cross-overlap decay from explicit BNT hypotheses). [Lean](#) [✓ Stmt](#) Let $(A_j)_{j=1}^r$ be a family in which each A_j is injective and left-canonical, $\sum_i (A_j^i)^\dagger A_j^i = \mathbb{1}$, and distinct equal-dimension blocks are not gauge-phase equivalent. Then $O_{A_j A_k}(N) \rightarrow 0$ for every pair $j \neq k$.

Proof. [✓ Proof](#) If the bond dimensions differ, Theorem 6.4.4 gives the decay. If the bond dimensions agree, the non-equivalence hypothesis excludes gauge-phase equivalence, so Theorem 6.5.5 gives the decay. $\square$

Lemma 8.9.1.4 (BNT from explicit blockwise hypotheses). Lean ✓ Stmt *Let $(\mu_k, A_k)_{k=1}^r$ be a weighted block family. Assume each A_k is injective and left-canonical, $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$, that $O_{A_k A_k}(N) \rightarrow 1$ for each k , and that distinct equal-dimension blocks are not gauge-phase equivalent. Then the blocks form a basis of normal tensors for the block-diagonal tensor $\bigoplus_k \mu_k A_k$.*

Proof. ✓ Proof Injective blocks are normal. The block-diagonal decomposition theorem gives the MPV expansion of $\bigoplus_k \mu_k A_k$ in terms of the block MPVs. The self-overlap limits are part of the hypotheses, and Theorem 8.9.1.3 supplies the off-diagonal decay. Therefore Lemma E.1.10 gives the eventual linear independence required in Definition 9.1.1. $\square$

The normal-canonical formulation encodes normality spectrally (primitive transfer map), the BNT definition algebraically (eventual block injectivity).

Remark 8.9.1.5 (Comparing the two normality conditions). The spectral normality of Definition 8.9.2 and the algebraic normality of Definition 2.4.4 agree: the Wielandt theory of Chapter 7 supplies the implication from primitivity to eventual block injectivity used here.

Lemma 8.9.1.6 (Restricted normal-CF-BNT hypotheses give a BNT). Lean ✓ Stmt *On this one-copy surface, if a family is in normal canonical form with BNT separation and each block is also normal in the algebraic sense, then the blocks form a basis of normal tensors for the associated block-diagonal tensor. The blocks here are distinct representatives.*

Proof. ✓ Proof The normal-canonical hypotheses already give the block-diagonal MPV expansion, the self-overlap normalization, and the eventual linear independence coming from irreducible trace-preserving overlap decay. Supplying algebraic normality for each block adds the remaining clause in Definition 9.1.1. $\square$

Remark 8.9.1.7. See Theorem 6.6.1 for the modulus-one eigenvalue rigidity statement for irreducible trace-preserving blocks.

8.9.2 Canonical form from primitivity

A normal block needs a primitive transfer map; an irreducible TP block may still have a cyclic peripheral spectrum. Raising the map to its period sends every peripheral eigenvalue to 1, but eigenvalue 1 can then have several cyclic corners and need not be simple on the full powered space. Period removal therefore proceeds in this order: power the map, obtain the adjoint-fixed cyclic projections, compress to their corners, prove each corner irreducible, and only then infer primitivity of the sector maps. This is the blocking step of [CPGSV16, Section 2.3].

Theorem 8.9.2.1 (Blocking yields a peripheral-spectrum primitive transfer map). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then there exists a blocking length $p > 0$ such that $\sigma_\partial(\mathcal{E}_{A^{[p]}}) = \{1\}$.*

Proof. ✓ Proof Pass to the conjugate-transposed Kraus family $K_i := (A^i)^\dagger$, which is unital with irreducible Kraus map. The TP fixed-point theorem for $\mathcal{E}_A$ provides a positive definite fixed point for the adjoint of K , so the corresponding theorem in the companion quantum-channel volume [LTC26, “Roots of unity via adjoint fixed point”] shows that every peripheral eigenvalue is a root of unity. Since $\sigma_\partial(\mathcal{E}_A)$ is finite (the corresponding theorem in the companion quantum-channel volume [LTC26, “Finiteness of peripheral eigenvalues”]), the corresponding lemma in the companion quantum-channel volume [LTC26, “Common exponent for a finite set of roots of unity”] gives a common exponent p with $\mu^p = 1$ for every $\mu \in \sigma_\partial(\mathcal{E}_A)$, and the corresponding lemma in the companion quantum-channel volume [LTC26, “Powering collapses peripheral eigenvalues”] gives $\sigma_\partial(\mathcal{E}_A^p) = \{1\}$. Blocking by p takes the p th power of the transfer

map, so $\mathcal{E}_{A^{[p]}}$ is primitive in the peripheral-spectrum sense of the corresponding definition in the companion quantum-channel volume [LTC26, “Primitive map”]. The blocked map need not remain irreducible: eigenvalue 1 may have several cyclic corners. The adjoint-fixed projections are therefore compressed before irreducibility and primitivity are asserted sector by sector in Theorem 8.3.1.3. This is the periodicity-removal step of [CPGSV16, Section 2.3] for an irreducible block already in TP gauge. $\square$

8.10 Left-canonical and dual diagonalization of irreducible blocks

Each irreducible block in trace-preserving gauge admits a unitary normalization with diagonal positive Perron data. This is the normalization component of CFII (Definition 8.8.3); the full CFII assertion also requires the CPSV canonical-form and normal-block hypotheses. The full PGVWC07 canonical form is Theorem 8.5.6.

Theorem 8.10.1 (Diagonal Perron data for an irreducible TP block). Lean ✓ Stmt *Let A be an irreducible MPS tensor in TP gauge, $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, with $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that:*

1. $B^i := U^\dagger A^i U$ is TP,
2. $\mathcal{E}_B(\Lambda) = \Lambda$,
3. Λ is diagonal and positive definite,
4. A and B generate the same MPV family.

These are the two normalization identities in the CFII definition of [CPGSV16, Appendix A]. The theorem does not by itself prove that A is in canonical form or that its block is normal. The unitary conjugation is performed inside TP gauge after TP normalization.

Proof. ✓ Proof Theorem 8.8.6 provides the unitary and the diagonal positive-definite fixed point, and unitary conjugation preserves trace preservation and the MPV family (Theorem D.1.1). $\square$

Theorem 8.10.2 (Dual diagonalization for irreducible unital blocks). Lean ✓ Stmt *Let A be an irreducible MPS tensor in unital gauge $\sum_i A^i (A^i)^\dagger = \mathbb{1}$, with $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that, for $B^i := U^\dagger A^i U$,*

$$\begin{aligned} \sum_i B^i (B^i)^\dagger &= \mathbb{1}, \\ \sum_i (B^i)^\dagger \Lambda B^i &= \Lambda. \end{aligned} \tag{8.57}$$

Moreover, A and B generate the same MPV family. This is the dual-map fixed-point and unitary-diagonalization step of [PGVWC07, Theorem 4].

Proof. ✓ Proof Apply Theorem 8.10.1 to the conjugate-transposed family $i \mapsto (A^i)^\dagger$: unitality of A is trace preservation for this family, whose transfer map is the dual of $\mathcal{E}_A$. Translating back gives (8.57). $\square$

8.11 Cyclic-sector isometries and primitive reduction

Theorem 8.11.1 (Spectral cyclic sectors with support isometries). [Lean] [✓ Stmt] *Let A be a left-canonical irreducible tensor, and suppose the adjoint transfer map $\mathcal{E}_{A^\dagger}$ has a positive-definite fixed point, is irreducible as a positive map, and has peripheral spectrum*

$$\sigma_\partial(\mathcal{E}_{A^\dagger}) = \{\gamma^j : j \in \mathbb{Z}/m\mathbb{Z}\}, \quad (8.58)$$

where γ is a primitive m th root of unity and $m \geq 1$. Then the spectral cyclic-sector decomposition of $A^{[m]}$ has compression maps which may be chosen with rectangular support isometries V_k satisfying

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ V_k V_k^\dagger &= P_k, \\ \varphi_k(X) &= V_k X V_k^\dagger. \end{aligned} \quad (8.59)$$

These isometries also realize the blocked corner letters $\varphi_k(C_k^\alpha) = P_k(A^{[m]})^\alpha P_k$.

Proof. [✓ Proof] Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”], applied to $\mathcal{E}_{A^\dagger}$, provides projections P_k satisfying $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$. Iterating m times gives $(\mathcal{E}_{A^\dagger})^m(P_k) = P_k$, and $(\mathcal{E}_{A^\dagger})^m = \mathcal{E}_{(A^{[m]})^\dagger}$ shows that each P_k is fixed by the adjoint transfer map of $A^{[m]}$. Applying Theorem D.5.3 to that blocked tensor gives the compressed sector tensors and the support isometries in (8.59). $\square$

8.11.1 Reduction to primitive blocks

This subsection completes the reduction after period removal. A TP-primitive-irreducible block is normal, and an arbitrary tensor becomes, after a positive blocking length, a weighted family of left-canonical primitive blocks whose total bond dimension is bounded by the original bond dimension. For two tensors with the same MPV family, the cyclic sectors can be expressed over one common blocked alphabet, which is the form used by the Fundamental Theorem.

Theorem 8.11.1.1 (TP-primitive irreducible tensor is normal). [Lean] [✓ Stmt] *Let A be a left-canonical MPS tensor with $D > 0$, tensor-irreducible, and with peripheral spectrum $\{1\}$. Tensor irreducibility implies irreducibility of the completely positive transfer map, so A is normal.*

Proof. [✓ Proof] The primitive peripheral spectrum, tensor irreducibility, and left-canonical normalization give primitive fixed-point data. The fixed point is positive definite by Lemma D.4.8. Theorem 7.7.4.2 then gives normality. $\square$

An arbitrary translation-invariant tensor reduces, after blocking, to a weighted direct sum of peripheral-spectrum primitive blocks. These globally blocked pieces need not remain irreducible; the cyclic-sector decomposition below refines them into primitive irreducible blocks.

Theorem 8.11.1.2 (Arbitrary tensor to primitive block decomposition). [Lean] [✓ Stmt] *For any MPS tensor A , there exist $p \geq 1$, nonzero weights $(\mu_k)_{k=1}^r$, and blocks $(B_k)_{k=1}^r$ such that, for every blocked σ of positive length N ,*

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma. \quad (8.60)$$

The retained block dimensions satisfy $\sum_{k=1}^r D_k \leq D$. Each block is left-canonical and primitive:

$$\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1},$$

$$\sigma_\partial(\mathcal{E}_{B_k}) = \{1\}.$$

Each bond dimension is positive.

Proof. ✓ **Proof** Theorem 8.2.1.1 supplies the left-canonical nontrivial blocks, the positive-length equality, and the bound $\sum_k D_k \leq D$. Apply Theorem D.4.2 directly to this finite irreducible family. Its proof first applies Theorem 8.9.2.1 to each block and then takes the least common multiple of the resulting periods through Theorem D.4.1. This gives one positive length p at which every blocked transfer map is primitive. Lemma D.4.16 keeps the blocked weights nonzero, and Lemma D.4.14 carries the positive-length equality through blocking, giving (8.60). $\square$

Theorem 8.11.1.3 (Unconditional common primitive block decompositions). Lean ✓ **Stmt** Let A and B have the same MPV family at every positive length. Then there is a positive blocking length at which both blocked tensors have the same positive-length MPV family as weighted direct sums of trace-preserving, primitive, tensor-irreducible blocks with positive bond dimensions and nonzero weights. The two weighted nonzero-sector families have the same positive-length MPV family.

Proof. ✓ **Proof** Apply Theorem 8.3.1.4 to choose the common blocking length and the two common cyclic-sector families. For either family, write G_k for its cyclic-sector blocks and (ν_x, C_x) for the weights and blocks of the flattened common-sector family from Lemma D.5.10. The common-alphabet nonzero-part identity gives, for every length N and blocked word σ ,

$$V^{(N)}\left(\bigoplus_k \mu_k^p G_k^{[p]}\right)_\sigma = V^{(N)}\left(\bigoplus_x \nu_x C_x\right)_\sigma.$$

Applying this identity on the two sides of the comparison yields a positive length p and weighted primitive block families such that, for every $N \geq 1$ and blocked word σ ,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_x \mu_x^A A_x\right)_\sigma,$$

$$V^{(N)}(B^{[p]})_\sigma = V^{(N)}\left(\bigoplus_y \mu_y^B B_y\right)_\sigma.$$

The two weighted nonzero parts are equal at every positive length. The structural properties and the nonvanishing of the weights come from the common-sector families. $\square$

Chapter 9

Basis of Normal Tensors

A *basis of normal tensors* (BNT) for an MPS tensor A is a finite family of normal blocks $\{A_j\}$ whose MPV states $\{|V^{(N)}(A_j)\rangle\}$ span $|V^{(N)}(A)\rangle$ at every positive length and are eventually linearly independent. Every coefficient of a canonical form is nonzero, so equal MPV families force a bijective gauge-phase matching $B_k = e^{i\phi_k} X_k A_{j_k} X_k^{-1}$, and the sector coefficients $c_N^{(j)} = \sum_q \mu_{j,q}^N$ recover the per-copy weight multisets by Newton–Girard. The references are [PGVWC07], [CPGSV16], and [CPGSV21].

9.1 Bases of normal tensors

Definition 9.1.1 (Basis of Normal Tensors (BNT)). Lean ✓ Stmt A *basis of normal tensors* (BNT) for a total tensor A_{tot} consists of g block tensors $(A_1, \dots, A_g)$ with bond dimensions $(D_1, \dots, D_g)$ such that each A_j is normal (Definition 2.4.4); at each system size $N \geq 1$ the MPV of A_{tot} lies in the span of the block MPVs $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$; and for sufficiently large N those block MPVs are linearly independent. Here normality means eventual block injectivity, as in Definition 2.4.4. Thus this is the algebraic form of [CPGSV21, Definition IV.2], rather than its primitive-channel formulation. In the canonical-form characterization of [CPGSV16, Section 2.3], a BNT is a minimal family of representatives for $\tilde{A}_k^i = e^{i\phi_k} X_k A_j^i X_k^{-1}$ among the normal blocks.

Definition 9.1.2 (Basis of normal tensors, coefficient form). Lean ✓ Stmt The coefficient form of a basis of normal tensors: each block A_j is a CPSV normal tensor (Definition 8.6.1), the MPV of A is given at every positive length by explicit coefficients $c_j^{(N)} \in \mathbb{C}$ with $|V^{(N)}(A)\rangle = \sum_{j=1}^g c_j^{(N)} |V^{(N)}(A_j)\rangle$, and the block MPVs $\{|V^{(N)}(A_j)\rangle\}$ are eventually linearly independent. This is the literal coefficient-form definition of [CPGSV16, Section 2.3]. It differs from Definition 9.1.1 only in the chosen normality predicate. Theorem 9.2.5 proves the implication between the two forms.

Theorem 9.1.3 (Distinct BNT members are not gauge-phase equivalent). Lean ✓ Stmt *Distinct members of a basis of normal tensors are not gauge-phase equivalent after identifying equal bond dimensions.*

Proof. ✓ Proof At every sufficiently large length the matrix product vectors of the basis members are linearly independent. A gauge-phase equivalence between two distinct members would give $|V^{(N)}(A_k)\rangle = \zeta^N |V^{(N)}(A_j)\rangle$ for a nonzero scalar ζ . Thus one vector would be a scalar multiple of the other, contradicting linear independence. $\square$

Theorem 9.1.4 (Characterization by canonical-form representatives). [Lean](#) [Lean](#) [Stmt](#) *Let A have a canonical-form MPV expansion*

$$|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(\tilde{A}_k)\rangle$$

at every positive length, where the tensors $\tilde{A}_k$ are normal tensors of positive bond dimension and every coefficient μ_k is nonzero. Let $(A_j)_{j=1}^g$ be tensors of positive bond dimension. Then $(A_j)_{j=1}^g$ is a basis of normal tensors for A if and only if:

1. every A_j is normal;
2. for every k there are an index j , an invertible matrix X_k , and a phase $e^{i\phi_k}$ such that $\tilde{A}_k^i = e^{i\phi_k} X_k A_j^i X_k^{-1}$ for every physical index i ;
3. no two distinct tensors A_j and $A_{j'}$ are related in this way.

This is [CPGSV16, Proposition 2.7], read with the nonzero-coefficient convention of Definition 8.6.7.

Proof. [Proof](#) Group the summands into the MPV phase classes of Definition 9.6.1.7, and choose one representative from each class. By Theorem 9.1.3, eventual linear independence of a BNT excludes gauge-phase equivalence between two distinct A_j .

Suppose that some phase-class representative is not gauge-phase equivalent to any A_j . Partition all phase classes into the unmatched classes T and the remaining classes. Every remaining representative is a nonzero scalar multiple of some $|V^{(N)}(A_j)\rangle$ at each positive length. Distinct unmatched representatives have vanishing mutual overlaps, and their overlaps with every A_j also vanish. Explicitly, for $t \neq t'$ in T and every j ,

$$\begin{aligned} O_{\tilde{A}_t \tilde{A}_{t'}}(N) &\longrightarrow 0, \\ O_{\tilde{A}_t A_j}(N) &\longrightarrow 0. \end{aligned}$$

By Lemma E.1.7, the combined family

$$\{|V^{(N)}(\tilde{A}_t)\rangle : t \in T\} \cup \{|V^{(N)}(A_j)\rangle : 1 \leq j \leq g\}$$

is linearly independent for all sufficiently large N . Comparing the canonical-form expansion with the BNT expansion gives a relation

$$\sum_{t \in T} c_{N,t} |V^{(N)}(\tilde{A}_t)\rangle + \sum_{j=1}^g \alpha_{N,j} |V^{(N)}(A_j)\rangle = 0.$$

Linear independence forces every $c_{N,t}$ to vanish. Hence the coefficient of every unmatched phase class vanishes for all sufficiently large N . For an unmatched class t , let q range over its canonical-form summands, write μ_q for the corresponding canonical-form weight, and write ζ_q for the phase relating that summand to the chosen representative. Then $c_{N,t} = \sum_{q \in t} (\zeta_q \mu_q)^N$. Every base $\zeta_q \mu_q$ is nonzero because every coefficient is nonzero. Eventual vanishing of this finite geometric sum would imply vanishing at $N = 0$, where its value is the positive number of summands in the class. This contradiction proves coverage. The normal-tensor phase theorem turns the resulting MPV phase relation into the stated gauge-phase relation.

Conversely, coverage gives, for each canonical-form summand, $|V^{(N)}(\tilde{A}_k)\rangle = \zeta_k^N |V^{(N)}(A_{j(k)})\rangle$. Hence

$$|V^{(N)}(A)\rangle = \sum_j \left(\sum_{k:j(k)=j} (\mu_k \zeta_k)^N \right) |V^{(N)}(A_j)\rangle.$$

This is the required BNT span at every positive length. Pairwise gauge-phase distinction and normality give eventual linear independence, completing the converse. $\square$

9.2 Normal blocks and overlap rigidity

This section relates spectral normality to eventual block injectivity and records the overlap dichotomy that detects bond-dimension and gauge-phase matching between normal blocks.

Theorem 9.2.1 (Normality forces positive bond dimension). [Lean](#) [✓ Stmt](#) *Every normal tensor has positive bond dimension.*

Proof. [✓ Proof](#) If the bond dimension were zero, the bond-matrix space and its endomorphism algebra would each contain only zero. The transfer operator would therefore have spectral radius zero, contradicting the spectral-radius-one clause in the definition of a normal tensor. $\square$

Theorem 9.2.2 (Pure Perron normalization at spectral radius one). [Lean](#) [✓ Stmt](#) *Let A be irreducible and suppose that the spectral radius of $\mathcal{E}_A$ is one. There is a positive definite matrix σ satisfying*

$$\sum_i (A^i)^\dagger \sigma A^i = \sigma. \quad (9.1)$$

The tensor

$$B^i = \sigma^{1/2} A^i \sigma^{-1/2} \quad (9.2)$$

is left-canonical and irreducible, and A and B are related by a pure invertible similarity.

Proof. [✓ Proof](#) Positive-definite right and left Perron eigenvectors have the same positive eigenvalue by the trace pairing. The spectral-radius identity makes this eigenvalue equal to one. Thus the left Perron eigenvector is an adjoint fixed point, so its square-root gauge is trace-preserving. Similarity preserves irreducibility. $\square$

Theorem 9.2.3 (Spectral normality gives a pure trace-preserving gauge). [Lean](#) [✓ Stmt](#) *Let A be a normal tensor in the sense of [CPGSV16, Section 2.2]. Its bond dimension is necessarily positive, and there is a positive definite matrix σ such that the gauge obtained from σ is trace-preserving, primitive, and irreducible. No scalar rescaling is needed.*

Proof. [✓ Proof](#) Pure Perron normalization gives a trace-preserving gauge by a pure similarity. Irreducibility and primitivity are invariant under this similarity. $\square$

Theorem 9.2.4 (Spectral normality gives eventual block injectivity). [Lean](#) [✓ Stmt](#) *Every normal tensor in the sense of [CPGSV16, Section 2.2] is injective after blocking by some finite length; its positive bond dimension follows from normality.*

Proof. [✓ Proof](#) Apply Theorem 9.2.3. The trace-preserving, primitive, irreducible representative is eventually block-injective, and gauge invariance transports this conclusion back to the original tensor. $\square$

Theorem 9.2.5 (A coefficient-form BNT is an algebraic BNT). [Lean](#) [✓ Stmt](#) If $(A_j)_{j=1}^g$ is a coefficient-form basis of normal tensors for A , then $(A_j)_{j=1}^g$ is an algebraic basis of normal tensors for A . Only this implication from spectral normality to algebraic normality is asserted.

Proof. [✓ Proof](#) Every block has positive bond dimension, so spectral normality implies eventual block injectivity for each block. The spanning identities and eventual linear independence are unchanged. $\square$

Theorem 9.2.6 (Non-decaying rectangular overlap determines the bond dimension). [Lean](#) [✓ Stmt](#) Let A and B be irreducible trace-preserving tensors of positive, possibly different, bond dimensions. If $|O_{AB}(N)| \rightarrow 1$, then their bond dimensions are equal.

Proof. [✓ Proof](#) If the bond dimensions differed, the rectangular overlap-decay theorem would give $O_{AB}(N) \rightarrow 0$, and hence $|O_{AB}(N)| \rightarrow 0$, contradicting the assumed limit. $\square$

Theorem 9.2.7 (A normal tensor has a nonzero letter). [Lean](#) [✓ Stmt](#) Let A be a CPSV normal tensor of positive bond dimension. Then $A^i \neq 0$ for some physical letter i .

Proof. [✓ Proof](#) If every letter vanished, then the transfer map would vanish and its spectral radius would be zero, contradicting the spectral-radius-one normalization of a CPSV normal tensor. $\square$

Theorem 9.2.8 (Phase-equivalent normal tensors are gauge-phase equivalent). [Lean](#) [Lean](#) [✓ Stmt](#) Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for some nonzero $\zeta \in \mathbb{C}$, $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ for $N \geq 0$. Then the bond dimensions are equal and there is an invertible matrix X and a scalar η of unit modulus such that $B^i = \eta X A^i X^{-1}$ for every physical letter i . This is the gauge recovery used in [CPGSV16, Appendix A].

Proof. [✓ Proof](#) Apply Theorem 9.2.3 to both tensors. Gauge invariance of matrix product vectors preserves the phase relation, whose self-overlap identities imply unit modulus for the phase. The resulting rectangular cross-overlap has norm tending to one. If the bond dimensions differed, Theorem 9.2.6 would give a contradiction. Thus the bond dimensions agree, and the unit-overlap rigidity theorem gives a gauge-phase equivalence between the trace-preserving representatives. Conjugating by the two Perron gauges gives the asserted relation for A and B . $\square$

Theorem 9.2.9 (Normal-tensor overlap dichotomy). [Lean](#) [✓ Stmt](#) Let A and B be normal tensors of positive, possibly different, bond dimensions. Their self-overlaps satisfy

$$\begin{aligned} O_{AA}(N) &\rightarrow 1, \\ O_{BB}(N) &\rightarrow 1. \end{aligned}$$

Moreover, either $O_{AB}(N) \rightarrow 0$, or their bond dimensions are equal; call the common value D . Then there are $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \in \mathbb{C}$, with $|\zeta| = 1$, such that $B^i = \zeta X A^i X^{-1}$ for every physical letter i . In the latter case, $|O_{AB}(N)| \rightarrow 1$. This is [CPGSV16, Lemma A.2].

Proof. [✓ Proof](#) Put both tensors in their pure trace-preserving gauges. If their bond dimensions differ, rectangular overlap decay gives $O_{AB}(N) \rightarrow 0$. If their dimensions agree but they are not gauge-phase equivalent, the equal-dimension overlap-decay theorem gives the same conclusion. In the remaining case, $B^i = \zeta X A^i X^{-1}$. By Theorem 9.2.8, $|\zeta| = 1$. Hence

$$\begin{aligned} O_{AB}(N) &= \bar{\zeta}^N O_{AA}(N), \\ |O_{AB}(N)| &= |O_{AA}(N)| \rightarrow 1. \end{aligned}$$

$\square$

Theorem 9.2.10 (Exact phase alternative for normal tensors). Lean ✓ Stmt *Let A and B be normal tensors of positive, possibly different, bond dimensions. Then either $O_{AB}(N) \rightarrow 0$, or there is a phase $\zeta \in \mathbb{C}$, with $|\zeta| = 1$, such that $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ for every chain length N . This is [CPGSV16, Corollary A.3].*

Proof. ✓ Proof Apply Theorem 9.2.9. In its second alternative, gauge-phase covariance gives $V^{(N)}(B)_\sigma = \zeta^N V^{(N)}(A)_\sigma$ for every configuration σ , and hence the asserted equality of vectors. $\square$

Theorem 9.2.11 (Unit overlap from eventual proportionality). Lean ✓ Stmt *Let A and B be tensors, possibly with different bond dimensions, whose self-overlaps tend to one. Suppose that, for all sufficiently large N , there is $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. Then $|O_{AB}(N)| \rightarrow 1$.*

Proof. ✓ Proof The proportionality relation gives

$$\begin{aligned} O_{AB}(N) &= c_N O_{BB}(N), \\ O_{AA}(N) &= |c_N|^2 O_{BB}(N). \end{aligned}$$

Hence $|c_N|^2 \rightarrow 1$ and therefore $|c_N| \rightarrow 1$. The first identity now gives $|O_{AB}(N)| \rightarrow 1$. $\square$

Theorem 9.2.12 (Geometric proportionality phase for normal tensors). Lean ✓ Stmt *Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for all sufficiently large N , their matrix product vectors are proportional by a nonzero scalar. Then there is $a \in \mathbb{C}$, with $|a| = 1$, such that $|V^{(N)}(A)\rangle = a^N |V^{(N)}(B)\rangle$ for every positive N . This is the one-normal-block consequence of [CPGSV16, Corollary A.3 and the proof of Theorem 2.10].*

Proof. ✓ Proof Normality gives $O_{AA}(N) \rightarrow 1$ and $O_{BB}(N) \rightarrow 1$, so Theorem 9.2.11 gives $|O_{AB}(N)| \rightarrow 1$. The first alternative in Theorem 9.2.10 gives $O_{AB}(N) \rightarrow 0$, hence $|O_{AB}(N)| \rightarrow 0$, a contradiction. In the remaining alternative, $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ with $|\zeta| = 1$; take $a = \zeta^{-1}$. $\square$

Theorem 9.2.13 (Separated normal tensors are eventually independent). Lean ✓ Stmt *Let $(A_j)_{j=1}^g$ be a finite family of positive-bond-dimension normal tensors, no two of which are gauge-phase equivalent after identifying equal bond dimensions. Then the matrix product vectors $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$ are linearly independent for all sufficiently large N , as in [CPGSV16, Appendix A].*

Proof. ✓ Proof Put every tensor in the pure trace-preserving gauge of Theorem 9.2.3. Distinct gauge-phase classes have overlaps tending to zero, while each self-overlap tends to one. The finite Gram matrix therefore tends to the identity and is invertible for all sufficiently large lengths. Gauge invariance of matrix product vectors gives the conclusion for the original tensors. $\square$

9.3 Block-injective consequences for BNT families

The overlap separation of normal blocks yields a simultaneous product-algebra span after blocking. We first state the fixed-length span condition and then apply it to coefficient-form and sector BNTs.

Definition 9.3.1 (Fixed-length block-injective canonical-form condition). Lean Lean ✓ Stmt For a fixed blocked length m , the fixed-length form of the block-injective canonical-form condition

is

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = m\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Equivalently,

$$\forall X = (X_j)_j \in \bigoplus_{j=1}^g M_{D_j}(\mathbb{C}), \quad \exists (c_\omega(X))_{|\omega|=m}, \quad (X_j)_{j=1}^g = \sum_{|\omega|=m} c_\omega(X) (A_1^\omega, \dots, A_g^\omega).$$

This is the fixed-length form of the block-injective canonical-form condition of [CPGSV21, lines 2113–2117].

Theorem 9.3.2 (Gauge invariance of simultaneous fixed-length product spans). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) Let $(A_j)_{j=1}^g$ and $(B_j)_{j=1}^g$ be tensor families with the same physical dimension and the same bond dimension D_j in each labelled block. Suppose that for every j there is an arbitrary invertible matrix $X_j \in \text{GL}_{D_j}(\mathbb{C})$ such that, for every physical index i ,

$$B_j^i = X_j A_j^i X_j^{-1}.$$

Then, at every fixed length L ,

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}) \iff \text{span}\{(B_1^\omega, \dots, B_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Both directions hold for general invertible gauges; no unitarity condition is imposed. In particular, a separation certificate proved after passing to left-canonical representatives can be transported back to a source unital gauge carrying a positive dual fixed point. This supplies only the simultaneous-span transport; the subsequent source-gauge boundary and parent-Hamiltonian arguments are separate.

Proof. [Stmt](#) [Proof](#) Given a target tuple $(M_j)_j$, conjugate it backwards to $(X_j^{-1} M_j X_j)_j$. If the word tuples of $(A_j)_j$ span the product algebra, one common coefficient family $(c_\omega)_\omega$ expresses this conjugated tuple. Gauge covariance of word evaluation then gives

$$\sum_{|\omega|=L} c_\omega B_j^\omega = X_j \left(\sum_{|\omega|=L} c_\omega A_j^\omega \right) X_j^{-1} = M_j \quad (9.3)$$

for every j . Thus the $(B_j)_j$ word tuples span the product algebra. Replacing every X_j by X_j^{-1} proves the reverse implication. $\square$

Lemma 9.3.3 (Direct-sum block injectivity at the source length). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) Let $(A_j)_{j=1}^g$ be irreducible left-canonical tensors with normalized self-overlaps and with no gauge-phase-equivalent distinct blocks. Suppose $g \geq 2$, $L_0 > 0$, and every A_j is block-injective at lengths L_0 , $L_0 + 1$, and $3(L_0 + 1)$. Then

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = 3(g-1)(L_0 + 1)\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the direct-sum induction length in [PGVWC07, lines 1346–1408].

Proof. ✓ Proof Lemma E.3.1 gives homogeneous pair-trace separation at $S = 3(L_0 + 1)$. For a chosen anchor block, pair separation prescribes an arbitrary matrix there and zero on one other block. Multiplying $g - 1$ such homogeneous combinations removes the other blocks one at a time. Induction on the number of removed blocks therefore realizes every tuple supported on the anchor at length $(g - 1)S$. Summing over the anchors gives the full product algebra at $3(g - 1)(L_0 + 1)$. $\square$

Theorem 9.3.4 (Simultaneous one-letter span after blocking a BNT). Lean Lean Lean Lean ✓ Stmt Let $(A_j)_{j=1}^g$ be a basis of normal tensors, with bond dimensions D_j . There is a positive integer L such that the one-letter evaluations of the blocked tensors span the full product algebra:

$$\text{span}_{\mathbb{C}} \{((A_1^{[L]})^i, \dots, (A_g^{[L]})^i) : 0 \leq i < d^L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Moreover, there is L_0 such that, for every $L \geq L_0$, the unblocked word tuples have the simultaneous span

$$\text{span}_{\mathbb{C}} \{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the existence form of simultaneous block injectivity associated with [CPGSV16, lines 317–345]; no numerical bound on L or L_0 is asserted here.

Proof. ✓ Proof Put each BNT representative in its trace-preserving Perron gauge. The gauged family is irreducible and left-canonical, has normalized self-overlaps, and remains gauge-phase separated. Choose a common positive length p at which every gauged block is block-injective. Propagation of homogeneous word spans gives block injectivity at $p + 1$ and $3(p + 1)$.

Lemma 9.3.3 gives pairwise separation at length $S = 3(p + 1)$. Write $g \dot{-} 1 = \max\{g - 1, 0\}$. Multiplying one separator for each of the other blocks gives a selector of length $(g \dot{-} 1)S$ for every block. Transport these selectors back from the Perron gauges. Gauge invariance also gives block injectivity of every original block at length p . Concatenating the injective prefix with the selectors yields the full product-algebra span at

$$L = p + (g \dot{-} 1)S = p + 3(g \dot{-} 1)(p + 1) > 0.$$

The truncated factor makes the same construction valid for empty and singleton families. The same span provides selectors and block injectivity at length L . For every sufficiently large n , concatenate an $(n - L)$ -letter injective prefix with these selectors to obtain the full product-algebra span at length n . Finally, length- L words of A_j are the one-letter evaluations of $A_j^{[L]}$, which proves the blocked statement. $\square$

Theorem 9.3.5 (Sharp simultaneous span under a total-dimension bound). Lean ✓ Stmt Let $(A_j)_{j=1}^g$ be a basis of normal tensors with bond dimensions D_j . If $K > 0$ and $\sum_j D_j \leq K$, then there is a positive integer $L \leq 3K^5$ such that

$$\text{span}_{\mathbb{C}} \{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Proof. ✓ Proof Put every representative in its trace-preserving Perron gauge. Since $D_j \leq \sum_k D_k \leq K$, quantum Wielandt and homogeneous-span propagation make every representative block-injective at K^4 , $K^4 + 1$, and $3(K^4 + 1)$. Conjugation preserves the simultaneous word span.

For $g = 0$, take $L = 1$; the product algebra has only its zero element. For $g = 1$, take $L = K^4$. If $g \geq 2$, the three-block separation theorem gives

$$L = 3(g-1)(K^4 + 1).$$

Positivity of the D_j gives $g = \sum_j 1 \leq \sum_j D_j \leq K$, and hence $L \leq 3(K-1)(K^4 + 1) \leq 3K^5$. $\square$

Theorem 9.3.6 (Total dimension of the BNT representatives). [Lean](#) [Stmt](#) *Let literal canonical-form data of ambient bond dimension D have phase classes j and chosen representatives of dimensions D_j . Then*

$$\sum_j D_j \leq D.$$

Proof. [Proof](#) In every phase class, choose its first copy. The class-copy equivalence sends these choices to distinct displayed block indices, and the first copy is the chosen representative. Thus the sum of the representative dimensions is bounded by the sum over all displayed blocks, which is at most D by the canonical-form dimension bound. $\square$

Theorem 9.3.7 (Sharp block-injective form for literal CPSV canonical form). [Lean](#) [Stmt](#) *Let A be in literal CPSV canonical form with positive ambient bond dimension D . There is a basis of normal tensors $(A_j)_{j=1}^g$ and a positive integer $L \leq 3D^5$ such that*

$$\text{span}_{\mathbb{C}} \left\{ ((A_1^{[L]})^i, \dots, (A_g^{[L]})^i) : 0 \leq i < d^L \right\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Hence A is in block-injective canonical form after blocking at most $3D^5$ sites. This is the proposition at [CPGSV16, lines 342–345].

Proof. [Proof](#) Choose one representative from every matrix-product-vector phase class. These representatives form a basis of normal tensors, and their total bond dimension is at most D by Theorem 9.3.6. Apply Theorem 9.3.5 with $K = D$. A word of length L in A_j is a one-letter evaluation of $A_j^{[L]}$, so the resulting length- L span is exactly the asserted one-letter span. $\square$

Let A be a tensor in canonical form, and let $A_1, \dots, A_g$ be a basis of normal tensors of A . For suitable coefficients $\mu_{j,q}$ and invertible matrices $X_{j,q}$, write $M_j = \text{diag}(\mu_{j,1}, \dots, \mu_{j,r_j})$ and $X = \bigoplus_{j=1}^g \bigoplus_{q=1}^{r_j} X_{j,q}$. Then

$$A^i = X \left[\bigoplus_{j=1}^g (M_j \otimes A_j^i) \right] X^{-1} = \bigoplus_{j=1}^g \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} A_j^i X_{j,q}^{-1}. \quad (9.4)$$

This is [CPGSV16, Section 2.3, Equations (20a)–(21)]. The decomposition (9.4) is used below to separate the normal blocks.

9.4 Permutation rigidity for bases of normal tensors

When two bases of normal tensors generate the same MPV subspace for all sufficiently large positive lengths, the blocks must agree up to a permutation and per-block gauge phases. The argument turns the eventual equality of spans into an invertible change-of-basis matrix, then analyses its asymptotics through the overlaps.

The change-of-basis matrix is now used to match the blocks. We first treat equal block counts, then remove that restriction.

Theorem 9.4.1 (Permutation rigidity for equal block counts). Lean ✓ Stmt Let $(A_j)_{j=1}^g$ and $(B_j)_{j=1}^g$ be injective families of positive bond dimension satisfying trace-preserving normalization, with self-overlaps tending to 1 and off-diagonal overlaps tending to 0 within each family. If the spans of the MPV states agree for all sufficiently large system sizes N , then there is a permutation π of $\{1, \dots, g\}$ such that A_j and $B_{\pi(j)}$ have the same bond dimension and are gauge-phase equivalent.

Proof. ✓ Proof For large N , Lemma E.2.2 gives an invertible matrix U_N relating the two MPV families. Fix j and write $|V^{(N)}(B_j)\rangle = \sum_i u_{ij}^{(N)} |V^{(N)}(A_i)\rangle$. With $G_A(N) = (O_{A_i A_\ell}(N))_{i,\ell}$ and $b_j(N) = (O_{A_i B_j}(N))_i$, this gives

$$G_A(N) \overline{u_j(N)} = b_j(N).$$

Moreover,

$$O_{B_j B_j}(N) = u_j(N)^\top G_A(N) \overline{u_j(N)}.$$

If all mixed overlaps $O_{A_i B_j}(N)$ tended to 0, then $b_j(N) \rightarrow 0$; since $G_A(N) \rightarrow \mathbb{1}$, one obtains $\overline{u_j(N)} \rightarrow 0$ and hence $u_j(N) \rightarrow 0$. Thus $O_{B_j B_j}(N) \rightarrow 0$, contradicting $O_{B_j B_j}(N) \rightarrow 1$. Thus each B_j has a matched block $A_{f(j)}$ with non-decaying mixed overlap. Theorems 6.4.4 and 6.5.5 then force equal bond dimension and gauge-phase equivalence for each match. Off-diagonal decay within the B -family makes f injective, hence bijective, and its inverse is the required permutation. $\square$

Lemma 9.4.2 (Permutation rigidity with asymptotically orthonormal overlaps). Lean ✓ Stmt Let $(A_j)_{j=1}^{g_A}$ and $(B_k)_{k=1}^{g_B}$ be two families of injective tensors of positive bond dimension satisfying the TP normalization $\sum_i (A_j^i)^\dagger A_j^i = \mathbb{1}$ and $\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$, whose self-overlaps converge to 1 and whose cross-overlaps within each family converge to 0. If, for all sufficiently large system sizes N ,

$$\text{span}_{\mathbb{C}}\{|V^{(N)}(A_j)\rangle : 1 \leq j \leq g_A\} = \text{span}_{\mathbb{C}}\{|V^{(N)}(B_k)\rangle : 1 \leq k \leq g_B\},$$

then $g_A = g_B$, and there is a permutation π of the common block index set such that A_j and $B_{\pi(j)}$ have the same bond dimension and are gauge-phase equivalent for each j .

Proof. ✓ Proof By Lemma E.1.10, both families are linearly independent for all sufficiently large N . Choose N large enough that both families are linearly independent and the assumed span equality holds. The common span then has the same dimension when computed from the A -family or the B -family, so $g_A = g_B$. After identifying the two index sets, Theorem 9.4.1 yields the permutation conclusion. $\square$

The two preceding results obtain the block matching from the asymptotic orthonormality of the overlaps. The same matching is obtained by the exact sector comparison of Section 9.8, which pairs the sectors of two canonical forms through a bijection fixed by their bond dimensions and gauge-phase equivalences. The proof of the Fundamental Theorem in Chapter 10 proceeds through the sector comparison.

The same overlap dichotomy also shows that the MPV states of distinct normal blocks are non-proportional at arbitrarily large lengths; this is an input to the exact sector matching later in the chapter.

Theorem 9.4.3 (Non-equivalent irreducible TP blocks are eventually non-proportional). [Lean](#)

[Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving blocks whose self-overlaps converge to 1. If they are not gauge-phase equivalent, then for every lower bound N_0 there is $N \geq N_0$ such that $|V^{(N)}(A)\rangle$ is not proportional to $|V^{(N)}(B)\rangle$. Consequently, in an already separated same-dimensional BNT block family, any two distinct blocks have such non-proportional MPV states at arbitrarily large lengths.*

Proof. [✓ Proof](#) Suppose that for all sufficiently large N one had $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. Then

$$O_{AA}(N) = |c_N|^2 O_{BB}(N), \quad (9.5)$$

$$O_{AB}(N) = c_N O_{BB}(N). \quad (9.6)$$

Since $O_{AA}(N) \rightarrow 1$ and $O_{BB}(N) \rightarrow 1$, the identity (9.5) gives $|c_N| \rightarrow 1$, and hence (9.6) gives $|O_{AB}(N)| \rightarrow 1$. The irreducible trace-preserving overlap decay theorem gives $O_{AB}(N) \rightarrow 0$ when $A \not\sim_{\text{gp}} B$, a contradiction. $\square$

9.5 Newton–Girard identities and power-sum recovery

The coefficients $c_N^{(j)} = \sum_q \mu_{j,q}^N$ that appear in the two-layer BNT expansion are power sums of the copy weights. Recovering the individual weights from these sums is a Newton–Girard problem: equal power sums determine equal characteristic polynomials, hence equal multisets of roots.

The general matrix statement is the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”]. It identifies a characteristic polynomial from the first n power-sum traces.

The matched-sector comparison below applies the power-sum identities to a single matched pair, after the BNT permutation and phase gauges have identified which basis tensors are compared: the coefficient $\sum_q \mu_{j,q}^N$ then equals $\sum_q (\nu_{j,q} e^{i\phi_j})^N$, and Newton–Girard recovers the weight multiset with multiplicity.

Theorem 9.5.1 (Finite-range equal power sums imply equal multisets). [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#) *Let $\alpha, \beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$. If $\sum_i \alpha_i^k = \sum_i \beta_i^k$ for $1 \leq k \leq n$, then the multisets $\{\alpha_i\}$ and $\{\beta_i\}$ are equal.*

Two variants relax the equal-cardinality assumption. If $\alpha : \{0, \dots, m-1\} \rightarrow \mathbb{C}$ and $\beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$ have no zero entries and their power sums agree for $1 \leq k \leq \max(m, n)$, then $m = n$ and the multisets are equal. Without a nonzero hypothesis, the list form $\sum_{k=1}^{x_a} \lambda_{a,k}^N = \sum_{k=1}^{x_b} \lambda_{b,k}^N$ for $1 \leq N \leq \max(x_a, x_b)$ gives equality of the nonzero multisets; if it also holds at $N = 0$ then $x_a = x_b$ and the full lists agree. The nonzero variant is the one applied to the nonzero coefficients of a canonical form, which is how [CPGSV16, Lemma A.5] is used.

Proof. [✓ Proof](#) Apply the finite-range Newton–Girard theorem to $\text{diag}(\alpha)$ and $\text{diag}(\beta)$ and extract the roots of $\prod_i (X - \alpha_i)$ and $\prod_i (X - \beta_i)$. For unequal cardinalities, pad both lists with zeros to length $\max(m, n)$; the nonzero hypothesis makes the padded-zero counts $\max(m, n) - m$ and $\max(m, n) - n$, forcing $m = n$. Without that hypothesis, delete the zero entries first (unchanged for positive N); the equality at $N = 0$ gives $x_a = x_b$. $\square$

Corollary 9.5.2 (A nonzero finite family has a nonvanishing power sum). [Lean](#) [✓ Stmt](#) *Let $m \geq 1$ and let $\lambda_0, \dots, \lambda_{m-1} \in \mathbb{C}$ be nonzero. Then there is an exponent N with $1 \leq N \leq m$ such that $\sum_{q=0}^{m-1} \lambda_q^N \neq 0$. This is the consequence of the Appendix power-sum lemma used in the proof of Lemma L [CPGSV16, Appendix C.3, lines 1851–1858]; the finite-range comparison is [CPGSV16, Appendix A, lines 1155–1163].*

The copy retains the original coefficient μ_k , and the representatives $(C_j)_j$ form a basis of normal tensors for A .

Enumerate $\mathcal{G}$. The induced equivalence Φ between the dependent grouped block coordinates and the original flattened listed coordinates gives

$$\sum_{x \in \mathcal{G}} D_x = \sum_{k=1}^r D_k. \quad (9.13)$$

For $x = (j, q)$, let $\hat{\mu}_x$ be the original weight of that copy and let $\widehat{B}_x^i = \zeta_x C_j^i$, with the required dimension identification. Define the grouped tensor by

$$T_{\text{grp}}^i = \text{reindex}_{\Phi} \left(\bigoplus_{x \in \mathcal{G}} \hat{\mu}_x \widehat{B}_x^i \right). \quad (9.14)$$

If $G = \bigoplus_k X_k$ is the block-diagonal gauge, then, letter by letter,

$$\bigoplus_{k=1}^r \mu_k A_k^i = G T_{\text{grp}}^i G^{-1}, \quad (9.15)$$

and

$$A^i = U^\dagger G T_{\text{grp}}^i G^{-1} U. \quad (9.16)$$

Proof. ✓ Proof Enumerate the block indices by their phase classes and choose one normal representative in each class. The normal-tensor phase theorem gives the dimension equality, unit phase, and invertible gauge in (9.12). The BNT characterization proves that the representatives form a basis of normal tensors. Reindexing the dependent block coordinates gives Φ and (9.13). The exact heterogeneous block-diagonal reindexing identity identifies the grouped tensor with the representative copies in the original retained-coordinate space. Block-diagonal conjugation then gives (9.15); substitution into (9.9) gives (9.16) with the original coisometry U . In the notation of [CPGSV16, Proposition 4.13], these identities are

$$\bigoplus_{k=1}^r \mu_k A_k^i = G T_{\text{grp}}^i G^{-1}, \quad (9.17)$$

$$A^i = U^\dagger G T_{\text{grp}}^i G^{-1} U, \quad (9.18)$$

$$U U^\dagger = \mathbb{1}. \quad (9.19)$$

□

Definition 9.6.1.10 (BNT sector presentation). Lean Lean Lean Lean Lean ✓ Stmt A *BNT sector presentation* of a tensor A is a nonempty sector decomposition P such that every representative has a positive number of copies, every copy has nonzero weight, every representative C_j is normal, the representative matrix product vectors are eventually linearly independent, and, for every $N > 0$,

$$V^{(N)}(A) = V^{(N)}(P). \quad (9.20)$$

This records the nonzero canonical summands constructed at [CPGSV16, lines 217–246]: repeated copies and their weights remain explicit.

Theorem 9.6.1.11 (Uniqueness of BNT presentations). [Lean](#) [✓ Stmt](#) *Two BNT presentations of the same positive-length matrix product vector family have equally many representatives. After a permutation, corresponding representatives have equal bond dimension and are related by an invertible gauge and a unit-modulus phase.*

Proof. [✓ Proof](#) Apply Theorem 9.1.4 first to the direct-sum tensor of the first presentation, using the representatives of the second as a BNT, and then with the presentations interchanged. This gives a matching map in each direction. Eventual linear independence excludes gauge-phase-equivalent distinct representatives, so both maps are injective. Since the index sets are finite, they have equal cardinality, and either matching map is a bijection. Normal-tensor gauge rigidity gives the asserted gauge-phase relation for each matched pair. $\square$

Theorem 9.6.1.12 (Representative sector decomposition). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *In the notation of Theorem 9.6.1.9, let $k(j, q)$ be the listed block corresponding to the class-copy pair (j, q) . The representatives C_j define a sector decomposition P_{rep} with copy weights*

$$\lambda_{j,q} = \nu_{k(j,q)} \zeta_{k(j,q)}. \quad (9.21)$$

Every copy weight in P_{rep} is nonzero. The representatives have simultaneous word-tuple separation at every sufficiently large length.

For every positive length N , the grouped tensor T_{grp} of Theorem 9.6.1.9 and the representative sector tensor generate the same matrix product vector. The original tensor has the same positive-length matrix product vectors as both:

$$V^{(N)}(T_{\text{grp}}) = V^{(N)}(P_{\text{rep}}). \quad (9.22)$$

$$V^{(N)}(A) = V^{(N)}(P_{\text{rep}}). \quad (9.23)$$

Hence P_{rep} is a BNT sector presentation of A whenever at least one listed block occurs.

Proof. [✓ Proof](#) The class-copy equivalence reindexes the listed blocks. (9.21) combines the original coefficient with the unit phase relating its block to C_j ; both factors are nonzero. The basis-of-normal-tensors condition and Theorem 9.3.4 give eventual simultaneous word-tuple separation for the representatives. Reindexing the finite sum by (j, q) gives (9.22). The original coisometric reconstruction and simultaneous block gauge preserve every positive-length closed-chain coefficient, yielding (9.23). $\square$

Definition 9.6.1.13 (BNT canonical form). [Lean](#) [✓ Stmt](#) *Given a sector decomposition P , these conditions define the normalized, left-canonical, aperiodic BNT surface used below. This is stronger than the abstract BNT of Definition 9.1.1 and the literal coefficient-form BNT of Definition 9.1.2: besides spanning and eventual linear independence, it fixes irreducible left-canonical representatives, their self-overlap normalization, gauge-phase separation, and the canonical-form weight normalization. It imposes no equal-modulus factorization or strict ordering on the raw weights. Explicitly, each basis block has positive bond dimension, is irreducible, and is left-canonical; the normalized self-overlap of every basis block converges to 1; the basis MPV family is eventually linearly independent; distinct same-dimension basis blocks are not gauge-phase equivalent after identifying equal bond dimensions; and the [CPGSV16, Section 2.3] canonical-form normalization holds,*

$$\begin{aligned} \forall (j, q), \quad & |\mu_{j,q}| \leq 1, \\ \exists (j_*, q_*), \quad & |\mu_{j_*,q_*}| = 1. \end{aligned}$$

The MPV expansion takes the raw two-layer form

$$V^{(N)}(P)_\sigma = \sum_{j=1}^g \left(\sum_{q=1}^{r_j} \mu_{j,q}^N \right) V^{(N)}(P_j)_\sigma, \quad (9.24)$$

matching [CPGSV16, Section 2.3] and [CPGSV21, Definition IV.2]. The canonical-form normalization admits every example of the source paper (in particular $C \oplus D$, $C \oplus (-C)$, $C \oplus (1/2)C$, and $C \oplus (-C) \oplus (1/2)C$) and is not derivable from the per-block normality hypotheses alone.

Definition 9.6.1.14 (Single-sector decomposition). [Lean](#) [✓ Stmt](#) For a tensor A , its single-sector decomposition has A as its only basis tensor, with one copy of weight 1.

Theorem 9.6.1.15 (Single-sector BNT canonical form). [Lean](#) [✓ Stmt](#) Let A have positive bond dimension, be irreducible and left-canonical, and satisfy $O_{AA}(N) \rightarrow 1$. Then its single-sector decomposition is a BNT canonical form.

Proof. [✓ Proof](#) The identity

$$O_{AA}(N) = \|V^{(N)}(A)\|^2 \rightarrow 1 \neq 0. \quad (9.25)$$

shows that $V^{(N)}(A)$ is nonzero for all sufficiently large N . A family consisting of one nonzero vector is linearly independent. All remaining conditions are immediate from the hypotheses and the unit weight. $\square$

Theorem 9.6.1.16 (Normal tensors have single-sector BNT presentations). [Lean](#) [✓ Stmt](#) Every normal tensor is gauge equivalent to a tensor whose single-sector decomposition is a BNT canonical form. The original tensor and the total tensor of this decomposition generate the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Choose the pure trace-preserving Perron gauge. Its tensor is irreducible and left-canonical, and primitivity gives self-overlap converging to one. The preceding theorem supplies the BNT canonical form. If P denotes the total tensor of its single-sector decomposition, then gauge invariance and the sector-decomposition formula give, for every $N > 0$ and word σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}(B)_\sigma = 1^N V^{(N)}(B)_\sigma = V^{(N)}(P)_\sigma. \quad (9.26)$$

$\square$

Lemma 9.6.1.17 (Unit-modulus weight witness from the canonical-form normalization). [Lean](#) [✓ Stmt](#) Under the BNT canonical form, the [CPGSV16, Section 2.3] normalization provides indices (j_*, q_*) with $|\mu_{j_*, q_*}| = 1$.

Proof. [✓ Proof](#) This is the unit-modulus existential in the canonical-form normalization. $\square$

Corollary 9.6.1.18 (Basis blocks of a BNT canonical form are normal). [Lean](#) [✓ Stmt](#) If P is in BNT canonical form, then every basis block P_j is normal in the algebraic sense of Definition 2.4.4.

Proof. [✓ Proof](#) Each basis block has positive bond dimension, is irreducible and left-canonical, and has normalized self-overlap converging to 1. Theorem 6.9.3 applies to each block. $\square$

Theorem 9.6.1.19 (A BNT canonical form determines an algebraic BNT). [Lean](#) [✓ Stmt](#) If P is in BNT canonical form, then its basis blocks form an algebraic basis of normal tensors for the total tensor associated with P . Only this implication from canonical form to algebraic BNT is asserted.

Proof. ✓ **Proof** The preceding corollary gives normality of every basis block. The sector coefficient formula (9.8) expresses the total MPV as a linear combination of the basis MPVs at each positive length, and the BNT sector data give their eventual linear independence. $\square$

Theorem 9.6.1.20 (Positive blocking preserves BNT canonical form). Lean ✓ **Stmt** *Let P be a BNT canonical form and let $p \geq 1$. Then the blocked sector decomposition $P^{[p]}$ is again a BNT canonical form.*

Proof. ✓ **Proof** For each representative A_j , normalized self-overlap gives peripheral period one, hence a primitive transfer map. Positive blocking therefore preserves irreducibility and left-canonical normalization. Moreover, $O_{A_j^{[p]}A_j^{[p]}}(N) = O_{A_jA_j}(Np) \rightarrow 1$. Let F_N be the linear isomorphism induced by concatenating blocked words, from the blocked configuration space at length N to the original one at length Np . For every representative, $F_N|V^{(N)}(A_j^{[p]})\rangle = |V^{(Np)}(A_j)\rangle$. Hence eventual linear independence is preserved. If two blocked representatives were gauge-phase equivalent, then for some nonzero ζ and every N , $|V^{(N)}(A_k^{[p]})\rangle = \zeta^N|V^{(N)}(A_j^{[p]})\rangle$. Writing these two vectors as f_k and f_j gives $1 \cdot f_k = \zeta^N \cdot f_j$ in an eventually linearly independent family, and therefore $k = j$. Finally, $|\mu_{j,q}^p| = |\mu_{j,q}|^p \leq 1$, and a unit-modulus weight remains of unit modulus after taking its p -th power. $\square$

Theorem 9.6.1.21 (Common exact quantum-Wielandt length for BNT representatives). Lean ✓ **Stmt** *Let P be a BNT canonical form of total bond dimension D , with representatives A_j of bond dimensions D_j . For every $0 \leq j < g$, the representative A_j is D^4 -block injective: $S_{D^4}(A_j) = M_{D_j}(\mathbb{C})$.*

Proof. ✓ **Proof** The BNT hypotheses make A_j irreducible and left-canonical, with $O_{A_jA_j}(N) \rightarrow 1$. The preceding theorem therefore makes A_j normal, and quantum Wielandt gives $S_{D_j^4}(A_j) = M_{D_j}(\mathbb{C})$. By Theorem E.2.6, $D_j^4 \leq D^4$. Write $D^4 = D_j^4 + k$ and apply positive-length propagation k times. $\square$

Theorem 9.6.1.22 (CPSV simultaneous block-injectivity bound). Lean Lean ✓ **Stmt** *Let P be a BNT canonical form with total bond dimension D . There is a positive integer $N \leq 3D^5$ such that its representatives have the simultaneous fixed-length span*

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = N\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the block-injectivity estimate in [CPGSV16, lines 340–345].

Proof. ✓ **Proof** Every representative is block-injective at D^4 , and homogeneous-span propagation gives the same conclusion at $D^4 + 1$ and $3(D^4 + 1)$. If $g = 1$, take $N = D^4$. If $g \geq 2$, apply Lemma 9.3.3 with $L_0 = D^4$ and take

$$N = 3(g - 1)(D^4 + 1).$$

The positive bond dimension of every representative gives $g \leq D$. Consequently $N \leq 3D^5$; the same estimate also covers the one-block case. The unit-modulus weight required by the BNT normalization ensures $D > 0$. $\square$

The matching arguments below read off three facts from a BNT canonical form: the raw coefficient identity, the cross-overlap decay between distinct basis blocks, and the combined-family linear independence.

Lemma 9.6.1.23 (Raw two-layer sector coefficient identity). Lean ✓ Stmt For every sector decomposition P , length N , and basis index j , $c_N^{(j)} = \sum_{q=1}^{r_j} \mu_{j,q}^N$. This is [CPGSV16, Section 2.3].

Proof. ✓ Proof The claim follows from the definition of the sector coefficient. $\square$

Lemma 9.6.1.24 (Cross-overlap decay between distinct basis blocks). Lean ✓ Stmt If P is a BNT canonical form and $j \neq k$, then $O_{P_j P_k}(N) \rightarrow 0$.

Proof. ✓ Proof Case on the bond dimensions. If they differ, the dimension-mismatch overlap-decay theorem applies. If they agree, the basis-distinctness clause rules out gauge-phase equivalence, so the equal-dimension overlap-decay theorem applies. $\square$

Lemma 9.6.1.25 (Combined-family eventual linear independence). Lean ✓ Stmt Let P and Q each be a BNT canonical form, and suppose every mixed overlap $O_{P_j Q_k}(N)$ tends to 0. Then the union family

$$\{|V^{(N)}(P_j)\}\}_{j=1}^{g_P} \cup \{|V^{(N)}(Q_k)\}\}_{k=1}^{g_Q}$$

is linearly independent for all sufficiently large N . This is [CPGSV16, Appendix A].

Proof. ✓ Proof Feed the per-family normalized self-overlaps, the within-family cross-overlap decay (Lemma 9.6.1.24), and the inter-family decay hypothesis into the two-family overlap-orthonormality criterion (Lemma E.1.9). $\square$

Example 9.6.1.26 ($C \oplus (1/2)C$, unequal-modulus admissibility). Lean ✓ Stmt For any tensor C , this construction forms a single-sector, two-copy decomposition with basis tensor C and raw weights $(1, 1/2)$. Thus the sector data themselves allow unequal-modulus copies. If C additionally has positive bond dimension, is irreducible and left-canonical, has normalized nonzero self-overlap, and satisfies the BNT independence clause, these data satisfy Definition 9.6.1.13; those hypotheses are not consequences of the construction alone.

9.6.2 Single-copy sector decompositions

A weighted block sum is the one-copy case of the sector decomposition used below. Equal or repeated sectors require the full coefficient formalism.

9.6.3 Prepared-block construction of BNT canonical forms

The construction below separates the after-blocking preparation of suitable blocks from the normalized-weight BNT construction: the prepared blocks are trace-preserving, primitive, irreducible, and injective; normalized weights then quotient gauge-phase-equivalent blocks into sectors to produce the canonical form of Definition 9.6.1.13.

Theorem 9.6.3.1 (Prepared BNT blocks after blocking). Lean ✓ Stmt Let A be an MPS tensor. Then there is a blocking length $p \geq 1$, nonzero weights μ_k , and blocks B_k over the p -blocked alphabet such that:

1. each block has positive bond dimension;
2. each block is trace-preserving, primitive, tensor-irreducible, and one-site injective;
3. $A^{[p]}$ and $\bigoplus_k \mu_k B_k$ generate the same MPV family at every positive length.

No weight-modulus normalization is asserted at this stage.

Proof. ✓ Proof Apply the common primitive irreducible block decomposition to (A, A) , giving a positive blocking length and a nonzero weighted family of trace-preserving primitive irreducible blocks with the positive-length MPV identity. Theorem E.3.2 supplies one common further blocking making every block one-site injective while preserving the other hypotheses; finally identify the iterated blocked alphabet with the direct one. $\square$

With the prepared blocks in hand, normalized weights produce a BNT canonical form by quotienting gauge-phase-equivalent blocks into sectors.

Theorem 9.6.3.2 (BNT canonical form of the phase-class quotient). Lean ✓ Stmt Let $(B_k)_{k=1}^r$ have positive bond dimensions and suppose that every B_k is trace-preserving, primitive, and irreducible. Assume $\mu_k \neq 0$ and $|\mu_k| \leq 1$ for every k , and $|\mu_{k_*}| = 1$ for some k_* . Partition the blocks into matrix-product-vector phase classes, choose one representative from each class, and retain every class member as a copy of its representative with the corresponding phase-adjusted weight. The resulting sector decomposition is in BNT canonical form.

Proof. ✓ Proof The chosen representatives inherit irreducibility and trace preservation, while primitivity gives normalized self-overlap. Distinct phase classes give gauge-phase-distinct representatives. The collapsed phase-class construction supplies the eventual linear independence and retains every original block as one copy. Its phase multipliers have unit modulus, so the adjusted weights preserve the assumed bounds and unit witness. These are precisely the fields of Definition 9.6.1.13. $\square$

Theorem 9.6.3.3 (BNT from normal blocks with normalized weights). Lean ✓ Stmt Let $r \in \mathbb{N}$, and for each $k = 1, \dots, r$ let $D_k \in \mathbb{N}$, $\mu_k \in \mathbb{C}$, and let B_k be an MPS tensor of physical dimension d and bond dimension D_k . Assume:

1. **positive bond dimensions:** $D_k > 0$ for all k ,
2. **TP normalization:** $\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$,
3. **primitivity:** $\mathcal{E}_{B_k}$ is primitive for each k ,
4. **irreducibility:** each B_k is irreducible,
5. **nonzero weights:** $\mu_k \neq 0$ for all k ,
6. **bounded weights:** $|\mu_k| \leq 1$ for all k ,
7. **global unit witness:** $|\mu_{k_*}| = 1$ for some k .

Then there exists a sector decomposition P such that P and $\bigoplus_{k=1}^r \mu_k B_k$ generate the same MPV family at every positive length, and P satisfies the basis-of-normal-tensors conditions (Definition 9.1.2) together with the multiplicity, span, and weight-bound conditions of Definition 9.6.1.13.

Proof. ✓ Proof The phase-class quotient groups gauge-phase-equivalent blocks into sectors with unit-modulus per-copy scalars. Under the TP, primitive, and irreducible hypotheses the phase-class construction applies, and for distinct representatives $B_j, B_{j'}$ ($j \neq j'$) the cross-overlap decays,

$$\begin{aligned} \lim_{N \rightarrow \infty} O_{B_j B_{j'}}(N) &= 0, \\ \lim_{N \rightarrow \infty} O_{B_j B_j}(N) &= 1, \end{aligned}$$

which gives the eventual linear-independence condition with no injectivity required. The resulting sector decomposition inherits the span and multiplicity conditions, and the bounded-weight and global-unit hypotheses give the weight-norm conditions. $\square$

The BNT construction preserves total bond dimension because CF summands assigned to the same BNT representative have equal bond dimension. Indeed, suppose trace-preserving primitive irreducible NTs X and Y of positive bond dimension have MPV families related by $|V^{(N)}(Y)\rangle = \zeta^N |V^{(N)}(X)\rangle$ with $\zeta \neq 0$. The normalized self-overlaps give $|O_{XX}(N)| \rightarrow 1$. The phase relation gives the cross-overlap identity

$$\begin{aligned} O_{YX}(N) &= \zeta^N O_{XX}(N), \\ |O_{YX}(N)| &= |\zeta|^N |O_{XX}(N)|. \end{aligned}$$

Since the same normalization forces $|\zeta| = 1$, this gives $|O_{YX}(N)| \rightarrow 1$. If their bond dimensions differed, the rectangular overlap-decay theorem would give $O_{YX}(N) \rightarrow 0$, a contradiction.

Theorem 9.6.3.4 (Gauge realization of prepared phase-equivalent NTs). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimensions. If their matrix-product-vector families differ by a nonzero scalar power, then their bond dimensions are equal and, after identifying the bond spaces, X and Y are gauge-phase equivalent. Neither positivity of the scalar nor unitarity of the gauge is asserted.*

Proof. [✓ Proof](#) Lemma E.3.4 identifies the bond dimensions. If the resulting same-dimensional blocks were not gauge-phase equivalent, Theorem 9.4.3 would give a positive length at which their matrix product vectors are not proportional. This contradicts the assumed scalar-power relation. The normalized self-overlaps required there follow from Theorem C.5.2. $\square$

Lemma 9.6.3.5 (Gauge realization with the fixed phase scalar). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimensions. Fix $\zeta \in \mathbb{C} \setminus \{0\}$ and suppose that, for every positive length N and every physical word σ ,*

$$V^{(N)}(Y)_\sigma = \zeta^N V^{(N)}(X)_\sigma. \quad (9.27)$$

Then, after identifying the bond spaces, there exists $G \in \text{GL}_{D_Y}(\mathbb{C})$ such that, for every physical index i ,

$$Y^i = \zeta G X^i G^{-1}. \quad (9.28)$$

Proof. [✓ Proof](#) Theorem 9.6.3.4 gives an invertible G and a nonzero scalar η such that $Y^i = \eta G X^i G^{-1}$. By Theorem 2.3.18, $V^{(N)}(Y)_\sigma = \eta^N V^{(N)}(X)_\sigma$. Theorem C.5.2 gives consecutive positive lengths M and $M+1$ with nonzero coefficients of $V^{(M)}(X)$ and $V^{(M+1)}(X)$. Comparing the two scalar-power identities gives $\zeta^M = \eta^M$ and $\zeta^{M+1} = \eta^{M+1}$. Hence $\zeta = \eta$, which proves (9.28). $\square$

Theorem 9.6.3.6 (Dimension-preserving BNT form of prepared blocks). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 9.6.3.2, there is a BNT canonical-form sector decomposition P such that*

$$\begin{aligned} \mathcal{V}^+(P) &= \mathcal{V}^+ \left(\bigoplus_{k=1}^r \mu_k B_k \right), \\ D_{\text{tot}}(P) &= \sum_{k=1}^r D_k. \end{aligned} \quad (9.29)$$

Proof. ✓ Proof Use the phase-class sector decomposition from Theorem 9.6.3.2. Its flattened copies are in bijection with the original blocks, and phase-equivalent blocks have equal bond dimension by Theorem 9.6.3.4. Summing the copy dimensions over this bijection gives (9.29). $\square$

Theorem 9.6.3.7 (Exact reindexing of heterogeneous weighted block sums). Lean Lean ✓ Stmt Let $e : \{1, \dots, r\} \simeq \{1, \dots, r'\}$ satisfy $D_k = D'_{e(k)}$. Let Q_k be the coordinate permutation whose rows are indexed by the primed block and whose columns are indexed by the unprimed block. Suppose that, for every block and physical index,

$$\mu_k A_k^i = Q_k^\dagger \left(\mu'_{e(k)} B_{e(k)}^i \right) Q_k. \quad (9.30)$$

The induced equivalence E of flattened dependent block coordinates has a permutation matrix Q_E , with rows indexed by the primed coordinates and columns indexed by the unprimed coordinates. For every physical index i ,

$$\bigoplus_{k=1}^r \mu_k A_k^i = Q_E^\dagger \left(\bigoplus_{\ell=1}^{r'} \mu'_\ell B_\ell^i \right) Q_E. \quad (9.31)$$

Proof. ✓ Proof Evaluate both sides on two flattened block coordinates. When their block labels agree, (9.30) gives the matrix entry. When the labels differ, both block-diagonal entries vanish. This proves (9.31). $\square$

Theorem 9.6.3.8 (Exact BNT reconstruction of prepared normal blocks). Lean ✓ Stmt Under the hypotheses of Theorem 9.6.3.2, there are a BNT canonical-form sector decomposition P , an equivalence E of flattened bond coordinates, and $X \in \text{GL}_{D_{\text{tot}}(P)}(\mathbb{C})$. Write Q_E for the permutation matrix whose rows are indexed by the coordinates of P and whose columns are indexed by the original block coordinates. The first identity below holds, and the second holds for every physical index i :

$$D_{\text{tot}}(P) = \sum_{k=1}^r D_k, \quad (9.32)$$

$$\bigoplus_{k=1}^r \mu_k B_k^i = Q_E^\dagger (X P^i X^{-1}) Q_E. \quad (9.33)$$

Proof. ✓ Proof Partition the prepared blocks into matrix-product-vector phase classes. For the copy q in class j , retain the precise class scalar $\zeta_{j,q}$ chosen in the weight $\zeta_{j,q} \mu_{j,q}$. The normal-tensor phase theorem supplies an invertible $G_{j,q}$ with

$$B_{j,q}^i = \zeta_{j,q} G_{j,q} C_j^i G_{j,q}^{-1}. \quad (9.34)$$

Thus the original weighted block equals the corresponding sector copy after conjugation, since its sector weight is $\zeta_{j,q} \mu_{j,q}$. Assemble the $G_{j,q}$ into a block-diagonal matrix X and apply Theorem 9.6.3.7. This gives (9.33); summing the matched copy dimensions gives (9.32). $\square$

Lemma 9.6.3.9 (Gauge equivalence of direct sums from per-block gauge equivalence). Lean ✓ Stmt If each pair (A_k, B_k) is gauge equivalent, then the weighted block-diagonal tensors $\bigoplus_k \mu_k A_k$ and $\bigoplus_k \mu_k B_k$ are gauge equivalent.

Proof. ✓ Proof Choose per-block gauge matrices via the hypothesis; then apply Lemma F.4.3. $\square$

Theorem 9.6.3.10 (Positive-length MPVs under coisometric reconstruction). [Lean](#) [✓ Stmt](#) Let A and B be MPS tensors, possibly with different bond dimensions. Suppose that U satisfies $UU^\dagger = \mathbb{1}$ and $A^i = U^\dagger B^i U$ for every physical index i . Then A and B generate the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Multiply the reconstruction identities around a positive-length cycle. Each adjacent pair $UU^\dagger$ cancels, and cyclicity of the trace removes the remaining outer factors. $\square$

Theorem 9.6.3.11 (Sector BNT form of normalized CPSV canonical data). [Lean](#) [✓ Stmt](#) Let A have CPSV canonical-form data satisfying the separate normalization condition (8.25), and assume $A \neq 0$. Then there is a BNT canonical-form sector decomposition P such that

$$\begin{aligned} \mathcal{V}^+(A) &= \mathcal{V}^+(P), \\ D_{\text{tot}}(P) &= \sum_{k=1}^r D_k. \end{aligned} \tag{9.35}$$

Thus the dimension is the total dimension of the blocks, not the ambient bond dimension D . This statement records positive-length MPV equality; the exact coisometric reconstruction is stated next.

Proof. [✓ Proof](#) Apply Theorem 9.6.3.12. Its reconstruction writes $A^i = U^\dagger B^i U$, where $UU^\dagger = \mathbb{1}$ and $B^i = X P^i X^{-1}$. Theorem 9.6.3.10 shows that A and B generate the same matrix product vectors at every positive length. Theorem 2.3.17 gives the same conclusion for B and P . Transitivity yields $\mathcal{V}^+(A) = \mathcal{V}^+(P)$, while the dimension identity is already part of the exact reconstruction theorem. $\square$

Theorem 9.6.3.12 (Exact coisometric BNT reconstruction). [Lean](#) [✓ Stmt](#) Let A have CPSV canonical-form data satisfying the separate normalization condition (8.25), and assume $A \neq 0$. There are a BNT canonical-form sector decomposition P , a coisometry $U \in \mathbb{C}^{D_{\text{tot}}(P) \times D}$, and $X \in \text{GL}_{D_{\text{tot}}(P)}(\mathbb{C})$. The first two identities below hold, and the third holds for every physical index i :

$$D_{\text{tot}}(P) = \sum_{k=1}^r D_k, \tag{9.36}$$

$$UU^\dagger = \mathbb{1}, \tag{9.37}$$

$$A^i = U^\dagger (X P^i X^{-1}) U. \tag{9.38}$$

Proof. [✓ Proof](#) The canonical-form data give the exact reconstruction

$$A^i = V^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) V, \tag{9.39}$$

$$VV^\dagger = \mathbb{1}. \tag{9.40}$$

Put every normal block into a trace-preserving primitive irreducible gauge Y , and apply Theorem 9.6.3.8. Let Q be the permutation matrix of its flattened coordinate equivalence, so $QQ^\dagger = \mathbb{1}$. Reindexing Y by the same equivalence gives an invertible matrix W on the bond space of P and the identities

$$QY = WQ, \tag{9.41}$$

$$Y^{-1}Q^\dagger = Q^\dagger W^{-1}. \tag{9.42}$$

If X_0 is the prepared-block gauge, set $X = W^{-1}X_0$ and $U = QV$. Substitution in (9.39) gives (9.38), while

$$UU^\dagger = Q(VV^\dagger)Q^\dagger = \mathbb{1}.$$

This proves (9.37). The block enumeration gives (9.36). $\square$

Theorem 9.6.3.13 (Equal-ambient gauge equivalence for CPSV canonical forms). [Lean](#) [Stmnt](#) *Let A and B have the same ambient bond dimension D . Suppose each tensor has CPSV canonical-form data, both data satisfy the separate weight normalization condition (8.25), neither tensor is zero, and $\mathcal{V}^+(A) = \mathcal{V}^+(B)$. Then A and B are gauge equivalent: there is $G \in \text{GL}_D(\mathbb{C})$ such that, for every physical index i ,*

$$B^i = GA^iG^{-1}. \quad (9.43)$$

Unused ambient coordinates are allowed. The conclusion requires the common ambient dimension D and makes no claim for canonical forms with different ambient bond dimensions.

The cited sources supply the retained canonical-form comparison. The passage back from the retained tensors to the ambient tensors is the coisometric lifting of Theorem 8.8.5.

Proof. [Stmnt](#) Apply Theorem 9.6.3.12 to obtain BNT canonical forms P, Q , coisometries U_A, U_B , and retained-space gauges X_A, X_B with

$$\begin{aligned} A^i &= U_A^\dagger X_A P^i X_A^{-1} U_A, \\ B^i &= U_B^\dagger X_B Q^i X_B^{-1} U_B. \end{aligned} \quad (9.44)$$

Theorem 9.6.3.10, gauge invariance, and the hypothesis give

$$\mathcal{V}^+(P) = \mathcal{V}^+(A) = \mathcal{V}^+(B) = \mathcal{V}^+(Q). \quad (9.45)$$

Theorem 10.4.2.1 therefore identifies the retained dimensions and gives Y such that $Q^i = YP^iY^{-1}$. After identifying the retained spaces, the two middle tensors in (9.44) are conjugate by $X_B Y X_A^{-1}$. Theorem 8.8.5 aligns the two coisometries inside the common ambient space and yields (9.43). $\square$

Combining the after-blocking preparation with the weight normalization of Theorem E.3.7 gives the arbitrary-input statement.

Theorem 9.6.3.14 (Normalized BNT form after blocking for a nonzero MPV family). [Lean](#) [Stmnt](#) *Let A be an MPS tensor whose matrix-product family has a nonzero coefficient at some positive length. Then there exist a blocking length $p \geq 1$, a positive real m , and a sector decomposition P over the blocked alphabet in the BNT canonical form of Definition 9.6.1.13, such that for every positive length N ,*

$$V^{(N)}(A^{[p]}) = m^N V^{(N)}(P). \quad (9.46)$$

The scalar m^N is the per-site normalization cost of the line-246 weight choice, which is realized rather than assumed. The nonzero-family hypothesis is the standing convention of [CPGSV16, Section 2.3]: the unit-modulus weight chosen there presupposes a nonempty weight family. The normalization and the proportional-MPV comparison are documented in [con26h].

Proof. ✓ Proof Choose a word w with $\text{tr}(A^w) \neq 0$. For every $p \geq 1$, there is a $q \geq 1$ such that $\text{tr}((A^w)^{pq}) \neq 0$. Otherwise Newton–Girard applied to $(A^w)^p$ and the zero matrix would show that $(A^w)^p$, and hence A^w , is nilpotent, contradicting $\text{tr}(A^w) \neq 0$. Repeating w exactly pq times gives a nonzero coefficient of the p -blocked tensor. Thus the prepared family at the blocking length supplied by Theorem 9.6.3.1 is nonempty.

Normalize its weights by their maximal modulus m (Theorem E.3.7). By the weight-scaling identity of Lemma E.3.8,

$$\begin{aligned} V^{(N)}(A^{[p]}) &= V^{(N)}\left(\bigoplus_k \mu_k B_k\right) = V^{(N)}\left(\bigoplus_k m \frac{\mu_k}{m} B_k\right), \\ &= m^N V^{(N)}\left(\bigoplus_k \frac{\mu_k}{m} B_k\right) = m^N V^{(N)}(P). \end{aligned}$$

where the last identity is the prepared supplier of Theorem 9.6.3.3 applied to the normalized weights, which gives the BNT canonical form P . $\square$

9.7 Coefficient comparison and copy-weight recovery

The exact matching of Section 9.8 produces, on each matched pair, a coefficient identity. This section turns that identity into equality of the per-copy weight multisets [CPGSV16, Appendix A]. No analytic non-decay estimate enters: the fixed-length linear independence isolates each sector coefficient exactly.

9.7.1 Matched-sector weight multiset equality

Given a matched pair $(j_0, \tilde{k}_0)$ and a nonzero scalar ζ with $c_N^{(j_0)}(P) = \zeta^N c_N^{(\tilde{k}_0)}(Q)$ past a threshold N_0 , this section records two conclusions: the multiset equality of the per-copy weights from power-sum recovery (Theorem E.2.5), and its upgrade to an explicit per-copy permutation. Together they give the $\mu_{j,q} = \nu_{j,q} e^{i\varphi_j}$ recovery of [CPGSV16, Appendix A]. In the equal-MPV case, Chapter 10 obtains this coefficient identity by expanding both total MPVs in the matched BNT basis and applying eventual linear independence (Lemma 10.2.2). For merely proportional total MPVs, a common length-dependent scalar remains in every matched coefficient identity and the scalar-free comparison need not hold; see Remark 10.3.6.

Lemma 9.7.1.1 (Matched-sector weight multiset equality). Lean ✓ Stmt *Let P, Q be sector decompositions with weights $\mu_{j,q}$ and $\nu_{k,q'}$, and fix indices $j_0, \tilde{k}_0$, and a nonzero scalar $\zeta \in \mathbb{C} \setminus \{0\}$. Suppose that for every $N > N_0$,*

$$c_N^{(j_0)}(P) = \zeta^N \cdot c_N^{(\tilde{k}_0)}(Q). \tag{9.47}$$

Then the per-sector multiplicities agree, $r_{j_0}(P) = r_{\tilde{k}_0}(Q)$, and the rescaled per-copy weight multisets coincide. This is [CPGSV16, Appendix A] read on a single matched sector.

Proof. ✓ Proof Expand both sides of the coefficient identity (9.47) into per-copy power sums $\sum_q \mu_{j_0,q}^N$ and $\sum_{q'} \nu_{\tilde{k}_0,q'}^N$, and distribute ζ^N through the right-hand sum to obtain, for every $N > N_0$,

$$\sum_q \mu_{j_0,q}^N = \sum_{q'} (\zeta \cdot \nu_{\tilde{k}_0,q'})^N.$$

Extend to all positive exponents and apply the unequal-cardinality multiset-recovery lemma (Theorem 9.5.1), using $\zeta \neq 0$ together with the weight non-vanishing assumption to ensure the input weights are nonzero. This yields the multiplicity match and the multiset equality. $\square$

Lemma 9.7.1.2 (Matched-sector weight-permutation extractor). Lean ✓ Stmt *Under the hypotheses of Lemma 9.7.1.1, the per-sector multiplicities agree and there is a permutation τ of the per-copy index set identifying the individual weights up to the gauge phase: for every q , $\nu_{k_0, \tau(q)} = \zeta^{-1} \cdot \mu_{j_0, q}$. This is the per-copy realisation of [CPGSV16, Appendix A]: the multiset equality from Lemma 9.7.1.1 is upgraded to a concrete indexing between the matched P - and Q -copies.*

Proof. ✓ Proof Invoke Lemma 9.7.1.1 to obtain the cardinality match and the multiset equality. From this multiset-map equality extract a permutation matching the elements pointwise (proved by induction on the cardinality). Compose with the cardinality identification and cancel ζ via $\zeta \neq 0$ to convert the gauge factor on the right of the pointwise identity to ζ^{-1} on the left. $\square$

9.8 Strong existential matching by exact linear independence

The exact matching reads [CPGSV16, Appendix A] as a statement at fixed length: equal MPV families force each basis sector of one form to pair with a basis sector of the other. The non-vanishing of the sector coefficients isolates the matched sector exactly, with no limit of a single-sector projection.

Theorem 9.8.1 (Exact single-sector matching). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family (at all positive lengths). For every sector j of P there exist a sector k of Q with matched bond dimension $\dim_P(j) = \dim_Q(k)$, a gauge-phase equivalence after identifying the matched bond dimensions, and a non-decaying cross-overlap. No per-sector unit-modulus hypothesis is required.*

Proof. ✓ Proof Equality at every positive length gives eventual nonzero proportionality with scalar 1. Apply Theorem E.2.1.1 to the chosen sector. $\square$

Theorem 9.8.2 (Strong existential matching, full-basis form). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family at every positive length. Then for every sector k of Q there exist a sector j of P with matched bond dimension $\dim_P(j) = \dim_Q(k)$, a gauge-phase equivalence after identifying the matched bond dimensions, and a non-decaying cross-overlap.*

Proof. ✓ Proof Convert equality to eventual nonzero proportionality with scalar 1 and apply Theorem E.2.1.2. $\square$

9.8.1 Bijective matching by symmetry

Applying the exact matching once more with P and Q swapped gives the reverse direction; the basis-distinctness clause forces injectivity both ways, so $g_a = g_b$ and the source relabelling $B_k = e^{i\phi_k} X_k A_{j_k} X_k^{-1}$ holds on the full basis [CPGSV16, Appendix A].

Theorem 9.8.1.1 (Bijective matching by symmetry). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family at every positive length. Then there is a bijection $\beta : J(Q) \simeq J(P)$ such that for every $k \in J(Q)$ the bond dimensions match, the basis tensors $P_{\beta(k)}$ and Q_k are gauge-phase equivalent after identifying the matched bond dimensions, and the basis cross-overlap does not tend to zero.*

Proof.  **Proof** Apply the full-basis matcher in both directions, using symmetry of the same-MPV relation. Lemma [E.2.1.3](#) converts these two families of matches into the required bijection. $\square$

Chapter 10

Proof of the Fundamental Theorem

For BNT canonical forms P and Q , write the weighted direct sums over normal-tensor sectors as $P^i = \bigoplus_{j,q} \mu_{j,q} P_j^i$ and $Q^i = \bigoplus_{k,q} \nu_{k,q} Q_k^i$. The proportional theorem matches the normal sectors by a bijection β , unit phases ζ_k , and invertible matrices X_k satisfying

$$Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}.$$

It does not determine the copy weights. In the equal-MPV case, eventual linear independence gives

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q).$$

Finite power-sum comparison then gives $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Thus the sector phases do not become equal: each phase cancels against the inverse phase in its matched copy weight. The remaining block conjugacies assemble into $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$, and a sector-coordinate permutation yields the literal global gauge.

Throughout, “eventually” means “for all sufficiently large N ”. At a fixed length in that range, BNT linear independence detects exact coefficient equalities even when a weight of modulus less than 1 makes its powers decay.

10.1 Matched block maps for product algebras

The Fundamental Theorem compares the simple matrix summands selected by a product-algebra equivalence. The following tensor-network-facing statements record the component maps and their dimension consequence.

Theorem 10.1.1 (Paired component maps for products of matrix algebras). [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#)

For a ring equivalence between products of full matrix algebras, the map induced between each pair of matched components is bijective.

Proof. [✓ Proof](#)

Apply Theorem [\[LTC26\]](#), “Paired component maps are bijections”] to the matched block ideals. □

Lemma F.1.2 applies to this MPV identity. Equivalently, the normalized self-overlaps satisfy

$$O_{Q_k Q_k}(N) = |\zeta_k|^{2N} O_{P_{\beta(k)} P_{\beta(k)}}(N),$$

and both outer sequences tend to 1, which forces $|\zeta_k| = 1$. $\square$

Lemma 10.2.2 (Full-basis coefficient identity from matched phases). [Lean](#) [✓ Stmt](#) *Let P be a BNT canonical form, let Q be a sector decomposition, and assume P_{tot} and Q_{tot} have the same MPV family at every positive length. Suppose that there is a bijection $\beta : J(Q) \simeq J(P)$ and scalars $\zeta_k \in \mathbb{C}$ such that, for every $k \in J(Q)$, every positive length N , and every spin configuration σ of length N ,*

$$V^{(N)}(Q_k)_\sigma = \zeta_k^N V^{(N)}(P_{\beta(k)})_\sigma. \quad (10.2)$$

Then for every $k \in J(Q)$ there is N_0 such that, for all $N > N_0$, $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$. This is the coefficient-comparison step of [CPGSV16, Appendix MPV proof, lines 1187–1188] in full-basis form.

Proof. [✓ Proof](#) For every N , expand the equality of total MPVs as

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = \sum_{k \in J(Q)} c_N^{(k)}(Q) |V^{(N)}(Q_k)\rangle.$$

The phase relation (10.2) gives $|V^{(N)}(Q_k)\rangle = \zeta_k^N |V^{(N)}(P_{\beta(k)})\rangle$, so reindexing the Q -sum by β onto the P -basis yields

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = \sum_{j \in J(P)} \zeta_{\beta^{-1}(j)}^N c_N^{(\beta^{-1}(j))}(Q) |V^{(N)}(P_j)\rangle.$$

For all sufficiently large N the P -basis MPV family $\{|V^{(N)}(P_j)\rangle\}_{j \in J(P)}$ is linearly independent, by the BNT eventual linear independence of P . Lemma E.1.11 then extracts the coefficient equalities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ from this eventual linear independence, for all sufficiently large N . $\square$

Definition 10.2.3 (Copy-weight matching for matched BNT sectors). [Lean](#) [✓ Stmt](#) Let the sectors of two decompositions P and Q be matched by $\beta : J(Q) \simeq J(P)$, with sector phases ζ_k . A copy-weight matching consists of copy permutations τ_k such that, for all sectors k and copies q , $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. This is the copy-level conclusion of [CPGSV16, Appendix MPV proof, line 1188].

Definition 10.2.4 (Copy-weight matching from matched-sector coefficient identities). [Lean](#) [✓ Stmt](#) The construction takes as input a sector matching $\beta : J(Q) \simeq J(P)$, nonzero sector phases ζ_k , and the eventual matched-sector coefficient identities

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q). \quad (10.3)$$

It returns the copy-weight matching obtained by choosing, in every sector, a copy permutation τ_k satisfying $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. This is the sectorwise construction in [CPGSV16, Appendix MPV proof, line 1188].

Proof. ✓ Proof Eventually the coefficient identity (10.3) in sector k reads

$$\sum_r \mu_{\beta(k),r}^N = \zeta_k^N \sum_q \nu_{k,q}^N = \sum_q (\zeta_k \nu_{k,q})^N.$$

Newton–Girard finite power-sum comparison (Lemma 9.7.1.2) applies sector by sector, producing a copy permutation τ_k with $\mu_{\beta(k),\tau_k(q)} = \zeta_k \nu_{k,q}$; since $\zeta_k \neq 0$ this is $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}$. $\square$

Theorem 10.2.5 (Equal-case sector and copy matching with differing bond dimensions). Lean

✓ Stmt *Let P and Q be BNT canonical forms, possibly with different sector counts and different bond dimensions before matching. If their total tensors generate the same MPV family at every positive length, then there is a bijection $\beta : J(Q) \simeq J(P)$ such that each matched basis pair has equal bond dimension and is gauge-phase equivalent. The matched copy multiplicities agree, and there are unit phases ζ_k and copy bijections τ_k satisfying*

$$\begin{aligned} Q_k^i &= \zeta_k X_k P_{\beta(k)}^i X_k^{-1}, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}. \end{aligned}$$

Only the global weight normalization in Definition 9.6.1.13 is assumed; no unit-modulus copy is required in each sector. This is the sector-data form of [CPGSV16, Corollary 2.11 and Appendix A, lines 1182–1188].

Proof. ✓ Proof First, Theorem 9.8.1.1 gives the sector bijection, the bond-dimension equalities, and the gauge-phase relations. Choose their phases ζ_k and use the normalized self-overlaps to obtain $|\zeta_k| = 1$. Second, Lemma 10.2.2 expands the equality of the two total MPVs in the matched P -basis. Eventual linear independence gives, for every k and all sufficiently large N ,

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q).$$

Third, expand both coefficients as finite power sums. Their eventual equality determines the two multisets of nonzero copy weights, so the multiplicities agree and a copy bijection τ_k realizes the identity $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}$. $\square$

10.3 The equal-MPV global gauge

For equal MPV families, the sector phases occur twice: in the matched tensor relation and with the inverse power in the matched copy weights. They therefore cancel before the block gauges are assembled.

Definition 10.3.1 (Flattened block-diagonal gauge). Lean Lean ✓ Stmt Given per-block gauges $X_k \in \text{GL}_{D_k}(\mathbb{C})$, form the block-diagonal element $\bigoplus_k X_k$ and then reindex the dependent block-coordinate space with the standard bond index $\{0, \dots, \sum_k D_k - 1\}$ of the assembled tensor. The result is an element of $\text{GL}_{\sum_k D_k}(\mathbb{C})$.

Theorem 10.3.2 (Global gauge from matched sectors and copy weights). Lean ✓ Stmt *Suppose that the sectors of two sector decompositions P and Q are matched by a bijection $\beta : J(Q) \simeq J(P)$, with bond-dimension equalities, copy permutations τ_k , nonzero phases ζ_k , and block gauges X_k . If, for every sector k , copy q , and physical index i ,*

$$\begin{aligned} Q_k^i &= \zeta_k X_k P_{\beta(k)}^i X_k^{-1}, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}, \end{aligned} \tag{10.4}$$

then, in the flattened Q -coordinates,

$$Q_{\text{flat}}^i = X P_{\text{matched}}^i X^{-1},$$

$$X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k).$$

This is the direct-sum gauge construction of [CPGSV16, Appendix MPV proof, lines 1189–1192], separated from the coefficient-comparison step that supplies the weight identities.

Proof. ✓ **Proof** For each matched copy,

$$\nu_{k,q} Q_k^i = (\zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}) (\zeta_k X_k P_{\beta(k)}^i X_k^{-1}) = \mu_{\beta(k), \tau_k(q)} X_k P_{\beta(k)}^i X_k^{-1}.$$

Thus every flattened weighted block is conjugated by its sector gauge X_k . Lemma F.4.3 combines these block conjugacies into the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$. $\square$

Lemma 10.3.3 (Equal-MPV matched copy-weight witnesses). Lean ✓ **Stmt** Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then there are a bijection $\beta : J(Q) \simeq J(P)$, bond-dimension equalities, unit-modulus phases ζ_k , and block gauges X_k such that $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. For the same phases there is a copy-weight matching: after a permutation of the copies inside each matched sector, $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$.

Proof. ✓ **Proof** The full-basis matching theorem gives the bijection and block gauges. The self-overlap comparison

$$\langle V_N(Q_k), V_N(Q_k) \rangle = |\zeta_k|^{2N} \langle V_N(P_{\beta(k)}), V_N(P_{\beta(k)}) \rangle$$

together with the BNT normalization limits on both sides forces $|\zeta_k| = 1$. The matched MPV phase identities also give the eventual coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$. Definition 10.2.4 applies the finite power-sum comparison in each sector and gives the copy-weight identities. $\square$

Theorem 10.3.4 (Full-basis global gauge in matched coordinates). Lean ✓ **Stmt** Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then there exist:

- a bijection $\beta : J(Q) \simeq J(P)$,
- bond-dimension equalities $\dim P_{\beta(k)} = \dim Q_k$,
- copy-number equalities $r_{\beta(k)}(P) = r_k(Q)$ and copy permutations τ_k ,
- unit-modulus phases ζ_k with $|\zeta_k| = 1$ and block gauges X_k ,

such that, for every sector k , copy q , and physical index i ,

$$Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1},$$

$$\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}.$$

Moreover, if $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$ is the flattened block-diagonal gauge, then, for every physical index i , the flattened Q -coordinates satisfy

$$Q_{\text{flat}}^i = X P_{\text{matched}}^i X^{-1}.$$

This is the explicit direct-sum gauge construction of [CPGSV16, Appendix MPV proof, lines 1189–1192].

Proof. ✓ Proof Lemma 10.3.3 gives β , the bond-dimension identifications, the phases ζ_k , the block gauges X_k , and copy permutations τ_k satisfying $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Theorem F.4.4 combines the block conjugacies into the flattened direct-sum conjugation by $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$. $\square$

Theorem 10.3.5 (Literal equal-MPV gauge after sector-coordinate permutation). Lean ✓ Stmt Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then the total bond dimensions agree; write their common value as D . There is

$$Y \in \text{GL}_D(\mathbb{C}) \quad (10.5)$$

such that, for every physical index i ,

$$Q_{\text{tot}}^i = Y P_{\text{tot}}^i Y^{-1},$$

where P_{tot}^i and Q_{tot}^i are both regarded as matrices in $M_D(\mathbb{C})$ using the total bond-dimension equality. This is the BNT form of [CPGSV16, Corollary 2.11 and Appendix A, lines 1189–1192].

Proof. ✓ Proof Apply Theorem 10.3.4 to obtain the flattened gauge after the sectors of P have been reindexed by the bijection between basis blocks of P and Q . The sector-coordinate permutation ρ , induced by this bijection, the copy permutations, and the bond-dimension equalities, identifies the reindexed P -side direct sum with the total tensor P_{tot} , transported along the total bond-dimension equality. Composing the permutation gauge for ρ with the sector-permuted gauge gives the single matrix Y of (10.5). $\square$

Remark 10.3.6 (Proportional sector matching does not determine copy weights). The proportional theorem determines the normal sectors, but it does not determine the copy weights or imply conjugacy of the total tensors. Let C be a normalized normal tensor and consider the one-sector BNT forms

$$P = C \oplus \tfrac{1}{2}C, \quad Q = C \oplus \tfrac{1}{3}C.$$

Their positive-length MPV families are

$$V^{(N)}(P) = (1 + 2^{-N})V^{(N)}(C), \quad V^{(N)}(Q) = (1 + 3^{-N})V^{(N)}(C).$$

They are nonzero and proportional with

$$c_N = \frac{1 + 2^{-N}}{1 + 3^{-N}},$$

but $(c_N)_N$ is not a geometric sequence ξ^N . For a one-dimensional choice of C , the total matrices have eigenvalue multisets $\{1, \frac{1}{2}\}$ and $\{1, \frac{1}{3}\}$, so they are not conjugate, even after multiplication by a common phase.

Thus the scalar-free coefficient identity used in copy-weight recovery is an additional hypothesis in the proportional setting, not a consequence awaiting an analytic estimate. CPSV16 Theorem 2.10 asserts the sector matching proved in Theorem 10.4.1. The copy-weight recovery and global block gauge belong to the equal-MPV Corollary 2.11, where the proportionality scalar is identically 1.

10.4 The proportional and equal cases

Theorem 10.4.1 (Multi-block fundamental theorem for normal tensors). [Lean](#) [✓ Stmt](#) *Let P and Q be BNT canonical forms (Definition 9.6.1.13), with bases of normal tensors $(P_j)_{j=1}^{g_a}$ and $(Q_k)_{k=1}^{g_b}$. Every listed sector has a positive number of copies, and every copy weight is nonzero. If the total tensors $P_{\text{tot}}, Q_{\text{tot}}$ generate eventually nonzero proportional MPV families — for all sufficiently large N there is a nonzero scalar c_N with $V^{(N)}(P_{\text{tot}}) = c_N V^{(N)}(Q_{\text{tot}})$ — then $g_a = g_b$. After relabelling the basis of normal tensors of P by a bijection $\beta : J(Q) \simeq J(P)$, there are scalars $\zeta_k \in \mathbb{C}$ with $|\zeta_k| = 1$ and invertible matrices Y_k such that, for every k and every physical index i ,*

$$Q_k^i = \zeta_k Y_k P_{\beta(k)}^i Y_k^{-1}.$$

Thus the paired normal tensors have the same bond dimension ($\dim P_{\beta(k)} = \dim Q_k$) and the same gauge-phase class.

Proof. [✓ Proof](#) Apply Theorem 10.2.1. Its partition-and-substitution argument gives a bijection $\beta : J(Q) \simeq J(P)$, equal bond dimensions, invertible block gauges X_k , and phases ζ_k of unit modulus. Taking $Y_k = X_k$ gives exactly the asserted relations. $\square$

10.4.1 Unitary gauges on normalized sectors

Lemma 10.4.1.1 (Phase normalization for an irreducible gauge). [Lean](#) [✓ Stmt](#) *Let A and B be left-canonical tensors of common positive bond dimension, with B irreducible. If X is invertible, $\zeta \neq 0$, and $B^i = \zeta X A^i X^{-1}$ for every i , then $|\zeta| = 1$.*

Proof. [✓ Proof](#)

Let ρ be a nonzero positive fixed point of $\mathcal{E}_A$ and set $\sigma = X\rho X^\dagger$. Then σ is nonzero and positive, and $\mathcal{E}_B(\sigma) = |\zeta|^2 \sigma$. The map $\mathcal{E}_B$ also has a nonzero positive fixed point. Its irreducibility therefore forces $|\zeta|^2 = 1$. $\square$

Lemma 10.4.1.2 (Unitary replacement with the same phase). [Lean](#) [✓ Stmt](#) *Let A and B be left-canonical irreducible tensors of common positive bond dimension satisfying $B^i = \zeta X A^i X^{-1}$ for an invertible X and a nonzero scalar ζ . There is a unitary matrix U , with the same scalar ζ , such that $|\zeta| = 1$ and $B^i = \zeta U A^i U^\dagger$ for every i .*

Proof. [✓ Proof](#) Lemma 10.4.1.1 gives $|\zeta| = 1$. Substitution into the left-canonical identities gives

$$\sum_i A^{i\dagger} (X^\dagger X) A^i = X^\dagger X.$$

Thus $X^\dagger X$ is a positive-semidefinite fixed point of the adjoint transfer map of A . Irreducibility gives $X^\dagger X = c\mathbb{1}$ for some $c > 0$, so $X^{-1} = c^{-1} X^\dagger$ and $U = c^{-1/2} X$ is unitary. Moreover,

$$U A^i U^\dagger = c^{-1} X A^i X^\dagger = X A^i X^{-1}.$$

$\square$

Corollary 10.4.1.3 (Canonical Form II unitary refinement of proportional matching). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 10.4.1, there exist a bijection $\beta : J(Q) \simeq J(P)$, unitary matrices $U_k \in \text{U}(\dim Q_k)$, and phases $\zeta_k \in \mathbb{C}$ with $|\zeta_k| = 1$, such that $\dim P_{\beta(k)} = \dim Q_k$ and, for every k and every physical index i ,*

$$Q_k^i = \zeta_k U_k P_{\beta(k)}^i U_k^\dagger.$$

This is the proportional part of [CPGSV16, Corollary A.6, lines 1197–1199].

Proof. ✓ Proof The proportional sector-matching theorem gives a bijection β , invertible matrices X_k , and unit-modulus phases ζ_k with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. The matched tensors are left-canonical and irreducible. Hence Lemma F.8.1 gives unitary matrices U_k satisfying $X_k P_{\beta(k)}^i X_k^{-1} = U_k P_{\beta(k)}^i U_k^\dagger$, and therefore $Q_k^i = \zeta_k U_k P_{\beta(k)}^i U_k^\dagger$. $\square$

10.4.2 The equal case

Theorem 10.4.2.1 (Equal BNT canonical forms are globally conjugate). Lean ✓ Stmt *Let P and Q be BNT canonical forms (Definition 9.6.1.13) whose total tensors P_{tot} , Q_{tot} generate the same MPV family for every positive system size N . Every listed sector has a positive number of copies and every copy weight is nonzero. Then their total bond dimensions coincide; writing the common value as D , there is one invertible matrix $X \in \text{GL}_D(\mathbb{C})$ such that, for every physical index i ,*

$$Q_{\text{tot}}^i = X P_{\text{tot}}^i X^{-1}.$$

Proof. ✓ Proof This is Theorem 10.3.5. The BNT comparison of Chapter 9 provides the basis relabelling, the per-block gauge-phase equivalences, and the copy-weight multiset matching, which combine into the block-diagonal global gauge. The exact linear-independence matching needs no per-sector unit-modulus hypothesis, and the BNT canonical form needs no one-site injectivity. $\square$

Corollary 10.4.2.2 (Canonical Form II unitary refinement in the equal case). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total tensors generate the same MPV family at every positive length. Assume every listed sector has a positive number of copies and every copy weight is nonzero, as in Definition 9.6.1.13. Their total bond dimensions agree; writing the common value as D , there is a unitary matrix $U \in \text{U}(D)$ such that, for every physical index i ,*

$$Q_{\text{tot}}^i = U P_{\text{tot}}^i U^\dagger.$$

Proof. ✓ Proof For each matched sector, retain the phase in both identities

$$\begin{aligned} Q_k^i &= \zeta_k U_k P_{\beta(k)}^i U_k^\dagger, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}. \end{aligned}$$

For every matched copy (k, q) ,

$$\nu_{k,q} \zeta_k = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)} \zeta_k = \mu_{\beta(k), \tau_k(q)}.$$

Hence $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$. Repeating U_k on its matched copies gives

$$\begin{aligned} X &= \bigoplus_{k,q} U_k, \\ X^\dagger X &= \mathbb{1}. \end{aligned}$$

If Π is the permutation matrix restoring the original sector and copy order, then $\Pi^\dagger \Pi = \mathbb{1}$. Hence

$$\begin{aligned} U &= X \Pi, \\ U^\dagger U &= \mathbb{1}, \end{aligned}$$

and the permutation of matched coordinates gives

$$Q_{\text{tot}}^i = U P_{\text{tot}}^i U^\dagger.$$

$\square$

Chapter 11

Symmetries, Physical String Order, and Virtual-Boundary Nondecay

This chapter collects the symmetry-theoretic consequences of the MPS Fundamental Theorem [PGVWC07], following the symmetry analysis of [PGWS⁺08]. For injective tensors, equality of periodic MPV families under a physical index action determines virtual gauges, and those gauges define projective representations on the bond space. We keep this virtual covariance distinct from symmetry of an infinite physical state, and distinguish physical endpoint string order from virtual-boundary nondecay.

11.1 Virtual Local Covariance and Projective Bond Actions

With a full group of on-site symmetries (rather than just a single unitary), the resulting gauges determine a projective representation on the bond space.

Definition 11.1.1 (Twisted tensor). [Lean](#) [✓ Stmt](#) Given a group representation $U : G \rightarrow \mathrm{GL}_d(\mathbb{C})$ on the physical index, the *g-twisted tensor* is defined, for each $i \in \{0, \dots, d-1\}$, by

$$\tilde{A}_g^i := \sum_{j=0}^{d-1} U(g)_{ij} A^j. \quad (11.1)$$

Definition 11.1.2 (MPV-family invariance under physical twists). [Lean](#) [✓ Stmt](#) The predicate called on-site symmetry requires $\mathcal{V}(A) = \mathcal{V}(\tilde{A}_g)$ for every $g \in G$, where $U : G \rightarrow \mathrm{GL}_d(\mathbb{C})$ acts on the physical index. This is equality of periodic MPV families and is the hypothesis used to construct virtual local gauges. It is not, by itself, the source notion that every reduced density operator of an infinite physical state is invariant under $U(g)^{\otimes N}$.

Lemma 11.1.3 (Functoriality of twisting). [Lean](#) [✓ Stmt](#) The twisted tensor satisfies the composition law

$$\tilde{A}_{gh} = \widetilde{(\tilde{A}_h)_g}, \quad (11.2)$$

i.e. twisting by gh is the same as first twisting by h and then by g .

Proof. ✓ Proof Expand both sides of (11.2) using (11.1) and $U(gh) = U(g)U(h)$, then swap the order of summation. □

Lemma 11.1.4 (Identity twist). Lean ✓ Stmt *Twisting by the identity group element is trivial: $\tilde{A}_1 = A$.*

Proof. ✓ Proof Since $U(1) = \mathbb{1}_d$, each component in (11.1) reduces to $\sum_j \delta_{ij} A^j = A^i$. □

Definition 11.1.5 (Blocked on-site action). Lean ✓ Stmt For a block of L physical sites, the induced action is the Kronecker-power representation

$$U^{[L]}(g) := U(g)^{\otimes L} \in \text{GL}_{d^L}(\mathbb{C}).$$

Lemma 11.1.6 (Twisting commutes with blocking). Lean ✓ Stmt *For every $g \in G$ and $L \geq 0$,*

$$\left(\widetilde{A^{[L]}}\right)_g^{U^{[L]}} = (\tilde{A}_g)^{[L]}. \quad (11.3)$$

Proof. ✓ Proof Decode a blocked physical index as a word of length L and use the word expansion from Lemma G.1.1.3. Reindexing the sum over words by the blocked index gives exactly the Kronecker-product matrix coefficients on the left of (11.3). □

Theorem 11.1.7 (MPV-family symmetry is preserved by blocking). Lean ✓ Stmt *If A has MPV-family invariance under U , then $A^{[L]}$ has MPV-family invariance under $U^{[L]}$ for every L .*

Proof. ✓ Proof Equation (11.3) identifies the twisted blocked tensor with the blocking of the twisted tensor. Equality of MPV families is preserved by blocking, so the defining equality for A yields the defining equality for $A^{[L]}$. □

Lemma 11.1.8 (Symmetric twists are gauge equivalent). Lean ✓ Stmt *If A is injective and on-site symmetric under U , then for every $g \in G$ the twisted tensor $\tilde{A}_g$ is gauge equivalent to A .*

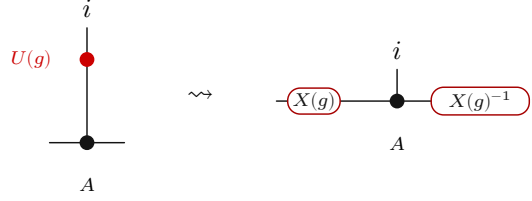
Proof. ✓ Proof On-site symmetry gives $\mathcal{V}(A) = \mathcal{V}(\tilde{A}_g)$, and the single-block Fundamental Theorem turns this equality of MPV families into gauge equivalence. □

Remark 11.1.9 (Fundamental-Theorem route for injective symmetry). The injective virtual-representation theorem is not proved from virtual-boundary nondecay. The on-site symmetry condition first identifies A and $\tilde{A}_g$ as the same MPV family; the single-block Fundamental Theorem gives the gauge in Lemma 11.1.8. Gauge uniqueness and the twist composition law then give Theorem 11.1.16. The virtual-boundary nondecay results later in this chapter use this virtual representation as an input.

Theorem 11.1.10 (Virtual symmetry equation). Lean ✓ Stmt *If A is injective and on-site symmetric under U , then for every $g \in G$ there exist an invertible matrix $X(g) \in \text{GL}_D(\mathbb{C})$ and a nonzero scalar $\varphi(g) \in \mathbb{C}^\times$ (in fact $\varphi = 1$ in the single-block case) such that, for every physical index i ,*

$$\tilde{A}_g^i = \varphi(g) X(g) A^i X(g)^{-1}. \quad (11.4)$$

Proof. ✓ Proof The physical action is transferred to the virtual level by the same local move that replaces the twisted tensor by a gauge-conjugated one:



Apply Lemma 11.1.8 to obtain $X(g)$, then set $\varphi(g) = 1$ in (11.4). $\square$

Lemma 11.1.11 (Gauge uniqueness up to scalar). [Lean](#) [✓ Stmt](#) *Let A be an injective tensor. If $X, Y \in \text{GL}_D(\mathbb{C})$ both satisfy $B^i = X A^i X^{-1} = Y A^i Y^{-1}$ for all i , then there exists a nonzero scalar $u \in \mathbb{C}^\times$ such that $Y = u \cdot X$.*

Proof. [✓ Proof](#) The matrix $Z = Y^{-1}X$ commutes with every A^i . Since A is injective, $\{A^i\}$ spans $M_D(\mathbb{C})$, so Z lies in the centre of $M_D(\mathbb{C})$. By Lemma 2.1.5, Z is a scalar matrix. Since Z is invertible, that scalar is nonzero, and $Y = u \cdot X$. $\square$

Theorem 11.1.12 (Gauge uniqueness from equality of MPV families). [Lean](#) [✓ Stmt](#) *Let A be an injective tensor and assume that A and B generate the same MPV family. If $X, Y \in \text{GL}_D(\mathbb{C})$ both satisfy $B^i = X A^i X^{-1} = Y A^i Y^{-1}$ for all i , then there is a nonzero scalar $u \in \mathbb{C}^\times$ such that $Y = u \cdot X$.*

Proof. [✓ Proof](#) This is exactly Lemma 11.1.11; the extra MPV hypothesis specifies the setting in which the two gauges are usually produced. $\square$

Definition 11.1.13 (Scalar 2-cochain). [Lean](#) [✓ Stmt](#) A scalar 2-cochain on a group G is a function $\omega : G \times G \rightarrow \mathbb{C}^\times$.

Definition 11.1.14 (Projective representation). [Lean](#) [✓ Stmt](#) Given a scalar 2-cochain ω , a projective representation of G on $\mathbb{C}^D$ is a map $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ satisfying, for all $g, h \in G$,

$$\rho(g)\rho(h) = \omega(g, h)\rho(gh). \quad (11.5)$$

Lemma 11.1.15 (Associativity forces the cocycle condition). [Lean](#) [✓ Stmt](#) *If ρ is a projective representation with factor system ω and $D > 0$, then ω satisfies the 2-cocycle identity*

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k). \quad (11.6)$$

Proof. [✓ Proof](#) Expand $\rho(g)(\rho(h)\rho(k))$ and $(\rho(g)\rho(h))\rho(k)$ using the projective multiplication law (11.5) and equate the results via matrix associativity. The hypothesis $D > 0$ allows cancellation of the invertible matrix factor, giving (11.6). $\square$

Theorem 11.1.16 (Virtual representation theorem for injective MPS). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor that is on-site symmetric under a group representation $U : G \rightarrow \text{GL}_d(\mathbb{C})$. Then there exist:*

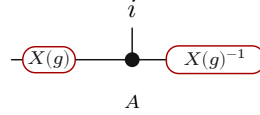
1. a scalar 2-cochain $\omega : G \times G \rightarrow \mathbb{C}^\times$, and
2. a map $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ satisfying, for all $g, h \in G$,

$$\rho(g)\rho(h) = \omega(g, h)\rho(gh).$$

such that, defining the gauge matrices $X(g) := \rho(g^{-1})$, one has, for every $g \in G$ and $i \in \{0, \dots, d-1\}$,

$$\tilde{A}_g^i = X(g) A^i X(g)^{-1}. \quad (11.7)$$

In tensor-network notation, for each fixed g this is the local gauge relation



with the black node denoting the tensor named in the surrounding formula and the two red side nodes acting on the virtual legs. In particular, ρ is a projective representation of G on the bond space $\mathbb{C}^D$.

Proof. ✓ Proof Since A is injective and on-site symmetric, each twisted tensor $\tilde{A}_g$ is gauge equivalent to A (Lemma 11.1.8). Choose $X(g) \in \text{GL}_D(\mathbb{C})$ satisfying (11.7) for all i .

By functoriality (Lemma 11.1.3), the product $X(h)X(g)$ also intertwines A with $\tilde{A}_{gh}$. Since $X(gh)$ does the same, gauge uniqueness (Lemma 11.1.11) gives a nonzero scalar $\omega'(h, g)$ such that

$$X(h)X(g) = \omega'(h, g)X(gh). \quad (11.8)$$

Defining $\rho(g) := X(g^{-1})$ converts (11.8) to the standard law (11.5), where $\omega(g, h) := \omega'(g^{-1}, h^{-1})$. □

Corollary 11.1.17 (2-cocycle identity for the factor system). Lean ✓ Stmt If $D > 0$, the factor system ω from Theorem 11.1.16 satisfies the 2-cocycle identity for all $g, h, k \in G$:

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k).$$

Proof. ✓ Proof Theorem 11.1.16 gives ρ and ω . By Lemma 11.1.15, the cocycle identity (11.6) follows from associativity of matrix multiplication and invertibility of the representation matrices. □

11.2 Cohomology Classes of Virtual Symmetry Cocycles

Two cocycles may differ by a coboundary yet represent the same projective-representation class. The following definitions and results make this precise and establish the quotient $H^2(G, \mathbb{C}^\times)$.

Definition 11.2.1 (Coboundary). Lean ✓ Stmt A scalar 2-cocycle ω is a *coboundary* if there exists $\varphi: G \rightarrow \mathbb{C}^\times$ such that, for all $g, h \in G$,

$$\omega(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}. \quad (11.9)$$

Definition 11.2.2 (Cohomologous cocycles). Lean ✓ Stmt Two cocycles ω_1, ω_2 are *cohomologous* (written $\omega_1 \sim \omega_2$) if there exists $\varphi: G \rightarrow \mathbb{C}^\times$ such that, for all $g, h \in G$,

$$\omega_1(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h). \quad (11.10)$$

Theorem 11.2.3 (Gauge independence of the cocycle class). [Lean](#) [✓ Stmt](#) If the bond dimension satisfies $D \geq 1$ and two projective representations ρ_1, ρ_2 (with cocycles ω_1, ω_2) both arise as virtual representations of the same injective MPS tensor A under an on-site symmetry U , then $\omega_2 \sim \omega_1$, and hence also $\omega_1 \sim \omega_2$ by symmetry.

Proof. [✓ Proof](#) Gauge uniqueness gives nonzero scalars $f(k)$ with $X_2(k) = f(k) X_1(k)$ for every physical twist k . Since the projective representation is $\rho_a(g) = X_a(g^{-1})$, setting $k = g^{-1}$ gives $\rho_2(g) = f(g^{-1})\rho_1(g)$. Equivalently, for each fixed g the two local gauges

$$\begin{array}{c} i \\ | \\ \boxed{X_1(g^{-1})} - \bullet - \boxed{X_1(g^{-1})^{-1}} \\ A \end{array} \quad \text{and} \quad \begin{array}{c} i \\ | \\ \boxed{X_2(g^{-1})} - \bullet - \boxed{X_2(g^{-1})^{-1}} \\ A \end{array}$$

describe the same twisted tensor, so they differ by a scalar. Set $\alpha(g) := f(g^{-1})$. Substituting $\rho_2(g) = \alpha(g)\rho_1(g)$ into the projective law (11.5) and cancelling the invertible matrix $X_1((gh)^{-1}) = \rho_1(gh)$ yields

$$\omega_2(g, h) = \alpha(g)\alpha(h)\alpha(gh)^{-1}\omega_1(g, h). \quad (11.11)$$

Thus ω_2 is the coboundary multiple of ω_1 with cochain α . This is precisely the orientation required by (11.10) for the stated relation $\omega_2 \sim \omega_1$. Equivalently,

$$\omega_1(g, h) = \alpha(g)^{-1}\alpha(h)^{-1}\alpha(gh)\omega_2(g, h),$$

so the inverse cochain also gives the symmetric conclusion $\omega_1 \sim \omega_2$. $\square$

Definition 11.2.4 (Multiplicative 2-cocycle condition). [Lean](#) [✓ Stmt](#) A scalar 2-cochain $\omega : G \times G \rightarrow \mathbb{C}^\times$ is a 2-cocycle if it satisfies, for all $g, h, k \in G$,

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k).$$

Theorem 11.2.5 (A projective factor system is a cocycle). [Lean](#) [✓ Stmt](#) If $D > 0$ and $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ obeys $\rho(g)\rho(h) = \omega(g, h)\rho(gh)$, then ω is a $\mathbb{C}^\times$ -valued 2-cocycle.

Proof. [✓ Proof](#) Associativity gives the scalar identity of Lemma 11.1.15. Because the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}$ is injective, the same identity holds in $\mathbb{C}^\times$, which is precisely Definition 11.2.4. $\square$

Definition 11.2.6 (Second cohomology quotient). [Lean](#) [✓ Stmt](#) The second cohomology set is the quotient

$$H^2(G, \mathbb{C}^\times) := Z^2(G, \mathbb{C}^\times)/\sim, \quad (11.12)$$

where $\sim$ is the coboundary-equivalence relation.

Definition 11.2.7 (Agreement of factor-system cohomology classes). [Lean](#) [✓ Stmt](#) The relation called projective equivalence holds precisely when the two factor systems are cohomologous. It compares their classes in $H^2(G, \mathbb{C}^\times)$; it does not assert an intertwining equivalence between the underlying representation spaces.

Theorem 11.2.8 (Factor-system class agreement). [Lean](#) [✓ Stmt](#) Let ρ_1, ρ_2 be projective representations with factor systems ω_1, ω_2 . Then

$$\rho_1 \sim_{\text{proj}} \rho_2 \iff \omega_1 \sim \omega_2. \quad (11.13)$$

Proof. [✓ Proof](#) This follows from the definition of projective equivalence as cohomology-class equality for factor systems. $\square$

11.2.1 Non-triviality via the commutator phase

For an abelian symmetry group, the antisymmetric commutator phase is invariant under a change of cocycle representative. A value not equal to 1 therefore witnesses non-triviality; no converse is asserted here.

Definition 11.2.1.1 (Commutator phase). [Lean](#) [✓ Stmt](#) The *commutator phase* of a 2-cochain ω at a pair g, h is $\beta_\omega(g, h) := \omega(g, h)\omega(h, g)^{-1}$. It is defined for any $\mathbb{C}^\times$ -valued 2-cochain; the cocycle condition is not needed.

Theorem 11.2.1.2 (The commutator phase is a class invariant). [Lean](#) [✓ Stmt](#) If $\omega_1 \sim \omega_2$ and $gh = hg$, then $\beta_{\omega_1}(g, h) = \beta_{\omega_2}(g, h)$.

Proof. [✓ Proof](#) Write $\omega_1(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h)$ for a coboundary witness φ . Substituting into the ratio and using $gh = hg$ and the commutativity of $\mathbb{C}^\times$ gives

$$\begin{aligned}\beta_{\omega_1}(g, h) &= \frac{\varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h)}{\varphi(h)\varphi(g)\varphi(hg)^{-1}\omega_2(h, g)} \\ &= \frac{\omega_2(g, h)}{\omega_2(h, g)} = \beta_{\omega_2}(g, h),\end{aligned}\tag{11.14}$$

where the φ -factors cancel since $\varphi(gh) = \varphi(hg)$. $\square$

Definition 11.2.1.3 (Non-trivial cohomology class). [Lean](#) [✓ Stmt](#) A factor system ω has a *non-trivial class* when it is not cohomologous to the constant cocycle 1.

Theorem 11.2.1.4 (Commutator-phase non-triviality test). [Lean](#) [✓ Stmt](#) If a commuting pair g, h has $\beta_\omega(g, h) \neq 1$, then ω has a non-trivial class.

Proof. [✓ Proof](#) The trivial cocycle has commutator phase 1 at every pair, so by Theorem 11.2.1.2 and (11.14), any cocycle cohomologous to it has $\beta = 1$ at every commuting pair. Contrapositively, $\beta_\omega(g, h) \neq 1$ for one commuting pair forces ω to be non-cohomologous to the trivial cocycle. $\square$

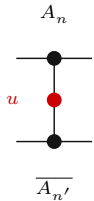
11.3 Physical Endpoint String Order and Virtual-Boundary Nondecay

The source notion of string order uses physical endpoint operators and a nonidentity unitary string [PGWS⁺08]. We also study a restricted virtual-boundary nondecay predicate and a virtual local-covariance relation. These notions are stated separately below; equivalence with physical state symmetry requires additional hypotheses.

Definition 11.3.1 (Twisted transfer map). [Lean](#) [✓ Stmt](#) Given an MPS tensor A and a physical-index matrix u , the *twisted transfer map* is

$$\mathcal{E}_u(X) := \sum_{n, n'} \langle n' | u | n \rangle A_n X A_{n'}^\dagger.\tag{11.15}$$

Diagrammatically one inserts the physical operator u on the doubled local transfer cell:

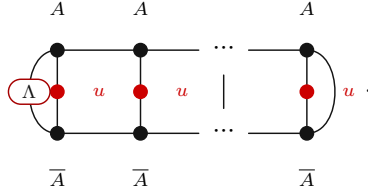


The upper and lower black nodes denote A_n and $A_n^\dagger$, and the red node labelled u couples the physical legs.

Definition 11.3.2 (Stationary-boundary block-twist functional). [Lean](#) [✓ Stmt](#) The *stationary-boundary block-twist functional* is

$$T_L(A, u, \Lambda) := \text{tr}(\Lambda \cdot \mathcal{E}_u^L(\mathbb{1})). \quad (11.16)$$

In tensor-network notation this is the length- L doubled chain with a copy of u inserted on each physical rung:



L sites

The boundary matrix Λ is contracted into the virtual boundary, the red nodes labelled u sit on the physical rungs, and the right boundary is traced.

Definition 11.3.3 (Periodic finite-chain symmetry overlap). [Not ready](#) For the periodic length- L matrix product vector $|\Psi_L(A)\rangle$, the source quantity is literally

$$R_L(u) := \langle \Psi_L(A) | u^{\otimes L} | \Psi_L(A) \rangle. \quad (11.17)$$

Identifying this periodic overlap with the normalized stationary-boundary functional $T_L(A, u, \Lambda)$ requires a thermodynamic-boundary and normalization hypothesis. Thus Definition 11.3.2 must not be read as the literal source definition of $R_L(u)$.

Definition 11.3.4 (Physical string correlator). [Lean](#) [✓ Stmt](#) For physical operators $x, y \in M_d(\mathbb{C})$, a twist $u \in M_d(\mathbb{C})$, and boundary state Λ , the transfer-form string correlator of [PGWS⁺08, display SOPMP, lines 176–181] is

$$S_N(x, y, u) := \text{tr}(\Lambda \mathcal{E}_x \mathcal{E}_u^N(\mathcal{E}_y(\mathbb{1}))). \quad (11.18)$$

This is the physical-endpoint expression corresponding to $\langle \Psi | x \otimes u^{\otimes N} \otimes y | \Psi \rangle$.

Definition 11.3.5 (Fixed-endpoint physical string order). [Lean](#) [✓ Stmt](#) For fixed physical operators x, y, u , the correlator has physical string order if there is $s > 0$ such that $|S_N(x, y, u)| \rightarrow s$.

Definition 11.3.6 (Physical endpoint string order). [Lean](#) [✓ Stmt](#) A tensor has physical string order in the sense of [PGWS⁺08, display SOP, lines 112–122] if there exist a unitary $u \neq \mathbb{1}$ and physical endpoint operators x, y for which Definition 11.3.5 holds. The comparison with arbitrary virtual boundary matrices is a separate endpoint- realizability statement.

Lemma 11.3.7 (Twisted spectral-radius bound and peripheral rigidity). [Not ready](#) This is Lemma 1 of [PGWS⁺08]. Let A represent a pure finitely correlated state in the canonical gauge $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, $\mathcal{E}_A^*(\Lambda) = \Lambda$, with $\Lambda > 0$ and $\text{tr}(\Lambda) = 1$. For every physical unitary u ,

$$\rho(\mathcal{E}_u) \leq 1.$$

Equality holds if and only if there are a unitary V and $\theta \in [0, 2\pi)$ satisfying the following identity for every j in an eigenbasis $u = \sum_j e^{i\theta_j} |\tilde{j}\rangle\langle\tilde{j}|$:

$$V^\dagger \tilde{A}_j = e^{i(\theta - \theta_j)} \tilde{A}_j V^\dagger. \quad (11.19)$$

Moreover, $\mathcal{E}_u$ has at most one eigenvalue of modulus one.

Proof. Let $\mathcal{E}_u(V) = \lambda V$. Multiplication by $\Lambda V^\dagger$, the trace, and Cauchy–Schwarz give

$$\begin{aligned} |\lambda| \operatorname{tr}(V \Lambda V^\dagger) &\leq \left(\sum_j \operatorname{tr}(V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger) \right)^{1/2} \left(\sum_j \operatorname{tr}(\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j) \right)^{1/2} \\ &= \operatorname{tr}(V \Lambda V^\dagger). \end{aligned} \quad (11.20)$$

Faithfulness of Λ yields $|\lambda| \leq 1$. Equality in Cauchy–Schwarz is equivalent to (11.19); conversely that relation makes $e^{i\theta}$ an eigenvalue. It also gives $\mathcal{E}_A(V^\dagger V) = V^\dagger V$. The fixed space of the pure canonical transfer map is one-dimensional, so $V^\dagger V = c\mathbb{1}$ for some $c > 0$. Replacing V by $c^{-1/2}V$ makes it unitary without changing the intertwining relation.

If V, V' are peripheral eigenmatrices with eigenvalues $e^{i\theta}, e^{i\theta'}$, respectively, the equality relations give the exact identity

$$\mathcal{E}_A(V^\dagger V') = e^{i(\theta' - \theta)} V^\dagger V'.$$

Purity leaves only the peripheral eigenvalue 1, hence $\theta' = \theta$ modulo 2π . The one-dimensional fixed space then gives $V^\dagger V' \in \mathbb{C}\mathbb{1}$, proving uniqueness of the peripheral eigenvalue and its eigendirection. $\square$

Theorem 11.3.8 (Physical endpoint criterion for string order). Not ready This is Theorem 1 of [PGWS⁺08]. A pure finitely correlated state has physical string order if and only if there exist a unitary $\tilde{u} \neq \mathbb{1}$, a unitary virtual matrix V , and physical indices n, m such that

$$\mathcal{E}_{\tilde{u}}(V) = V, \quad \operatorname{tr}(V \Lambda A_n A_m^\dagger) \neq 0. \quad (11.21)$$

The transfer criteria below assume one-site injectivity. The literal physical equivalence additionally requires that sufficiently long physical endpoint operators realize the needed virtual boundary matrices.

Proof. Lemma 11.3.7 shows that a nonzero limit requires a unique peripheral eigenvalue $e^{i\theta}$. If V and Y are the corresponding right and left eigenmatrices, then $Y = \Lambda V^\dagger$, and the leading term retains its length-dependent phase:

$$S_N(x, y, u) = e^{iN\theta} \operatorname{tr}(Y \mathcal{E}_y(\mathbb{1})) \operatorname{tr}(\Lambda \mathcal{E}_x(V)) + o(1).$$

In the eigenbasis $u = \sum_j e^{i\theta_j} |\tilde{j}\rangle\langle\tilde{j}|$, shift the string unitary by the peripheral phase,

$$\tilde{u} := e^{-i\theta} u = \sum_j e^{i(\theta_j - \theta)} |\tilde{j}\rangle\langle\tilde{j}|.$$

Then $\mathcal{E}_{\tilde{u}}(V) = V$. Choosing $y = x^\dagger \tilde{u}$ reduces nonvanishing to the second condition in (11.21); if it holds for n, m , take $x = |n\rangle\langle m|$. The reverse implication follows from the same phase-retaining factorization and the endpoint-realizability hypothesis. $\square$

Theorem 11.3.9 (Physical state symmetry and twisted spectral radius). Not ready This is Theorem 2 of [PGWS⁺08]. For a pure finitely correlated state and a unitary $u \neq \mathbb{1}$, the physical state is invariant under $u^{\otimes N}$ on every N -site reduced density operator if and only if

$$\rho(\mathcal{E}_u) = 1.$$

The theorem below gives the corresponding virtual local-covariance criterion for a one-site-injective tensor. Passing between that condition and invariance of every physical reduced state is an additional step.

Proof. If $\rho(\mathcal{E}_u) = 1$, Lemma 11.3.7 gives a unitary V satisfying the local intertwining relation. Substitution of that relation into the adjoint transfer map, together with $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$, gives $\mathcal{E}_A^\dagger(V\Lambda V^\dagger) = V\Lambda V^\dagger$. This positive fixed point has trace one, so uniqueness gives $V\Lambda V^\dagger = \Lambda$, equivalently $V^\dagger\Lambda V = \Lambda$, which yields the physical symmetry. Conversely, physical invariance gives, for length-independent boundary tensors L, R ,

$$D^{-2} \leq \text{tr}(\varrho_N^2) = \text{tr}[L(\mathcal{E}_u \otimes \mathcal{E}_{u^\dagger})^N(R)].$$

Exponential decay would contradict this bound, so $\rho(\mathcal{E}_u) = 1$. □

Definition 11.3.10 (Virtual local covariance with boundary invariance). Lean ✓ Stmt The virtual local-covariance predicate holds for A, u, Λ if there exist a unitary matrix V and a phase μ with $|\mu| = 1$ such that $V^\dagger\Lambda V = \Lambda$ and, for every physical index i ,

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger. \quad (11.22)$$

The unitary V intertwines the physical action of u with a virtual conjugation, and the boundary state Λ is invariant under V .

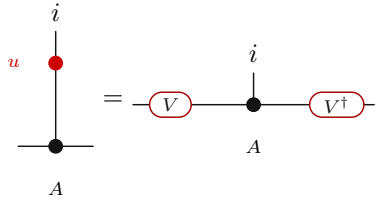
Definition 11.3.11 (Virtual-boundary nondecay). Lean ✓ Stmt For boundary matrices $X, Y \in M_D(\mathbb{C})$, set explicitly

$$R_L(u; X, Y) := \text{tr}(\Lambda X \mathcal{E}_u^L(Y)).$$

The virtual-boundary nondecay predicate holds for u and Λ if there exist X, Y and a positive constant $c > 0$ such that $c \leq \|R_L(u; X, Y)\|$ for every L . Thus some virtual-boundary sequence does not decay to zero.

Virtual-boundary nondecay is an existence statement over the virtual matrices X, Y , for a fixed twist u ; no claim is made for any particular boundary choice. Definition 11.3.6 records the physical-endpoint string-order notion from the source, which quantifies existentially over u, x, y and requires a positive limit rather than a lower bound uniform in the length. That paper also notes that the correlator for a particular endpoint choice can vanish even when the physical symmetry is present.

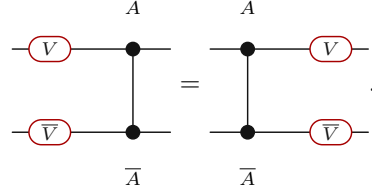
Definition 11.3.12 (Local physical intertwining). Lean ✓ Stmt The local physical intertwining relation is $\sum_j u_{ij} A^j = V A^i V^\dagger$. Locally, the physical action of u can therefore be pushed to the virtual legs:



Definition 11.3.13 (Transfer-map covariance). [Lean](#) [✓ Stmt](#) Transfer-map covariance is the identity that the transfer map commutes with virtual conjugation:

$$\mathcal{E}(VXV^\dagger) = V \mathcal{E}(X) V^\dagger. \quad (11.23)$$

In doubled-layer notation, the virtual operators slide through the local transfer cell:

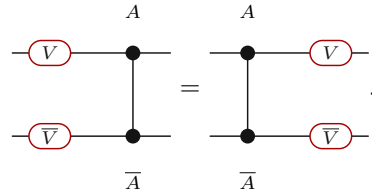


Definition 11.3.14 (Commutation in the doubled transfer picture). [Lean](#) [✓ Stmt](#) In the transfer-map language, the commutation of the doubled transfer matrix $E = \sum_i A_i \otimes \bar{A}_i$ with $V \otimes \bar{V}$ is

$$V \mathcal{E}_1(X) V^\dagger = \mathcal{E}_1(VXV^\dagger). \quad (11.24)$$

Theorem 11.3.15 (Transfer-map covariance and doubled commutation are equivalent). [Lean](#) [✓ Stmt](#) Transfer-map covariance and commutation in the doubled transfer picture are equivalent for any matrix V .

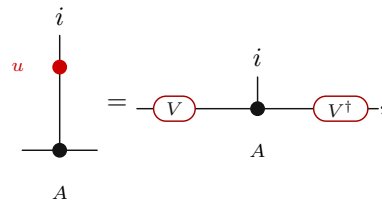
Proof. [✓ Proof](#) Both sides express the same local picture with opposite orientation:



The two formulas are the same identity read from opposite sides. □

Theorem 11.3.16 (Local physical intertwining implies transfer-map covariance). [Lean](#) [✓ Stmt](#) If V and u are unitary and the local physical intertwining relation holds, then the transfer map is covariant.

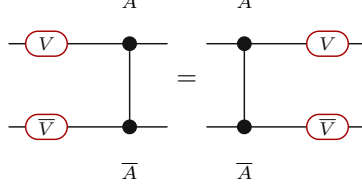
Proof. [✓ Proof](#) Starting from



one inserts $V^\dagger V = \mathbb{1}$ so that each local transfer cell is written in terms of conjugated Kraus operators, then replaces those operators using the intertwining relation. After that, the physical u -boxes disappear by the unitary Kraus-mixing identity. □

Theorem 11.3.17 (Transfer-map covariance implies local physical intertwining). [Lean](#) [✓ Stmt](#) Assume that A is one-site injective, $VV^\dagger = \mathbb{1}$, and the transfer map is covariant. Then there is u with $uu^\dagger = \mathbb{1}$ satisfying the local physical intertwining relation. In fact, covariance and $VV^\dagger = \mathbb{1}$ alone imply the conclusion; the injectivity hypothesis is unnecessary.

Proof. [✓ Proof](#) Set $B^i := V A^i V^\dagger$. The transfer-map covariance relation



implies that the two d -element Kraus families B and A define the same transfer map. Rectangular Kraus freedom (the corresponding theorem in the companion quantum-channel volume [LTC26, “Rectangular Kraus freedom”]) converts $\mathcal{E}_B = \mathcal{E}_A$ into a unitary Kraus mixing matrix u with $B^i = \sum_j u_{ij} A^j$. $\square$

Theorem 11.3.18 (Twisted-transfer eigenvalues have modulus at most one). [Lean](#) [✓ Stmt](#) Let A be an injective MPS tensor with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and let u satisfy $uu^\dagger = \mathbb{1}$. If $\mathcal{E}_u(X) = \lambda X$ for some nonzero $X \in M_D(\mathbb{C})$, then $|\lambda| \leq 1$.

Proof. [✓ Proof](#) By Lemma G.4.2, the twisted transfer map is the mixed transfer operator F_{A,B_u} , as in (G.6). Because B_u is obtained from A by unitary mixing of the Kraus operators, the transfer maps of A and B_u coincide, so the two adjoint channels share a common positive definite fixed point. Gauging both families by that fixed point puts them in a common trace-preserving gauge. Theorem 6.3.1 then applies to the mixed transfer operator in that gauge, and similarity preserves the eigenvalue λ . $\square$

Theorem 11.3.19 (Modulus-one twisted eigenvalue implies gauge-phase equivalence). [Lean](#) [✓ Stmt](#) Let A be an injective MPS tensor with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and let u satisfy $uu^\dagger = \mathbb{1}$. If

$$\mathcal{E}_u(X) = \lambda X \tag{11.25}$$

for some nonzero $X \in M_D(\mathbb{C})$ with $|\lambda| = 1$, then A is gauge-phase equivalent to the companion tensor B_u .

Proof. [✓ Proof](#) By Lemma G.4.2, the twisted transfer map is the mixed transfer operator F_{A,B_u} . Because B_u is obtained from A by unitary mixing of the Kraus operators, the transfer maps of A and B_u coincide, so the two adjoint channels share a common positive definite fixed point. Gauging both families by that fixed point puts them in a common trace-preserving gauge. In that gauge, (11.25) becomes a modulus-one peripheral eigenvalue for an irreducible mixed transfer operator, so Theorem 6.6.1 yields gauge-phase equivalence. Undoing the common trace-preserving gauge gives the claim for A and B_u . $\square$

Theorem 11.3.20 (Virtual-boundary nondecay gives companion gauge-phase equivalence). [Lean](#) [✓ Stmt](#) Let A be injective, let $\Lambda \in M_D(\mathbb{C})$, and assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If virtual-boundary nondecay holds for u with boundary parameter Λ , then A is gauge-phase equivalent to the companion tensor B_u .

Proof. ✓ Proof If $\rho(\mathcal{E}_u) < 1$, Lemma G.6.1 forces every virtual-boundary sequence to decay to zero, contradicting Definition 11.3.11. Hence $\rho(\mathcal{E}_u) \geq 1$. Theorem 11.3.18 shows that every eigenvalue has modulus at most 1, so some peripheral eigenvalue has modulus exactly 1. Applying Theorem 11.3.19 then gives the gauge-phase equivalence. $\square$

Theorem 11.3.21 (Gauge-phase equivalence with the companion tensor yields a virtual unitary). Lean ✓ Stmt *Let A be injective, assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If A is gauge-phase equivalent to the companion tensor B_u , then there exist a unitary V and a phase μ with $|\mu| = 1$ such that, for every physical index i ,*

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger. \quad (11.26)$$

Proof. ✓ Proof Start from an invertible intertwiner X and a nonzero gauge scalar ζ between A and B_u . The positive matrix $Q := XX^\dagger$ satisfies $\mathcal{E}_A(Q) = |\zeta|^2 Q$. Since $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, irreducibility forces the two positive eigenvalues to agree, so $|\zeta| = 1$ and Q is a positive fixed point of $\mathcal{E}_A$. Uniqueness of positive fixed points for an injective normalized tensor then makes Q a positive scalar multiple of the identity. Rescaling X by the square root of that scalar produces a unitary V , and the original intertwining relation becomes (11.26), with $|\mu| = 1$. $\square$

Theorem 11.3.22 (Virtual unitary gives a twisted-transfer eigenmatrix). Lean ✓ Stmt *Assume $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$ and that, for every physical index i , $\sum_j u_{ij} A^j = \mu V A^i V^\dagger$ with V unitary. Then V is an eigenmatrix of the twisted transfer map:*

$$\mathcal{E}_u(V) = \mu V. \quad (11.27)$$

Proof. ✓ Proof Expand $\mathcal{E}_u(V)$ using (11.15), insert the intertwining relation into each local term, and use $V^\dagger V = \mathbb{1}$ together with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. $\square$

Theorem 11.3.23 (Boundary-state invariance of the virtual unitary). Lean ✓ Stmt *Let A be injective, assume $uu^\dagger = \mathbb{1}$, and let Λ be a positive definite boundary state with $\text{tr}(\Lambda) = 1$ and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. If V is a unitary and μ a phase satisfying, for every physical index i ,*

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger, \quad (11.28)$$

then

$$V^\dagger \Lambda V = \Lambda. \quad (11.29)$$

Proof. ✓ Proof Relation (11.28) first shows that $V \Lambda V^\dagger$ is another positive fixed point of the adjoint transfer map. It has the same trace as Λ , so uniqueness of the normalized positive fixed point gives $V \Lambda V^\dagger = \Lambda$. Multiplying by $V^\dagger$ on the left and by V on the right yields (11.29). $\square$

Lemma 11.3.24 (Virtual local covariance implies virtual-boundary nondecay). Lean ✓ Stmt *Assume $\text{tr}(\Lambda) = 1$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If A, u, Λ satisfy virtual local covariance, then virtual-boundary nondecay holds for u .*

Proof. ✓ Proof Choose the boundary matrices $X = V^\dagger$ and $Y = V$, where V is the unitary from the virtual local-covariance datum. By Theorem 11.3.22, one has $\mathcal{E}_u^L(V) = \mu^L V$. Therefore

$$R_L(u; V^\dagger, V) = \mu^L \text{tr}(\Lambda) = \mu^L,$$

whose norm is constantly 1. $\square$

Theorem 11.3.25 (Virtual local covariance iff a unitary peripheral eigenmatrix exists). [Lean](#) [✓ Stmt](#) Let A be injective, and assume $uu^\dagger = \mathbb{1}$, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then virtual local covariance holds if and only if there exist a unitary $V \in M_D(\mathbb{C})$ and a phase μ with $|\mu| = 1$ such that $\mathcal{E}_u(V) = \mu V$. By Theorem 11.3.18, this witness form is equivalent to the peripheral-spectrum condition $\rho(\mathcal{E}_u) = 1$.

Proof. [✓ Proof](#) The forward direction is Theorem 11.3.22. For the reverse direction, start from a unitary peripheral eigenmatrix. Theorem 11.3.19 gives gauge-phase equivalence with the companion tensor; then Theorem 11.3.21 yields a virtual unitary and phase satisfying the local covariance relation, and Theorem 11.3.23 gives the boundary-state invariance needed for virtual local covariance. $\square$

Theorem 11.3.26 (Virtual-boundary nondecay $\Leftrightarrow$ virtual local covariance). [Lean](#) [✓ Stmt](#) For an injective MPS, assume $uu^\dagger = \mathbb{1}$, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then virtual-boundary nondecay for u holds if and only if A, u, Λ satisfy virtual local covariance.

Proof. [✓ Proof](#) If virtual-boundary nondecay holds, Theorem 11.3.20 gives a gauge-phase equivalence with the companion tensor. Theorem 11.3.21 extracts a virtual unitary and phase from that equivalence, and Theorem 11.3.23 shows that the same unitary preserves the boundary state, so one obtains virtual local covariance. Conversely, Lemma 11.3.24 turns virtual local covariance into a nondecaying virtual-boundary sequence. $\square$

Theorem 11.3.27 (Virtual unitary from virtual-boundary nondecay). [Lean](#) [✓ Stmt](#) For an injective MPS, let $\Lambda \in M_D(\mathbb{C})$, and assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If virtual-boundary nondecay holds for u with boundary parameter Λ , then there exists a virtual unitary V and a unit-modulus scalar μ such that $\sum_j u_{ij} A^j = \mu V A^i V^\dagger$ for all i . Diagrammatically this is the same local intertwining relation, but only up to an overall unit scalar on the virtual side:

The diagram shows an equality between two tensor network expressions. On the left, a vertical line with a red dot at the top (labeled i) and a black dot at the bottom (labeled A) is connected to a horizontal line with a black dot at the bottom (labeled A). A red dot on the vertical line is labeled u . On the right, the expression is μ multiplied by a box labeled V , followed by a horizontal line with a black dot (labeled i at the top and A at the bottom), and then a box labeled $V^\dagger$.

Proof. [✓ Proof](#) Virtual-boundary nondecay first gives gauge-phase equivalence with the companion tensor by Theorem 11.3.20. Theorem 11.3.21 then normalizes the resulting intertwiner to a unitary and produces the phase μ . $\square$

11.4 SPT Phase Labels and Virtual-Boundary Nondecay

The predicate in Definition 11.4.1 compares virtual cocycle labels. The source classification below concerns symmetric paths that remain within the MPS and corresponding parent-Hamiltonian framework, with the endpoint representations embedded into a common physical space as in [SPGC11].

Definition 11.4.1 (Equality of virtual cocycle labels). [Lean](#) [✓ Stmt](#) For tensors A, B and a fixed physical representation $U : G \rightarrow \text{GL}_d(\mathbb{C})$, the predicate called *same SPT phase* asserts the existence of factor systems $\omega_A, \omega_B : G \times G \rightarrow \mathbb{C}^\times$ and projective bond representations ρ_A, ρ_B

such that

$$\begin{aligned}\widetilde{A}_g^i &= \rho_A(g^{-1})A^i\rho_A(g^{-1})^{-1}, \\ \widetilde{B}_g^i &= \rho_B(g^{-1})B^i\rho_B(g^{-1})^{-1},\end{aligned}\tag{11.30}$$

exactly, and $\omega_A \sim \omega_B$. Thus the predicate is equality of virtual factor-system classes. It neither quantifies over Hamiltonian paths nor states physical phase equivalence.

Remark 11.4.2 (Scalar and character conventions). These factor systems take values in $\mathbb{C}^\times$. For unitary bond representations one may choose unit-modulus representatives and obtain the physical classification label in $H^2(G, \text{U}(1))$; this normalization is not an equality $\mathbb{C}^\times = \text{U}(1)$. Moreover, Definition 11.4.1 requires exact tensor equations. A physical state symmetry may instead act on an MPS vector by a one-dimensional character $\chi(g)$, producing a scalar factor in the local tensor equation. Exact MPV-family equality and such a character are distinct hypotheses.

Theorem 11.4.3 (Classification along symmetric MPS parent-Hamiltonian paths). Not ready
For $p = 0, 1$, let A_p be an injective MPS tensor whose parent Hamiltonian H_p has a unique ground state and is symmetric under a unitary representation U^p of G on $\mathcal{H}_p$. Choose the endpoint phase gauges. After a common finite blocking, the embedded endpoints admit a continuous symmetric MPS path with corresponding translationally invariant, finite-range parent Hamiltonians that vary smoothly and remain uniformly gapped if and only if their virtual projective representations determine the same class in $H^2(G, \text{U}(1))$.

More precisely, for the forward construction there exist a path-sector Hilbert space $\mathcal{H}_{\text{path}}$ and a linear representation U^{path} such that the endpoints embed into

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_{\text{path}}, \quad U_g = U_g^0 \oplus U_g^1 \oplus U_g^{\text{path}}.$$

The converse quantifies over smooth paths whose unique ground states remain MPS and whose Hamiltonians are the corresponding parent Hamiltonians, as in [SPGC11, Section II.F, lines 929–953]. It does not assert invariance along an arbitrary gapped Hamiltonian path.

Proof. First deform each blocked injective MPS to its isometric form; the deformation preserves the symmetry relation

$$U_g \mathcal{P} = \mathcal{P}(V_g \otimes \overline{V}_g).$$

Along any continuous symmetric MPS path quantified in the theorem, the tensor and its induced virtual action V_g vary continuously. The cohomology class is discrete, so it cannot change along that path. This proves necessity within the MPS/parent-Hamiltonian framework.

Conversely, let the endpoint isometric forms have virtual actions V_g^0 and V_g^1 in the same cohomology class. Embed their bond spaces into the direct sum and interpolate between the maximally entangled bonds by

$$|\omega(\gamma)\rangle = (1 - \gamma) \sum_{i=1}^{D_0} |i, i\rangle + \gamma \sum_{i=D_0+1}^{D_0+D_1} |i, i\rangle.$$

The commuting parent Hamiltonians along this interpolation remain gapped. Their symmetry is

$$\begin{aligned}(V_g^0 \oplus V_g^1) \otimes (\overline{V}_g^0 \oplus \overline{V}_g^1) &= \widehat{U}_g^0 \oplus \widehat{U}_g^1 \oplus \widehat{U}_g^{\text{path}}, \\ \widehat{U}_g^{\text{path}} &= (V_g^0 \otimes \overline{V}_g^1) \oplus (V_g^1 \otimes \overline{V}_g^0).\end{aligned}\tag{11.31}$$

Equality of the cohomology classes is exactly what makes the cross-term action $\widehat{U}^{\text{path}}$ a linear representation. Conjugating back by the endpoint isometries gives the common direct-sum physical representation and the required symmetric gapped path. $\square$

Theorem 11.4.4 (Virtual-boundary nondecay for injective MPV-symmetric tensors). [Lean](#)

[✓ Stmt](#) *Let A be an injective tensor, symmetric under a unitary representation U . Assume $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then the virtual-boundary nondecay predicate holds for every group element g .*

Proof. [✓ Proof](#) The virtual representation theorem produces $\rho(g^{-1}) \in \text{GL}_D(\mathbb{C})$ satisfying $\sum_j U(g)_{ij} A^j = \rho(g^{-1}) A^i \rho(g^{-1})^{-1}$. A direct computation using $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$ shows that $\mathcal{E}_{U(g)}(\rho(g^{-1})) = \rho(g^{-1})$, so $\rho(g^{-1})$ is a nonzero eigenvector with eigenvalue 1. Theorem 11.3.19 gives gauge-phase equivalence with the companion tensor, and Theorem 11.3.21 extracts a virtual unitary and phase. By Theorem 11.3.23, this unitary preserves the boundary state, so Lemma 11.3.24 gives virtual-boundary nondecay. $\square$

Remark 11.4.5 (Virtual-boundary nondecay does not determine the SPT label). Theorem 11.4.4 applies to every injective symmetric tensor with canonical normalisations, irrespective of its cocycle class, so tensors in different SPT phases all satisfy virtual-boundary nondecay for every group element. This nondecay is equivalent to virtual local covariance (Theorem 11.3.26); the invariant that separates SPT phases is the cohomology class of the virtual representation cocycle (Theorem 11.4.3).

Chapter 12

Parent-Hamiltonian Foundations: Injective and Normal Ground Spaces

For a translation-invariant matrix product state with tensor A , the *parent Hamiltonian* is the translation-invariant sum of local terms, each the orthogonal projector onto the complement of the length- L local space $G_L(A) = \text{im}(\Gamma_L)$. This is the space $\mathcal{G}_L^A$ of [PGVWC07]; we write $\Gamma_L(X)(\sigma) = \text{tr}(A^\sigma X)$, which agrees with the source convention $\text{tr}(XA^\sigma)$ by cyclicity of the trace. Following [PGVWC07, CPGSV21], we identify the N -site space with functions $\{0, \dots, d-1\}^N \rightarrow \mathbb{C}$ on the computational basis.

The first part proves the intersection property for the local spaces and uses it to identify the periodic ground space of an injective tensor. For $N \geq 2$ and $1 < L \leq N$, the parent Hamiltonian is frustration-free and the periodic matrix product vector $V^{(N)}(A)$ is its unique ground state. The periodic-boundary comparison reduces an arbitrary boundary matrix to the commutant of the tensor.

For a normal tensor, injectivity is available after a finite blocking length. The remaining sections develop the inverting-and-growing-back argument at this length and the closure property needed to pass from open intervals to the periodic ring, including the comparison in both directions. The next two chapters study commutation and spectral gaps, followed by the degenerate ground spaces arising from several normal blocks.

12.1 The N -site Hilbert space

Definition 12.1.1 (N -site Hilbert space). [Lean](#) [✓ Stmt](#) For local dimension d and chain length N , the N -site Hilbert space is $\mathcal{H}_N^{(d)} := \{0, \dots, d-1\}^N \rightarrow \mathbb{C}$. This is the computational-basis identification used throughout the chapter for periodic-chain states.

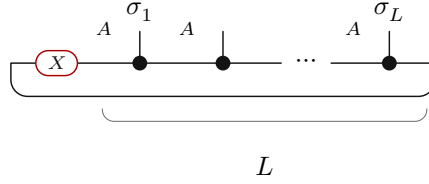
12.2 Local ground space $G_L(A)$

Fix an MPS tensor $A = \{A^i\}_{i=0}^{d-1}$ with $A^i \in M_D(\mathbb{C})$ and a block length $L \geq 1$. For a word $w = (i_1, \dots, i_L)$, write $A^w := A^{i_1} \dots A^{i_L}$. The boundary-condition parametrization is

$$\begin{aligned} \Gamma_L : M_D(\mathbb{C}) &\longrightarrow (\{0, \dots, d-1\}^L \rightarrow \mathbb{C}), \\ \Gamma_L(X)(\sigma) &:= \text{tr}(A^\sigma X). \end{aligned} \quad (12.1)$$

Here A^σ abbreviates $A^{\sigma(0)} \dots A^{\sigma(L-1)}$. By cyclicity of the trace, this is the same as the convention of [PGVWC07], namely $\text{tr}(XA^\sigma)$ for the local space $\mathcal{G}_L^A$.

Definition 12.2.1 (Ground-space map). [Lean](#) [✓ Stmt](#) The map Γ_L in (12.1) is a $\mathbb{C}$ -linear map from $M_D(\mathbb{C})$ to $(\{0, \dots, d-1\}^L \rightarrow \mathbb{C})$. In tensor-network notation,



The chain denotes the word tensor A^σ with open physical indices $\sigma_1, \dots, \sigma_L$, and the red capsule denotes the boundary matrix X inserted on the closing virtual bond before taking the trace. The brace records the chain length L .

Lemma 12.2.2 (Evaluation formula). [Lean](#) [✓ Stmt](#) For every boundary matrix $X \in M_D(\mathbb{C})$ and every length- L word σ , one has $\Gamma_L(X)(\sigma) = \text{tr}(A^\sigma X)$.

Definition 12.2.3 (Local ground space). [Lean](#) [✓ Stmt](#) The local ground space is the image $G_L(A) := \text{range}(\Gamma_L) \subseteq (\{0, \dots, d-1\}^L \rightarrow \mathbb{C})$.

Lemma 12.2.4 (Dimension bound). [Lean](#) [✓ Stmt](#) For every L , one has $\dim G_L(A) \leq D^2$.

Proof. [✓ Proof](#) Since $G_L(A)$ is the range of a linear map from $M_D(\mathbb{C})$, one has $\dim G_L(A) \leq \dim M_D(\mathbb{C}) = D^2$. $\square$

Lemma 12.2.5 (Ambient-space dimension). [Lean](#) [✓ Stmt](#) The above coefficient space satisfies $\dim(\{0, \dots, d-1\}^L \rightarrow \mathbb{C}) = d^L$.

Lemma 12.2.6 (Nontriviality criterion). [Lean](#) [✓ Stmt](#) If $d^L > D^2$, then $G_L(A)$ is a proper subspace of the local Hilbert space, namely $G_L(A) \neq (\{0, \dots, d-1\}^L \rightarrow \mathbb{C})$.

Proof. [✓ Proof](#) If $G_L(A)$ were the whole space, its dimension would be d^L by Lemma 12.2.5; Lemma 12.2.4 would then give the contradiction $d^L \leq D^2$. $\square$

12.3 Parent interaction and chain Hamiltonian

The canonical parent interaction at range L is

$$\begin{aligned} h_L(A) &= \mathbb{1} - \Pi_{G_L(A)} = \Pi_{G_L(A)^\perp}, \\ \ker h_L(A) &= G_L(A). \end{aligned} \quad (12.2)$$

The sources also allow any positive local interaction with this kernel; the orthogonal projector is the canonical representative used here.

Definition 12.3.1 (Local ground space under the canonical ℓ^2 identification). [Lean](#) [✓ Stmt](#) Let $\iota_L: (\{0, \dots, d-1\}^L \rightarrow \mathbb{C}) \simeq \ell^2(\{0, \dots, d-1\}^L)$ be the canonical computational-basis identification. The local ground space under this ℓ^2 identification is $G_L^{\text{ES}}(A) := \iota_L(G_L(A)) \subseteq \ell^2(\{0, \dots, d-1\}^L)$.

Lemma 12.3.2 (Membership under the canonical ℓ^2 identification). [Lean](#) [✓ Stmt](#) For $v \in \ell^2(\{0, \dots, d-1\}^L)$, one has $v \in G_L^{\text{ES}}(A)$ if and only if $\iota_L^{-1}v \in G_L(A)$.

Proof. [✓ Proof](#) This is the defining equation $G_L^{\text{ES}}(A) = \iota_L(G_L(A))$. $\square$

Definition 12.3.3 (Hilbert-space boundary map). [Lean](#) [✓ Stmt](#) Let $U: M_D(\mathbb{C}) \rightarrow \ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$ be Frobenius vectorization in column-row coordinates. The Hilbert-space boundary map is

$$\Gamma_L^{\text{ES}} := \iota_L \circ \Gamma_L \circ U^{-1}: \ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\}) \longrightarrow \ell^2(\{0, \dots, d-1\}^L). \quad (12.3)$$

Thus Γ_L^{ES} has a Euclidean coordinate-space domain, whereas the original map Γ_L has matrix domain $M_D(\mathbb{C})$.

Theorem 12.3.4 (Boundary-map transport under Frobenius vectorization). [Lean](#) [✓ Stmt](#) Let $U: M_D(\mathbb{C}) \rightarrow \ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$ be Frobenius vectorization, with the target indexed in column-row order, and let ι_L be the computational-basis identification on the physical coefficient space. Then, for every $X \in M_D(\mathbb{C})$, $\Gamma_L^{\text{ES}}(U(X)) = \iota_L(\Gamma_L(X))$.

Proof. [✓ Proof](#) By definition, Γ_L^{ES} is the composite of U^{-1} , the original boundary map Γ_L , and ι_L . Applying this composite to $U(X)$ and cancelling $U^{-1}U$ gives the claimed identity. $\square$

Theorem 12.3.5 (Boundary map on a matrix unit). [Lean](#) [✓ Stmt](#) Let $e_{(b,a)}$ denote the Frobenius coordinate vector representing the virtual matrix unit E_{ab} . Then $\Gamma_L^{\text{ES}}(e_{(b,a)}) = \iota_L(\Gamma_L(E_{ab}))$.

Proof. [✓ Proof](#) Frobenius vectorization records a matrix entry (a, b) at the column-row coordinate (b, a) , so $U(E_{ab}) = e_{(b,a)}$. Substitute this identity into the boundary-map transport theorem. $\square$

Theorem 12.3.6 (Exact range of the Hilbert-space boundary map). [Lean](#) [✓ Stmt](#) The range of Γ_L^{ES} is exactly $G_L^{\text{ES}}(A)$.

Proof. [✓ Proof](#) Every virtual vector is $U(X)$ for a unique matrix X , and the transport identity sends it to $\iota_L(\Gamma_L(X))$. These are exactly the vectors in $\iota_L(\text{range } \Gamma_L) = G_L^{\text{ES}}(A)$, proving both inclusions. $\square$

Theorem 12.3.7 (Hilbert-space boundary-map injectivity). [Lean](#) [✓ Stmt](#) If A is L -block injective, then Γ_L^{ES} is injective.

Proof. [✓ Proof](#) Write each virtual vector uniquely as $U(X)$ by Frobenius vectorization. The transport identity reduces equality of their images under Γ_L^{ES} to equality of the images of the corresponding matrices under Γ_L . Block injectivity makes the latter map injective. $\square$

Theorem 12.3.8 (Long-length Hilbert-space boundary-map injectivity). [Lean](#) [✓ Stmt](#) Suppose that $L_0 > 0$, that A is L_0 -block injective, and that $L \geq L_0$. Then Γ_L^{ES} is injective.

Proof. [✓ Proof](#) Positive-length propagation makes A block injective at length L . The preceding theorem then gives injectivity of Γ_L^{ES} . $\square$

Definition 12.3.9 (Ground-space Gram operator). [Lean](#) [✓ Stmt](#) The boundary Gram operator is

$$\mathcal{K}_L := (\Gamma_L^{\text{ES}})^\dagger \Gamma_L^{\text{ES}}. \quad (12.4)$$

Definition 12.3.10 (Operator-level Gram reshuffling). [Lean](#) [Lean](#) [✓ Stmt](#)

For a complex-linear map $\mathcal{L}: M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$, let $R(\mathcal{L})$ be the matrix with entries

$$R(\mathcal{L})_{(b,a),(e,c)} := J(\mathcal{L})_{(c,e),(a,b)}. \quad (12.5)$$

The operator $\mathcal{R}(\mathcal{L})$ on $\ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$ is represented by the matrix $R(\mathcal{L})$ in the standard orthonormal basis. Thus the row-column pair $((b, a), (e, c))$ is sent explicitly to the Choi row-column pair $((c, e), (a, b))$.

Theorem 12.3.11 (Gram reshuffling commutes with subtraction). [Lean](#) [✓ Stmt](#) For complex-linear maps $\mathcal{L}, \mathcal{M}: M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$,

$$\mathcal{R}(\mathcal{L} - \mathcal{M}) = \mathcal{R}(\mathcal{L}) - \mathcal{R}(\mathcal{M}). \quad (12.6)$$

Proof. [✓ Proof](#) The defining matrix entries of the Choi reshuffling are linear in the map. $\square$

Theorem 12.3.12 (Matrix-unit coordinates of Gram reshuffling). [Lean](#) [✓ Stmt](#) Let $\mathcal{L}: M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ be complex-linear and let $\mathbf{e}_{(j,i)}$ be the standard unit vector at (j, i) . For all $a, b, c, e \in \{0, \dots, D-1\}$,

$$\langle \mathbf{e}_{(b,a)}, \mathcal{R}(\mathcal{L}) \mathbf{e}_{(e,c)} \rangle_{\ell^2} = J(\mathcal{L})_{(c,e),(a,b)}. \quad (12.7)$$

This is the coordinate identity in (12.5); it is not a Hilbert–Schmidt pairing of E_{ab} with $\mathcal{L}(E_{ce})$.

Proof. [✓ Proof](#) Evaluate the matrix of $\mathcal{R}(\mathcal{L})$ on the two standard basis vectors. $\square$

Theorem 12.3.13 (Euclidean operator norm bounded by Frobenius mass). [Lean](#) [✓ Stmt](#) Let I be a finite index set and let $B = (B_{ij})_{i,j \in I}$ be a complex matrix. Then the squared operator norm of B on $\ell^2(I)$ satisfies

$$\|B\|_{\ell^2(I) \rightarrow \ell^2(I)}^2 \leq \sum_{(i,j) \in I \times I} |B_{ij}|^2. \quad (12.8)$$

Proof. [✓ Proof](#) Apply Cauchy–Schwarz to each row of Bx , sum over the rows, and take the supremum over unit vectors $x \in \ell^2(I)$. $\square$

Theorem 12.3.14 (Squared mass of one column). [Lean](#) [✓ Stmt](#) Let I be a finite index set, let $A = (A_{ij})_{i,j \in I}$ be a complex matrix, and let $j \in I$. Then

$$\sum_{i \in I} |A_{ij}|^2 \leq \|A\|_{\ell^2(I) \rightarrow \ell^2(I)}^2. \quad (12.9)$$

Proof. [✓ Proof](#) Apply A to the unit vector supported at j . Its image is the j th column of A , while the vector itself has Euclidean norm one. $\square$

Theorem 12.3.15 (Entrywise mass bounded by the L^2 operator norm). [Lean](#) [✓ Stmt](#) Let I be a finite index set and let $A = (A_{ij})_{i,j \in I}$ be a complex matrix. Then

$$\sum_{(i,j) \in I \times I} |A_{ij}|^2 \leq |I| \|A\|_{\ell^2(I) \rightarrow \ell^2(I)}^2. \quad (12.10)$$

Proof. [✓ Proof](#) Sum Theorem 12.3.14 over all columns. $\square$

Theorem 12.3.16 (L^2 operator norm of a matrix unit). [Lean](#) [✓ Stmt](#) Let I be a finite index set. For every $i, j \in I$, the matrix unit E_{ij} satisfies

$$\|E_{ij}\|_{\ell^2(I) \rightarrow \ell^2(I)} = 1. \quad (12.11)$$

Proof. [✓ Proof](#) The matrix unit sends the unit vector at j to the unit vector at i and sends every vector to a scalar multiple of that unit vector. $\square$

Theorem 12.3.17 (Entrywise Frobenius invariance of Gram reshuffling). [Lean](#) [✓ Stmt](#) For every complex-linear map $\mathcal{L}: M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$, the sum of squared entry norms of $R(\mathcal{L})$ equals that of the rectangular Choi matrix:

$$\sum_{p: (\{0, \dots, D-1\} \times \{0, \dots, D-1\}) \times (\{0, \dots, D-1\} \times \{0, \dots, D-1\})} \|R(\mathcal{L})_{p_1, p_2}\|^2 = \sum_{p: (\{0, \dots, D-1\} \times \{0, \dots, D-1\}) \times (\{0, \dots, D-1\} \times \{0, \dots, D-1\})} \|J(\mathcal{L})_{p_1, p_2}\|^2. \quad (12.12)$$

The two matrices differ only by a reshuffling of indices; the equality is entrywise Frobenius, not operator norm.

Proof. [✓ Proof](#) The map $((b, a), (e, c)) \mapsto ((c, e), (a, b))$ is an involution (bijection) on the product index space. Reindexing the sum of squared norms by this involution yields the claimed equality. $\square$

Theorem 12.3.18 (Operator bound for a reshuffled Choi matrix). [Lean](#) [✓ Stmt](#) Let I be a finite index set and let $\mathcal{L}: \mathbb{C}^{I \times I} \rightarrow \mathbb{C}^{I \times I}$ be complex-linear. Then

$$\|\mathcal{R}(\mathcal{L})\|_{\ell^2(I \times I) \rightarrow \ell^2(I \times I)}^2 \leq \sum_{u, v \in I \times I} |J(\mathcal{L})_{u, v}|^2. \quad (12.13)$$

Proof. [✓ Proof](#) Apply Theorem 12.3.13 to $R(\mathcal{L})$, then use Theorem 12.3.17 to rewrite its entrywise Frobenius mass as that of $J(\mathcal{L})$. $\square$

Theorem 12.3.19 (Dimension bound for Gram reshuffling). [Lean](#) [✓ Stmt](#) Let I be a finite index set, equip $\mathbb{C}^{I \times I}$ with its L^2 operator norm, and let $\mathcal{L}: \mathbb{C}^{I \times I} \rightarrow \mathbb{C}^{I \times I}$ be complex-linear. Denote the operator norm of $\mathcal{L}$ for these matrix norms by $\|\mathcal{L}\|_{\text{op}, L^2}$. Then

$$\|\mathcal{R}(\mathcal{L})\|_{\ell^2(I \times I) \rightarrow \ell^2(I \times I)}^2 \leq |I|^3 \|\mathcal{L}\|_{\text{op}, L^2}^2. \quad (12.14)$$

Proof. [✓ Proof](#) Use Theorem 12.3.18 and expand the Choi matrix in matrix units. For each of the $|I|^2$ input matrix units, Theorem 12.3.15 contributes a factor $|I|$. The operator-norm estimate for $\mathcal{L}$ and Theorem 12.3.16 bound the remaining matrix norm. $\square$

Theorem 12.3.20 (Geometric power bound below unit spectral radius). [Lean](#) [✓ Stmt](#) Let V be a complex Banach space and let $T: V \rightarrow V$ be a bounded linear operator. If $\rho_{\text{spec}}(T) < 1$, then there are real constants C, r with $C > 0$ and $0 < r < 1$ such that

$$\|T^n\| \leq Cr^n \quad (12.15)$$

for every $n \in \mathbb{N}$.

Proof. [✓ Proof](#) Choose r strictly between the spectral radius and one. Gelfand's formula gives $\|T^n\| < r^n$ for all sufficiently large n ; enlarging the constant C absorbs the finitely many remaining powers. $\square$

Theorem 12.3.21 (Pointwise geometric power bound below unit spectral radius). [Lean](#) [✓ Stmt](#) Let V be a complex Banach space and let $T: V \rightarrow V$ be a bounded linear operator. If $\rho_{\text{spec}}(T) < 1$, then there are real constants C, r with $C > 0$ and $0 < r < 1$ such that

$$\|T^n x\| \leq Cr^n \|x\| \quad (12.16)$$

for every $n \in \mathbb{N}$ and $x \in V$.

Proof. [✓ Proof](#) Apply Theorem 12.3.20 and the defining operator-norm inequality to $T^n x$. $\square$

Theorem 12.3.22 (Transfer–Gram Choi reshuffling). [Lean](#) [✓ Stmt](#) Let $\mathbf{e}_{(j,i)} := \text{single}_{(j,i)}(1)$ denote the standard unit vector in $\ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$, and let $\mathcal{E}_A$ be the transfer map. Then

$$\langle \mathbf{e}_{(b,a)}, \mathcal{K}_L(\mathbf{e}_{(e,c)}) \rangle_{\ell^2} = J(\mathcal{E}_A^L)_{(c,e),(a,b)}. \quad (12.17)$$

Under Frobenius vectorization, $\mathbf{e}_{(b,a)} = U(E_{ab})$ and $\mathbf{e}_{(e,c)} = U(E_{ce})$, explaining the reversed coordinate order. Thus the Gram operator is the indicated Choi reshuffling; in particular it is not obtained by replacing the right-hand side with $\langle E_{ab}, \mathcal{E}_A^L(E_{ce}) \rangle_{\text{HS}}$.

Proof. [✓ Proof](#) Move the adjoint in $\mathcal{K}_L = (\Gamma_L^{\text{ES}})^\dagger \Gamma_L^{\text{ES}}$ across the inner product and expand the two boundary-map coefficients. For each physical word σ , the resulting summand is $(A^\sigma)_{ec} \overline{(A^\sigma)_{ba}}$. On the other hand, expanding the transfer power on E_{ca} gives

$$(\mathcal{E}_A^L(E_{ca}))_{eb} = \sum_{\sigma} (A^\sigma)_{ec} \overline{(A^\sigma)_{ba}}. \quad (12.18)$$

The defining Choi indexing $J(\mathcal{L})_{(c,e),(a,b)} = (\mathcal{L}(E_{ca}))_{eb}$ now yields the stated reshuffling. $\square$

Theorem 12.3.23 (Ground-space Gram operator as a reshuffled Choi matrix). [Lean](#) [✓ Stmt](#) The ground-space Gram operator is exactly the operator-level reshuffling of the Choi matrix of the iterated transfer map:

$$\mathcal{K}_L = \mathcal{R}(\mathcal{E}_A^L). \quad (12.19)$$

The equality is an equality of operators on $\ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$. Its entries are those in (12.7), rather than Hilbert–Schmidt pairings with $\mathcal{E}_A^L$.

Proof. [✓ Proof](#) The two operators have equal matrix coefficients on every pair of standard basis vectors. Hence they are equal. $\square$

Theorem 12.3.24 (Geometric convergence of the ground-space Gram operator). [Lean](#) [✓ Stmt](#) Let A be an MPS tensor of bond dimension D , let $\rho \in M_D(\mathbb{C})$ satisfy $\text{tr}(\rho) \neq 0$, and let P_ρ be the fixed-point projection

$$P_\rho(X) = \frac{\text{tr}(X)}{\text{tr}(\rho)} \rho. \quad (12.20)$$

Suppose that $\mathcal{E}_A$ is trace-preserving, $\mathcal{E}_A(\rho) = \rho$, and

$$\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1. \quad (12.21)$$

Then there are real constants C, r with $C > 0$ and $0 < r < 1$ such that, for every $n \geq 1$,

$$\|\mathcal{K}_n - \mathcal{R}(P_\rho)\|^2 \leq D^3 (Cr^n)^2. \quad (12.22)$$

No positivity, injectivity, or uniqueness hypothesis is imposed.

Proof. ✓ Proof Write $N = \mathcal{E}_A - P_\rho$. The power decomposition gives $\mathcal{E}_A^n = P_\rho + N^n$ for $n \geq 1$, while Theorem 12.3.20 gives $\|N^n\| \leq Cr^n$. Use the exact Gram reshuffling identity and Theorem 12.3.11 to rewrite the left-hand side as $\|\mathcal{R}(N^n)\|^2$, then apply Theorem 12.3.19. $\square$

Theorem 12.3.25 (Primitive-MPS geometric Gram convergence). Lean ✓ Stmt *Let A be a primitive MPS tensor with fixed-point witness ρ . Then there are real constants C, r with $C > 0$ and $0 < r < 1$ such that, for every $n \geq 1$,*

$$\|\mathcal{K}_n - \mathcal{R}(P_\rho)\|^2 \leq D^3(Cr^n)^2. \quad (12.23)$$

Proof. ✓ Proof Positive semidefiniteness and nonvanishing of ρ imply $\text{tr}(\rho) \neq 0$. The trace-preserving identity, fixed-point equation, and complementary spectral-radius bound are the remaining fields of the primitive-MPS hypothesis, so Theorem 12.3.24 applies. $\square$

Theorem 12.3.26 (Ground-space Gram convergence). Lean ✓ Stmt *Let A be an MPS tensor of bond dimension D , let $\rho \in M_D(\mathbb{C})$ satisfy $\text{tr}(\rho) \neq 0$, and let P_ρ be the corresponding fixed-point projection. If $\mathcal{E}_A$ is trace-preserving, $\mathcal{E}_A(\rho) = \rho$, and $\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1$, then*

$$\mathcal{K}_n \rightarrow \mathcal{R}(P_\rho) \quad (12.24)$$

in operator norm as $n \rightarrow \infty$.

Proof. ✓ Proof The right-hand side of (12.22) tends to zero because $0 < r < 1$. Taking square roots gives the asserted norm convergence. $\square$

Theorem 12.3.27 (Primitive-MPS ground-space Gram convergence). Lean ✓ Stmt *If A is a primitive MPS tensor with fixed-point witness ρ , then*

$$\mathcal{K}_n \rightarrow \mathcal{R}(P_\rho) \quad (12.25)$$

in operator norm as $n \rightarrow \infty$.

Proof. ✓ Proof The primitive-MPS data supply the nonzero trace, trace preservation, fixed-point equation, and complementary spectral-radius hypothesis of Theorem 12.3.26. $\square$

The limit in Theorem 12.3.26 is the reshuffled fixed-point projection $\mathcal{R}(P_\rho)$, not the identity in unweighted Frobenius coordinates. The limiting metric is generally nonidentity. Under a positive-definite fixed-point hypothesis, the inverse estimates below perturb around $\mathcal{R}(P_\rho)$ rather than around the identity. They do not yet compare the physical range projectors on overlapping windows.

Theorem 12.3.28 (Exact limiting Gram metric). Lean Lean Lean ✓ Stmt *Let $\rho \in M_D(\mathbb{C})$ have nonzero trace and define $P_\rho(X) := (\text{tr } X / \text{tr } \rho) \rho$. Then*

$$R(P_\rho) = (\text{tr } \rho)^{-1}(\rho^\top \otimes I_D), \quad (12.26)$$

$$\mathcal{R}(P_\rho)U(X) = U((\text{tr } \rho)^{-1}X\rho). \quad (12.27)$$

Proof. ✓ Proof For matrix units E_{ca} , direct evaluation gives

$$R(P_\rho)_{(b,a),(e,c)} = (\text{tr } \rho)^{-1} \rho_{eb} \delta_{ca}. \quad (12.28)$$

These are exactly the entries in (12.26). Applying that Kronecker product to the column-row vectorization of X gives (12.27). $\square$

Theorem 12.3.29 (Invertibility of the limiting Gram metric). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $D > 0$ and $\rho \in M_D(\mathbb{C})$ is positive definite, then $R(P_\rho)$ is positive definite and $\mathcal{R}(P_\rho)$ is invertible, hence injective.

Proof. [✓ Proof](#) Positive definiteness gives $\text{tr } \rho > 0$. By (12.26),

$$R(P_\rho) = (\text{tr } \rho)^{-1}(\rho^\top \otimes I_D) > 0. \quad (12.29)$$

Hence its associated operator $\mathcal{R}(P_\rho)$ is invertible. $\square$

Theorem 12.3.30 (Arbitrary-base Neumann inverse bound). [Lean](#) [Lean](#) [✓ Stmt](#) Let K_0 be invertible in a normed ring with convergent geometric series and $\|I\| = 1$. If $a < 1$ and

$$\|K_0^{-1}(K - K_0)\| \leq a, \quad (12.30)$$

then K is invertible and

$$\|K^{-1}\| \leq (1 - a)^{-1} \|K_0^{-1}\|. \quad (12.31)$$

The same conclusion follows from $\|K_0^{-1}\| \|K - K_0\| \leq a$.

Proof. [✓ Proof](#) Put $E := K_0^{-1}(K - K_0)$. Then $K = K_0(I + E)$ and $\|E\| < 1$, so

$$K^{-1} = (I + E)^{-1} K_0^{-1}, \quad (12.32)$$

$$\|K^{-1}\| \leq \|(I + E)^{-1}\| \|K_0^{-1}\| \leq (1 - a)^{-1} \|K_0^{-1}\|. \quad (12.33)$$

Submultiplicativity reduces the product-form hypothesis to (12.30). $\square$

Theorem 12.3.31 (Arbitrary-base inverse-displacement bound). [Lean](#) [Lean](#) [✓ Stmt](#) Let K_0 be invertible in a normed ring with convergent geometric series and $\|I\| = 1$. If $a < 1$ and (12.30) holds, then

$$\|K^{-1} - K_0^{-1}\| \leq (1 - a)^{-1} a \|K_0^{-1}\|. \quad (12.34)$$

The same estimate follows from $\|K_0^{-1}\| \|K - K_0\| \leq a$.

Proof. [✓ Proof](#) With $E := K_0^{-1}(K - K_0)$, the resolvent identity is

$$K^{-1} - K_0^{-1} = -(I + E)^{-1} E K_0^{-1}. \quad (12.35)$$

Since $\|(I + E)^{-1}\| \leq (1 - a)^{-1}$ and $\|E\| \leq a$, taking norms gives (12.34). Submultiplicativity again gives the product-form variant. $\square$

Theorem 12.3.32 (Continuity of inversion at an invertible limit). [Lean](#) [✓ Stmt](#) Let K_i be a net in a normed ring with convergent geometric series. If $K_i \rightarrow K_0$ and K_0 is invertible, then

$$K_i^{-1} \rightarrow K_0^{-1}. \quad (12.36)$$

Proof. [✓ Proof](#) Write K_0 as the image of a unit. Continuity of inversion on the unit group, followed by the convergence $K_i \rightarrow K_0$, gives the result. $\square$

Definition 12.3.33 (Inverse Gram operator). [Lean](#) [✓ Stmt](#) For an injective continuous linear map T with finite-dimensional domain, let S_T be the continuous endomorphism induced by the inverse linear equivalence of the Gram endomorphism $T^\dagger T$.

Theorem 12.3.34 (Inverse-Gram identities). Lean Lean Lean Lean Lean Lean ✓ Stmt Let $\mathbb{K}$ be a real or complex scalar field, let E be a finite-dimensional complete $\mathbb{K}$ -Hilbert space, and let F be a complete $\mathbb{K}$ -Hilbert space. If $T: E \rightarrow F$ is an injective continuous linear map, then the inverse Gram operator is a two-sided inverse of the Gram endomorphism:

$$S_T(T^\dagger T) = I_E, \quad (12.37)$$

$$(T^\dagger T)S_T = I_E. \quad (12.38)$$

Consequently, S_T is both the canonical inverse and the ring inverse of $T^\dagger T$. The two pseudoinverse factors satisfy

$$\|TS_T\| = \sqrt{\|S_T\|}, \quad (12.39)$$

$$\|S_T T^\dagger\| = \sqrt{\|S_T\|}. \quad (12.40)$$

Proof. ✓ Proof By definition, S_T is induced by the inverse linear equivalence of the injective endomorphism $T^\dagger T$. The inverse-equivalence identities give (12.37) and (12.38). Together these equations characterize the canonical inverse of a continuous linear endomorphism, which agrees with its inverse in the endomorphism ring. For the norm identities, use the C^* -identity and the two inverse equations to compute the squares of $\|TS_T\|$ and $\|S_T T^\dagger\|$. $\square$

The next three results give quantitative inverse bounds for the nonidentity limiting metric $\mathcal{R}(P_\rho)$. They require the fixed-point witness ρ to be positive definite and preserve the squared-bound normalization of Theorem 12.3.25. They do not estimate a contraction between physical projectors on overlapping windows.

Theorem 12.3.35 (Pointwise geometric Gram inverse bounds). Lean ✓ Stmt Let A be a primitive MPS tensor of nonzero bond dimension whose fixed-point witness ρ is positive definite. Put

$$\mathcal{K}_\infty = \mathcal{R}(P_\rho), \quad \mathcal{J}_\infty = \mathcal{K}_\infty^{-1}. \quad (12.41)$$

There are constants $g, c, r > 0$ with $r < 1$ and $c = \|\mathcal{J}_\infty\|g$ such that, for every $n \geq 1$,

$$\|\mathcal{K}_n - \mathcal{K}_\infty\| \leq gr^n. \quad (12.42)$$

Whenever $cr^n < 1$, the operator $\mathcal{K}_n$ is invertible and

$$\|\mathcal{K}_n^{-1}\| \leq \frac{\|\mathcal{J}_\infty\|}{1 - cr^n}, \quad (12.43)$$

$$\|\mathcal{K}_n^{-1} - \mathcal{J}_\infty\| \leq \frac{cr^n}{1 - cr^n} \|\mathcal{J}_\infty\|. \quad (12.44)$$

Proof. ✓ Proof Start from the squared estimate $\|\mathcal{K}_n - \mathcal{K}_\infty\|^2 \leq D^3(Cr^n)^2$. Taking square roots gives $g = \sqrt{D^3}C$ without any additional dimension loss, and hence (12.42). Positive definiteness makes $\mathcal{K}_\infty$ invertible. The relative error is at most $\|\mathcal{J}_\infty\|gr^n = cr^n$, so the two arbitrary-base Neumann estimates give (12.43) and (12.44). $\square$

Theorem 12.3.36 (Pointwise geometric inverse-Gram bounds). Lean ✓ Stmt Under the hypotheses of Theorem 12.3.35, there are constants $g, c, r > 0$ with $r < 1$ and $c = \|\mathcal{J}_\infty\|g$ such that (12.42) holds for every $n \geq 1$. If $cr^n < 1$, then the physical boundary map Γ_n^{ES} is injective, its inverse Gram operator equals $\mathcal{K}_n^{-1}$, and

$$\|S_{\Gamma_n^{\text{ES}}}\| \leq \frac{\|\mathcal{J}_\infty\|}{1 - cr^n}, \quad (12.45)$$

$$\|S_{\Gamma_n^{\text{ES}}} - \mathcal{J}_\infty\| \leq \frac{cr^n}{1 - cr^n} \|\mathcal{J}_\infty\|. \quad (12.46)$$

Proof. ✓ Proof Invertibility of $(\Gamma_n^{\text{ES}})^\dagger \Gamma_n^{\text{ES}} = \mathcal{K}_n$ implies injectivity of Γ_n^{ES} . The inverse-Gram identities identify $S_{\Gamma_n^{\text{ES}}}$ with $\mathcal{K}_n^{-1}$, so the preceding pointwise bounds apply unchanged. $\square$

Theorem 12.3.37 (Geometric smallness at the C3 correction lengths). Lean ✓ Stmt Let $c > 0$, let $0 < r < 1$, and let $l, K \in \mathbb{N}$. If $cr^l < 1$, then

$$cr^{l+1} < 1, \quad (12.47)$$

$$cr^{K+l} < 1, \quad (12.48)$$

$$cr^{K+l+1} < 1. \quad (12.49)$$

Proof. ✓ Proof Since $0 < r < 1$, the sequence r^m is nonincreasing. Each of $l+1$, $K+l$, and $K+l+1$ is at least l , so every displayed inequality follows from $cr^l < 1$. $\square$

Theorem 12.3.38 (Eventual inverse-Gram control in the limiting metric). Lean ✓ Stmt Let A be a primitive MPS tensor whose fixed-point witness ρ is positive definite, and put $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$. For every $0 < a < 1$, all sufficiently large n satisfy

$$\begin{aligned} \mathcal{K}_n &\in \text{GL}, \\ \|\mathcal{K}_n^{-1} - \mathcal{K}_\infty^{-1}\| &\leq \frac{a}{1-a} \|\mathcal{K}_\infty^{-1}\|. \end{aligned} \quad (12.50)$$

Proof. ✓ Proof The positive-definite fixed point makes $\mathcal{K}_\infty$ invertible. Gram convergence implies that eventually

$$\|\mathcal{K}_\infty^{-1}\| \|\mathcal{K}_n - \mathcal{K}_\infty\| \leq a. \quad (12.51)$$

Factoring $\mathcal{K}_n$ around the invertible base $\mathcal{K}_\infty$ and applying the arbitrary-base Neumann estimate gives (12.50). $\square$

Theorem 12.3.39 (Convergence of finite-volume inverse Gram operators). Lean ✓ Stmt Under the same primitive-MPS and positive-definite fixed-point hypotheses,

$$\mathcal{K}_n^{-1} \longrightarrow \mathcal{K}_\infty^{-1} \quad (12.52)$$

in operator norm.

Proof. ✓ Proof Inversion is continuous at the invertible operator $\mathcal{K}_\infty$, so the claim follows from $\mathcal{K}_n \rightarrow \mathcal{K}_\infty$. $\square$

Theorem 12.3.40 (Eventual injectivity and inverse-Gram control). Lean ✓ Stmt Under the same hypotheses, for every $0 < a < 1$ and all sufficiently large n , the physical boundary map Γ_n^{ES} is injective and its inverse Gram operator is $\mathcal{K}_n^{-1}$. Moreover,

$$\|S_{\Gamma_n^{\text{ES}}} - \mathcal{K}_\infty^{-1}\| \leq \frac{a}{1-a} \|\mathcal{K}_\infty^{-1}\|. \quad (12.53)$$

Proof. ✓ Proof Eventual invertibility of $(\Gamma_n^{\text{ES}})^\dagger \Gamma_n^{\text{ES}}$ implies eventual injectivity of Γ_n^{ES} . The inverse-Gram identities then identify $S_{\Gamma_n^{\text{ES}}}$ with $\mathcal{K}_n^{-1}$, and the preceding estimate applies. $\square$

Definition 12.3.41 (Injective range-projector candidate). Lean ✓ Stmt For injective T , define the operator $Q_T := TS_T T^\dagger$ on its codomain.

Theorem 12.3.42 (Injective range-projector formula). [Lean](#) [✓ Stmt](#) If T is an injective map from a finite-dimensional Hilbert space into a complete Hilbert space, then

$$P_{\text{range } T} = T(T^\dagger T)^{-1}T^\dagger. \quad (12.54)$$

Proof. [✓ Proof](#) The vector $Q_T x$ lies in $\text{range } T$. Moreover, $T^\dagger Q_T x = (T^\dagger T)S_T T^\dagger x = T^\dagger x$, so for every y the adjoint identity gives $\langle x - Q_T x, Ty \rangle = 0$. Thus $x - Q_T x$ is orthogonal to $\text{range } T$, and the range-orthogonal decomposition characterizing the orthogonal projector gives $Q_T = P_{\text{range } T}$. $\square$

Theorem 12.3.43 (Physical ground-space range projector). [Lean](#) [✓ Stmt](#) If A is L -block injective, then the injective range projector of Γ_L^{ES} is the physical orthogonal projector onto the local ground space:

$$Q_{\Gamma_L^{\text{ES}}} = P_{G_L^{\text{ES}}(A)}. \quad (12.55)$$

In particular, this is the physical range projector on the local Hilbert space, not the virtual transfer fixed-point projector.

Proof. [✓ Proof](#) The general injective range-projector theorem identifies $Q_{\Gamma_L^{\text{ES}}}$ with the orthogonal projector onto $\text{range}(\Gamma_L^{\text{ES}})$. The exact-range theorem identifies this range with $G_L^{\text{ES}}(A)$. $\square$

Theorem 12.3.44 (Physical ground-space inverse-Gram projector formula). [Lean](#) [✓ Stmt](#) Suppose that $L_0 > 0$, that A is L_0 -block injective, and that $L \geq L_0$. Then the physical orthogonal projector onto the local ground space is

$$P_{G_L^{\text{ES}}(A)} = \Gamma_L^{\text{ES}} ((\Gamma_L^{\text{ES}})^\dagger \Gamma_L^{\text{ES}})^{-1} (\Gamma_L^{\text{ES}})^\dagger. \quad (12.56)$$

Thus the displayed operator is the physical range projector on $\ell^2(\{0, \dots, d-1\}^L)$.

Proof. [✓ Proof](#) Positive-length propagation makes A block injective at length L . Apply the physical range-projector identity at that length and unfold its inverse-Gram expression. $\square$

Theorem 12.3.45 (Neumann inverse estimate). [Lean](#) [✓ Stmt](#) If $\|t\| \leq a < 1$ in a normed ring with convergent geometric series and $\|1\| = 1$, then

$$\|(1-t)^{-1}\| \leq (1-a)^{-1}. \quad (12.57)$$

Proof. [✓ Proof](#) Since $\|t\| \leq a < 1$, the Neumann series converges and $(1-t)^{-1} = \sum_{n \geq 0} t^n$. The triangle inequality and submultiplicativity give

$$\left\| \sum_{n \geq 0} t^n \right\| \leq \sum_{n \geq 0} \|t\|^n = (1 - \|t\|)^{-1} \leq (1-a)^{-1}, \quad (12.58)$$

where the last step uses monotonicity of the reciprocal on the positive reals. $\square$

Theorem 12.3.46 (Quantitative inverse-Gram estimate). [Lean](#) [✓ Stmt](#) Let $\mathbb{K}$ be a real or complex scalar field, let E be a finite-dimensional complete $\mathbb{K}$ -Hilbert space, and let F be a complete $\mathbb{K}$ -Hilbert space. Let $T: E \rightarrow F$ be an injective continuous linear map. If $a \in \mathbb{R}$ satisfies $a < 1$ and

$$\|I_E - T^\dagger T\| \leq a, \quad (12.59)$$

then

$$\|S_T\| \leq (1-a)^{-1}. \quad (12.60)$$

Proof. ✓ Proof If E is trivial, then $S_T = 0$ and the bound follows from $a < 1$. Otherwise, identify S_T with the ring inverse of $T^\dagger T$ and apply Theorem 12.3.45 to $t = I_E - T^\dagger T$. $\square$

Definition 12.3.47 (Canonical parent interaction). Lean ✓ Stmt The parent interaction at range L is the canonical orthogonal projector specified in (12.2).

Lemma 12.3.48 (Parent interaction is idempotent). Lean ✓ Stmt The local parent interaction is a projector: $h_L(A)^2 = h_L(A)$.

Proof. ✓ Proof This is the idempotency of the orthogonal projector onto $G_L(A)^\perp$, transported from the Hilbert-space realization of the local coefficient space. $\square$

Lemma 12.3.49 (Parent interaction annihilates ground-space elements). Lean ✓ Stmt For any $v \in G_L(A)$, one has $h_L(A)v = 0$.

Proof. ✓ Proof Since $h_L(A)$ is the orthogonal projector onto $G_L(A)^\perp$, it annihilates every element of $G_L(A)$. $\square$

Definition 12.3.50 (Restriction to a cyclic interval). Lean ✓ Stmt For $\sigma \in \{0, \dots, d-1\}^N$ and a starting site i , write $\sigma|_{[i, i+L-1]} \in \{0, \dots, d-1\}^L$ for the word defined by $(\sigma|_{[i, i+L-1]})(r) = \sigma(i+r \bmod N)$ for $0 \leq r < L$.

Definition 12.3.51 (Prescribing a cyclic interval). Lean ✓ Stmt If $L \leq N$ and $\tau \in \{0, \dots, d-1\}^L$, write $\sigma^{[i, i+L-1] \leftarrow \tau}$ for the configuration defined by

$$\sigma^{[i, i+L-1] \leftarrow \tau}(k) = \begin{cases} \tau(r), & k \equiv i+r \pmod{N} \text{ for some } 0 \leq r < L, \\ \sigma(k), & \text{otherwise.} \end{cases}$$

The index r in the first case is unique because $L \leq N$.

Lemma 12.3.52 (Extracting a replaced window). Lean ✓ Stmt If $L \leq N$, then restricting to the same cyclic interval after prescribing its values recovers the inserted word: $(\sigma^{[i, i+L-1] \leftarrow \tau})|_{[i, i+L-1]} = \tau$.

Lemma 12.3.53 (Replacing an extracted window). Lean ✓ Stmt If $L \leq N$, then prescribing on a cyclic interval the values already carried by σ leaves σ unchanged: $\sigma^{[i, i+L-1] \leftarrow \sigma|_{[i, i+L-1]}} = \sigma$.

Proof. ✓ Proof Case-split on whether the offset $(k-i+N) \bmod N$ lies in $[0, L)$: inside the cyclic interval the prescribed value is the original value of σ ; outside the interval both sides already agree with σ . $\square$

Lemma 12.3.54 (Replacing the same window twice). Lean ✓ Stmt If $L \leq N$, then two prescriptions on the same cyclic interval reduce to the second prescription:

$$(\sigma^{[i, i+L-1] \leftarrow \tau})^{[i, i+L-1] \leftarrow v} = \sigma^{[i, i+L-1] \leftarrow v}.$$

Proof. ✓ Proof On the cyclic interval both sides take the value prescribed by v . Away from that interval neither second prescription changes the value. $\square$

For a periodic chain of length N , the parent Hamiltonian is the sum of translated copies of the local interaction.

Definition 12.3.55 (Translated local term). [Lean](#) [✓ Stmt](#) For each site index $i \in \{0, \dots, N-1\}$, the translated local term is determined by

$$(h_i \psi)(\sigma) = \begin{cases} [h_L(A) (\tau \mapsto \psi(\sigma^{[i, i+L-1] \leftarrow \tau}))] (\sigma|_{[i, i+L-1]}), & L \leq N, \\ 0, & L > N. \end{cases} \quad (12.61)$$

Lemma 12.3.56 (Pointwise form of a translated local term). [Lean](#) [✓ Stmt](#) If $L \leq N$, then the translated term is obtained by applying the local parent interaction on the chosen cyclic window:

$$(h_i(A, L) \psi)(\sigma) = [h_L(A) (\tau \mapsto \psi(\sigma^{[i, i+L-1] \leftarrow \tau}))] (\sigma|_{[i, i+L-1]}).$$

Proof. [✓ Proof](#) This is the first case in (12.61). □

Lemma 12.3.57 (Translated local terms are idempotent). [Lean](#) [✓ Stmt](#) For every site i , one has $h_i(A, L)^2 = h_i(A, L)$.

Proof. [✓ Proof](#) If $L > N$, the translated term is zero. Suppose $L \leq N$ and put $f_{\sigma, i}(\tau) := \psi(\sigma^{[i, i+L-1] \leftarrow \tau})$. The pointwise formula for translated terms, together with Lemmas 12.3.52 and 12.3.54, gives

$$(h_i(A, L)^2 \psi)(\sigma) = (h_L(A)^2 f_{\sigma, i})(\sigma|_{[i, i+L-1]}) = (h_L(A) f_{\sigma, i})(\sigma|_{[i, i+L-1]}) = (h_i(A, L) \psi)(\sigma).$$

Here the second equality follows from Lemma 12.3.48. □

Definition 12.3.58 (Parent Hamiltonian). [Lean](#) [✓ Stmt](#) The chain Hamiltonian is

$$H_N(A, L) := \sum_{i=0}^{N-1} h_i. \quad (12.62)$$

Definition 12.3.59 (Frustration-free condition). [Lean](#) [✓ Stmt](#) A vector ψ is a frustration-free ground state of the parent Hamiltonian when each local interaction annihilates it, that is, when $h_i \psi = 0$ for every $i \in \{0, \dots, N-1\}$.

Lemma 12.3.60 (Periodic MPS vector window membership). [Lean](#) [✓ Stmt](#) For any window of L consecutive sites starting at position i in an N -site periodic chain with $L \leq N$, the restriction of the periodic MPS vector $V^{(N)}(A)$ to that window lies in $G_L(A)$.

Proof. [✓ Proof](#) The boundary matrix is the product of the A -matrices on the complement sites. Trace cyclicity rotates the full N -site product so that the window indices come first, matching the definition of the ground-space map. □

Lemma 12.3.61 (Local term annihilates the periodic MPS vector). [Lean](#) [✓ Stmt](#) For each site i , one has $h_i V^{(N)}(A) = 0$.

Proof. [✓ Proof](#) This combines periodic-MPS-vector window membership with the fact that the parent interaction annihilates ground-space elements. □

Lemma 12.3.62 (Parent Hamiltonian annihilates the periodic MPS vector). [Lean](#) [✓ Stmt](#) For every N with $L \leq N$, the periodic MPS vector satisfies $H_N(A, L) V^{(N)}(A) = 0$.

Proof. [✓ Proof](#) This is a sum of zeros, since each local term annihilates the periodic MPS vector. □

Lemma 12.3.63 (Frustration-freeness of the periodic MPS vector). [Lean](#) [✓ Stmt](#) The periodic MPS vector is a frustration-free ground state: for every site i , one has $h_i V^{(N)}(A) = 0$.

Proof. [✓ Proof](#) The claim follows from Lemma 12.3.61. □

12.4 Intersection property and unique ground state

This section develops the intersection property of local ground spaces for injective MPS and the resulting uniqueness of the ground state on the periodic chain.

12.4.1 Restriction maps

Given a state ψ on $L + 1$ sites, we define two restriction maps.

Remark 12.4.1.1 (Word and restriction identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

✓ **Stmnt** Appending a last letter to a word corresponds to right multiplication by the corresponding tensor letter, while prepending a first letter corresponds to left multiplication. The restriction maps are the associated linear maps on the state spaces under the computational-basis identification, together with their pointwise formulas.

Definition 12.4.1.2 (Left restriction). Lean ✓ Stmt Fixing the last physical index j gives the state $\sigma \mapsto \psi(\sigma_1, \dots, \sigma_L, j)$.

Definition 12.4.1.3 (Right restriction). Lean ✓ Stmt Fixing the first physical index i gives the state $\sigma \mapsto \psi(i, \sigma_1, \dots, \sigma_L)$.

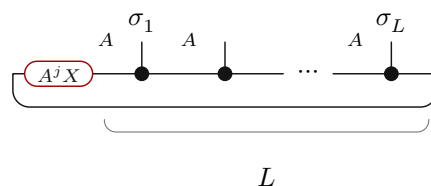
Definition 12.4.1.4 (Left ground membership). [Lean](#) [Stm](#) A state ψ on $L + 1$ sites satisfies the left ground condition if, for every j , the state obtained by fixing the last index to j lies in $G_L(A)$.

Definition 12.4.1.5 (Right ground membership). [Lean] [✓ Stmt] A state ψ on $L + 1$ sites satisfies the right ground condition if, for every i , the state obtained by fixing the first index to i lies in $G_L(A)$.

12.4.2 Forward direction: ground space restricts

Lemma 12.4.2.1 (Left restriction). [Lean] [✓ Stmt] If $\psi \in G_{L+1}(A)$, i.e. if $\psi(\sigma) = \text{tr}(A^\sigma X)$ for some X , then, for each j , the state obtained by fixing the last index to j lies in $G_L(A)$ with witness $A^j X$.

Proof. ✔ **Proof** The restriction simply absorbs the last tensor into the boundary matrix:



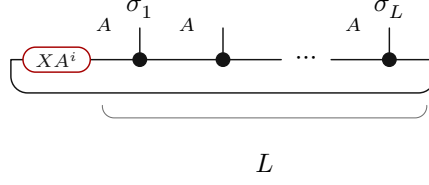
Direct computation gives

$$\psi(\sigma_1, \dots, \sigma_L, j) = \text{tr}(A^{\sigma_1} \dots A^{\sigma_L} A^j X) = \Gamma_L(A^j X)(\sigma).$$

☐

Lemma 12.4.2.2 (Right restriction). Lean ✓ Stmt *If $\psi \in G_{L+1}(A)$ with $\psi(\sigma) = \text{tr}(A^\sigma X)$, then, for each i , the state obtained by fixing the first index to i lies in $G_L(A)$ with witness XA^i .*

Proof. ✓ Proof On the right restriction, one first rotates the trace and then reads the resulting boundary matrix on the shortened window:



By trace cyclicity,

$$\psi(i, \sigma_1, \dots, \sigma_L) = \text{tr}(A^i A^\sigma X) = \text{tr}(A^\sigma X A^i) = \Gamma_L(X A^i)(\sigma).$$

□

12.4.3 Injectivity of the ground-space map

Theorem 12.4.3.1 (Ground-space map injectivity). [Lean](#) [✓ Stmt](#) *If A is injective and $L \geq 1$, then Γ_L is injective.*

Proof. [✓ Proof](#) If $\Gamma_L(X) = 0$, then $\text{tr}(A^\sigma X) = 0$ for every length- L word σ . Since A is injective, the products $\{A^\sigma\}_\sigma$ span $M_D(\mathbb{C})$, so the trace pairing gives $X = 0$. □

Theorem 12.4.3.2 (Ground-space map injectivity from word span). [Lean](#) [✓ Stmt](#) *If the length- L products $\{A^\sigma : |\sigma| = L\}$ span $M_D(\mathbb{C})$, then Γ_L is injective.*

Proof. [✓ Proof](#) If $\Gamma_L(X) = 0$, then $\text{tr}(A^\sigma X) = 0$ for every word σ of length L . The spanning hypothesis turns this into $\text{tr}(MX) = 0$ for every $M \in M_D(\mathbb{C})$, and the trace pairing gives $X = 0$. □

Theorem 12.4.3.3 (Ground-space map injectivity after blocking). [Lean](#) [✓ Stmt](#) *If A is L_0 -block injective, then Γ_{L_0} is injective.*

Proof. [✓ Proof](#) Lemma 7.1.1.20 identifies L_0 -block injectivity with the word-span hypothesis of Theorem 12.4.3.2. □

Theorem 12.4.3.4 (Ground-space dimension). [Lean](#) [✓ Stmt](#) *If A is injective and $L \geq 1$, then $\dim G_L(A) = D^2$.*

Proof. [✓ Proof](#) The upper bound $\dim G_L(A) \leq D^2$ is Lemma 12.2.4. Injectivity of Γ_L gives $\dim G_L(A) = \dim M_D(\mathbb{C}) = D^2$. □

12.4.4 The intersection property

Theorem 12.4.4.1 (One inclusion of the intersection property). [Lean](#) [✓ Stmt](#) *If A is injective and $L \geq 2$, then the nontrivial inclusion in the intersection property holds:*

$$G_L^{\text{left}} \cap G_L^{\text{right}} \subseteq G_{L+1}(A).$$

Proof. [✓ Proof](#) The “inverting and growing back” argument compares the left and right restrictions on their common $(L-1)$ -site overlap and then reconstructs the full $(L+1)$ -site word from a single boundary matrix:

The restriction to the cyclic interval beginning at i is

$$(\text{Res}_{i,L}^\rho \psi)(\omega) = \psi(\rho_{i,L,\omega}^{\text{cyc}}),$$

with all site labels read modulo N . If $L \leq N$, this is the same configuration as $\rho^{[i,i+L-1] \leftarrow \omega}$. If two boundary conditions agree whenever $\delta_i(k) \geq L$, their restrictions are equal. In particular, filling the interval inside the outside configuration leaves the restriction unchanged:

$$\text{Res}_{i,L}^{\rho_{i,L,\omega}^{\text{cyc}}} \psi = \text{Res}_{i,L}^\rho \psi.$$

For every $r < L$, the inserted word is recovered as $\rho_{i,L,\omega}^{\text{cyc}}(i+r) = \omega(r)$. Here site labels are taken modulo N . The remaining listed identities say that deleting the final site of an $(L+1)$ -interval gives the L -interval beginning at i ; if $L+1 \leq N$, deleting the first site gives the L -interval beginning at $i+1$; and the cyclic formula agrees with the contiguous formula when the interval does not wrap around the ring.

Theorem 12.4.5.3 (One-site restrictions of cyclic-window boundary conditions). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *If a cyclic $(L+1)$ -window is represented by a boundary matrix Y , then deleting one endpoint gives*

$$\begin{aligned} \Gamma_{L+1}(Y)|_{\text{last}=b} &= \Gamma_L(A^b Y), \\ \Gamma_{L+1}(Y)|_{\text{first}=a} &= \Gamma_L(Y A^a). \end{aligned}$$

Hence two adjacent cyclic windows whose restrictions agree on the common length- L overlap have boundary matrices satisfying $Y_1 A^a = A^b Y_2$. Consequently, for a finite chain of adjacent overlaps indexed by $0 \leq r < n$, satisfying $Y_r A^{a_r} = A^{b_r} Y_{r+1}$, one has $Y_0 A^{a_0} \dots A^{a_{n-1}} = A^{b_0} \dots A^{b_{n-1}} Y_n$. In particular, if the two endpoint words are named by the equations $a_r = \rho(i_0 + r)$ and $b_r = \rho(i_0 + r + L + 1)$, with site labels read modulo N , then the same transport identity is written as $Y_0 A^a = A^b Y_n$.

Proof. [✓ Proof](#) The first two identities are the cyclic first- and last-site deletion identities, followed by the corresponding ground-space restriction formulas. For adjacent windows, equality of the completed outside configurations gives $\Gamma_L(Y_0 A^a) = \Gamma_L(A^b Y_n)$. Injectivity of Γ_L gives $Y_0 A^a = A^b Y_n$. Iterating the adjacent identity gives the word-product identity. $\square$

Lemma 12.4.5.4 (Iterated contiguous-window intersection). [Lean](#) [✓ Stmt](#) *If an N -site state satisfies the L -site ground condition on every non-wrapping contiguous window, with $L \geq 2$, then it lies in $G_N(A)$.*

Proof. [✓ Proof](#) The induction repeatedly merges two neighbouring admissible windows $[i, i+L-1]$ and $[i+1, i+L]$ into the larger window $[i, i+L]$ by applying Theorem 12.4.4.1 to their common overlap $[i+1, i+L-1]$. $\square$

Definition 12.4.5.5 (Suffix restriction). [Lean](#) [✓ Stmt](#) Fix a prefix $u \in \{0, \dots, d-1\}^K$. The suffix restriction of an $(K+L)$ -site state ψ is the L -site state given by $(R_u^{\text{tail}} \psi)(\sigma) = \psi(u, \sigma)$.

Definition 12.4.5.6 (Suffix ground membership). [Lean](#) [✓ Stmt](#) A state ψ on $K+L$ sites lies in the suffix ground condition if every fixed prefix u gives a suffix state in $G_L(A)$.

Lemma 12.4.5.7 (Ground-space suffix restriction). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *If $\psi \in G_{K+L}(A)$, then every fixed-prefix suffix restriction of ψ lies in $G_L(A)$. For a vector in $G_{L+1}(A)$, fixing either endpoint gives the expected left or right multiplication of its boundary matrix.*

Proof. ✓ Proof Write $\psi(\rho) = \text{tr}(A^\rho X)$ for a boundary matrix X . Fixing the final site gives the boundary matrix $A^j X$, while fixing the first site gives $X A^i$ by cyclicity of the trace. Fixing a prefix u gives $\psi(u, \sigma) = \text{tr}(A^u A^\sigma X) = \text{tr}(A^\sigma X A^u)$, so the suffix restriction again has the form of a ground-space vector. $\square$

Remark 12.4.5.8 (Ground-space contiguous-window restriction). The two-sided window restriction is recorded in the martingale chapter as Theorem 13.3.5.6: for sites s, L, N with $s + L \leq N$, a full configuration τ supplying the complementary values, and a boundary matrix X , the non-wrapping restriction of the open-boundary ground-space vector $\Gamma_N(X)$ to the window $[s, s + L - 1]$ is again an open-boundary ground-space vector,

$$\text{Res}_{s,L}^\tau \Gamma_N(X) \in G_L(A).$$

Both outside words are absorbed into the boundary matrix: writing $w_L = (\tau_0, \dots, \tau_{s-1})$ and $w_R = (\tau_{s+L}, \dots, \tau_{N-1})$, word multiplicativity and cyclicity of the trace give, for any window word ω ,

$$\text{tr}(A^{w_L} A^\omega A^{w_R} X) = \text{tr}(A^\omega A^{w_R} X A^{w_L}),$$

so the restriction is $\Gamma_L(A^{w_R} X A^{w_L})$. This two-sided absorption generalizes Lemma 12.4.5.7, whose suffix restriction absorbs only the prefix.

Lemma 12.4.5.9 (Left-tail compatibility). Lean ✓ Stmt Assume ψ on $K + L_0 + 1$ sites satisfies the left ground condition on the first $K + L_0$ sites and the suffix ground condition on every fixed prefix of length K . Then there exist matrices $(Z_j)_j$ and $(Y_u)_u$ with the two trace representations below and, for all j and u , the compatibility relation

$$\begin{aligned} \psi(\sigma, j) &= \text{tr}(A^\sigma Z_j), \\ \psi(u, \tau) &= \text{tr}(A^\tau Y_u), \\ Z_j A^u &= A^j Y_u. \end{aligned}$$

Proof. ✓ Proof Choose Z_j from the long left-window condition and Y_u from the suffix ground condition. The two restrictions give the same vector in $G_{L_0}(A)$, namely $\Gamma_{L_0}(Z_j A^u) = \Gamma_{L_0}(A^j Y_u)$. Injectivity of Γ_{L_0} , obtained from L_0 -block injectivity, gives $Z_j A^u = A^j Y_u$. $\square$

Theorem 12.4.5.10 (Inverting step for word compatibility). Lean ✓ Stmt Assume A is L_0 -block-injective with $L_0 > 0$. If matrices $(Z_j)_j$ satisfy $Z_j A^\sigma = A^j Y_\sigma$ for every word σ of length K , then there exists a matrix X such that $Z_j = A^j X$ for all j .

Proof. ✓ Proof For $K = 1$, the hypothesis gives $Z_j A^i = A^j Y_i$ for every letter i . Every blocked word of length L_0 is nonempty, so this identity extends to the coefficients of the blocked tensor $A^{[L_0]}$. Since $A^{[L_0]}$ is injective, the identity matrix is a linear combination of those blocked coefficients, and substituting this decomposition yields $Z_j = A^j X$ for a common matrix X . For larger K , strip the first letter: for each i , the matrices $Z_j A^i$ satisfy the same compatibility hypothesis with words of length $K - 1$, so the induction hypothesis reduces the problem to the case $K = 1$. $\square$

Theorem 12.4.5.11 (Growing-back step at the injectivity length). Lean ✓ Stmt Assume A is L_0 -block-injective with $L_0 > 0$. If an element ψ on $K + L_0 + 1$ sites satisfies the left ground condition on the first $K + L_0$ sites and the suffix ground condition on every prefix of length K , then $\psi \in G_{K+L_0+1}(A)$.

Proof. [✓ Proof](#) Lemma 12.4.5.9 produces matrices $(Z_j)_j$ and $(Y_u)_u$ with $Z_j A^u = A^j Y_u$ for every prefix word u of length K . Theorem 12.4.5.10 gives a common right factor X such that $Z_j = A^j X$ for all j . Therefore, each last-site restriction of ψ agrees with the corresponding last-site restriction of the ground-space vector determined by X . Since every configuration is obtained by adjoining its final letter to its initial $(K + L_0)$ -tuple, the two states coincide, so ψ lies in $G_{K+L_0+1}(A)$. $\square$

Definition 12.4.5.12 (Open-chain left and tail ground subspaces). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every $L \in \mathbb{N}$, let $\mathcal{U}_L(A)$ be the subspace of the $(L+1)$ -site Hilbert space whose corresponding functions satisfy the left ground condition: every fixed-final-site restriction belongs to $G_L(A)$. For every $K, L \in \mathbb{N}$, let $\mathcal{V}_{K,L}(A)$ be the subspace of the $(K+L)$ -site Hilbert space whose corresponding functions satisfy the tail ground condition: every fixed-prefix restriction belongs to $G_L(A)$. These definitions include $L = 0$. Let $P_L^{\text{left}}(A)$ and $P_{K,L}^{\text{tail}}(A)$ be the orthogonal projectors onto $\mathcal{U}_L(A)$ and $\mathcal{V}_{K,L}(A)$, respectively.

Theorem 12.4.5.13 (Membership in the open-chain left ground subspace). [Lean](#) [✓ Stmt](#) *Under the canonical ℓ^2 identification, a Hilbert-space vector belongs to $\mathcal{U}_L(A)$ exactly when the corresponding function satisfies the left ground condition: every fixed-final-site restriction belongs to $G_L(A)$.*

Proof. [✓ Proof](#) Transport through the canonical ℓ^2 equivalence and unfold the left-site restriction condition. $\square$

Theorem 12.4.5.14 (Membership in the open-chain tail ground subspace). [Lean](#) [✓ Stmt](#) *Under the canonical ℓ^2 identification, a Hilbert-space vector belongs to $\mathcal{V}_{K,L}(A)$ exactly when the corresponding function satisfies the tail ground condition: every fixed-prefix restriction belongs to $G_L(A)$.*

Proof. [✓ Proof](#) Transport through the canonical ℓ^2 equivalence and unfold the tail restriction condition. $\square$

Theorem 12.4.5.15 (Open-chain left–tail ground-space intersection). [Lean](#) [✓ Stmt](#) *If $D \geq 1$, A is L_0 -block-injective, and $L_0 > 0$, then*

$$\mathcal{U}_{K+L_0}(A) \cap \mathcal{V}_{K,L_0+1}(A) = \mathcal{G}_{K+L_0+1}^{\text{ES}}(A).$$

In Nachtergaele’s notation, with $n = K + L_0$ and $l = L_0$, this is the common range identity for G_{Λ_n} and $G_{\Lambda_{n+1} \setminus \Lambda_{n-l}}$ inside Λ_{n+1} .

Proof. [✓ Proof](#) The forward inclusion is the growing-back step at the injectivity length. Conversely, a vector in the longer ground space remains in the ground space after every left or tail restriction. $\square$

Theorem 12.4.5.16 (Open-chain iteration of the intersection property). [Lean](#) [✓ Stmt](#) *Let A be L_0 -block-injective with $D \geq 1$ and $L_0 > 0$. If an N -site state satisfies the ground-space constraint on every non-wrapping contiguous interval of length $L_0 + 1$, for every fixed choice of the complementary sites, with $N \geq L_0 + 1$, then it lies in $G_N(A)$.*

Proof. [✓ Proof](#) Induct on the extra length beyond $L_0 + 1$. The induction hypothesis gives the long left-window ground condition after deleting the last site, while Definition 12.4.5.5 and the contiguous-interval restriction identities above identify every fixed-prefix suffix with one of the assumed length- $L_0 + 1$ intervals. Theorem 12.4.5.11 then grows back the rightmost site. $\square$

Lemma 12.4.5.17 (Open-chain iteration for a sequence of subspaces). [Lean](#) [✓ Stmt](#) Let $S_m \subseteq (\mathbb{C}^d)^{\otimes m}$ be subspaces, and let $0 < L \leq N$. Suppose that every non-wrapping length- L interval of ψ , for every fixed choice of the complementary sites, lies in S_L . Suppose also that, for every $m \geq L$, $(\mathbb{C}^d \otimes S_m) \cap (S_m \otimes \mathbb{C}^d) = S_{m+1}$. Then $\psi \in S_N$.

Proof. [✓ Proof](#) Induct on the length m of a non-wrapping interval. The case $m = L$ is the assumed local condition. If all length- m intervals lie in S_m , then the two one-site restrictions of a length- $(m+1)$ interval are adjacent length- m intervals. Hence the length- $(m+1)$ interval lies in $(\mathbb{C}^d \otimes S_m) \cap (S_m \otimes \mathbb{C}^d)$, and the displayed identity places it in S_{m+1} . Taking $m = N$ and the interval starting at the first site gives $\psi \in S_N$. $\square$

Lemma 12.4.5.18 (Periodic constraints and propagated subspaces). [Lean](#) [✓ Stmt](#) Let $S_m \subseteq (\mathbb{C}^d)^{\otimes m}$ be subspaces, and let $0 < L \leq N$. Suppose that $G_L(A) \subseteq S_L$. Suppose also that, for every $m \geq L$, $(\mathbb{C}^d \otimes S_m) \cap (S_m \otimes \mathbb{C}^d) = S_{m+1}$. Then the periodic local constraints imply $\mathcal{G}_{N,L}(A) \subseteq S_N$.

Proof. [✓ Proof](#) If $\psi \in \mathcal{G}_{N,L}(A)$, then every length- L cyclic interval of ψ lies in $G_L(A)$, and hence in S_L . A non-wrapping interval is the cyclic interval starting at the same site. Therefore, all non-wrapping length- L intervals lie in S_L , and Lemma 12.4.5.17 gives $\psi \in S_N$. $\square$

Lemma 12.4.5.19 (Cyclic-window range antitonicity). [Lean](#) [Lean](#) [✓ Stmt](#) On a nonempty periodic chain, longer cyclic-window constraints imply all shorter cyclic-window constraints: if $L' \leq L \leq N$, then $\mathcal{G}_{N,L}(A) \subseteq \mathcal{G}_{N,L'}(A)$.

Proof. [✓ Proof](#) For every cyclic window vector $v \in G_{L+1}(A)$ and every last letter b , one has $v|_{\text{last}=b} \in G_L(A)$. Thus $\mathcal{G}_{N,L+1}(A) \subseteq \mathcal{G}_{N,L}(A)$, and iteration gives the general antitonicity. $\square$

Theorem 12.4.5.20 (Open-chain containment from cyclic local constraints). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$. If an N -site periodic-chain state satisfies all cyclic ground-space constraints at some range $L_0 + 1 \leq L \leq N$, then it lies in the open-chain ground space $G_N(A)$.

Proof. [✓ Proof](#) Lemma 12.4.5.19 reduces the cyclic hypotheses to range $L_0 + 1$. On every non-wrapping interval, the cyclic and contiguous restriction maps agree, so Theorem 12.4.5.16 applies. $\square$

12.5 Transport under physical blocking

For a tensor $A = (A^i)_{i=0}^{d-1}$ and a block length p , write $A^{[p]}$ for the tensor whose physical matrices are the consecutive products $A^{i_1} \cdots A^{i_p}$. We keep the number N of blocked sites distinct from the number Np of original sites. The exact Hilbert-space transport identities below are elementary consequences of this definition and of the local-space construction; they are not attributed to a source theorem.

Definition 12.5.1 (Blocked configuration isometry). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Grouping each consecutive p -tuple of physical indices defines a canonical computational-basis unitary satisfying

$$W_{N,p} : \ell^2(\{0, \dots, d^p - 1\}^N) \longrightarrow \ell^2(\{0, \dots, d - 1\}^{Np}). \quad (12.63)$$

Its forward and inverse coordinate formulas are the corresponding reindexings by the blocked-configuration equivalence and its inverse.

Lemma 12.5.2 (Boundary-map transport under blocking). [Lean](#) [Lean](#) [✓ Stmt](#) For every $x \in \ell^2(\{0, \dots, D-1\} \times \{0, \dots, D-1\})$, the Euclidean boundary maps intertwine with the blocked-configuration isometry:

$$W_{N,p} \Gamma_N^{\text{ES}, A^{[p]}}(x) = \Gamma_{Np}^{\text{ES}, A}(x). \quad (12.64)$$

Proof. [✓ Proof](#) For a blocked configuration $J = (J_1, \dots, J_N)$, its flattened word σ , and a boundary matrix $X \in M_D(\mathbb{C})$, blocked-word evaluation gives

$$(A^{[p]})^J = A^{\sigma_1} \dots A^{\sigma_{Np}}, \quad (12.65)$$

$$\Gamma_N^{A^{[p]}}(X)(J) = \text{tr}(A^{\sigma_1} \dots A^{\sigma_{Np}} X) = \Gamma_{Np}^A(X)(\sigma). \quad (12.66)$$

Applying the Frobenius transport formula to the coefficient identity yields (12.64). $\square$

Theorem 12.5.3 (Local ground-space transport under blocking). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The Euclidean local space of the blocked tensor is carried exactly to the original Euclidean local space:

$$W_{N,p} G_N^{\text{ES}}(A^{[p]}) = G_{Np}^{\text{ES}}(A). \quad (12.67)$$

Hence their orthogonal projectors satisfy

$$P_{G_{Np}^{\text{ES}}(A)} = W_{N,p} P_{G_N^{\text{ES}}(A^{[p]})} W_{N,p}^*. \quad (12.68)$$

These statements include $N = 0$ or $p = 0$; in those cases the same explicit configuration isometry and boundary-map calculation apply. In particular, the projector identity holds for three blocked sites and $3p$ original sites. These identities concern the full local ground spaces and their orthogonal projectors under blocking.

Proof. [✓ Proof](#) The exact range identification for the Euclidean boundary map and (12.64) give

$$W_{N,p} \text{range}\left(\Gamma_N^{\text{ES}, A^{[p]}}\right) = \text{range}\left(\Gamma_{Np}^{\text{ES}, A}\right), \quad (12.69)$$

which is (12.67). For every subspace K , orthogonal projection under the unitary $W_{N,p}$ satisfies

$$P_{W_{N,p}(K)} = W_{N,p} P_K W_{N,p}^*. \quad (12.70)$$

Taking $K = G_N^{\text{ES}}(A^{[p]})$ gives (12.68). $\square$

12.6 Periodic-boundary comparison for injective tensors

On the periodic ring, the two cyclic supports that cross the periodic-boundary cut expose the same complementary word, and matching their boundary matrices forces X to commute with every A^j . This section derives the boundary-crossing compatibilities $C_\tau^+ A^j X = Y_\tau A^j$ and $X A^j C_\tau^- = A^j Y_\tau$ and combines them into the commutation $X A^j = A^j X$.

Lemma 12.6.1 (Boundary-crossing compatibility at the last site). [Lean](#) [✓ Stmt](#) Assume A is L_0 -block-injective with $L_0 > 0$ and $M \geq L_0$. If the reduced cyclic interval crossing the last site has boundary matrices Y_τ , then, for every physical letter j , one has

$$C_\tau^+ A^j X = Y_\tau A^j. \quad (12.71)$$

Here C_τ^+ is the complementary word seen by that cyclic interval.

Proof. [✓ Proof](#) The trace identity for this cyclic interval is tested against all words of length L_0 . Injectivity of Γ_{L_0} , which follows from L_0 -block injectivity, turns these test-word identities into (12.71). $\square$

Lemma 12.6.2 (Boundary-crossing compatibility at the second boundary position). [Lean](#) [✓ Stmt](#) Assume A is L_0 -block-injective with $L_0 > 0$ and $M \geq L_0$. If the second boundary-crossing reduced cyclic interval has boundary matrices Y_τ , then $XA^jC_\tau^- = A^jY_\tau$ for every physical letter j , where C_τ^- is the complementary word seen from the second cyclic position.

Proof. [✓ Proof](#) Factor the cyclic word at the second cyclic position, rotate the trace, and use injectivity of Γ_{L_0} to remove the length- L_0 test word. $\square$

Theorem 12.6.3 (The two boundary-crossing compatibilities). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$. Suppose $\psi \in \mathcal{G}_{N,L}(A)$, $N \geq 2$, $L_0 < L \leq N$, and $\psi = \Gamma_N(X)$ for a boundary matrix X . Then there are two families of boundary matrices Y_τ^+ and Y_τ^- , indexed by fixed complement values τ , such that, for every physical letter j ,

$$\begin{aligned} C_\tau^+ A^j X &= Y_\tau^+ A^j, \\ X A^j C_\tau^- &= A^j Y_\tau^-. \end{aligned} \tag{12.72}$$

Here C_τ^+ and C_τ^- are the complementary words exposed by the two boundary-crossing cyclic intervals, both of reduced length $L_0 + 1$.

Proof. [✓ Proof](#) First use Lemma 12.4.5.19 to reduce the cyclic constraints from range L to range $L_0 + 1$. Then apply the two cyclic-interval compatibility lemmas at the two boundary positions to obtain the one-sided identities in (12.72). $\square$

Lemma 12.6.4 (Boundary-crossing equations for $\Gamma_{M+1}(X)$). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $D \geq 1$, and $L_0 \leq M$. Suppose $\psi = \Gamma_{M+1}(X)$, and suppose matrices $Y_i(\tau)$ represent the length- $(L_0 + 1)$ cyclic restrictions $\text{Res}_{i,L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Then, for every physical letter j and boundary condition τ ,

$$\begin{aligned} A^{\tau_{L_0}} \dots A^{\tau_{M-1}} A^j X &= Y_M(\tau) A^j, \\ X A^j A^{\tau_1} \dots A^{\tau_{M-L_0}} &= A^j Y_{M+1-L_0}(\tau). \end{aligned} \tag{12.73}$$

Proof. [✓ Proof](#) Substitute $\psi = \Gamma_{M+1}(X)$ in the two cyclic restrictions at the last site and at the second boundary-crossing support. Lemmas 12.6.1 and 12.6.2 then give (12.73). $\square$

Lemma 12.6.5 (One-sided products for a common complement word). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $D \geq 1$, and $L_0 \leq M$. Suppose $\psi = \Gamma_{M+1}(X)$, and suppose matrices $Y_i(\tau)$ represent the length- $(L_0 + 1)$ cyclic restrictions $\text{Res}_{i,L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. For every boundary letter η , every physical letter j , and every complementary word μ ,

$$\begin{aligned} Y_M(\tau_\eta^+(\mu)) A^j &= A^\mu A^j X, \\ X A^j A^\mu &= A^j Y_{M+1-L_0}(\tau_\eta^-(\mu)). \end{aligned}$$

Proof. [✓ Proof](#) Substitute $\tau_\eta^+(\mu)$ and $\tau_\eta^-(\mu)$ into Lemma 12.6.4. The exposed-complement identities of Lemma 12.6.6 identify both complementary products with A^μ . $\square$

Lemma 12.6.6 (Common complement word for the two boundary supports). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) On a chain of length $N = M + 1$, with $L_0 \leq M$, fix a physical letter η used outside the complementary sites and a word μ of length $M + 1 - (L_0 + 1)$. Write $\tau_\eta^+(\mu)$ for the boundary condition at the support crossing the last site and $\tau_\eta^-(\mu)$ for the boundary condition at the second boundary-crossing support. Their exposed complements are both exactly μ .

Proof. ✓ Proof Put the same letter η on the remaining sites. Fill the physical sites $L_0, \dots, M-1$ with μ for the support crossing the last site, and fill the sites $1, \dots, M-L_0$ with μ for the second boundary-crossing support. The two complement-extraction identities follow from the index arithmetic. The same outside-site calculation gives $\text{Res}_{M, L_0+1}^\rho \psi = \text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)} \psi$ for any ρ with $\rho_{k+L_0} = \mu_k$, and $\text{Res}_{M+1-L_0, L_0+1}^\rho \psi = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)} \psi$ for any ρ with $\rho_{k+1} = \mu_k$. $\square$

Lemma 12.6.7 (Boundary equations from a common complement word). Lean Lean Lean Lean Lean ✓ Stmt Let A be L_0 -block-injective, with $L_0 > 0$, and let $L_0 \leq M$. Suppose two length- $(L_0 + 1)$ restrictions of a state ψ at the same cyclic support are represented by $\text{Res}_{i, L_0+1}^\rho \psi = \Gamma_{L_0+1}(Y_\rho)$ and $\text{Res}_{i, L_0+1}^\tau \psi = \Gamma_{L_0+1}(Y_\tau)$. If these two restrictions are equal, then $Y_\rho A^j = Y_\tau A^j$ for every physical letter j . In particular, the last-site boundary-crossing condition and the second boundary-crossing condition satisfy, respectively,

$$\begin{aligned} \rho_{k+L_0} = \mu_k &\implies Y_\rho A^j = Y_{\tau_\eta^+(\mu)} A^j, \\ \rho_{k+1} = \mu_k &\implies Y_\rho A^j = Y_{\tau_\eta^-(\mu)} A^j. \end{aligned}$$

Consequently, for every word σ of length L_0 , both implications remain true after right multiplication by A^σ .

Proof. ✓ Proof Apply the first-letter restriction to both equal length- $(L_0 + 1)$ restrictions. The two resulting length- L_0 vectors are represented by $\Gamma_{L_0}(Y_\rho A^j)$ and $\Gamma_{L_0}(Y_\tau A^j)$, and block injectivity makes Γ_{L_0} injective. The two displayed special cases follow from Lemma 12.6.6. The length- L_0 word forms are obtained by multiplying these equations on the right by A^σ . $\square$

Lemma 12.6.8 (Reduction to compatible boundary conditions). Lean ✓ Stmt Let A be L_0 -block-injective, with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let the matrices $Y_i(\tau)$ represent its length- $(L_0 + 1)$ cyclic restrictions, $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Fix a boundary letter η and a complementary word μ . Let ρ^+ and ρ^- be two boundary conditions satisfying $\rho_{k+L_0}^+ = \mu_k$ and $\rho_{k+1}^- = \mu_k$. If the adjacent-window comparison gives, for every physical letter j and every word σ of length L_0 , $Y_M(\rho^+) A^j A^\sigma = Y_{M+1-L_0}(\rho^-) A^j A^\sigma$, then the same product equation holds for $\tau_\eta^+(\mu)$ and $\tau_\eta^-(\mu)$:

$$Y_M(\tau_\eta^+(\mu)) A^j A^\sigma = Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma.$$

Proof. ✓ Proof By Lemma 12.6.7,

$$\begin{aligned} Y_M(\rho^+) A^j &= Y_M(\tau_\eta^+(\mu)) A^j, \\ Y_{M+1-L_0}(\rho^-) A^j &= Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j. \end{aligned}$$

Multiplying on the right by A^σ and applying the hypothesis gives the displayed product equation. $\square$

Theorem 12.6.9 (Pointwise reduction to compatible boundary conditions). Lean ✓ Stmt Let A be L_0 -block-injective, with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let the matrices $Y_i(\tau)$ represent its length- $(L_0 + 1)$ cyclic restrictions, $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Fix a boundary letter η and a complementary word μ . Suppose that, for each physical letter j and word σ of length L_0 , boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$ satisfy, for every complementary position k ,

$$\begin{aligned} \rho_{j, \sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j, \sigma}^-(k + 1) &= \mu_k. \end{aligned}$$

If, for every j and σ ,

$$Y_M(\rho_{j,\sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j,\sigma}^-)A^jA^\sigma,$$

then, for every j and σ ,

$$Y_M(\tau_\eta^+(\mu))A^jA^\sigma = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma.$$

Proof. ✓ **Proof** Fix j and σ . Lemma 12.6.7 gives

$$Y_M(\rho_{j,\sigma}^+)A^j = Y_M(\tau_\eta^+(\mu))A^j, \quad (12.74)$$

$$Y_{M+1-L_0}(\rho_{j,\sigma}^-)A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j. \quad (12.75)$$

Multiply (12.74) and (12.75) on the right by A^σ and use the assumed product equation for the chosen pair (j, σ) . $\square$

Lemma 12.6.10 (Endpoint words for the periodic-boundary windows). Lean ✓ **Stmt** Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$. Suppose that matrices $Y_r(\rho)$, for $0 \leq r \leq L_0-1$, represent the length- (L_0+1) cyclic restrictions beginning at the sites $M+1-L_0+r$: $\text{Res}_{M+1-L_0+r, L_0+1}^\rho \psi = \Gamma_{L_0+1}(Y_r(\rho))$. Then

$$Y_0(\rho)A^{\rho_{M+1-L_0}} \dots A^{\rho_{M-1}} = A^{\rho_1} \dots A^{\rho_{L_0-1}} Y_{L_0-1}(\rho).$$

For $L_0 = 1$, both products are empty.

Proof. ✓ **Proof** Use Theorem 12.4.5.3 for the L_0-1 adjacent restrictions starting at $i_0 = M+1-L_0$. For $0 \leq r < L_0-1$, the two endpoint words are determined by

$$\begin{aligned} i_0 + r &= M+1-L_0+r, \\ i_0 + r + L_0 + 1 &\equiv r+1 \pmod{M+1}. \end{aligned}$$

Substituting these two words in the iterated transport identity gives the displayed product equation. $\square$

Lemma 12.6.11 (Boundary-condition product for the periodic-boundary windows). Lean Lean ✓ **Stmt** Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and suppose matrices $Y_i(\rho)$ represent the length- (L_0+1) cyclic restrictions $\text{Res}_{i, L_0+1}^\rho(\psi) = \Gamma_{L_0+1}(Y_i(\rho))$. Then, for every boundary condition ρ ,

$$Y_{M+1-L_0}(\rho)A^{\rho_{M+1-L_0}} \dots A^{\rho_{M-1}} = A^{\rho_1} \dots A^{\rho_{L_0-1}} Y_M(\rho).$$

For $L_0 = 1$, both products are empty. The same equation remains true after multiplying both sides on the right by any matrix R .

Proof. ✓ **Proof** Apply Theorem 12.4.5.3 to the L_0-1 restrictions beginning at $M+1-L_0+r$, with $Y_r(\rho) = Y_{M+1-L_0+r}(\rho)$. With $i_0 = M+1-L_0$, the site labels satisfy

$$\begin{aligned} i_0 + r &= M+1-L_0+r, \\ i_0 + r + L_0 + 1 &\equiv r+1 \pmod{M+1}. \end{aligned}$$

Hence, for $r = 0, \dots, L_0-2$,

$$Y_{M+1-L_0+r}(\rho)A^{\rho_{M+1-L_0+r}} = A^{\rho_{r+1}} Y_{M+2-L_0+r}(\rho).$$

Multiplying these L_0-1 equations from left to right gives the displayed product equation. The right-multiplied form follows by multiplying this equality by R . $\square$

Lemma 12.6.12 (Transport followed by the one-sided boundary equation). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, and let $L_0 \leq M$. Let $\psi = \Gamma_{M+1}(X)$, and suppose that matrices $Y_i(\rho)$ represent all length- $(L_0 + 1)$ cyclic restrictions of ψ . Then, for every boundary condition ρ and every physical letter j ,

$$Y_{M+1-L_0}(\rho) A^{\rho_{M+1-L_0}} \dots A^{\rho_{M-1}} A^j = A^{\rho_1} \dots A^{\rho_{L_0-1}} A^{\rho_{L_0}} \dots A^{\rho_{M-1}} A^j X.$$

For $\rho = \tau_\eta^-(\mu)$, this is the transport identity supplied by the adjacent-window argument, with the single factor A^j following $Y_{M+1-L_0}(\tau_\eta^-(\mu))$. The padded identity, whose corresponding factor is $A^j A^\sigma$, is supplied separately.

Proof. [✓ Proof](#) Lemma 12.6.11 gives

$$Y_{M+1-L_0}(\rho) A^{\rho_{M+1-L_0}} \dots A^{\rho_{M-1}} A^j = A^{\rho_1} \dots A^{\rho_{L_0-1}} Y_M(\rho) A^j.$$

The one-sided equation for the window beginning at M gives $Y_M(\rho) A^j = A^{\rho_{L_0}} \dots A^{\rho_{M-1}} A^j X$. Substitution gives the displayed formula. $\square$

Lemma 12.6.13 (Long boundary-condition product for the periodic-boundary windows). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M + 1$, and suppose matrices $Y_i(\rho)$ represent the length- $(L_0 + 1)$ cyclic restrictions $\text{Res}_{i, L_0+1}^\rho(\psi) = \Gamma_{L_0+1}(Y_i(\rho))$. Then, for every boundary condition ρ ,

$$Y_M(\rho) A^{\rho_M} A^{\rho_0} \dots A^{\rho_{M-L_0}} = A^{\rho_{L_0}} \dots A^{\rho_M} A^{\rho_0} Y_{M+1-L_0}(\rho).$$

Each side has $M + 2 - L_0$ one-site factors, with site labels read modulo $M + 1$.

Proof. [✓ Proof](#) Apply Theorem 12.4.5.3 to the $M + 2 - L_0$ restrictions beginning at $M + r$ modulo $M + 1$. With $i_0 = M$, the site labels satisfy

$$\begin{aligned} i_0 + r &\equiv M + r \pmod{M + 1}, \\ i_0 + r + L_0 + 1 &\equiv L_0 + r \pmod{M + 1}. \end{aligned}$$

The final window is $i_0 + M + 2 - L_0 \equiv M + 1 - L_0 \pmod{M + 1}$. Substituting these site labels in the iterated product identity gives the displayed equation. $\square$

Lemma 12.6.14 (One matrix Y_μ from compared boundary matrices). [Lean](#) [✓ Stmt](#) Fix a physical letter η used outside the complementary sites. Suppose the two one-sided boundary matrix families Y^+ and Y^- have been extracted from the cyclic windows. If, for every word μ on the complementary sites, these boundary matrices agree on the boundary conditions built from η , $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$, then there is a single family Y_μ satisfying

$$\begin{aligned} A^\mu A^b X &= Y_\mu A^b, \\ X A^a A^\mu &= A^a Y_\mu \end{aligned}$$

for all letters a, b .

Proof. [✓ Proof](#) Define Y_μ to be the first boundary matrix evaluated on the boundary condition $\tau_\eta^+(\mu)$. Lemma 12.6.6 rewrites the first cyclic-window complement as μ , while the assumed comparison at $\tau_\eta^-(\mu)$ rewrites the second boundary matrix to the same Y_μ . $\square$

Lemma 12.6.15 (Commutation from two one-sided equations). [Lean](#) [✓ Stmt](#) Suppose a single family of boundary matrices Y_μ , indexed by words μ on the complementary sites, satisfies the two one-sided identities

$$\begin{aligned} A^\mu A^b X &= Y_\mu A^b, \\ X A^a A^\mu &= A^a Y_\mu \end{aligned} \tag{12.76}$$

for every pair of physical letters a, b . Then X commutes with every word product $A^a A^\mu A^b$ obtained by adjoining one letter on each side of the word A^μ .

Proof. [✓ Proof](#) Multiply the second identity in (12.76) on the right by A^b and the first identity on the left by A^a : $X A^a A^\mu A^b = A^a Y_\mu A^b = A^a A^\mu A^b X$. $\square$

Theorem 12.6.16 (Boundary matrix commutation from periodic closure). [Lean](#) [✓ Stmt](#) Let A be injective with $D \geq 1$, and let $\psi = \Gamma_N(X)$ for some boundary matrix $X \in M_D(\mathbb{C})$. If every cyclic restriction of length L of ψ lies in $G_L(A)$, with $N \geq 2$ and $1 < L \leq N$, then $X A^j = A^j X$ for every $j = 0, \dots, d-1$.

Proof. [✓ Proof](#) The periodic boundary condition produces matrix identities of the form $XMW = MY$ for arbitrary test matrices M and complementary words W . Spanning first over the length- $(L-1)$ tails and then over the complementary words, injectivity forces $(XM - MX)W = 0$ for every W , hence $XM = MX$ for every matrix M . In particular, X commutes with each letter A^j . $\square$

12.7 Periodic unique ground state

This section gives three periodic ground-space facts: the span of the periodic MPS vector $V^{(N)}(A) = \Gamma_N(\mathbb{1})$, its membership in every cyclic window constraint $\mathcal{G}_{N,L}(A)$, and its nonvanishing for block-injective tensors. Together with the boundary-matrix commutation $X A^j = A^j X$ of the previous section, these establish that the periodic ground space is spanned by $V^{(N)}(A)$; the one-dimensionality conclusion is completed downstream.

Definition 12.7.1 (Span of the periodic MPS vector). [Lean](#) [✓ Stmt](#) The subspace spanned by the MPS vector $V^{(N)}(A)(\sigma) = \text{tr}(A^{\sigma_0} \dots A^{\sigma_{N-1}})$.

Lemma 12.7.2 (The periodic MPS vector is the ground-space map at the identity). [Lean](#) [✓ Stmt](#) For every chain length N , one has $V^{(N)}(A) = \Gamma_N(\mathbb{1})$.

Lemma 12.7.3 (The periodic MPS vector lies in the local ground space). [Lean](#) [✓ Stmt](#) $V^{(N)}(A) \in G_N(A)$ for every N , with boundary matrix $X = \mathbb{1}$.

Proof. [✓ Proof](#) For every σ , $V^{(N)}(A)(\sigma) = \text{tr}(A^\sigma) = \text{tr}(A^\sigma \mathbb{1}) = \Gamma_N(\mathbb{1})(\sigma)$. $\square$

Definition 12.7.4 (One-dimensional ground space). [Lean](#) [✓ Stmt](#) A finite-dimensional ground space S is one-dimensional if $\dim S = 1$.

Theorem 12.7.5 (One-dimensionality criterion). [Lean](#) [✓ Stmt](#) For finite-dimensional S , $\dim S = 1$ if and only if there exists a nonzero $\psi_0 \in S$ such that every $\psi \in S$ is of the form $c \psi_0$.

Proof. [✓ Proof](#) Apply the standard finite-dimensional criterion that a nonzero space has dimension one exactly when all of its vectors are proportional to a single nonzero vector. $\square$

Definition 12.7.6 (Periodic parent-Hamiltonian ground space). [Lean](#) [✓ Stmt](#) For $N > 0$ and $L \leq N$, define the *periodic parent-Hamiltonian ground space*

$$\mathcal{G}_{N,L}(A) := \{\psi \in (\{0, \dots, d-1\}^N \rightarrow \mathbb{C}) : \forall i \in \{0, \dots, N-1\}, \psi|_{[i, i+L-1]} \in G_L(A)\}.$$

Here the condition $\psi|_{[i, i+L-1]} \in G_L(A)$ means that, for every choice of physical indices outside the window, the restriction of ψ to that window lies in $G_L(A)$. The one-index notation $G_L(A)$ denotes the local ground space $\mathcal{G}_L$ of [\[CPGSV21, Section IV.C\]](#). The two-index notation $\mathcal{G}_{N,L}(A)$ used here denotes the periodic intersection of those local constraints over all translated length- L windows on the N -site ring. When $N = 0$ or $L > N$, set $\mathcal{G}_{N,L}(A) := \top$ by convention.

Lemma 12.7.7 (Cyclic-window representation matrices inside the periodic ground space). [Lean](#) [✓ Stmt](#) If $0 < N$, $L \leq N$, and $\psi \in \mathcal{G}_{N,L}(A)$, then, for every cyclic window and every outside configuration, there is a boundary matrix Y such that the corresponding restricted L -site state is $\Gamma_L(Y)$.

Proof. [✓ Proof](#) This is the defining cyclic-window condition for membership in $\mathcal{G}_{N,L}(A)$, together with the definition of the local ground space as the range of Γ_L . $\square$

Lemma 12.7.8 (The periodic MPS vector satisfies every cyclic window constraint). [Lean](#) [✓ Stmt](#) If $0 < N$ and $L \leq N$, then $V^{(N)}(A) \in \mathcal{G}_{N,L}(A)$.

Theorem 12.7.9 (Periodic MPS vector nonvanishing for block-injective tensors). [Lean](#) [✓ Stmt](#) If A is L_0 -block-injective with $L_0 > 0$, $D \geq 1$, and $N \geq L_0 + 1$, then $V^{(N)}(A) \neq 0$ in the N -site space.

Proof. [✓ Proof](#) Suppose for contradiction that $V^{(N)}(A) = 0$, i.e., $\text{tr}(A^\sigma) = 0$ for every word σ of length N . For any words w and u with $|w| = N - L_0$ and $|u| = L_0$, we have $\text{tr}(A^w A^u) = 0$. Since A is L_0 -block-injective, $\{A^u : |u| = L_0\}$ spans $M_D(\mathbb{C})$. By nondegeneracy of the trace pairing, $A^w = 0$ for every $|w| = N - L_0$. Repeatedly strip a length- L_0 suffix: if every product of length $m + L_0$ vanishes, then the spanning property of the length- L_0 products implies that every product of length m vanishes. Eventually all products of some length $r < L_0$ vanish. If $r = 0$, this gives $1 = 0$. If $r > 0$, every length- L_0 product has a zero length- r prefix, so all such products vanish, contradicting that they span $M_D(\mathbb{C})$. $\square$

12.8 Injectivity-length reduction for the closure property

This section implements the inverting-and-growing-back reduction from [\[CPGSV21, Section IV.C\]](#): a commutation or annihilation relation for length- m word products is reduced to the injectivity length L_0 , then used in the periodic-boundary closure-property step. Throughout, $A^w := A^{i_1} \dots A^{i_L}$ denotes the product along a word $w = (i_1, \dots, i_L)$, Γ_N sends a boundary matrix to its open-chain vector, and A being L_0 -block-injective means $\text{span}\{A^u : |u| = L_0\} = M_D(\mathbb{C})$, so a relation valid on every length- L_0 product extends by linearity to all of $M_D(\mathbb{C})$.

Theorem 12.8.1 (Common right factor from suffix equations). [Lean](#) [✓ Stmt](#) Assume A is L_0 -block-injective with $L_0 > 0$. Let $\{Z_u\}_{|u|=L_0}$ be a family indexed by words of length L_0 . If, for some $K \geq 0$ and every suffix word w of length K , there exists $Y_w \in M_D(\mathbb{C})$ such that $Z_u A^w = A^u Y_w$ for every word u of length L_0 , then there exists $X \in M_D(\mathbb{C})$ with $Z_u = A^u X$ for every word u of length L_0 .

Proof. ✓ Proof Induct on K . For $K = 0$, the empty suffix gives $Z_u = A^u Y_\emptyset$. For the successor step, fix the first physical index i and apply the induction hypothesis to the shortened suffix. This gives a matrix X_i satisfying $Z_u A^i = A^u X_i$ for every word u of length L_0 . Extending this letter equation to all length- L_0 words and decomposing $\mathbb{1}$ in the spanning family $\{A^u : |u| = L_0\}$ yields a common factor X . $\square$

Theorem 12.8.2 (Common boundary matrix from spanning word identities). Lean ✓ Stmt *Let A be a tensor whose length- K word products span $M_D(\mathbb{C})$. Let $\{F_b\}_{b \in B}$ and $\{Z_b\}_{b \in B}$ be two families of matrices. If, for every length- K word w , there is a matrix Y_w such that $Z_b A^w = F_b Y_w$ for every $b \in B$, then there is a single matrix Y such that $Z_b = F_b Y$ for every $b \in B$.*

Proof. ✓ Proof Suppose $Z_b M_1 = F_b Y_1$ and $Z_b M_2 = F_b Y_2$ for every $b \in B$. Then $Z_b(M_1 + M_2) = Z_b M_1 + Z_b M_2 = F_b(Y_1 + Y_2)$. For a scalar c , one similarly has $Z_b(cM_1) = cZ_b M_1 = F_b(cY_1)$. Thus the identity holds for every matrix in the span of the length- K word products. Since this span is all of $M_D(\mathbb{C})$, apply it to the identity matrix. $\square$

Theorem 12.8.3 (Injectivity-length commutation from length- m commutation). Lean ✓ Stmt *If A is L_0 -block-injective with $L_0 > 0$, $m \geq L_0$, and $X \in M_D(\mathbb{C})$ commutes with every length- m product A^w , then X already commutes with every length- L_0 product A^u .*

Proof. ✓ Proof Write each length- m word as a concatenation uw with $|u| = L_0$ and $|w| = m - L_0$. The hypothesis gives $XA^u A^w = A^u A^w X = A^u(A^w X)$. Thus the family $Z_u := XA^u$ satisfies the hypothesis of Theorem 12.8.1. One obtains $XA^u = A^u R$ for all $|u| = L_0$. Decomposing $\mathbb{1}$ in the span of the length- L_0 words shows $R = X$, hence $XA^u = A^u X$. $\square$

Theorem 12.8.4 (Centrality from length- m commutation). Lean ✓ Stmt *If A is L_0 -block-injective with $L_0 > 0$, $m \geq L_0$, and $X \in M_D(\mathbb{C})$ commutes with every length- m word product, then X commutes with every matrix in $M_D(\mathbb{C})$.*

Proof. ✓ Proof By Theorem 12.8.3, X commutes with every length- L_0 word product. These products span $M_D(\mathbb{C})$ by block injectivity, so linearity gives $XM = MX$ for every $M \in M_D(\mathbb{C})$. $\square$

Lemma 12.8.5 (Amplifying fixed-length word commutation). Lean ✓ Stmt *If a boundary matrix X commutes with every word product of length m , then it commutes with every word product whose length is a multiple qm .*

Proof. ✓ Proof Proceed by induction on q . For $q = 0$, the claim is trivial. For the inductive step, let a word of length $(q+1)m$ be given and write it as a length- m prefix followed by a length- qm suffix. The prefix commutes with X by hypothesis, and the suffix commutes by the induction hypothesis. $\square$

Lemma 12.8.6 (Padded complement annihilation). Lean Lean ✓ Stmt *If A is L_0 -block-injective, $q \geq 1$, and $k \leq qL_0$, then the two one-sided annihilation statements are*

$$\begin{aligned} ZA^w = 0 \quad (|w| = k) &\implies Z = 0, \\ A^w Z = 0 \quad (|w| = k) &\implies Z = 0. \end{aligned}$$

The second implication is the left-handed form.

Proof. ✓ Proof The hypothesis $\text{span}\{A^u : |u| = L_0\} = M_D(\mathbb{C})$ implies the same spanning statement at every positive multiple qL_0 . Each length- qL_0 word factors as a length- k prefix followed by padding, so Z annihilates all generators of the full span and hence annihilates the identity. For the left-handed form, factor each length- qL_0 word as a prefix followed by a length- k suffix and use the same spanning argument. $\square$

Lemma 12.8.7 (One-sided boundary matrices are unique). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$. If two boundary matrices have the same product with every one-site tensor on one fixed side, then the two boundary matrices are equal.

Proof. [✓ Proof](#) If $Y_1 A^j = Y_2 A^j$ for every physical index j , then induction on the word gives $Y_1 A^w = Y_2 A^w$ for every nonempty word w . Since A is L_0 -block-injective, $\text{span}\{A^w : |w| = L_0\} = M_D(\mathbb{C})$. Thus the left multiplication maps $M \mapsto Y_1 M$ and $M \mapsto Y_2 M$ are equal on $M_D(\mathbb{C})$, and evaluating them on the identity gives $Y_1 = Y_2$. The other one-sided statement follows by the same argument, with right multiplication in place of left multiplication. $\square$

Lemma 12.8.8 (Generator commutation from length- m commutation). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 12.8.4, one has $XA^i = A^i X$ for every physical index i .

Proof. [✓ Proof](#) Apply Theorem 12.8.4 with $M = A^i$. $\square$

Theorem 12.8.9 (MPS-line containment from length- m commutation). [Lean](#) [✓ Stmt](#) Let A be injective after blocking $L_0 > 0$ sites, and let $m \geq L_0$. If a boundary matrix X commutes with every length- m word product, then $\Gamma_N(X) \in \text{span}\{V^{(N)}(A)\}$.

Proof. [✓ Proof](#) Theorem 12.8.4 makes X commute with the whole matrix algebra. The center of $M_D(\mathbb{C})$ consists of scalar matrices, so $X = c\mathbf{1}$, and therefore $\Gamma_N(X) = cV^{(N)}(A)$. $\square$

Theorem 12.8.10 (MPS-line containment from positive-length commutation). [Lean](#) [✓ Stmt](#) Let A be injective after blocking $L_0 > 0$ sites. If X commutes with every word product of some positive length m , then $\Gamma_N(X) \in \text{span}\{V^{(N)}(A)\}$.

Proof. [✓ Proof](#) Lemma 12.8.5 amplifies the length- m commutation hypothesis to length $L_0 m \geq L_0$. Then Theorem 12.8.9 applies. $\square$

Theorem 12.8.11 (MPS-line containment from two one-sided equations). [Lean](#) [✓ Stmt](#) Let A be injective after blocking $L_0 > 0$ sites. Suppose there are matrices Y_μ such that

$$\begin{aligned} A^\mu A^b X &= Y_\mu A^b, \\ X A^a A^\mu &= A^a Y_\mu, \end{aligned}$$

for every pair of physical letters a, b . Then $\Gamma_N(X) \in \text{span}\{V^{(N)}(A)\}$.

Proof. [✓ Proof](#) The identities in (12.76) give commutation with all words of length $|\mu| + 2$, a positive length. The positive-length containment theorem then applies. $\square$

Theorem 12.8.12 (Boundary equality implies containment in the MPS line). [Lean](#) [✓ Stmt](#) Let A be injective after blocking $L_0 > 0$ sites, and fix a physical letter η used outside the complementary sites. Suppose that, for every boundary condition τ and every physical letter j ,

$$\begin{aligned} C_\tau^+ A^j X &= Y_\tau^+ A^j, \\ X A^j C_\tau^- &= A^j Y_\tau^-, \end{aligned}$$

where C_τ^+ and C_τ^- are the complementary words exposed by the two boundary-crossing cyclic windows. Suppose also that the two boundary matrices satisfy $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$ for every word μ on the complementary sites. Then $\Gamma_N(X) \in \text{span}\{V^{(N)}(A)\}$.

Proof. ✓ Proof Lemma 12.6.14 gives a family Y_μ satisfying

$$\begin{aligned} A^\mu A^b X &= Y_\mu A^b, \\ X A^a A^\mu &= A^a Y_\mu. \end{aligned}$$

Theorem 12.8.11 applies. □

Theorem 12.8.13 (Closure-property step for periodic chains). Lean ✓ Stmt *Let A be injective after blocking $L_0 > 0$ sites and let $D \geq 1$. Assume $N \geq 2$ and $L_0 < L \leq N$. Suppose that, for every physical letter η used in the two boundary conditions and every boundary matrix obtained from an N -site chain ground state, the identities*

$$\begin{aligned} A^\mu A^j X &= Y_{\tau_\eta^+(\mu)}^+ A^j, \\ X A^j A^\mu &= A^j Y_{\tau_\eta^-(\mu)}^-, \end{aligned}$$

hold for every physical letter j and every complementary word μ , and are supplemented, for every μ , by $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$. Then $\mathcal{G}_{N,L}(A) \subseteq \text{span}\{V^{(N)}(A)\}$.

Proof. ✓ Proof The cyclic-to-open-chain reduction writes any chain ground state as $\Gamma_N(X)$. The two boundary-crossing cyclic windows give

$$\begin{aligned} A^\mu A^j X &= Y_{\tau_\eta^+(\mu)}^+ A^j, \\ X A^j A^\mu &= A^j Y_{\tau_\eta^-(\mu)}^-, \\ Y_{\tau_\eta^+(\mu)}^+ &= Y_{\tau_\eta^-(\mu)}^-. \end{aligned}$$

Theorem 12.8.12 finishes. □

12.9 Periodic-boundary comparison for normal tensors

This section isolates the closure-property part of the periodic-boundary argument for normal tensors. In the source proof, after the intersection property has grown the open interval, the same inverting-and-growing-back argument is applied when closing the boundaries [CPGSV21, Section IV.C]. For a vector $\psi = \Gamma_{M+1}(X)$, the closure property asks that the two boundary-crossing restrictions $\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi)$ and $\text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi)$ agree. The symbols $\tau_\eta^\pm(\mu)$ index these restrictions, and the matrices $Y_i(\tau)$ below satisfy $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Once the one-sided boundary products are known, equality of the restrictions follows from the boundary-matrix commutation relation $X A^j = A^j X$ for every physical letter j . The block-window identity $X A^\alpha A^\nu = A^\alpha Y_\nu$ is derived from the cyclic-window constraints crossing the periodic cut.

Lemma 12.9.1 (Block-to-letter translation of the boundary block matrix equation). Lean ✓ Stmt *Let A be an MPS tensor and let $L_0, K \geq 0$. Let X be a matrix, and let Y_{c_b} be a matrix for each iterated block index c_b of the complement. Suppose that, for every block letter b of the alphabet $\{0, \dots, d-1\}^{L_0}$ and every iterated block index c_b of the complement, with $w(b)$ the length- L_0 word of b and $\widetilde{w(c_b)}$ the length- $L_0 K$ complement word obtained by concatenating the blocks of c_b , $X A^{w(b)} A^{\widetilde{w(c_b)}} = A^{w(b)} Y_{c_b}$. Then there is a family Y'_c , indexed by the length- $L_0 K$ words c , with $X A^s A^c = A^s Y'_c$ for every length- L_0 word s and every length- $L_0 K$ word c .*

Proof. ✓ **Proof** Each length- L_0 word s is the word $w(b)$ of its block letter b , and each length- $L_0 K$ word c regroups into K blocks, recovering it as the concatenated complement word $\widetilde{w(c_b)}$ of an iterated block index c_b ; both identifications are bijective. Substituting these representations into the hypothesis and setting $Y'_c = Y_{c_b}$ gives the asserted equation for every s and c . $\square$

Lemma 12.9.2 (Boundary matrix commutation from block-window equations). Lean ✓ **Stmt** *Let A be L_0 -block-injective with $L_0 > 0$. Let X be a boundary matrix, and let Y_v be a matrix for each complementary word v of length K . Suppose that, for every word u of length L_0 and every word v of length K , $XA^u A^v = A^u Y_v$. Then, for every physical letter j , $XA^j = A^j X$. This is the boundary-matrix commutation step obtained from the equations $XA^u A^v = A^u Y_v$, for all length- L_0 words u and all length- K words v , as in [CPGSV21, Section IV.C, lines 2049–2090].*

Proof. ✓ **Proof** By Lemma 7.1.1.20, $\text{span}\{A^u : |u| = L_0\} = M_D(\mathbb{C})$. Hence, for every $M \in M_D(\mathbb{C})$ and every complementary word v of length K , the displayed hypothesis extends by linearity in the length- L_0 block to $XMA^v = MY_v$. Taking $M = \mathbb{1}$ gives

$$Y_v = XA^v. \quad (12.77)$$

Taking $M = A^j$ and substituting (12.77) gives $(XA^j - A^j X)A^v = 0$ for every word v of length K . Since $K \leq (K+1)L_0$ and $K+1 \geq 1$, Lemma 12.8.6 gives $XA^j - A^j X = 0$. $\square$

Lemma 12.9.3 (Trace identity from the boundary-crossing cyclic window). Lean ✓ **Stmt** *Let A be a tensor with virtual dimension at least one, and let $L_0 > 0$ and $L_0 < M$. Let $\psi = \Gamma_{M+1}(X)$ and assume that every cyclic restriction of length $L_0 + 1$ belongs to $G_{L_0+1}(A)$. Then there are boundary matrices Y_ν , indexed by nonempty complementary words ν of length $M+1 - (L_0+1)$, such that, for every physical letter j and every word α of length L_0 ,*

$$\text{tr}(A^j X A^\alpha A^\nu) = \text{tr}(A^j A^\alpha Y_\nu).$$

Proof. ✓ **Proof** Let Y_ν be the matrix representing the cyclic restriction whose window begins at the last site. The cyclic word is $\alpha \nu j$ after deleting the first letter from the window and reading the complement, so the ground-space representation and cyclicity of the trace give

$$\text{tr}(A^j X A^\alpha A^\nu) = \text{tr}(A^\alpha A^\nu A^j X) = \text{tr}(A^j A^\alpha Y_\nu).$$

$\square$

Lemma 12.9.4 (One-site-injective boundary block-window identity). Lean ✓ **Stmt** *Let A be a tensor with virtual dimension at least one whose single-site matrices span the full matrix algebra. Let $L_0 > 0$ and $L_0 < M$, and let $\psi = \Gamma_{M+1}(X)$. If every cyclic restriction of length $L_0 + 1$ belongs to $G_{L_0+1}(A)$, then there are boundary matrices Y_ν , indexed by nonempty complementary words ν of length $M+1 - (L_0+1)$, such that, for every word α of length L_0 , $XA^\alpha A^\nu = A^\alpha Y_\nu$.*

Proof. ✓ **Proof** For every physical letter j , Lemma 12.9.3 gives

$$\text{tr}(A^j X A^\alpha A^\nu) = \text{tr}(A^j A^\alpha Y_\nu). \quad (12.78)$$

Put $Z = XA^\alpha A^\nu - A^\alpha Y_\nu$. For every physical letter j , Then (12.78) gives $\text{tr}(A^j Z) = 0$. Since the matrices A^j span the full matrix algebra, the trace pairing gives $(\forall j, \text{tr}(A^j Z) = 0) \implies Z = 0$. Hence $XA^\alpha A^\nu = A^\alpha Y_\nu$. $\square$

Theorem 12.9.5 (Trace identities with length- L_0 probes from cyclic-window constraints).

[Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ and assume that every cyclic restriction of length $L_0 + 1$ belongs to $G_{L_0+1}(A)$. Then there are boundary matrices Y_ν , indexed by nonempty complementary words ν of length $M + 1 - (L_0 + 1)$, such that, for every word α of length L_0 and every word β of length L_0 ,

$$\text{tr}(A^\beta X A^\alpha A^\nu) = \text{tr}(A^\beta A^\alpha Y_\nu).$$

Proof. [✓ Proof](#) Choose representation matrices for every cyclic restriction of length $L_0 + 1$. Fix α and ν . The boundary-crossing equations move the matrix X through the first boundary letter. In local boundary notation, this has the form

$$X A^{\rho_0} A^{\rho_1 \cdots \rho_{M-L_0}} = A^{\rho_0} Y_{M+1-L_0}(\rho).$$

The adjacent boundary-window product then transports this comparison through the remaining $L_0 - 1$ boundary letters:

$$Y_{M+1-L_0}(\rho) A^{\rho_{M+1-L_0} \cdots \rho_{M-1}} = A^{\rho_1 \cdots \rho_{L_0-1}} Y_M(\rho).$$

Finally, the outside-label uniqueness lemma identifies the final boundary matrix with the matrix Y_ν attached to the complementary word ν . Thus $X A^\alpha A^\nu = A^\alpha Y_\nu$. Multiplying on the left by A^β and taking the trace gives the asserted identity for every length- L_0 word β . $\square$

Theorem 12.9.6 (Length-word trace separation for the boundary block window). [Lean](#) [✓ Stmt](#)

Let A be L_0 -block-injective. Let X be a boundary matrix, and let Y_ν be a matrix for each complementary word ν of length K . Suppose that, for every word α of length L_0 , every word β of length L_0 , and every complementary word ν ,

$$\text{tr}(A^\beta X A^\alpha A^\nu) = \text{tr}(A^\beta A^\alpha Y_\nu).$$

Then, for every word α of length L_0 and every complementary word ν , $X A^\alpha A^\nu = A^\alpha Y_\nu$.

Proof. [✓ Proof](#) The trace identities say exactly that, for each fixed pair (α, ν) ,

$$\Gamma_{L_0}(X A^\alpha A^\nu) = \Gamma_{L_0}(A^\alpha Y_\nu).$$

Since A is L_0 -block-injective, Γ_{L_0} is injective by Theorem 12.4.3.3. Hence $X A^\alpha A^\nu = A^\alpha Y_\nu$. $\square$

Lemma 12.9.7 (Matrix identity $X A^\alpha A^\nu = A^\alpha Y_\nu$ from cyclic-window constraints). [Lean](#) [✓ Stmt](#)

Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ and assume that every cyclic restriction of length $L_0 + 1$ belongs to $G_{L_0+1}(A)$. Then there are boundary matrices Y_ν , indexed by nonempty complementary words ν of length $M + 1 - (L_0 + 1)$, such that, for every word α of length L_0 , $X A^\alpha A^\nu = A^\alpha Y_\nu$.

Proof. [✓ Proof](#) Theorem 12.9.5 gives, for every word β of length L_0 ,

$$\text{tr}(A^\beta X A^\alpha A^\nu) = \text{tr}(A^\beta A^\alpha Y_\nu).$$

Theorem 12.9.6 then gives the displayed matrix equation. $\square$

Lemma 12.9.8 (Boundary restrictions from commutation and one-sided products). [Lean](#) [✓ Stmt](#)

Let A be L_0 -block-injective with $L_0 > 0$. Let X be a boundary matrix, let W_η and V_η be matrix

families indexed by a physical letter η , and let μ be a word. Suppose that, for every physical letter j , $XA^j = A^jX$. Suppose also that, for every η and every j ,

$$\begin{aligned} W_\eta A^j &= A^\mu A^j X, \\ XA^j A^\mu &= A^j V_\eta. \end{aligned}$$

Then, for every η and every j , $W_\eta A^j = V_\eta A^j$.

Proof. [✓ Proof](#) The commutation relation $XA^j = A^jX$ extends by induction to $XA^w = A^wX$ for every word w . Hence the second one-sided equation gives, for every k , $A^k V_\eta = XA^k A^\mu = A^k XA^\mu = A^k A^\mu X$. Lemma 12.8.7 gives $V_\eta = A^\mu X$. The first one-sided equation then gives $W_\eta A^j = A^\mu A^j X = A^\mu XA^j = V_\eta A^j$. $\square$

Lemma 12.9.9 (Products with one-site tensors determine the boundary matrix). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$. Fix two matrix families Y^+ and Y^- and the two reindexed boundary conditions $\tau_\eta^+(\mu)$ and $\tau_\eta^-(\mu)$. If, for every physical letter j , $Y_{\tau_\eta^+(\mu)}^+ A^j = Y_{\tau_\eta^-(\mu)}^- A^j$, then $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$.

Proof. [✓ Proof](#) Apply Lemma 12.8.7 to the two matrices $Y_{\tau_\eta^+(\mu)}^+$ and $Y_{\tau_\eta^-(\mu)}^-$. $\square$

Lemma 12.9.10 (Outside labels determine boundary-restriction matrices). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $L_0 \leq M$. If an outside configuration ρ has the same word μ as $\tau_\eta^+(\mu)$ on the sites outside the last-site cyclic window, then the two matrices representing that restriction agree: $Y_\rho = Y_{\tau_\eta^+(\mu)}$. The same assertion holds for the second boundary-crossing window: if ρ has the same outside word as $\tau_\eta^-(\mu)$, then $Y_\rho = Y_{\tau_\eta^-(\mu)}$. This is the boundary-matrix independence step used to choose one representative outside configuration for each outside word in the closure-property matrix comparison.

Proof. [✓ Proof](#) Equality of the cyclic restrictions gives, for every physical letter j , $Y_\rho A^j = Y_{\tau_\eta^\pm(\mu)}^\pm A^j$. Lemma 12.8.7 identifies the two matrices from these one-site products. $\square$

Lemma 12.9.11 (First-letter restrictions determine a vector). [Lean](#) [✓ Stmt](#) Let R_j be restriction to first letter j :

$$(R_j \phi)(\sigma_1, \dots, \sigma_L) = \phi(j, \sigma_1, \dots, \sigma_L).$$

For $\phi, \psi \in (\mathbb{C}^d)^{\otimes(L+1)}$,

$$(\forall j, R_j \phi = R_j \psi) \implies \phi = \psi.$$

Proof. [✓ Proof](#) At $(\sigma_0, \sigma_1, \dots, \sigma_L)$, the required equality is

$$\phi(\sigma_0, \sigma_1, \dots, \sigma_L) = (R_{\sigma_0} \phi)(\sigma_1, \dots, \sigma_L) = (R_{\sigma_0} \psi)(\sigma_1, \dots, \sigma_L) = \psi(\sigma_0, \sigma_1, \dots, \sigma_L).$$

$\square$

Lemma 12.9.12 (Auxiliary restriction equality for the closure property). [Lean](#) [✓ Stmt](#) Let $L_0 > 0$ and $L_0 \leq M$, and set

$$\begin{aligned} B_{\eta, \mu}^+(\psi) &= \text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi), \\ B_{\eta, \mu}^-(\psi) &= \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi). \end{aligned}$$

With R_j as in Lemma 12.9.11,

$$(\forall j, R_j B_{\eta, \mu}^+(\psi) = R_j B_{\eta, \mu}^-(\psi)) \implies B_{\eta, \mu}^+(\psi) = B_{\eta, \mu}^-(\psi).$$

Proof. ✓ Proof Apply Lemma 12.9.11 to the two length- $(L_0 + 1)$ restrictions, using the displayed family of first-letter equalities. $\square$

Lemma 12.9.13 (Boundary-crossing restriction from right products). Lean ✓ Stmt Let A be an MPS tensor, let $L_0 > 0$, and let $L_0 \leq M$. Suppose the two boundary-crossing restrictions of length $L_0 + 1$ are represented by boundary matrices $Y_{\eta, \mu}^+$ and $Y_{\eta, \mu}^-$:

$$\begin{aligned} B_{\eta, \mu}^+(\psi) &= \Gamma_{L_0+1}(Y_{\eta, \mu}^+), \\ B_{\eta, \mu}^-(\psi) &= \Gamma_{L_0+1}(Y_{\eta, \mu}^-). \end{aligned}$$

If, for every physical letter j , $Y_{\eta, \mu}^+ A^j = Y_{\eta, \mu}^- A^j$, then $B_{\eta, \mu}^+(\psi) = B_{\eta, \mu}^-(\psi)$.

Proof. ✓ Proof Fix j and take the first-letter restriction of the two displayed length- $(L_0 + 1)$ restrictions. These are $\Gamma_{L_0}(Y_{\eta, \mu}^+ A^j)$ and $\Gamma_{L_0}(Y_{\eta, \mu}^- A^j)$. The one-site product equality gives, for every j ,

$$\Gamma_{L_0}(Y_{\eta, \mu}^+ A^j) = \Gamma_{L_0}(Y_{\eta, \mu}^- A^j).$$

Lemma 12.9.11 identifies the two original restrictions. $\square$

Lemma 12.9.14 (Boundary restrictions from first-letter products). Lean ✓ Stmt Let $L_0 > 0$ and $L_0 \leq M$. Let $Y_i(\tau)$ represent the length- $(L_0 + 1)$ cyclic restrictions of ψ . Fix a complementary word μ . If, for every boundary letter η and physical letter j , $Y_M(\tau_\eta^+(\mu)) A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j$, then, for every η ,

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Proof. ✓ Proof Apply Lemma 12.9.13 for each boundary letter η , using the corresponding first-letter product equations. $\square$

Lemma 12.9.15 (Equality of the two restrictions crossing the periodic cut). Lean ✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, let $L_0 < M$, and let

$$\begin{aligned} \psi &\in \mathcal{G}_{M+1, L_0+1}(A), \\ \psi &= \Gamma_{M+1}(X). \end{aligned}$$

For every boundary letter η and complementary word μ , the two restrictions crossing the periodic cut satisfy

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Proof. ✓ Proof The cyclic-window representation lemma gives matrices $Y_M(\tau_\eta^+(\mu))$ and $Y_{M+1-L_0}(\tau_\eta^-(\mu))$ such that

$$\begin{aligned} \text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) &= \Gamma_{L_0+1}(Y_M(\tau_\eta^+(\mu))), \\ \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi) &= \Gamma_{L_0+1}(Y_{M+1-L_0}(\tau_\eta^-(\mu))). \end{aligned}$$

Theorem 12.9.1.8 gives, for every j , $Y_M(\tau_\eta^+(\mu)) A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j$. Lemma 12.9.13 identifies the two length- $(L_0 + 1)$ restrictions. $\square$

Lemma 12.9.16 (Boundary-matrix product from boundary-crossing restrictions). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Suppose that, for every outside letter η ,

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Then there are boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$ such that, for every complementary position k ,

$$\begin{aligned}\rho_{j, \sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j, \sigma}^-(k + 1) &= \mu_k,\end{aligned}$$

and, for every physical letter j and every word σ of length L_0 ,

$$Y_M(\rho_{j, \sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j, \sigma}^-)A^jA^\sigma.$$

Proof. [✓ Proof](#) For the product indexed by j and σ , specialize the preceding equality at the outside letter $\eta = j$. Take $\rho_{j, \sigma}^+ = \tau_j^+(\mu)$ and $\rho_{j, \sigma}^- = \tau_j^-(\mu)$. The complement equations are the two identities of Lemma 12.6.6. Fix j . Applying the first-letter restriction to the displayed equality gives

$$\Gamma_{L_0}(Y_M(\tau_j^+(\mu))A^j) = \Gamma_{L_0}(Y_{M+1-L_0}(\tau_j^-(\mu))A^j),$$

where the two sides are identified by Theorem 12.4.5.3. By Theorem 12.4.3.3, $Y_M(\tau_j^+(\mu))A^j = Y_{M+1-L_0}(\tau_j^-(\mu))A^j$. Right multiplication by A^σ gives the asserted product equation. $\square$

Lemma 12.9.17 (Boundary-matrix product from right products). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Fix a complementary word μ of length $M+1 - (L_0+1)$. Suppose that, for every boundary letter η and every physical letter j , $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$. Then there are boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$ such that, for every complementary position k ,

$$\begin{aligned}\rho_{j, \sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j, \sigma}^-(k + 1) &= \mu_k,\end{aligned}$$

and, for every physical letter j and every word σ of length L_0 ,

$$Y_M(\rho_{j, \sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j, \sigma}^-)A^jA^\sigma.$$

Proof. [✓ Proof](#) For each boundary letter η , Lemma 12.9.13 gives

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Applying these equalities, Lemma 12.9.16 gives the displayed boundary conditions and product equation. $\square$

Lemma 12.9.18 (Boundary-crossing comparison after multiplication by words). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Fix a complementary word μ , a matrix X , and a family of matrices $Y_i(\tau)$. Suppose that, for every boundary letter η , every physical letter j , and every word σ of length L_0 ,

$$Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma = A^\mu A^j X A^\sigma.$$

Then, for every η and j , $Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j = A^\mu A^j X$.

Proof. ✓ Proof Fix η and j , and set $Z_{\eta,j} = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j - A^\mu A^j X$. The hypothesis gives $Z_{\eta,j}A^\sigma = 0$ for every σ with $|\sigma| = L_0$. Since A is L_0 -block-injective, the products A^σ with $|\sigma| = L_0$ span the full matrix algebra. Applying Lemma 12.8.6 gives $Z_{\eta,j} = 0$. $\square$

Lemma 12.9.19 (Boundary-crossing comparison after left multiplication by words). Lean
✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Fix a complementary word μ , a matrix X , and a family of matrices $Y_i(\tau)$. Suppose that, for every boundary letter η , every physical letter j , and every pair of words σ, α of length L_0 ,

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j A^\sigma) = A^\alpha (A^\mu A^j X A^\sigma).$$

Then, for every η, j , and σ ,

$$Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j A^\sigma = A^\mu A^j X A^\sigma.$$

Proof. ✓ Proof Fix η, j , and σ , and set

$$Z_{\eta,j,\sigma} = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j A^\sigma - A^\mu A^j X A^\sigma.$$

The hypothesis says that $A^\alpha Z_{\eta,j,\sigma} = 0$ for every word α of length L_0 . The left-handed form of Lemma 12.8.6 gives $Z_{\eta,j,\sigma} = 0$. $\square$

Lemma 12.9.20 (Boundary-matrix product from the boundary-crossing comparison). Lean
✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i,L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Fix a complementary word μ and a matrix X . Suppose that, for every boundary letter η and every physical letter j , $Y_M(\tau_\eta^+(\mu))A^j = A^\mu A^j X$. Suppose also that, for every η, j , and every word σ of length L_0 ,

$$Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j A^\sigma = A^\mu A^j X A^\sigma.$$

Then there are boundary conditions $\rho_{j,\sigma}^+$ and $\rho_{j,\sigma}^-$ such that, for every complementary position k ,

$$\begin{aligned} \rho_{j,\sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j,\sigma}^-(k + 1) &= \mu_k, \end{aligned}$$

and, for every physical letter j and every word σ of length L_0 ,

$$Y_M(\rho_{j,\sigma}^+)A^j A^\sigma = Y_{M+1-L_0}(\rho_{j,\sigma}^-)A^j A^\sigma.$$

Proof. ✓ Proof Lemma 12.9.18 gives, for every η and j , $Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j = A^\mu A^j X$. Combining this equation with $Y_M(\tau_\eta^+(\mu))A^j = A^\mu A^j X$ gives $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$. Lemma 12.9.17 then gives the displayed boundary conditions and product equation. $\square$

Lemma 12.9.21 (Boundary-matrix product from the left-multiplied boundary-crossing comparison). Lean ✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$, and let $L_0 \leq M$. Let ψ be a state on the chain of length $M+1$, and let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i,L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. Fix a complementary word μ and a matrix X . Suppose that, for every boundary letter η and every physical letter j , $Y_M(\tau_\eta^+(\mu))A^j = A^\mu A^j X$. Suppose also that, for every η, j , and every pair of words σ, α of length L_0 ,

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j A^\sigma) = A^\alpha (A^\mu A^j X A^\sigma).$$

Then there are boundary conditions $\rho_{j,\sigma}^+$ and $\rho_{j,\sigma}^-$ satisfying the complementary-word equations

$$\begin{aligned}\rho_{j,\sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j,\sigma}^-(k + 1) &= \mu_k,\end{aligned}$$

and the product equation

$$Y_M(\rho_{j,\sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j,\sigma}^-)A^jA^\sigma.$$

Proof. [✓ Proof](#) Lemma 12.9.19 gives

$$Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma = A^\mu A^j X A^\sigma.$$

Lemma 12.9.20 then gives the displayed boundary conditions and product equation. $\square$

Theorem 12.9.22 (Boundary-matrix product from left-multiplied equations). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ be an open-chain representation, and let $Y_i(\tau)$ represent the length- $(L_0 + 1)$ cyclic restrictions of ψ . Fix a complementary word μ . Suppose that, for every boundary letter η , physical letter j , and length- L_0 words σ, α ,

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma) = A^\alpha (A^\mu A^j X A^\sigma).$$

Then there are boundary conditions $\rho_{j,\sigma}^+$ and $\rho_{j,\sigma}^-$ satisfying the complementary-word equations

$$\begin{aligned}\rho_{j,\sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j,\sigma}^-(k + 1) &= \mu_k,\end{aligned}$$

and the product equation

$$Y_M(\rho_{j,\sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j,\sigma}^-)A^jA^\sigma.$$

Proof. [✓ Proof](#) Lemma 12.6.5 gives the equation for the boundary-crossing window beginning at M , $Y_M(\tau_\eta^+(\mu))A^j = A^\mu A^j X$. The assumed left-multiplied coordinate equation and Lemma 12.9.21 then give the displayed boundary conditions and product equation. $\square$

Theorem 12.9.23 (Boundary restrictions from the left-multiplied comparison). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $L_0 \leq M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ be an open-chain representation, and let $Y_i(\tau)$ represent the length- $(L_0 + 1)$ cyclic restrictions of ψ . Fix a complementary word μ . Suppose that, for every boundary letter η , physical letter j , and length- L_0 words α, σ ,

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma) = A^\alpha (A^\mu A^j X A^\sigma).$$

Then, for every boundary letter η ,

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Proof. [✓ Proof](#) The assumed equations and Lemma 12.9.19 give, for every η, j , and σ ,

$$Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma = A^\mu A^j X A^\sigma.$$

Lemma 12.9.18 removes the right word: $Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j = A^\mu A^j X$. The equation for the boundary-crossing window beginning at M , from Lemma 12.6.5, gives $Y_M(\tau_\eta^+(\mu))A^j = A^\mu A^j X$. Hence the first-letter products agree, and Lemma 12.9.14 identifies the two restrictions. $\square$

Theorem 12.9.24 (Equality of the boundary-crossing restrictions). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ be an open-chain representation. Assume that, for all i and τ , $\text{Res}_{i, L_0+1}^\tau(\psi) \in G_{L_0+1}(A)$. In local coordinate notation, fix a nonempty word μ on the sites outside the two cyclic windows. For every outside letter η , choose the two outside configurations so that, for each index k of this outside word,

$$\begin{aligned}\tau_\eta^+(\mu)_{k+L_0} &= \mu_k, \\ \tau_\eta^-(\mu)_{k+1} &= \mu_k,\end{aligned}$$

and put the letter η on the remaining sites. The notation $\tau_\eta^\pm(\mu)$ is a coordinate parametrization for the comparison used at the periodic boundary in [\[CPGSV21, Section IV.C, lines 2078–2079\]](#); it is not notation from the source. The corresponding boundary-crossing restriction equality is

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Proof. [✓ Proof](#) Choose matrices $Y_i(\tau)$ representing the length- (L_0+1) restrictions. The one-sided boundary-product lemma gives, for every outside letter η and physical letter j ,

$$\begin{aligned}Y_M(\tau_\eta^+(\mu))A^j &= A^\mu A^j X, \\ XA^j A^\mu &= A^j Y_{M+1-L_0}(\tau_\eta^-(\mu)).\end{aligned}$$

Lemma 12.9.7 gives matrices Y_ν satisfying, for every word α of length L_0 and every complementary word ν , $XA^\alpha A^\nu = A^\alpha Y_\nu$. Lemma 12.9.2 then gives, for every physical letter k , the commutation identity $XA^k = A^k X$. Lemma 12.9.8 gives $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$. Lemma 12.9.14 then identifies the two boundary-crossing restrictions, giving the periodic-boundary comparison of [\[CPGSV21, lines 2078–2079\]](#). $\square$

12.9.1 Reverse comparison for boundary-crossing restrictions

This subsection runs the comparison in the reverse direction: from equality of the two cyclic restrictions it recovers the first-letter product equation $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$ and the left-multiplied boundary comparison.

Lemma 12.9.1.1 (First-letter products from equal boundary restrictions). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $L_0 \leq M$. Let ψ be a vector on $M+1$ sites, and suppose the two length- (L_0+1) boundary restrictions are represented by $\Gamma_{L_0+1}(Y_M(\tau_\eta^+(\mu)))$ and $\Gamma_{L_0+1}(Y_{M+1-L_0}(\tau_\eta^-(\mu)))$. If these two restrictions are equal, then for every physical letter j , $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$. Thus the first-letter restriction of the preceding equality is precisely the displayed product equation.

Proof. [✓ Proof](#) Restricting both sides to the first physical letter j gives

$$\Gamma_{L_0}(Y_M(\tau_\eta^+(\mu))A^j) = \Gamma_{L_0}(Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j).$$

Since A is L_0 -block-injective, Γ_{L_0} is injective, and the displayed product equality follows. $\square$

Lemma 12.9.1.2 (Left-multiplied comparison from equal boundary restrictions). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $L_0 \leq M$. Let ψ be a vector on $M+1$ sites, and let $Y_i(\tau)$ represent the length- (L_0+1) cyclic restrictions of ψ . Assume that for every boundary letter η ,

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Then for every boundary letter η , physical letter j , and length- L_0 words α, σ ,

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma) = A^\alpha (Y_M(\tau_\eta^+(\mu)) A^j A^\sigma).$$

Proof. ✓ Proof The preceding lemma gives $Y_M(\tau_\eta^+(\mu)) A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j$. Multiplying on the left by A^α and on the right by A^σ gives the desired comparison. $\square$

Theorem 12.9.1.3 (Left-multiplied comparison of boundary matrices). Lean ✓ Stmt *Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ be an open-chain representation, and let $Y_i(\tau)$ represent the length- (L_0+1) cyclic restrictions of ψ . Fix a nonempty complementary word μ . Then for every boundary letter η , physical letter j , and length- L_0 words α, σ ,*

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma) = A^\alpha (Y_M(\tau_\eta^+(\mu)) A^j A^\sigma).$$

Proof. ✓ Proof The matrices $Y_i(\tau)$ give the local ground-space representations needed in the preceding lemma. Once the preceding boundary-restriction equality is proved, it gives

$$\text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

Applying the left-multiplied comparison for equal boundary restrictions gives

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma) = A^\alpha (Y_M(\tau_\eta^+(\mu)) A^j A^\sigma).$$

$\square$

Theorem 12.9.1.4 (Left-multiplied coordinate comparison at the periodic boundary). Lean ✓ Stmt *Let A be L_0 -block-injective with $L_0 > 0$, $L_0 < M$, and $D \geq 1$. Let $\psi = \Gamma_{M+1}(X)$ be an open-chain representation, and let $Y_i(\tau)$ represent the length- (L_0+1) cyclic restrictions of ψ . Fix a nonempty complementary word μ . Then for every boundary letter η , physical letter j , and length- L_0 words α, σ ,*

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma) = A^\alpha (A^\mu A^j X A^\sigma).$$

Proof. ✓ Proof Once the comparison of the two cyclic restrictions is available, it gives

$$A^\alpha (Y_{M+1-L_0}(\tau_\eta^-(\mu)) A^j A^\sigma) = A^\alpha (Y_M(\tau_\eta^+(\mu)) A^j A^\sigma).$$

The one-sided equation for the boundary-crossing window beginning at M gives $Y_M(\tau_\eta^+(\mu)) A^j = A^\mu A^j X$. Substitution gives the displayed identity. $\square$

Lemma 12.9.1.5 (Letterwise comparison of boundary-crossing restrictions). Lean ✓ Stmt *Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, let $L_0 < M$, and let $\psi \in \mathcal{G}_{M+1, L_0+1}(A)$ with $\psi = \Gamma_{M+1}(X)$. Let R_j denote restriction to the first physical letter j . For every outside letter η , complementary word μ , and physical letter j , one has*

$$R_j \text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) = R_j \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi).$$

This is the one-letter product form of the local comparison corresponding to [CPGSV21, lines 2078–2079].

Proof. ✓ Proof Let $Y_i(\tau)$ be the representation matrices supplied by Lemma 12.7.7. By Theorem 12.9.1.8, $Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j$. Fixing the first physical letter in the two length- $(L_0 + 1)$ windows gives

$$\begin{aligned} R_j \text{Res}_{M, L_0+1}^{\tau_\eta^+(\mu)}(\psi) &= \Gamma_{L_0}(Y_M(\tau_\eta^+(\mu))A^j), \\ R_j \text{Res}_{M+1-L_0, L_0+1}^{\tau_\eta^-(\mu)}(\psi) &= \Gamma_{L_0}(Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j). \end{aligned}$$

The two restrictions are therefore equal. □

Theorem 12.9.1.6 (Boundary-matrix product for the two boundary-crossing restrictions).

Lean ✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$. Let ψ be a state on the chain of length $M + 1$ with open-chain representation $\psi = \Gamma_{M+1}(X)$. Let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. If $L_0 < M$, then for every nonempty complementary word μ there are boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$, depending on the physical letter j and the word σ of length L_0 , such that, for every complementary position k ,

$$\begin{aligned} \rho_{j, \sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j, \sigma}^-(k + 1) &= \mu_k, \end{aligned}$$

and, for every j and σ ,

$$Y_M(\rho_{j, \sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j, \sigma}^-)A^jA^\sigma.$$

Proof. ✓ Proof Theorem 12.9.1.4 gives, for every boundary letter η , physical letter j , and words α, σ of length L_0 ,

$$A^\alpha(Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma) = A^\alpha(A^\mu A^j X A^\sigma).$$

Theorem 12.9.22 applies this left-multiplied comparison together with the one-sided boundary products and gives the displayed boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$, including

$$Y_M(\rho_{j, \sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j, \sigma}^-)A^jA^\sigma.$$

□

Theorem 12.9.1.7 (Product equality at the periodic boundary). Lean ✓ Stmt Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$. Let $\psi \in \mathcal{G}_{M+1, L_0+1}(A)$ have an open-chain representation $\psi = \Gamma_{M+1}(X)$. Let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. If $L_0 < M$, then for every boundary letter η , complementary word μ , physical letter j , and word σ of length L_0 ,

$$Y_M(\tau_\eta^+(\mu))A^jA^\sigma = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma.$$

Proof. ✓ Proof Theorem 12.9.1.6 gives boundary conditions $\rho_{j, \sigma}^+$ and $\rho_{j, \sigma}^-$ satisfying

$$\begin{aligned} \rho_{j, \sigma}^+(k + L_0) &= \mu_k, \\ \rho_{j, \sigma}^-(k + 1) &= \mu_k, \end{aligned}$$

and

$$Y_M(\rho_{j, \sigma}^+)A^jA^\sigma = Y_{M+1-L_0}(\rho_{j, \sigma}^-)A^jA^\sigma.$$

Theorem 12.6.9 replaces these auxiliary conditions by the displayed boundary conditions:

$$Y_M(\tau_\eta^+(\mu))A^jA^\sigma = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^jA^\sigma,$$

which is the product equality needed for the periodic-boundary comparison. □

Theorem 12.9.1.8 (Right-product equality at the periodic boundary). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$. Let $\psi \in \mathcal{G}_{M+1, L_0+1}(A)$ have an open-chain representation $\psi = \Gamma_{M+1}(X)$. Let $Y_i(\tau)$ be matrices satisfying $\text{Res}_{i, L_0+1}^\tau(\psi) = \Gamma_{L_0+1}(Y_i(\tau))$. If $L_0 < M$, then for every letter η used in the two boundary conditions, every word μ on the complementary sites, and every physical letter j ,

$$Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j.$$

This is the one-site product equality used to compare the two boundary matrices in the periodic-boundary comparison.

Proof. [✓ Proof](#) Fix j and set

$$Z_j = Y_M(\tau_\eta^+(\mu))A^j - Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j.$$

Theorem 12.9.1.7 gives, after subtracting the two sides, $Z_j A^\sigma = 0$ for every word σ of length L_0 . Since A is L_0 -block-injective, $\text{span}\{A^\sigma : |\sigma| = L_0\} = M_D(\mathbb{C})$. Applying Lemma 12.8.6 gives $Z_j = 0$, hence

$$Y_M(\tau_\eta^+(\mu))A^j = Y_{M+1-L_0}(\tau_\eta^-(\mu))A^j.$$

□

Theorem 12.9.1.9 (Equality of the two boundary-matrix families). [Lean](#) [✓ Stmt](#) Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, let $N \geq 2$, $L_0 + 1 < N$, and $L_0 < L \leq N$, and let $\psi \in \mathcal{G}_{N, L}(A)$ have an open-chain representation $\psi = \Gamma_N(X)$. Suppose Y^+ and Y^- are two families of matrices obtained from the two boundary-crossing cyclic-window constraints, and that they satisfy, for every physical letter j and every boundary condition τ ,

$$\begin{aligned} A^{\tau_{L_0} \cdots \tau_{N-2}} A^j X &= Y_\tau^+ A^j, \\ X A^j A^{\tau_1 \cdots \tau_{N-L_0-1}} &= A^j Y_\tau^-. \end{aligned}$$

For every physical letter $\eta \in \{0, \dots, d-1\}$ and every word μ on the complementary sites, define two boundary conditions by

$$\begin{aligned} \tau_\eta^+(\mu)_i &= \begin{cases} \mu_{i-L_0}, & L_0 \leq i < N-1, \\ \eta, & \text{otherwise,} \end{cases} \\ \tau_\eta^-(\mu)_i &= \begin{cases} \mu_{i-1}, & 1 \leq i < N-L_0, \\ \eta, & \text{otherwise.} \end{cases} \end{aligned}$$

Then the two families agree on these two boundary conditions: $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$. This is the boundary-matrix comparison required for equality of the two restrictions crossing the periodic cut.

Proof. [✓ Proof](#) The reduced cyclic-window compatibilities at the boundary give, for every physical letter j ,

$$\begin{aligned} A^\mu A^j X &= Y_{\tau_\eta^+(\mu)}^+ A^j, \\ X A^j A^\mu &= A^j Y_{\tau_\eta^-(\mu)}^-. \end{aligned}$$

By Lemma 12.9.9, it is enough to prove, for every j ,

$$Y_{\tau_\eta^+(\mu)}^+ A^j = Y_{\tau_\eta^-(\mu)}^- A^j.$$

Write $N = M + 1$. The two boundary-crossing supports start at M and $M + 1 - L_0$. Let $Y_i(\tau)$ denote the cyclic-window representation matrix given by Lemma 12.7.7 at the window beginning at i . The first step identifies the two abstract boundary-condition families with the two boundary-restriction matrices:

$$\begin{aligned} Y_{\tau_\eta^+(\mu)}^+ &= Y_M(\tau_\eta^+(\mu)), \\ Y_{\tau_\eta^-(\mu)}^- &= Y_{M+1-L_0}(\tau_\eta^-(\mu)). \end{aligned}$$

Together with the preceding identifications, Theorem 12.9.1.8 gives, for every physical letter j ,

$$Y_{\tau_\eta^+(\mu)}^+ A^j = Y_{\tau_\eta^-(\mu)}^- A^j.$$

Lemma 12.9.9 then gives the desired equality of boundary matrices. $\square$

Theorem 12.9.1.10 (One-step full-ring boundary comparison). [Lean] [✓ Stmt] *Let A be L_0 -block-injective with $L_0 > 0$. If*

$$T_{L_0}(\Gamma_{L_0+1}(X)) = \Gamma_{L_0+1}(Y),$$

then $XA^j = A^jY$ for every physical letter j .

Proof. [✓ Proof] Evaluate the translated equality on a word βj , where $|\beta| = L_0$. Cyclic translation exchanges the two parts of the word, and cyclicity of the trace gives

$$\Gamma_{L_0}(XA^j)(\beta) = \Gamma_{L_0}(A^jY)(\beta).$$

Injectivity of Γ_{L_0} gives $XA^j = A^jY$. $\square$

Theorem 12.9.1.11 (Full-ring boundary comparison). [Lean] [✓ Stmt] *Let A be L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, and let $\psi \in \mathcal{G}_{L_0+1, L_0+1}(A)$ satisfy $\psi = \Gamma_{L_0+1}(X)$. Then $XA^j = A^jX$ for every physical letter j .*

Proof. [✓ Proof] Write s for the cut shift by L_0 and Y_i for the boundary matrix of the translated state at cut i . Translation invariance of the chain space, the cyclic open-chain reduction, and the composition law supply compatible matrices at successive cuts. Theorem 12.9.1.10 gives the single-letter comparison, and induction on the word w gives

$$Y_i A^w = A^w Y_{|w|s+i}.$$

Injectivity of the propagated length- $(L_0 + 1)$ boundary map gives $Y_0 = X$. Since $(L_0 + 1)s = 0$, every full-ring word ω satisfies $XA^\omega = A^\omega X$. Lemma 12.8.8 then gives $XA^j = A^jX$ for every letter j . $\square$

Theorem 12.9.1.12 (Minimal-ring ground-space containment). [Lean] [✓ Stmt] *If A is L_0 -block-injective with $L_0 > 0$ and $D \geq 1$, then*

$$\mathcal{G}_{L_0+1, L_0+1}(A) \subseteq \text{span}\{V^{(L_0+1)}(A)\}.$$

Proof. [✓ Proof] Let $\psi \in \mathcal{G}_{L_0+1, L_0+1}(A)$. The cyclic open-chain reduction gives $\psi = \Gamma_{L_0+1}(X)$. Theorem 12.9.1.11 shows that X commutes with every letter, and hence with every word of length L_0 . The long-word commutation theorem then places ψ in $\text{span}\{V^{(L_0+1)}(A)\}$. $\square$

12.9.2 Normal unique-ground-state consequences

In the injective case, for $N \geq 2$ and $1 < L \leq N$, the chain ground space equals $\text{span}\{V^{(N)}(A)\}$, so the parent Hamiltonian has a unique ground state in that length range. In the normal case the same conclusion is the payoff of the section, given the boundary-crossing comparison and the length hypotheses $N \geq 2$, $L_0 + 1 \leq N$, and $L_0 < L \leq N$. At equality, the full-ring change-of-cut theorem replaces the complementary-word comparison.

Theorem 12.9.2.1 (Chain ground space is spanned by the MPS vector in the injective case).

[Lean](#) [✓ Stmt](#) *If A is injective, $D \geq 1$, $N \geq 2$, and $1 < L \leq N$, then $\mathcal{G}_{N,L}(A) = \text{span}\{V^{(N)}(A)\}$.*

Proof. [✓ Proof](#) Let $\psi \in \mathcal{G}_{N,L}(A)$. The non-wrapping window conditions and Lemma 12.4.5.4 imply $\psi \in G_N(A)$, so $\psi = \Gamma_N(X)$ for some boundary matrix X . The periodic boundary condition now applies Theorem 12.6.16 and forces X to commute with every letter A^j . Since A is injective, the letters generate the full matrix algebra, hence X is scalar. Therefore ψ is proportional to $V^{(N)}(A)$, while the reverse inclusion is Lemma 12.7.8. $\square$

Theorem 12.9.2.2 (Normal tensor: $\mathcal{G}_{N,L}(A) \subseteq \text{span}\{V^{(N)}(A)\}$). [Lean](#) [✓ Stmt](#) *If A is normal, $D \geq 1$, and injective after blocking $L_0 > 0$ sites, with $N \geq 2$, $L_0 + 1 \leq N$, and $L_0 < L \leq N$, then $\mathcal{G}_{N,L}(A) \subseteq \text{span}\{V^{(N)}(A)\}$.*

Proof. [✓ Proof](#) If $N = L_0 + 1$, range antitonicity gives

$$\mathcal{G}_{N,L}(A) \subseteq \mathcal{G}_{N,L_0+1}(A),$$

and Theorem 12.9.1.12 applies to the minimal ring. If $N > L_0 + 1$, the open-chain containment gives $\psi = \Gamma_N(X)$. The two boundary-crossing cyclic-window constraints give

$$\begin{aligned} A^\mu A^j X &= Y_{\tau_\eta^+(\mu)}^+ A^j, \\ X A^j A^\mu &= A^j Y_{\tau_\eta^-(\mu)}^-. \end{aligned}$$

The boundary-crossing comparison is $Y_{\tau_\eta^+(\mu)}^+ = Y_{\tau_\eta^-(\mu)}^-$. These equations imply $\psi \in \text{span}\{V^{(N)}(A)\}$. $\square$

Theorem 12.9.2.3 (Chain ground space is spanned by the MPS vector in the normal case).

[Lean](#) [✓ Stmt](#) *If A is normal, $D \geq 1$, and injective after blocking $L_0 > 0$ sites, with $N \geq 2$, $L_0 + 1 \leq N$, and $L_0 < L \leq N$, then $\mathcal{G}_{N,L}(A) = \text{span}\{V^{(N)}(A)\}$.*

Theorem 12.9.2.4 (Unique ground state throughout the block-injective interaction range).

[Lean](#) [✓ Stmt](#) *If $D \geq 1$, A is injective after blocking $L_0 > 0$ sites, and $L_0 < L \leq N$, then the parent Hamiltonian of interaction range L has a unique periodic ground state. These inequalities already imply $N \geq 2$ and $N \geq L_0 + 1$. Thus this strengthens the finite-size range $N \geq 2L_0$ printed in PGVWC07, Theorem *uniqueGS* [PGVWC07].*

Proof. [✓ Proof](#) Block injectivity at the positive length L_0 makes A normal. Hence Theorem 12.9.2.3 gives

$$\mathcal{G}_{N,L}(A) = \text{span}\{V^{(N)}(A)\}. \quad (12.79)$$

The block-injectivity nonvanishing theorem gives

$$V^{(N)}(A) \neq 0. \quad (12.80)$$

Therefore

$$\dim \mathcal{G}_{N,L}(A) = \dim \text{span}\{V^{(N)}(A)\} = 1. \quad (12.81)$$

□

Theorem 12.9.2.5 (Unique ground state on the periodic chain). [Lean](#) [✓ Stmt](#) *For an injective tensor A with $D \geq 1$, if $N \geq 2$ and $1 < L \leq N$, then the periodic chain ground space is one-dimensional.*

Proof. [✓ Proof](#) By Theorem 12.9.2.1, the ground space is the line $\text{span}\{V^{(N)}(A)\}$. It remains to show that the MPV is nonzero. If $V^{(N)}(A) = 0$, then Lemma 12.7.2 gives $\Gamma_N(\mathbb{1}) = \Gamma_N(0)$. Since A is injective and $N > 0$, Theorem 12.4.3.1 forces $\mathbb{1} = 0$, a contradiction. Therefore the spanning line is one-dimensional. □

Theorem 12.9.2.6 (Unique ground state after blocking). [Lean](#) [✓ Stmt](#) *If A becomes injective after blocking $L_0 > 0$ sites and $D \geq 1$, the parent Hamiltonian with interaction range $2L_0$ has a unique ground state on every periodic chain with $N \geq 2L_0$. In particular, this includes $L_0 = 1$ and $N = 2$.*

Proof. [✓ Proof](#) Since $L_0 > 0$, one has $L_0 < 2L_0$. The result is therefore the case $L = 2L_0$ of Theorem 12.9.2.4. □

Theorem 12.9.2.7 (Unique ground state for normal tensors at range $L_0 + 1$). [Lean](#) [✓ Stmt](#) *If A is normal, becomes injective after blocking $L_0 > 0$ sites, and $D \geq 1$, then the parent Hamiltonian with interaction range $L_0 + 1$ has a unique ground state on every periodic chain with $L_0 + 1 \leq N$.*

Proof. [✓ Proof](#) The inequalities $L_0 < L_0 + 1 \leq N$ put this statement in the range of Theorem 12.9.2.4. □

12.10 Half-chain Schmidt spectrum and the dual fixed point

This section carries out the Schmidt-decomposition step behind the half-chain interpretation of the positive diagonal fixed point Λ of [PGVWC07, Theorem 6]; it does not establish the full statement of that theorem. On a ring of $2L$ sites the matrix product vector splits at the two boundaries into a pair of half-chain families, the overlaps of the half-chain vectors are matrix elements of the L -fold transfer map, and the unnormalized half-chain reduced density matrix factors exactly through the half-chain Gram matrix. Away from zero, its spectrum is that of a boundary comparison matrix of size D^2 , and under the generic spectral condition of the source that comparison matrix converges entrywise, geometrically in L , to the diagonal matrix with entries $\lambda_\alpha \lambda_\beta$ up to the trace normalization of Λ . Two conclusions of the source remain outside this section: the passage from entrywise convergence of the comparison matrix to convergence of its eigenvalues, and the normalization of the finite-size reduced density matrix to unit trace. The finite-size convention throughout is the ring of $N = 2L$ sites split into two halves of L sites each.

The source states its canonical form with a trace-preserving dual map and a positive diagonal fixed point Λ of that dual. Here the trace-preserving direction is placed on the transfer map itself and Λ is its fixed point; the two conventions are exchanged by passing to the conjugate-transposed tensor.

Definition 12.10.1 (Half-chain boundary families). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a tensor A and a half-chain length L , the *half-chain family* is the matrix Ψ with rows indexed by configurations $\sigma \in \{0, \dots, d-1\}^L$ and columns indexed by boundary pairs (α, β) , whose column at (α, β) is the half-chain vector

$$|\Psi_{\alpha, \beta}\rangle = \sum_{\sigma} \langle \alpha | A^{\sigma} | \beta \rangle |\sigma\rangle. \quad (12.82)$$

The *complementary family* Φ is the matrix with rows indexed by boundary pairs and columns by configurations, whose row at (α, β) is the vector $|\Psi_{\beta, \alpha}\rangle$ with the two boundary indices swapped.

Definition 12.10.2 (Half-chain coefficient kernel). [Lean](#) [Lean](#) [✓ Stmt](#) The *coefficient kernel* of the ring of $2L$ sites is the matrix over the two half-chain configuration spaces whose entry at (σ, τ) is the matrix product vector component $V^{(2L)}(A)_{\sigma\tau} = \text{tr}(A^{\sigma} A^{\tau})$ of the joined configuration.

Lemma 12.10.3 (Splitting of the ring state at the two boundaries). [Lean](#) [✓ Stmt](#) The *coefficient kernel factors as the product $\Psi\Phi$ of the half-chain family with its complementary family*. Equivalently,

$$|\psi\rangle = \sum_{\alpha, \beta} |\Psi_{\alpha, \beta}\rangle |\Psi_{\beta, \alpha}\rangle. \quad (12.83)$$

Proof. [✓ Proof](#) Insert the resolution of the identity $\sum_{\beta} |\beta\rangle \langle \beta|$ at the two cut bonds of the ring trace: $\text{tr}(A^{\sigma} A^{\tau}) = \sum_{\alpha, \beta} \langle \alpha | A^{\sigma} | \beta \rangle \langle \beta | A^{\tau} | \alpha \rangle$. $\square$

Definition 12.10.4 (Half-chain reduced density matrix). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The *unnormalized half-chain reduced density matrix* of the ring state on $2L$ sites is $\rho_L = K K^{\dagger}$ with K the coefficient kernel. Entrywise this is the partial trace of the unnormalized ring state over the complementary half,

$$(\rho_L)_{\sigma\sigma'} = \sum_{\tau} V^{(2L)}(A)_{\sigma\tau} \overline{V^{(2L)}(A)_{\sigma'\tau}}, \quad (12.84)$$

and ρ_L is positive semidefinite.

Definition 12.10.5 (Half-chain Gram matrix). [Lean](#) [✓ Stmt](#) The *half-chain Gram matrix* is $G = \Psi^{\dagger} \Psi$, the matrix of overlaps $G_{(\alpha, \beta), (\alpha', \beta')} = \langle \Psi_{\alpha, \beta} | \Psi_{\alpha', \beta'} \rangle$ of the half-chain vectors.

Lemma 12.10.6 (Half-chain overlaps through the transfer map). [Lean](#) [✓ Stmt](#) The *half-chain overlaps are matrix elements of the L -fold transfer map on matrix units*:

$$\langle \Psi_{\alpha, \beta} | \Psi_{\alpha', \beta'} \rangle = \langle \alpha' | \mathcal{E}_A^L(|\beta'\rangle \langle \beta|) | \alpha \rangle. \quad (12.85)$$

Proof. [✓ Proof](#) Expand the overlap as a sum over configurations of products $\langle \alpha' | A^{\sigma} | \beta' \rangle \overline{\langle \alpha | A^{\sigma} | \beta \rangle}$ and recognize the entry of $\sum_{\sigma} A^{\sigma} | \beta' \rangle \langle \beta | (A^{\sigma})^{\dagger}$, which is the L -fold transfer map applied to the matrix unit. $\square$

Theorem 12.10.7 (Half-chain Schmidt factorization). [Lean](#) [Lean](#) [✓ Stmt](#) Let S denote the swap of the two boundary indices. The *unnormalized half-chain reduced density matrix factors exactly as*

$$\rho_L = \Psi (S G^{\top} S) \Psi^{\dagger}, \quad (12.86)$$

where $S G^{\top} S$ is the Gram matrix of the complementary family.

Proof. ✓ Proof By the splitting $K = \Psi\Phi$ of Lemma 12.10.3, $\rho_L = \Psi(\Phi\Phi^\dagger)\Psi^\dagger$, and the Gram matrix $\Phi\Phi^\dagger$ of the complementary family is the transpose of the boundary-swapped half-chain Gram matrix. $\square$

Theorem 12.10.8 (Boundary comparison of the half-chain spectrum). Lean ✓ Stmt *The characteristic polynomials of the half-chain reduced density matrix and of the boundary comparison matrix $(SG^\top S)G$ of size D^2 agree after matching the trivial kernel factors:*

$$X^{D^2} \chi_{\rho_L}(X) = X^{d^L} \chi_{(SG^\top S)G}(X). \quad (12.87)$$

In particular the nonzero spectrum of ρ_L , with multiplicities, is that of the boundary comparison matrix.

Proof. ✓ Proof Apply the rectangular cyclicity of the characteristic polynomial to the factorization (12.86), regrouping the product as Ψ times $(SG^\top S)\Psi^\dagger$; the reversed product is $(SG^\top S)\Psi^\dagger\Psi = (SG^\top S)G$. $\square$

Theorem 12.10.9 (Geometric transfer error of the half-chain overlaps). Lean ✓ Stmt *Let A have trace-preserving transfer map $\mathcal{E}_A$ with fixed point Λ , $\text{tr}(\Lambda) \neq 0$, and suppose*

$$\rho_{\text{spec}}(\mathcal{E}_A - P_\Lambda) < 1, \quad (12.88)$$

where P_Λ is the fixed-point projection. Then there are constants $C > 0$ and $0 < r < 1$ such that for every $L \geq 1$ and all boundary pairs,

$$\left| G_{(\alpha, \beta), (\alpha', \beta')} - \langle \alpha' | P_\Lambda (|\beta'\rangle\langle\beta|) | \alpha \rangle \right| \leq Cr^L. \quad (12.89)$$

The hypothesis (12.88) expresses the generic condition C2 of [PGVWC07] in the trace-preserving gauge: apart from the simple eigenvalue one, the spectrum of $\mathcal{E}_A$ lies strictly inside the unit disc. The proof chooses r strictly between the spectral radius of $\mathcal{E}_A - P_\Lambda$ and one. Without a semisimplicity assumption on the subleading spectrum, the theorem does not assert the same estimate with r equal to that spectral radius.

Proof. ✓ Proof Write $\mathcal{E}_A^L = P_\Lambda + (\mathcal{E}_A - P_\Lambda)^L$ for $L \geq 1$ by the power decomposition. By (12.85) the deviation of a Gram entry from the fixed-point projection entry is an entry of $(\mathcal{E}_A - P_\Lambda)^L$ applied to a matrix unit, bounded by the operator norm of the L -th power, which decays geometrically below unit spectral radius. $\square$

Lemma 12.10.10 (Diagonal limit of the half-chain overlaps). Lean ✓ Stmt *For a diagonal fixed point $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_D)$ with $\text{tr}(\Lambda) \neq 0$, the limiting overlaps form a diagonal matrix in the boundary pairs:*

$$\langle \alpha' | P_\Lambda (|\beta'\rangle\langle\beta|) | \alpha \rangle = \frac{\lambda_\alpha}{\text{tr}(\Lambda)} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}. \quad (12.90)$$

For $\text{tr}(\Lambda) = 1$ this is the display $\lambda_\alpha \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}$ of [PGVWC07].

Proof. ✓ Proof Evaluate the fixed-point projection on the matrix unit: its trace is $\delta_{\beta, \beta'}$, and the diagonal entry of Λ at (α', α) is $\lambda_\alpha \delta_{\alpha, \alpha'}$. $\square$

Theorem 12.10.11 (Geometric convergence to the doubled fixed-point spectrum). Lean ✓ Stmt *Let A have trace-preserving transfer map with diagonal fixed point $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_D)$, $\text{tr}(\Lambda) \neq$*

0, satisfying the spectral condition (12.88). Then there are constants $C > 0$ and $0 < r < 1$ such that for every $L \geq 1$ the boundary comparison matrix satisfies, entrywise,

$$\left| ((SG^T S)G)_{(\alpha, \beta), (\alpha', \beta')} - \frac{\lambda_\alpha \lambda_\beta}{\text{tr}(\Lambda)^2} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} \right| \leq Cr^L. \quad (12.91)$$

The limiting matrix is the diagonal matrix $\Lambda \otimes \Lambda$ up to the trace normalization. Together with Theorem 12.10.8, the nonzero spectrum of the half-chain reduced density matrix on the ring of $2L$ sites is that of a $D^2 \times D^2$ matrix converging entrywise, geometrically in L , to $\Lambda \otimes \Lambda / \text{tr}(\Lambda)^2$. The remaining passage to the convergence of the eigenvalues themselves is the continuity of the spectrum of a matrix in its entries and is not part of this section.

Proof. ✓ Proof Both factors of the comparison matrix are entrywise within Cr^L of their diagonal limits by Theorem 12.10.9 and Lemma 12.10.10, the swapped-transposed factor because swapping and transposing permute entries and exchange the two diagonal weights. The limit entries are bounded, so the product deviates from the product of the limits by at most a constant multiple of r^L , with constant $D^2(C(B+C) + BC) + 1$ for an entry bound B on the limits. $\square$

Chapter 13

Commuting Parent Hamiltonians and Spectral Gaps

For normal matrix product states, the first sections compare commutation of neighboring parent terms with renormalization fixed points and zero correlation length under the stated hypotheses. Decorrelation is expressed through commuting support projectors and their idempotent product.

The remaining sections establish spectral-gap estimates for frustration-free sums of local projections. The anticommutator form of the martingale criterion and cyclic norm-compression bounds turn control of overlapping ground spaces into a uniform lower bound above the ground space.

13.1 Commuting Local Terms for Parent Hamiltonians

This section concerns the commutation equation for local terms in a periodic chain. For normal tensors it relates renormalization fixed points to the tensors whose length-two local terms commute, $h_i(A, 2)h_j(A, 2) = h_j(A, 2)h_i(A, 2)$; the implication from commuting terms back to a renormalization fixed point uses a left-canonical normalization.

The nearest-neighbor commuting parent Hamiltonian condition in [CPGSV16, Definition 3.9 and Theorem 3.10] is distinct from the parent commuting Hamiltonian condition of [CPGSV16, Appendix D.2], which is stated for local projectors Q_{AX} and Q_{XB} . The latter appears in the decorrelation section that follows. In [CPGSV16, Definition 3.9] the parent-Hamiltonian condition also says that, for $N > L$, the ground space is spanned by the periodic MPS vectors associated to the BNT components; nearest-neighbor means the specialization $L = 2$.

The proof of the forward implication in [CPGSV16, Theorem 3.10] is the single sentence at source lines 1305–1307 saying that it follows immediately from the structural characterization in Theorem 3.11. It does not invoke Definition D.2. The three-site kernel intersection proved below is instead a separate local specialization of that Appendix D definition.

Definition 13.1.1 (Three-coordinate identification). [Lean](#) [✓ Stmt](#) For every type α , evaluation at the three coordinates gives the canonical equivalence

$$(\{0, \dots, 3-1\} \rightarrow \alpha) \simeq \alpha \times (\alpha \times \alpha). \quad (13.1)$$

Its forward map is

$$x \mapsto (x(0), (x(1), x(2))). \quad (13.2)$$

Proof. ✓ Proof On the first pair, the final coordinate is unchanged, which contributes $\delta_{b,b'}$ to the matrix coefficient. On the final pair, the first coordinate is unchanged, which contributes $\delta_{a,a'}$. $\square$

Theorem 13.1.4 (Three-site translated local terms). Lean Lean ✓ Stmt Put $q(A) = 1 - P_{G_2(A)}$ for the canonical two-site parent interaction. On the three-site space $\mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$, with sites 0, 1, 2 labelled A, X, B , the two adjacent translated length-two parent interactions are the two local actions

$$\begin{aligned} h_0^{(3)}(A, 2) &= q(A)_{AX}, \\ h_1^{(3)}(A, 2) &= q(A)_{XB}. \end{aligned}$$

This identifies the two translated parent interactions with the two supports appearing in [CPGSV16, Definition D.2]; it does not assert that they commute.

Proof. ✓ Proof Let $\sigma = (a, x, b)$ be a three-site configuration. For the first translated term, the length-two window and the corresponding replacement satisfy

$$\begin{aligned} \sigma_{\{0,1\}} &= (a, x) = \sigma_{AX}, \\ \sigma^{\{0,1\} \leftarrow (\alpha, \xi)} &= (\alpha, \xi, b) = \sigma^{AX \leftarrow (\alpha, \xi)}. \end{aligned}$$

For the second translated term,

$$\begin{aligned} \sigma_{\{1,2\}} &= (x, b) = \sigma_{XB}, \\ \sigma^{\{1,2\} \leftarrow (\xi, \beta)} &= (a, \xi, \beta) = \sigma^{XB \leftarrow (\xi, \beta)}. \end{aligned}$$

Substituting these two pairs of identities into the definition of the translated parent term gives the two displayed equations. $\square$

Theorem 13.1.5 (Three-site commutation from overlapping supports). Lean ✓ Stmt Suppose the canonical two-site parent interaction $q(A) = 1 - P_{G_2(A)}$ satisfies the overlapping AX/XB commutation condition of Definition 13.1.2. Then, on the three-site chain, $h_0^{(3)}(A, 2)h_1^{(3)}(A, 2) = h_1^{(3)}(A, 2)h_0^{(3)}(A, 2)$. This only identifies the first two translated local terms with the lifted AX and XB actions; the construction of the [CPGSV16, Appendix D.2] projectors associated with the [CPGSV16, Appendix B] basic-vector form is a separate step.

Proof. ✓ Proof By Theorem 13.1.4, the two translated local terms are $q(A)_{AX}$ and $q(A)_{XB}$. The assumed overlapping support condition says exactly that these two lifted endomorphisms commute. $\square$

Theorem 13.1.6 (Three-site commutation from two-site representatives). Lean ✓ Stmt Let Q_{AX} and Q_{XB} be two-site projectors whose lifted AX and XB actions commute. Suppose that $q(A) = Q_{AX}$ and $q(A) = Q_{XB}$. Then the first two translated length-two local terms on the three-site chain commute.

Proof. ✓ Proof The lift identities identify the translated terms with the lifted actions of the two projectors:

$$\begin{aligned} h_0^{(3)}(A, 2) &= (Q_{AX})_{AX}, \\ h_1^{(3)}(A, 2) &= (Q_{XB})_{XB}. \end{aligned}$$

The overlapping-support hypothesis is the equation $(Q_{AX})_{AX}(Q_{XB})_{XB} = (Q_{XB})_{XB}(Q_{AX})_{AX}$. Substituting the two lifted identities gives the desired commutation of $h_0^{(3)}(A, 2)$ and $h_1^{(3)}(A, 2)$. $\square$

Theorem 13.1.7 (Three-site commutation from source projectors). [Lean](#) [✓ Stmt](#) Suppose the projectors Q_{AX} and Q_{XB} satisfy the [CPGSV16, Definition D.2] local parent-commuting condition on AXB , and suppose that the canonical two-site parent interaction is identified, as a two-site operator, with the two source projectors: $q(A) = Q_{AX}$ and $q(A) = Q_{XB}$. After lifting these two equalities to the AX and XB faces, the first two translated length-two local terms on the three-site chain commute. Thus, $h_0^{(3)}(A, 2)h_1^{(3)}(A, 2) = h_1^{(3)}(A, 2)h_0^{(3)}(A, 2)$.

Proof. [✓ Proof](#) The lift identities of Theorem 13.1.4 identify the translated terms with $(Q_{AX})_{AX}$ and $(Q_{XB})_{XB}$. After forgetting the kernel-intersection equation, the [CPGSV16, Definition D.2] condition says exactly that these two lifted projectors commute. $\square$

Theorem 13.1.8 (Three-site commutation from complementary Appendix projectors). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose the Appendix D.2 projectors Q_{AX} and Q_{XB} satisfy the local parent-commuting condition on AXB , and suppose that the canonical two-site parent interaction is their complement on both faces: $q(A) = 1 - Q_{AX} = 1 - Q_{XB}$. Then the first two translated length-two local terms on the three-site chain commute, $h_0^{(3)}(A, 2)h_1^{(3)}(A, 2) = h_1^{(3)}(A, 2)h_0^{(3)}(A, 2)$.

Proof. [✓ Proof](#) The lift identities of Theorem 13.1.4 identify the translated terms with $(1 - Q_{AX})_{AX}$ and $(1 - Q_{XB})_{XB}$. The Appendix D.2 condition implies the overlapping two-site commutation condition after forgetting the kernel-intersection equation. Definition 13.1.2 then gives commutation of the complementary lifted projectors, hence the displayed commutation relation. $\square$

Definition 13.1.9 (Commuting local terms for a parent Hamiltonian). [Lean](#) [✓ Stmt](#) The parent Hamiltonian with block length L on N sites has *commuting* local terms if $h_i h_j = h_j h_i$ for all $i, j \in \{0, \dots, N-1\}$. This records the translated commutation equation $[\tau_j(P_L), P_L] = 0$ from [CPGSV16, Definition 3.9]; the same source definition also requires the ground space to be spanned, for $N > L$, by the periodic MPS vectors of the BNT components.

Definition 13.1.10 (Nearest-neighbor commuting parent Hamiltonian). [Lean](#) [✓ Stmt](#) The nearest-neighbor commuting condition is the length-2 specialization of the preceding commutation equation. For every $i, j \in \{0, \dots, N-1\}$, one has $h_i h_j = h_j h_i$. The source definition [CPGSV16, Definition 3.9] also requires the corresponding parent Hamiltonian to have $L = 2$ and to have the prescribed ground-space spanning property for $N > 2$.

Lemma 13.1.11 (Overlap reduction for commuting local terms). [Lean](#) [Lean](#) [✓ Stmt](#) Let $L \leq N$. Suppose that $h_i h_j = h_j h_i$ whenever the length- L cyclic supports of the two local terms meet. Then the parent Hamiltonian has commuting local terms: $h_i h_j = h_j h_i$ for all i, j . In the case $L = 2$, this gives the nearest-neighbor commuting condition.

Proof. [✓ Proof](#) If the two cyclic supports meet, the hypothesis gives the commutation relation. If they are disjoint, the corresponding local terms commute by locality. $\square$

Lemma 13.1.12 (Length-two cyclic-overlap alternatives). [Lean](#) [✓ Stmt](#) For $i, j \in \mathbb{Z}/N\mathbb{Z}$, if the two length-two cyclic supports starting at i and j overlap, then

$$\begin{aligned} & j = i, \\ \text{or} \quad & j \equiv i + 1 \pmod{N}, \\ \text{or} \quad & i \equiv j + 1 \pmod{N}. \end{aligned} \tag{13.3}$$

Proof. ✓ Proof Expanding each length-two cyclic support gives shifts $r_i, r_j \in \{0, 1\}$ and an equality of sites $i + r_i \equiv j + r_j \pmod{N}$. The four possible pairs (r_i, r_j) give exactly the three alternatives in (13.3). $\square$

Lemma 13.1.13 (Adjacent two-site translates suffice for nearest-neighbor commutation). Lean

Lean ✓ Stmt Let $N \geq 2$. Suppose that, for every $i \in \mathbb{Z}/N\mathbb{Z}$, the translated length-two local term commutes with its cyclic right neighbor:

$$h_i(A, 2)h_{i+1}(A, 2) = h_{i+1}(A, 2)h_i(A, 2). \quad (13.4)$$

Then all translated length-two local terms commute, and the family $p_i = h_i(A, 2)$ satisfies the finite-chain local-projector hypotheses used in the conditional Appendix B extraction.

Proof. ✓ Proof By Lemma 13.1.12, overlapping length-two cyclic supports force the three alternatives in (13.3). The equal-start case gives $h_i(A, 2) = h_j(A, 2)$. The two nearest-neighbor cases are (13.4) and its transpose. The overlap-reduction lemma supplies the remaining disjoint pairs by locality. $\square$

Lemma 13.1.14 (Local-projector hypotheses from overlapping two-site commutation). Lean

✓ Stmt Let $N \geq 2$. Suppose that the translated length-two local terms commute for every pair of overlapping cyclic windows: $h_i(A, 2)h_j(A, 2) = h_j(A, 2)h_i(A, 2)$. Then the family $p_i = h_i(A, 2)$ gives the finite-chain local-projector hypotheses used in the conditional Appendix B extraction.

Proof. ✓ Proof Lemma 13.1.11 turns the overlapping-pair commutation equations into commutation for all pairs. The idempotency equation is Lemma 12.3.57. $\square$

Definition 13.1.15 (Commutation and zero-energy equations for nearest-neighbor local terms).

Lean ✓ Stmt This condition records the commutation equation and the zero-energy equation for the nearest-neighbor local terms:

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(A) &= 0. \end{aligned}$$

This is the zero-energy part of the condition appearing in [CPGSV16, Theorem 3.10(iii)]: the MPS vector has zero energy for the translated two-site local terms, and those terms commute. The missing source assertion is that the whole parent-Hamiltonian ground space is spanned by the corresponding periodic MPS vectors from the BNT components, as recorded in Definition 13.1.16.

Definition 13.1.16 (Parent-Hamiltonian ground-space spanning condition). Lean ✓ Stmt

Let B be a canonical-form tensor with BNT components $A_1, \dots, A_g$. The source parent-Hamiltonian spanning condition is that, for every $N > L$, $\ker H_L^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}$. This is the ground-space clause in [CPGSV16, Definition 3.9]. It is separate from the commutation equation $[\tau_j(P_L), P_L] = 0$ and from the zero-energy equation for a single periodic MPS vector.

Definition 13.1.17 (Nearest-neighbor commuting parent-Hamiltonian ground-space condition).

Lean ✓ Stmt For a fixed chain length N , this condition is the conjunction of the nearest-neighbor commutation equation, the zero-energy equation for the MPS vector, and the ground-space spanning equation:

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(B) &= 0, \\ \ker H_2^{(N)}(B) &= \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{aligned}$$

The source statement in [CPGSV16, Theorem 3.10(iii)] quantifies this condition over all $N > 2$.

Definition 13.1.18 (All-chain nearest-neighbor commuting parent-Hamiltonian ground-space condition). [Lean](#) [✓ Stmt](#) This is the source quantification in [CPGSV16, Theorem 3.10(iii)]. For every $N > 2$, the fixed-chain condition of Definition 13.1.17 holds:

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(B) &= 0, \\ \ker H_2^{(N)}(B) &= \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{aligned}$$

This condition records the ground-space spanning equation, but does not prove it for a concrete canonical-form tensor.

Remark 13.1.19 (Elementary consequences of the commuting definition). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The nearest-neighbor condition is the length-2 specialization of the general commuting condition. The ground-vector condition above contains this commuting equation together with the frustration-free equation for $V^{(N)}(A)$; it does not assert the source ground-space spanning property. For $N \geq 2$, Lemma 12.3.63 with $L = 2$ gives $h_i V^{(N)}(A) = 0$ for every site i . The fixed-chain ground-space condition also contains $\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}$, and quantifying this condition over all $N > 2$ gives the nearest-neighbor case of the spanning condition above. Conversely, the all-chain condition is exactly the conjunction of the all-chain ground-vector equations and the nearest-neighbor instance of the spanning condition above. This is precisely the condition named in Definition 13.1.18. Since the displayed zero-energy equation follows from the parent-Hamiltonian construction for $N > 2$, independently of the commutation and spanning equations, this is equivalently

$$\begin{aligned} \forall N > 2, \quad & \left[\begin{array}{l} \forall i, j, \quad h_i^{(N)} h_j^{(N)} = h_j^{(N)} h_i^{(N)}, \\ \forall i, \quad h_i^{(N)} V^{(N)}(B) = 0, \\ \ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\} \end{array} \right] \\ \iff & \left[\begin{array}{l} \forall N > 2, \quad \forall i, j, \quad h_i^{(N)} h_j^{(N)} = h_j^{(N)} h_i^{(N)}, \\ \forall N > 2, \quad \ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{array} \right]. \end{aligned}$$

Theorem 13.1.20 (Chain ground-space vectors are frustration-free). [Lean](#) [✓ Stmt](#) Assume $N > 0$ and $L \leq N$. Let ψ be a vector in the periodic chain ground space $G_L^{(N)}(A)$. Then ψ satisfies all local zero-energy equations: for every site i , one has $h_i(A, L)\psi = 0$.

Proof. [✓ Proof](#) Write $R_{i,\eta}\psi$ for the cyclic length- L restriction of ψ at site i , with the outside configuration fixed by η . Membership in $G_L^{(N)}(A)$ gives $R_{i,\eta}\psi \in G_L(A) = \ker h_L(A)$ for every i and η . Hence $h_L(A)R_{i,\eta}\psi = 0$. For a full configuration σ , taking $\eta = \sigma|_{\{i, \dots, i+L-1\}^c}$ gives

$$(h_i(A, L)\psi)(\sigma) = (h_L(A)R_{i,\eta}\psi)(\sigma|_{\{i, \dots, i+L-1\}}) = 0.$$

Thus $h_i(A, L)\psi = 0$ for every i . □

Lemma 13.1.21 (Zero-energy BNT vectors lie in the parent-Hamiltonian kernel). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let B be a canonical-form tensor with BNT components $A_1, \dots, A_g$. If every component vector has zero energy for the parent Hamiltonian, $H_L^{(N)}(B)V^{(N)}(A_j) = 0$ for $1 \leq j \leq g$, then

$$\text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\} \subseteq \ker H_L^{(N)}(B).$$

It is enough to assume the local frustration-free equations $h_i V^{(N)}(A_j) = 0$ for every i and j . For $N > 0$ and $L \leq N$, in the block-diagonal case $B = \bigoplus_j \mu_j A_j$, with every $\mu_j \neq 0$, the same inclusion follows because each local space $G_L(A_j)$ is contained in $G_L(B)$. This is the easy inclusion in the source ground-space spanning equation of [CPGSV16, Definition 3.9]; the reverse inclusion is the remaining spanning direction.

Proof. ✓ Proof The kernel is a linear subspace. Hence it is enough to check the generators $V^{(N)}(A_j)$. The first hypothesis gives this directly. Under the local frustration-free equations,

$$H_L^{(N)}(B)V^{(N)}(A_j) = \sum_i h_i V^{(N)}(A_j) = 0.$$

For a block-diagonal tensor, write $R_{i,\tau}$ for the cyclic length- L restriction at site i , with the outside configuration fixed by τ . Then $R_{i,\tau} V^{(N)}(A_j) \in G_L(A_j) \subseteq G_L(B)$. The preceding theorem therefore gives $h_i(B, L)V^{(N)}(A_j) = 0$ for every i . $\square$

Lemma 13.1.22 (Kernel containment from chain-space splitting). Lean ✓ Stmt Let $B = \bigoplus_{j=0}^{r-1} \mu_j A_j$. Suppose $N > 0$, $L \leq N$, and the periodic chain constraints for B split into the block constraints:

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=0}^{r-1} \mathcal{G}_{N,L}(A_j).$$

If every block chain space is contained in its periodic MPS line,

$$\mathcal{G}_{N,L}(A_j) \subseteq \mathbb{C}V^{(N)}(A_j) \quad (0 \leq j < r),$$

then the zero-energy space of the parent Hamiltonian is contained in the BNT vector span:

$$\ker H_L^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\}.$$

Proof. ✓ Proof Theorem 13.3.14 gives $\ker H_L^{(N)}(B) \subseteq \mathcal{G}_{N,L}(B)$. Together with the splitting hypothesis,

$$\ker H_L^{(N)}(B) \subseteq \mathcal{G}_{N,L}(B) \subseteq \bigvee_j \mathcal{G}_{N,L}(A_j).$$

The blockwise containments give

$$\bigvee_j \mathcal{G}_{N,L}(A_j) \subseteq \bigvee_j \mathbb{C}V^{(N)}(A_j),$$

and Lemma 14.1.2 gives

$$\bigvee_j \mathbb{C}V^{(N)}(A_j) = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\}.$$

$\square$

Lemma 13.1.23 (Kernel containment from the block-diagonal boundary comparison). Lean

✓ Stmt Let $B = \bigoplus_{j=0}^{r-1} \mu_j A_j$, with every $\mu_j \neq 0$. Suppose the physical alphabet and all block dimensions are nonempty. Assume that the block family is irreducible, left-canonical, has normalized self-overlap, and has no two blocks gauge-phase equivalent. Suppose also that each A_j

is L_0 -block-injective, that $L_0 > 0$, and that $\sum_a A_a^j (A_a^j)^\dagger = I$ ($0 \leq j < r$). Let $N > 0$, $L > 0$, $L \leq N$, $L + L_0 \leq N$, and $(L_0 + 1) + (r - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1))) + 1 \leq L$. Assume the opened-boundary C^j, D^j comparison: whenever $\psi \in \mathcal{G}_{N,L}(B)$ and $\psi = \Gamma_N^B(\bigoplus_j X_j)$, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every ρ , and every β , $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. If every block chain space is contained in its periodic MPS line,

$$\mathcal{G}_{N,L}(A_j) \subseteq \text{CV}^{(N)}(A_j) \quad (0 \leq j < r),$$

then

$$\ker H_L^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r - 1\}.$$

Proof. [✓ Proof](#) Lemma 14.3.2.9 gives $\mathcal{G}_{N,L}(B) = \bigvee_{j=0}^{r-1} \mathcal{G}_{N,L}(A_j)$ from the stated block hypotheses and the opened-boundary comparison. Lemma 13.1.22 then applies to this equality and to the blockwise containments. $\square$

Lemma 13.1.24 (Kernel containment from boundary-crossing comparisons). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=0}^{r-1} \mu_j A_j$, with every $\mu_j \neq 0$. Suppose the physical alphabet and all block dimensions are nonempty. Assume that the block family is irreducible, left-canonical, has normalized self-overlap, and has no two blocks gauge-phase equivalent. Suppose also that each A_j is L_0 -block-injective, that $L_0 > 0$, and that $\sum_a A_a^j (A_a^j)^\dagger = I$ ($0 \leq j < r$). Let $N > 0$, $L > 0$, $L \leq N$, $L + L_0 \leq N$, and $(L_0 + 1) + (r - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1))) + 1 \leq L$. Assume that, for every boundary-crossing interval beginning at i , the simultaneous products $w \mapsto (A_w^1, \dots, A_w^r)$, $|w| = N - i$, span the product algebra $\prod_j M_{D_j}(\mathbb{C})$. If every block chain space is contained in its periodic MPS line,

$$\mathcal{G}_{N,L}(A_j) \subseteq \text{CV}^{(N)}(A_j) \quad (0 \leq j < r),$$

then

$$\ker H_L^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r - 1\}.$$

Proof. [✓ Proof](#) Lemma 14.3.2.11 gives $\mathcal{G}_{N,L}(B) = \bigvee_{j=0}^{r-1} \mathcal{G}_{N,L}(A_j)$ from the block hypotheses, the local boundary-crossing constraints, and the stated tail-word span. Lemma 13.1.22 then applies to this equality and to the blockwise containments. $\square$

Lemma 13.1.25 (Block-diagonal spanning is equivalent to the reverse inclusion). [Lean](#) [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=0}^{r-1} \mu_j A_j$, with every $\mu_j \neq 0$. Then, for every $N > L$, the source parent-Hamiltonian spanning condition $\ker H_L^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r - 1\}$ is equivalent to the reverse containment, for the same N :

$$\ker H_L^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r - 1\}.$$

In the nearest-neighbor case $L = 2$, the all-chain source condition of Definition 13.1.18 is equivalent to the conjunction

$$\begin{aligned} & (\forall N > 2, \forall i, j \in \{0, \dots, N - 1\}, h_i(B, 2)h_j(B, 2) = h_j(B, 2)h_i(B, 2)) \\ & \wedge \left(\forall N > 2, \ker H_2^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r - 1\} \right). \end{aligned}$$

Proof. [✓ Proof](#) The forward direction of the displayed equality gives the containment by rewriting the kernel. Conversely, for $N > L$ the easy inclusion above gives

$$\text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\} \subseteq \ker H_L^{(N)}(B),$$

and the assumed reverse containment gives equality. For $L = 2$, the all-chain nearest-neighbor condition of Definition 13.1.18 decomposes as the conjunction

$$\begin{aligned} & (\forall N > 2, \forall i, j \in \{0, \dots, N-1\}, h_i(B, 2)h_j(B, 2) = h_j(B, 2)h_i(B, 2)) \\ & \wedge (\forall N > 2, \ker H_2^{(N)}(B) \subseteq \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\}). \end{aligned}$$

The second conjunct is the reverse containment just identified with the source spanning clause. $\square$

Lemma 13.1.26 (Chain-space comparison gives the block-diagonal spanning equation). [Lean](#)

[Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=0}^{r-1} \mu_j A_j$, where $\mu_j \neq 0$. Suppose that, for every $N > L$, $G_L^{(N)}(B) = \bigvee_{j=0}^{r-1} \text{span}\{V^{(N)}(A_j)\}$. Then the source parent-Hamiltonian spanning equation holds: $\ker H_N(B, L) = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\}$. It is enough to prove the two equations

$$\begin{aligned} G_L^{(N)}(B) &= \bigvee_{j=0}^{r-1} G_L^{(N)}(A_j), \\ G_L^{(N)}(A_j) &= \text{span}\{V^{(N)}(A_j)\}. \end{aligned}$$

Proof. [✓ Proof](#) The finite-volume identity gives $\ker H_N(B, L) = G_L^{(N)}(B)$. The assumed chain-space comparison therefore rewrites the kernel as $\bigvee_{j=0}^{r-1} \text{span}\{V^{(N)}(A_j)\}$. The latter supremum is the BNT vector span. Conversely, the easy inclusion of Lemma 13.1.21 gives $\text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\} \subseteq \ker H_N(B, L)$, so the displayed equation is the source spanning clause. If the comparison is first stated as $G_L^{(N)}(B) = \bigvee_j G_L^{(N)}(A_j)$, the single-block equations $G_L^{(N)}(A_j) = \text{span}\{V^{(N)}(A_j)\}$ give the previous hypothesis. $\square$

Lemma 13.1.27 (Gauge invariance of fixed points and parent interactions). [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A virtual similarity preserves the physical blocking relation and hence transfer idempotence. It also preserves every local MPS space, so the canonical parent interaction and all its translated local terms are unchanged.

Proof. [✓ Proof](#) Suppose $B^i = XA^iX^{-1}$ for every physical letter. If the blocking relation for A is $A^{i_1}A^{i_2} = \sum_j U_{i_1, i_2}^j A^j$, then conjugation by X gives the same relation for B . Hence transfer idempotence is invariant under the similarity. At every length L , the two tensors have the same local MPS space, so $G_L(A) = G_L(B)$ and $h_i^{(N)}(A) = h_i^{(N)}(B)$. The parent-interaction identities and the commutation equations follow. $\square$

Theorem 13.1.28 (Renormalization fixed points have commuting nearest-neighbor local terms).

[Lean](#) [✓ Stmt](#) If A has nonzero virtual dimension, is a normal tensor in the sense of [CPGSV16], and is a renormalization fixed point, then for every chain length $N > 2$ the two-site local terms commute. This theorem records the commutation part of the nearest-neighbor source condition, not the ground-space spanning part.

Proof. ✓ Proof Put A into its trace-preserving gauge. This is a pure similarity, since the normal-tensor normalization fixes the transfer spectral radius at one. Similarity preserves the physical blocking relation and the local MPS spaces, hence also the canonical parent interactions. In the trace-preserving gauge, the Appendix B virtual-bond projectors give $h_0^{(3)} h_1^{(3)} = h_1^{(3)} h_0^{(3)}$. Cyclic transport gives, for every $N > 2$ and every site i , $h_i^{(N)} h_{i+1}^{(N)} = h_{i+1}^{(N)} h_i^{(N)}$. For every pair i, j whose two-site supports are disjoint, locality gives $h_i^{(N)} h_j^{(N)} = h_j^{(N)} h_i^{(N)}$. These cases exhaust all pairs of two-site terms on the periodic chain. Transporting the parent interactions back proves the claim. $\square$

Theorem 13.1.29 (Renormalization fixed points satisfy the NNCPH commutation and zero-energy equations). Lean ✓ Stmt *If A has nonzero virtual dimension, is a normal tensor in the sense of [CPGSV16], and is a renormalization fixed point, then for every chain length $N > 2$ the MPS vector $V^{(N)}(A)$ satisfies the commutation and zero-energy equations for nearest-neighbor local terms:*

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(A) &= 0. \end{aligned}$$

The zero-energy equation is the usual parent-Hamiltonian frustration-freeness statement. The separate ground-space spanning clause of the source definition is not asserted here.

Proof. ✓ Proof The first equation is Theorem 13.1.28. The second is Lemma 12.3.63 with $L = 2$. $\square$

Theorem 13.1.30 (Nearest-neighbor parent ground spaces for a single normal RFP tensor). Lean Lean ✓ Stmt *Let A have positive bond dimension and be a normal renormalization fixed-point tensor in the sense of [CPGSV16]. If the canonical-form tensor consists of the single normal sector A , then for every $N > 2$, $\ker H_2^{(N)}(A) = \text{span}\{V^{(N)}(A)\}$. Its translated two-site interactions also commute and annihilate $V^{(N)}(A)$. Hence the singleton family satisfies the all-chain nearest-neighbor commuting parent-Hamiltonian ground-space condition.*

This statement is restricted to one normal sector. It does not assert the corresponding result for canonical-form tensors with several sectors or repeated copies.

Proof. ✓ Proof Spectral normality gives eventual block injectivity. The fixed-point equation then implies that the one-site matrices already span the full matrix algebra. Thus A is injective. For $N > 2$, the finite-volume kernel identity and injective periodic-chain theorem give

$$\ker H_2^{(N)}(A) = \mathcal{G}_{N,2}(A) = \text{span}\{V^{(N)}(A)\}.$$

The span of the matrix product vectors of the singleton family is the last space in this equality. Theorem 13.1.29 supplies commutation and zero energy on every chain of length greater than two. $\square$

Theorem 13.1.31 (Commuting nearest-neighbor terms for distinct multiplicity-one RFP sectors). Lean ✓ Stmt *Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and let $B = \bigoplus_{j=1}^g A_j$ contain one copy of every basis tensor. If B is a renormalization fixed point, then for every $N > 2$ its translated two-site parent interactions commute pairwise. Thus B satisfies the nearest-neighbor commuting parent-Hamiltonian condition.*

This statement treats one copy of every distinct sector. It does not include repeated copies or arbitrary raw sector weights.

Proof. ✓ Proof The fixed-point equation gives residual tensors U_j whose matrix-entry columns form one orthonormal family on the disjoint union of the sector matrix spaces. On the tensor square of this disjoint union, let Q be the sum of the sector bond projectors, extended by zero between unequal sectors. Write L_j and R_j for the left and right overlapping lifts of the sector- j bond projector. The single-sector relation and the mixed-sector orthogonality give

$$\begin{aligned} L_j R_j &= R_j L_j, \\ L_j R_k &= 0 = R_j L_k \quad (j \neq k), \end{aligned}$$

where the latter identities follow from the shared one-site factor $E_j^* E_k = 0$ for the canonical summand inclusions E_j . Summing over the sectors shows that the two adjacent placements of Q commute.

If U is the common isometry, then transport gives the identity $(U \otimes U)Q(U^* \otimes U^*) = P_{G_2(B)}$, since the range on either side is $\bigvee_j G_2(A_j) = G_2(B)$. Hence it is the complement of the canonical two-site parent interaction of B . The two adjacent three-site parent terms therefore commute. Cyclic transport gives adjacent commutation on every chain with $N > 2$; disjoint local terms commute by locality, which proves the claim. $\square$

Theorem 13.1.32 (Nearest-neighbor parent ground spaces for distinct multiplicity-one RFP sectors). Lean ✓ Stmt *Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and let $B = \bigoplus_{j=1}^g A_j$ contain one copy of every basis tensor. If B is a renormalization fixed point, then for every $N > 2$,*

$$\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}.$$

This is the multiplicity-one distinct-sector case of [CPGSV16, Theorem 3.10]. It does not include repeated copies of a basis tensor or arbitrary raw sector weights.

Proof. ✓ Proof The residual isometry equations imply that the simultaneous one-site matrices $(A_j^i)_j$ span the product of the block matrix algebras. Hence the same is true for every positive word length. The one-step restriction intersection therefore gives $\mathcal{G}_{N,2}(B) \subseteq \bigvee_j G_N(A_j)$. Comparing this block decomposition before and after moving the periodic cut by one site shows that every block boundary matrix commutes with the corresponding one-site matrices. Thus each block component lies in $\mathcal{G}_{N,2}(A_j)$, and $\mathcal{G}_{N,2}(B) = \bigvee_j \mathcal{G}_{N,2}(A_j)$. Each block is a normal renormalization fixed point, so one-site injectivity and the injective periodic-chain theorem identify its chain space with the line spanned by $V^{(N)}(A_j)$. The finite-volume kernel identity completes the proof. $\square$

Theorem 13.1.33 (Forward NNCPH ground spaces for distinct multiplicity-one RFP sectors). Lean ✓ Stmt *Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and let $B = \bigoplus_{j=1}^g A_j$ contain one copy of every basis tensor. If B is a renormalization fixed point, then for every $N > 2$ its translated two-site parent interactions commute and*

$$\begin{aligned} H_2^{(N)}(B)V^{(N)}(B) &= 0, \\ \ker H_2^{(N)}(B) &= \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{aligned}$$

Hence B satisfies the all-chain nearest-neighbor commuting parent-Hamiltonian ground-space condition. This is the multiplicity-one distinct-sector forward implication in [CPGSV16, Definition 3.9 and Theorem 3.10]. It does not include repeated copies or arbitrary raw sector weights.

Proof. [✓ Proof](#) Theorem 13.1.31 gives pairwise commutation of the translated two-site interactions for every $N > 2$. The parent-Hamiltonian construction gives the zero-energy equation. Theorem 13.1.32 identifies its kernel with the span of the periodic matrix product vectors of the distinct sectors. These are the three clauses of the all-chain condition. $\square$

Theorem 13.1.34 (Cyclic trace expansion for a closed MPS chain). [Lean](#) [✓ Stmt](#) *For a nonempty closed chain, the trace of the word $A^{\sigma_0} \dots A^{\sigma_{L-1}}$ is the sum over cyclic virtual-index configurations:*

$$\mathrm{tr}(A^{\sigma_0} \dots A^{\sigma_{L-1}}) = \sum_{\alpha_0, \dots, \alpha_{L-1}} \prod_{t=0}^{L-1} (A^{\sigma_t})_{\alpha_t, \alpha_{t+1}}, \quad \alpha_L = \alpha_0.$$

Proof. [✓ Proof](#) The trace is the diagonal sum:

$$\mathrm{tr}(A^{\sigma_0} \dots A^{\sigma_{L-1}}) = \sum_{\alpha_0} (A^{\sigma_0} \dots A^{\sigma_{L-1}})_{\alpha_0, \alpha_0}.$$

For prescribed endpoints, the matrix product entry expands as

$$(A^{\sigma_0} \dots A^{\sigma_{L-1}})_{\alpha_0, \alpha_L} = \sum_{\alpha_1, \dots, \alpha_{L-1}} \prod_{t=0}^{L-1} (A^{\sigma_t})_{\alpha_t, \alpha_{t+1}}.$$

Substituting $\alpha_L = \alpha_0$ in the trace gives the cyclic formula. $\square$

Definition 13.1.35 (Physical isometry in the basic-vector expression). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that A is equipped with Appendix B structural data (X, Λ, U) : the matrix X is invertible, the diagonal entries of Λ are strictly positive, $A^i = X \Lambda U^i X^{-1}$, and U satisfies the pair-index isometry equation below. Let $\mathcal{H}_a$ and $\mathcal{H}_b$ have bases indexed by $\alpha, \beta \in \{0, \dots, D-1\}$. The coefficients $U_{\alpha, \beta}^i$ define maps

$$U: \mathcal{H}_a \otimes \mathcal{H}_b \longrightarrow \mathcal{H}_{\mathrm{phys}},$$

$$(Uv)_i = \sum_{\alpha, \beta} U_{\alpha, \beta}^i v_{\alpha, \beta},$$

and

$$(U^* \psi)_{\alpha, \beta} = \sum_i \overline{U_{\alpha, \beta}^i} \psi_i.$$

Theorem 13.1.36 (The physical map is an isometric embedding). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be equipped with Appendix B structural data (X, Λ, U) as in Definition 13.1.35. Its pair-index isometry equation*

$$\sum_i \overline{U_{\alpha, \beta}^i} U_{\alpha', \beta'}^i = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}$$

implies $U^* U = 1$. In particular, U is injective.

Proof. [✓ Proof](#) Expanding the coefficient of $U^* U v$ at (α, β) and interchanging the two finite sums gives

$$(U^* U v)_{\alpha, \beta} = \sum_{\alpha', \beta'} \left(\sum_i \overline{U_{\alpha, \beta}^i} U_{\alpha', \beta'}^i \right) v_{\alpha', \beta'} = v_{\alpha, \beta}.$$

A map with a left inverse is injective. $\square$

Definition 13.1.37 (Tensor powers of the physical isometry). [Lean](#) [Lean](#) [✓ Stmt](#) For every $N \geq 0$, the coefficients of U define

$$U^{\otimes N}: (\mathcal{H}_a \otimes \mathcal{H}_b)^{\otimes N} \longrightarrow \mathcal{H}_{\text{phys}}^{\otimes N},$$

with

$$(U^{\otimes N}v)_\sigma = \sum_{p_0, \dots, p_{N-1}} \left(\prod_{t=0}^{N-1} U_{(p_t)_a, (p_t)_b}^{\sigma_t} \right) v_{p_0, \dots, p_{N-1}}.$$

The conjugate coefficient map is

$$(U^{*\otimes N}\psi)_p = \sum_{\sigma} \left(\prod_{t=0}^{N-1} \overline{U_{(p_t)_a, (p_t)_b}^{\sigma_t}} \right) \psi_{\sigma}.$$

This is the tensor power occurring in the basic-vector expression [\[CPGSV16, Equation \(3.17\)\]](#).

Theorem 13.1.38 (The tensor powers are isometric embeddings). [Lean](#) [Lean](#) [✓ Stmt](#) For every $N \geq 0$, one has $U^{*\otimes N}U^{\otimes N} = 1$. Hence $U^{\otimes N}$ is injective.

Proof. [✓ Proof](#) For virtual-pair configurations p and q , interchange the two finite sums and factor the sum over physical configurations site by site:

$$\sum_{\sigma} \prod_t \overline{U_{(p_t)_a, (p_t)_b}^{\sigma_t}} U_{(q_t)_a, (q_t)_b}^{\sigma_t} = \prod_t \sum_i \overline{U_{(p_t)_a, (p_t)_b}^i} U_{(q_t)_a, (q_t)_b}^i.$$

The pair-index isometry equation turns the last product into $\prod_t \delta_{p_t, q_t} = \delta_{p, q}$. Therefore, the coefficient at p is v_p . A map with a left inverse is injective. $\square$

Theorem 13.1.39 (Appendix B basis change and the parent interaction). [Lean](#) [✓ Stmt](#) Let $A^i = X\Lambda U^i X^{-1}$ be the Appendix B structural form, and write $A_{\text{core}}^i = \Lambda U^i$. The canonical local parent interaction is unchanged by this basis change: $q_L(A) = q_L(A_{\text{core}})$.

Proof. [✓ Proof](#) The basis change gives a gauge equivalence between A_{core} and A , hence equality of their length- L local ground spaces. The parent interaction is the orthogonal complement projector of this local ground space. $\square$

Definition 13.1.40 (Appendix B two-site basic support). [Lean](#) [✓ Stmt](#) For Appendix B structural data, the two-site basic support is $G_2(A_{\text{core}}) = G_2(\Lambda U)$.

Definition 13.1.41 (Two-site bond insertion and basic-vector embedding). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Put $|\varphi\rangle = \sum_b \lambda_b |b, b\rangle$. For outer-boundary coefficients $v_{a,c}$, insertion of this vector on the middle virtual bond is the map I_{φ} defined by

$$(I_{\varphi}v)_{(a,b),(b',c)} = \delta_{b,b'} \lambda_b v_{a,c}.$$

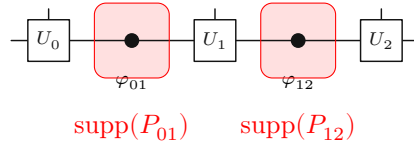
The two-site physical map is $U^{\otimes 2}I_{\varphi}$. A boundary matrix Y determines outer-boundary coefficients by $(v_Y)_{a,c} = \lambda_a Y_{c,a}$. Opening the outer virtual bond in the two-site contraction of [\[CPGSV16, Equations \(3.17\)–\(3.18\)\]](#) gives these coefficients.

Definition 13.1.42 (Virtual bond projector and its adjacent placements). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Write $s = \langle \varphi, \varphi \rangle = \sum_b \lambda_b^2$. The orthogonal projector onto $\mathbb{C}\varphi$ has pair-index coefficients

$$(P_{\varphi}v)_{b,b'} = \delta_{b,b'} \frac{\lambda_b}{s} \sum_k \lambda_k v_{k,k}.$$

On three virtual site pairs, let P_{01} place this projector on the bond joining the right index of site zero to the left index of site one, and let P_{12} place it on the next bond.

Theorem 13.1.43 (Commutation of the adjacent virtual bond projectors). [Lean](#) [✓ Stmt](#) *The two adjacent placements commute: $P_{01}P_{12} = P_{12}P_{01}$. The following support-location schematic marks the two bonds on which the projectors act; it does not depict the doubled operator network for the equality above.*



Proof. [✓ Proof](#) The contracted bond coordinates are disjoint. Thus replacing the first bond by the diagonal basis vector $|k, k\rangle$ and the second by $|\ell, \ell\rangle$ is independent of the order. Expanding both rank-one projectors gives the same double sum over (k, ℓ) . $\square$

Theorem 13.1.44 (Surjectivity of the weighted boundary parametrization). [Lean](#) [✓ Stmt](#) *Strict positivity of the λ_a makes $Y \mapsto v_Y$ surjective.*

Proof. [✓ Proof](#) For prescribed coefficients $v_{a,c}$, take $Y_{c,a} = \lambda_a^{-1}v_{a,c}$. $\square$

Theorem 13.1.45 (The two-site basic-vector image). [Lean](#) [Lean](#) [✓ Stmt](#) *Opening the outer virtual bond in the two-site contraction of [CPGSV16, Equations (3.17)–(3.18)] gives the following local-support family. For every boundary matrix Y , $U^{\otimes 2}I_\varphi(v_Y) = \Gamma_2(\Lambda U)(Y)$. Consequently, $\text{ran}(U^{\otimes 2}I_\varphi) = G_2(\Lambda U)$.*

Proof. [✓ Proof](#) At a two-site physical configuration (i, j) , the bond insertion gives $\sum_{a,b,c} U_{a,b}^i U_{b,c}^j \lambda_b \lambda_a Y_{c,a}$. Expanding $\text{tr}((\Lambda U^i)(\Lambda U^j)Y)$ gives the same sum, so the two maps agree after the weighted boundary parametrization. Since that parametrization is surjective, their ranges are equal. $\square$

Definition 13.1.46 (Two-site basic support projector). [Lean](#) [✓ Stmt](#) Let P_2 be the orthogonal projector onto $G_2(\Lambda U)$, as in the local parent-space construction of [CPGSV16, Definition 3.9].

Theorem 13.1.47 (Range and complement of the two-site basic support projector). [Lean](#) [✓ Stmt](#) *The projector P_2 satisfies*

$$\begin{aligned} P_2^2 &= P_2, \\ \text{ran}(P_2) &= \text{ran}(U^{\otimes 2}I_\varphi) = G_2(\Lambda U), \\ P_2 &= 1 - q_2(\Lambda U). \end{aligned}$$

In particular, $q_2(\Lambda U)$ annihilates the image of $U^{\otimes 2}I_\varphi$, while P_2 fixes that image pointwise.

Proof. [✓ Proof](#) Orthogonal projection onto $G_2(\Lambda U)$ has that subspace as its range. The preceding theorem identifies this range with $\text{ran}(U^{\otimes 2}I_\varphi)$. Orthogonal projection onto the complementary subspace is $q_2(\Lambda U)$, hence $P_2 = 1 - q_2(\Lambda U)$. Since $q_2(\Lambda U)^2 = q_2(\Lambda U)$,

$$P_2^2 = (1 - q_2(\Lambda U))^2 = 1 - q_2(\Lambda U) = P_2.$$

If ψ lies in $\text{ran}(U^{\otimes 2}I_\varphi)$, then $\psi \in G_2(\Lambda U) = \ker q_2(\Lambda U)$, and therefore

$$\begin{aligned} q_2(\Lambda U)\psi &= 0, \\ P_2\psi &= (1 - q_2(\Lambda U))\psi = \psi. \end{aligned}$$

$\square$

Definition 13.1.48 (Two-site support of a residual isometry family). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\mathcal{V} = \bigsqcup_j \{j\} \times [D_j] \times [D_j]$ be the disjoint union of the within-sector virtual pairs, and let $V: \mathcal{V} \rightarrow \mathbb{C}^d$ be the common physical map whose matrix elements are $V_{i,(j,\alpha,\beta)} = (U_j^i)_{\alpha,\beta}$. For each sector insert the vector $|\varphi_j\rangle = \sum_b \lambda_{j,b} |b, b\rangle$ on the middle virtual bond, apply V on both sites, and sum the resulting physical vectors. This defines the equal-sector two-site embedding

$$I_{\text{res}}: \prod_j \mathbb{C}^{D_j \times D_j} \rightarrow (\mathbb{C}^d)^{\otimes 2}.$$

Unequal-sector virtual bonds do not occur. The corresponding physical support operator $P_{2,\text{res}}$ is the orthogonal projector onto $G_2(\bigoplus_j B_j)$.

Theorem 13.1.49 (Range of the residual-family bond embedding). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The common physical map is an isometry, $V^*V = 1$. Moreover, the equal-sector bond embedding and the physical support projector satisfy

$$\begin{aligned} \text{ran}(I_{\text{res}}) &= \sum_j G_2(B_j) = G_2\left(\bigoplus_j B_j\right), \\ \text{ran}(P_{2,\text{res}}) &= \text{ran}(I_{\text{res}}), \end{aligned}$$

and $P_{2,\text{res}} = 1 - q_2(\bigoplus_j B_j)$. If the B_j form a basis of normal tensors and their direct sum is a renormalization fixed point, the sectorwise Appendix B decompositions and the common residual isometry required above exist.

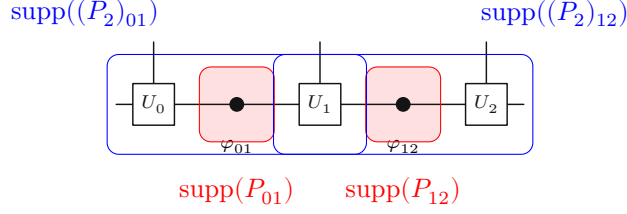
Proof. [✓ Proof](#) The first identity is the one-letter Gram identity on the disjoint union of the within-sector virtual pairs. In each sector, the bond insertion has range $G_2(B_j)$; allowing independent outer-boundary coefficients and summing over sectors therefore gives $\sum_j G_2(B_j)$. The local ground space of a direct sum is this sum, which proves the two range identities. Orthogonal projection onto that space is complementary to its parent interaction. Finally, the residual-isometry form of the cross-block fixed-point structure supplies the simultaneous sector decompositions. $\square$

Theorem 13.1.50 (Commutation of adjacent two-site basic support projectors). [Lean](#) [✓ Stmt](#) Let $(P_2)_{01}$ and $(P_2)_{12}$ be the two adjacent placements of the two-site basic support projector on three physical sites. Then $(P_2)_{01}(P_2)_{12} = (P_2)_{12}(P_2)_{01}$. This is the local three-site consequence of the parent-space construction and the basic-vector form in [CPGSV16, Definition 3.9, Equations (3.16)–(3.18), and Appendix B]. It does not assert commutation for every chain length.

Proof. [✓ Proof](#) The one-site map U is an isometry. The preceding range identity therefore identifies the physical projector with the isometric transport of the virtual bond projector: $P_2 = U^{\otimes 2} P_\varphi (U^*)^{\otimes 2}$. On three sites, multiplication of the two adjacent transports cancels the middle factors by $U^*U = 1$. Consequently their products are

$$\begin{aligned} U^{\otimes 3} P_{01} P_{12} (U^*)^{\otimes 3}, \\ U^{\otimes 3} P_{12} P_{01} (U^*)^{\otimes 3}, \end{aligned}$$

respectively. They agree by Theorem 13.1.43. The following support-location schematic records the one-site embedding and the two overlapping physical supports. It does not depict the doubled $U \cdots U^*$ operator networks in the identities above.



□

Definition 13.1.51 (Appendix B AX coefficient representative). [Lean](#) [✓ Stmt](#) The AX coefficient-space representative is, before the AX -lift, the canonical two-site parent interaction of the core tensor, $\widehat{Q}_{AX} = q_2(A_{\text{core}})$. The source projector Q_{AX} of Definition D.2 acts on $\mathcal{H}_A \otimes \mathcal{H}_X$. The displayed operator is the two-site coefficient representative that will be placed on the AX face; it is not by itself the construction of that source projector.

Definition 13.1.52 (Appendix B XB coefficient representative). [Lean](#) [✓ Stmt](#) The XB coefficient-space representative is, before the XB -lift, the same coefficient-space representative, $\widehat{Q}_{XB} = \widehat{Q}_{AX}$. The source projector Q_{XB} of Definition D.2 acts on $\mathcal{H}_X \otimes \mathcal{H}_B$. The displayed operator is the coefficient representative later placed on the XB face.

Theorem 13.1.53 (The (AX) factor action). [Lean](#) [✓ Stmt](#) On the homogeneous three-single-site coefficient space $(\mathbb{C}^d)^{\otimes 3}$, the hatted Appendix B (AX) representative acts as $[\widehat{Q}_{AX}]_{AX} = [\widehat{Q}_{AX}] \otimes I_d$. This equation gives the coefficient-factor specialization of the left local action in [CPGSV16, Definition D.2]. It does not construct a projector for an arbitrary factorization $\mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$.

Proof. [✓ Proof](#) Apply the left identity in Theorem 13.1.3 to $\widehat{Q}_{AX}$. □

Theorem 13.1.54 (The (XB) factor action). [Lean](#) [✓ Stmt](#) On the homogeneous three-single-site coefficient space $(\mathbb{C}^d)^{\otimes 3}$, the hatted Appendix B (XB) representative acts as $[\widehat{Q}_{XB}]_{XB} = I_d \otimes [\widehat{Q}_{XB}]$. This equation gives the coefficient-factor specialization of the right local action in [CPGSV16, Definition D.2]. It does not construct a projector for an arbitrary factorization $\mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$.

Proof. [✓ Proof](#) Apply the right identity in Theorem 13.1.3 to $\widehat{Q}_{XB}$. □

Lemma 13.1.55 (Appendix B coefficient representatives and the parent interaction). [Lean](#) [Lean](#) [✓ Stmt](#) As coefficient-space representatives, $\widehat{Q}_{AX} = q_2(A)$ and $\widehat{Q}_{XB} = q_2(A)$.

Proof. [✓ Proof](#) Both operators are $q_2(A_{\text{core}})$, and the Appendix B basis change leaves q_2 unchanged. □

Lemma 13.1.56 (Appendix B two-site representative idempotents). [Lean](#) [Lean](#) [✓ Stmt](#) The two Appendix B two-site coefficient representatives are idempotent:

$$\begin{aligned}\widehat{Q}_{AX}^2 &= \widehat{Q}_{AX}, \\ \widehat{Q}_{XB}^2 &= \widehat{Q}_{XB}.\end{aligned}$$

Proof. [✓ Proof](#) By the preceding identification, both representatives are the canonical two-site parent interaction $q_2(A)$. The latter is an orthogonal projector, hence is idempotent. □

Theorem 13.1.57 (Appendix B overlapping two-site commutation). [Lean](#) [✓ Stmt](#) *The two Appendix B coefficient representatives satisfy*

$$\begin{aligned}\widehat{Q}_{AX}^2 &= \widehat{Q}_{AX}, \\ \widehat{Q}_{XB}^2 &= \widehat{Q}_{XB},\end{aligned}$$

and their adjacent actions on the common three-site coefficient space commute:

$$(\widehat{Q}_{AX})_{AX}(\widehat{Q}_{XB})_{XB} = (\widehat{Q}_{XB})_{XB}(\widehat{Q}_{AX})_{AX}.$$

Thus they satisfy the algebraic overlapping two-site condition of Definition 13.1.2. This is the coefficient-space analog of the projector-and-commutator part of [CPGSV16, Definition D.2]. The hatted maps have not yet been identified with the source projectors on the explicitly factored spaces. By itself the result does not prove the kernel-intersection equation in that definition and makes no assertion for every chain length. The coefficient-space kernel equation is established in the next two results.

Proof. [✓ Proof](#) Both coefficient representatives are the complement $1 - P_2$ of the two-site basic support projector. Their idempotency was established in Lemma 13.1.56. Since the two adjacent lifts of P_2 commute, their complementary lifts commute as well. $\square$

Theorem 13.1.58 (Three-site intersection of adjacent parent kernels). [Lean](#) [Lean](#) [✓ Stmt](#) *If A is injective, then $G_3(A) = \ker(q_2(A)_{AX}) \cap \ker(q_2(A)_{XB})$. More generally, on $L+1$ sites, simultaneous annihilation by the two adjacent length- L parent interactions is equivalent to membership in $G_{L+1}(A)$ whenever A is injective and $L > 1$.*

Proof. [✓ Proof](#) By Theorem 12.4.4.1, a three-site vector belongs to $G_3(A)$ exactly when its restrictions to both adjacent pairs belong to $G_2(A)$. Since $\ker q_2(A) = G_2(A)$, this is equivalent to annihilation by both adjacent terms. Theorem 13.1.4 identifies these terms with the two displayed lifts. $\square$

Theorem 13.1.59 (Appendix B coefficient-space analog of the Appendix D kernel datum). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be injective and let an Appendix B structural datum be fixed. Put*

$$\begin{aligned}\widehat{K}_{AXB} &= G_3(A), \\ \widehat{Q}_{AX} &= q_2(A), \\ \widehat{Q}_{XB} &= q_2(A),\end{aligned}$$

with the two copies of $q_2(A)$ lifted to the indicated faces of the three-site coefficient space. Then

$$\begin{aligned}[(\widehat{Q}_{AX})_{AX}, (\widehat{Q}_{XB})_{XB}] &= 0, \\ \widehat{K}_{AXB} &= \ker((\widehat{Q}_{AX})_{AX}) \cap \ker((\widehat{Q}_{XB})_{XB}).\end{aligned}$$

These are the coefficient-space analogs of the idempotence, commutation, and kernel-intersection equations in [CPGSV16, Definition D.2]. The hatted operators are not identified here with the source projectors on $\mathcal{H}_A \otimes \mathcal{H}_X$ and $\mathcal{H}_X \otimes \mathcal{H}_B$. For a normal left-canonical RFP tensor, injectivity follows from the RFP structural theorem and is not an additional hypothesis.

Proof. [✓ Proof](#) Idempotency and commutation are Theorem 13.1.57. The two coefficient representatives are $q_2(A)$, and Theorem 13.1.58 supplies the remaining kernel equation. $\square$

Theorem 13.1.60 (Transport of the three-site commutator). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose the first two length-two parent interactions commute on the three-site chain: $h_0^{(3)}h_1^{(3)} = h_1^{(3)}h_0^{(3)}$. Then, for every $N > 2$ and every $i \in \mathbb{Z}/N\mathbb{Z}$, the two adjacent translates commute: $h_i^{(N)}h_{i+1}^{(N)} = h_{i+1}^{(N)}h_i^{(N)}$. For each exterior configuration τ , let $R_{i,\tau}^{(3)}$ restrict to the cyclic three-site window beginning at i . It intertwines the two N -site terms with the two three-site terms:

$$\begin{aligned} R_{i,\tau}^{(3)}h_i^{(N)} &= h_0^{(3)}R_{i,\tau}^{(3)}, \\ R_{i,\tau}^{(3)}h_{i+1}^{(N)} &= h_1^{(3)}R_{i,\tau}^{(3)}. \end{aligned}$$

Thus the same local AXB calculation is placed at every cyclic translate through the two restriction identities above. The hypothesis $N > 2$ is essential to this transport: at $N = 2$ the two cyclic windows contain the same sites in opposite orders. This case is not part of [CPGSV16, Theorem 3.10].

Proof. [✓ Proof](#) Restrict an N -site vector to the cyclic window $(i, i+1, i+2)$ while holding all other physical indices fixed. Window extraction and replacement identify the two restricted local terms with $h_0^{(3)}$ and $h_1^{(3)}$. Apply the three-site commutator to the restricted vector and then evaluate at the original three-site configuration. Replacing a window by the values extracted from that configuration recovers the original N -site configuration. $\square$

Theorem 13.1.61 (Appendix B adjacent commutation on periodic chains). [Lean](#) [✓ Stmt](#) For every Appendix B structural datum and every periodic chain of length $N > 2$, the adjacent translated length-two parent interactions commute.

Proof. [✓ Proof](#) Theorem 13.1.57 gives the three-site commutator, and Theorem 13.1.60 places it at every adjacent pair on the periodic chain. $\square$

Definition 13.1.62 (Appendix B local-projector family). [Lean](#) [✓ Stmt](#) For an Appendix B structural datum and a periodic chain of length $N > 2$, set $p_i = h_i^{(N)}$ for $i \in \mathbb{Z}/N\mathbb{Z}$, where $h_i^{(N)}$ is the translated length-two parent interaction. These maps are idempotent and commute pairwise: adjacent commutation follows from Theorem 13.1.61, while disjoint length-two interactions commute by locality.

Theorem 13.1.63 (Left-canonical RFP commutation and zero energy). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A normal left-canonical renormalization fixed-point tensor of nonzero virtual dimension has pairwise commuting length-two parent interactions on every periodic chain of length $N > 2$, and its periodic matrix product vector is annihilated by every such interaction. The removal of the left-canonical hypothesis and the ground-space spanning equation in [CPGSV16, Definition 3.9] remain separate.

Proof. [✓ Proof](#) The Appendix B structural theorem supplies the datum used in Definition 13.1.62. Its commuting idempotents are precisely the translated parent interactions. The standard frustration-free identity shows that each interaction annihilates the periodic matrix product vector. $\square$

Lemma 13.1.64 (Appendix B overlapping condition from the lifted commutator). [Lean](#) [✓ Stmt](#) Suppose the lifted AX and XB representatives commute:

$$(\widehat{Q}_{AX})_{AX}(\widehat{Q}_{XB})_{XB} = (\widehat{Q}_{XB})_{XB}(\widehat{Q}_{AX})_{AX}.$$

Then $\widehat{Q}_{AX}$ and $\widehat{Q}_{XB}$ satisfy the algebraic overlapping two-site condition of Definition 13.1.2.

Proof. ✓ Proof The idempotency clauses are supplied by Lemma 13.1.56. The only remaining clause in Definition 13.1.2 is exactly the displayed lifted commutator. $\square$

Lemma 13.1.65 (Definition D.2 projectors identified with Appendix B representatives). Lean

Lean ✓ Stmt Let Q_{AX} and Q_{XB} be the two-site coefficient-space transports of source projectors satisfying the Definition D.2 local condition on AXB . If these transported projectors agree with the Appendix B coefficient representatives,

$$\begin{aligned} Q_{AX} &= \widehat{Q}_{AX}, \\ Q_{XB} &= \widehat{Q}_{XB}, \end{aligned}$$

then $\widehat{Q}_{AX}$ and $\widehat{Q}_{XB}$ satisfy the algebraic overlapping two-site condition of Definition 13.1.2. Consequently, $h_0^{(3)}(A, 2)h_1^{(3)}(A, 2) = h_1^{(3)}(A, 2)h_0^{(3)}(A, 2)$.

Proof. ✓ Proof Rewriting the two projectors in the Definition D.2 condition as $\widehat{Q}_{AX}$ and $\widehat{Q}_{XB}$ gives the algebraic overlapping condition. The three-site commutation is then Lemma 13.1.67. $\square$

Lemma 13.1.66 (Three-site lifts of the Appendix B coefficient representatives). Lean Lean

✓ Stmt On the three-site A, X, B window, the two adjacent length-two local terms are the two local lifts

$$\begin{aligned} h_0^{(3)}(A, 2) &= (\widehat{Q}_{AX})_{AX}, \\ h_1^{(3)}(A, 2) &= (\widehat{Q}_{XB})_{XB}. \end{aligned}$$

Proof. ✓ Proof Apply the three-site local-term identities to $q_2(A)$, and replace $q_2(A)$ by $\widehat{Q}_{AX}$ or $\widehat{Q}_{XB}$ using the preceding two-site identifications. $\square$

Lemma 13.1.67 (Appendix B overlapping condition gives three-site local-term commutation).

Lean ✓ Stmt If the Appendix B AX and XB coefficient representatives satisfy the overlapping two-site commutation datum

$$\begin{aligned} \widehat{Q}_{AX}^2 &= \widehat{Q}_{AX}, \\ \widehat{Q}_{XB}^2 &= \widehat{Q}_{XB}, \\ (\widehat{Q}_{AX})_{AX}(\widehat{Q}_{XB})_{XB} &= (\widehat{Q}_{XB})_{XB}(\widehat{Q}_{AX})_{AX}, \end{aligned}$$

then the corresponding first two translated length-two local terms on the three-site window commute: $h_0^{(3)}(A, 2)h_1^{(3)}(A, 2) = h_1^{(3)}(A, 2)h_0^{(3)}(A, 2)$.

Proof. ✓ Proof By Lemma 13.1.66, the two translated local terms are exactly the AX and XB lifts of $\widehat{Q}_{AX}$ and $\widehat{Q}_{XB}$. The displayed equality is therefore the commutation clause of the overlapping two-site datum. $\square$

Lemma 13.1.68 (Appendix B lifted commutator gives three-site local-term commutation).

Lean Lean ✓ Stmt If the Appendix B AX and XB coefficient representatives satisfy the lifted Definition D.2 commutator

$$(\widehat{Q}_{AX})_{AX}(\widehat{Q}_{XB})_{XB} = (\widehat{Q}_{XB})_{XB}(\widehat{Q}_{AX})_{AX},$$

as coefficient-space representatives. On the three-site window their two lifts give the local identifications

$$\begin{aligned} h_0^{(3)}(A, 2) &= (\widehat{Q}_{AX})_{AX}, \\ h_1^{(3)}(A, 2) &= (\widehat{Q}_{XB})_{XB}. \end{aligned}$$

Consequently, for every chain length $L \geq 1$, the original tensor has the same cyclic coefficient formula: $V^{(L)}(A)_\sigma = C_L(\sigma)$. In particular, the corresponding two-site amplitude is

$$\phi_2(a, b) = V^{(2)}(A)(a, b) = V^{(2)}(\Lambda U)(a, b).$$

Thus the one-pair case of the factorization is already verified for the core tensor: $V^{(2)}(\Lambda U)(\sigma) = \phi_2(\sigma_0, \sigma_1)$. The even-chain physical-pair factorization is the following condition, for a positive number n of pairs,

$$V^{(2n)}(A)(\sigma) = \prod_{p=0}^{n-1} \phi_2(\sigma_{2p}, \sigma_{2p+1}).$$

The case $n = 0$ is not part of this condition: the empty-chain MPV coefficient is the virtual bond dimension, whereas the empty product on the right-hand side is 1. The first equality, $V^{(L)}(\Lambda U)_\sigma = C_L(\sigma)$, is the trace expansion of the closed matrix product and is the coefficient form of $U^{\otimes L} \varphi_j^{\otimes L}$. The remaining coefficient step is to relate this source expression, with the physical word σ fixed, to the even-chain physical-pair factorization above. On a three-site chain, the two adjacent length-two local terms are the AX and XB actions of the canonical two-site parent interaction. Theorem 13.1.57 constructs the adjacent physical support projectors from the basic-vector expression and proves the overlapping relation

$$(\widehat{Q}_{AX})_{AX}(\widehat{Q}_{XB})_{XB} = (\widehat{Q}_{XB})_{XB}(\widehat{Q}_{AX})_{AX}.$$

Theorem 13.1.60 transports this identity around a periodic chain. Consequently, for every $N > 2$ and every pair of sites $i, j \in \mathbb{Z}/N\mathbb{Z}$, $h_i(A, 2)h_j(A, 2) = h_j(A, 2)h_i(A, 2)$. Definition 13.1.62 collects these maps into a commuting idempotent family; idempotency is Lemma 12.3.57. Together with Lemma 12.3.63, this gives the zero-energy equation for $V^{(N)}(A)$. Neither conclusion requires the even-chain physical-pair factorization displayed above. That factorization remains a separate conditional coefficient statement. The full source implication additionally requires removal of the left-canonical hypothesis and the ground-space spanning equation. The coefficient-space analog of the local kernel-intersection equation is Theorem 13.1.59; identifying its hatted maps with the source projectors remains separate and is not needed for the chain commutator.

Theorem 13.1.70 (Even-chain factorization and ground-space spanning give all-chain NNCPH).

[Lean](#) [Lean](#) [✓ Stmt](#) Suppose the conditional even-chain hypotheses give commuting two-site parent terms for a tensor B . If the nearest-neighbor parent Hamiltonian also satisfies the ground-space spanning equation of Definition 13.1.16, with BNT components $A_1, \dots, A_g$, then B satisfies the all-chain nearest-neighbor commuting parent-Hamiltonian ground-space condition of Definition 13.1.18.

In particular, if a normal left-canonical RFP tensor has nonzero bond dimension, the same conditional Appendix B hypotheses, and the same ground-space spanning equation, then it satisfies the full all-chain condition in [CPGSV16, Theorem 3.10(iii)].

Proof. [✓ Proof](#) The conditional hypotheses give $h_i h_j = h_j h_i$ for all translated two-site local terms. Lemma 12.3.63 gives $h_i V^{(N)}(B) = 0$. For every $N > 2$, the spanning hypothesis supplies $\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}$. These are exactly the three clauses in Definition 13.1.17, uniformly in N . $\square$

Theorem 13.1.71 (Eventual dimension of nearest-neighbor parent ground spaces). [Lean](#) [✓ Stmt](#) Let P be a sector decomposition with representatives $A_1, \dots, A_g$ and total tensor B . Suppose that the vectors $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$ are linearly independent for all sufficiently large N , and that B satisfies the all-chain nearest-neighbor commuting parent-Hamiltonian ground-space condition with these representatives. Then there is N_0 such that, for every $N > N_0$, $\dim_{\mathbb{C}} \ker H_2^{(N)}(B) = g$. Thus the ground-state degeneracy is eventually constant, as required in [\[Bei12, Section IV\]](#).

Proof. [✓ Proof](#) Choose $N_0 \geq 2$ beyond the linear-independence threshold. For $N > N_0$, the ground-space spanning equation gives $\ker H_2^{(N)}(B) = \text{span}\{|V^{(N)}(A_j)\rangle : j = 1, \dots, g\}$. Identifying these vectors with their coefficient functions preserves linear independence. The dimension of their span is therefore g . $\square$

Definition 13.1.72 (Two-site parent-interaction matrix). [Lean](#) [✓ Stmt](#) Let A be a matrix product tensor. Write its canonical two-site parent interaction in the ordered-pair basis of the two-site physical Hilbert space, and denote the resulting matrix by Q_A .

Theorem 13.1.73 (Hermitian commuting overlapping parent interactions). [Lean](#) [✓ Stmt](#) Suppose that the canonical length-two parent terms of A commute on a three-site chain. Then Q_A is Hermitian, and its natural placements on the first and last pairs of a three-fold tensor product commute:

$$(Q_A \otimes \mathbb{1})(\mathbb{1} \otimes Q_A) = (\mathbb{1} \otimes Q_A)(Q_A \otimes \mathbb{1}).$$

Proof. [✓ Proof](#) The parent interaction is the orthogonal projector onto the orthogonal complement of the local matrix product space, hence is Hermitian. Transporting the adjacent three-site parent-term commutator to the common left-associated tensor-product basis gives the displayed identity. $\square$

Theorem 13.1.74 (Spatial block actions of the two-site parent interaction). [Lean](#) [✓ Stmt](#) Under the preceding hypothesis, there are positive integers d_q, m_q , a unitary identification

$$\mathcal{H} \cong \bigoplus_q \mathcal{H}_{q,l} \otimes \mathcal{H}_{q,r},$$

and Hermitian matrices R_q on $\mathcal{H} \otimes \mathcal{H}_{q,l}$ and S_q on $\mathcal{H}_{q,r} \otimes \mathcal{H}$ such that, after conjugating the middle site by this unitary, the first identity below holds on the first pair and the second on the last pair:

$$\begin{aligned} Q_A &= \bigoplus_q (R_q \otimes \mathbb{1}_{q,r}), \\ Q_A &= \bigoplus_q (\mathbb{1}_{q,l} \otimes S_q). \end{aligned}$$

This is the spatial decomposition of [\[Bei12, Lemma 2.1\]](#) for the canonical parent interaction.

Proof. [✓ Proof](#) Apply the finite-dimensional spatial decomposition for two Hermitian overlapping operators to Q_A in both positions, using Theorem [13.1.73](#). $\square$

Definition 13.1.75 (Finite sector graph). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that a one-site unitary identifies the physical Hilbert space with a finite direct sum

$$\mathcal{H} \cong \bigoplus_a \mathcal{H}_{a,l} \otimes \mathcal{H}_{a,r},$$

and that in these coordinates the complementary two-site interaction is $\bigoplus_{a,b} \mathbb{1}_{a,l} \otimes P_{a,b} \otimes \mathbb{1}_{b,r}$, where each $P_{a,b}$ is an orthogonal projector. The directed edge $a \rightarrow b$ is present precisely when $\text{ran } P_{a,b}$ is nonzero. An ordered cycle of length N is a sequence $(a_0, \dots, a_{N-1})$ for which every cyclically adjacent edge $a_n \rightarrow a_{n+1}$ is present. Cyclic rotations are not identified. This is the graph following equation (2) of [Bei12, Section III].

Theorem 13.1.76 (Symmetric spatial form of a commuting positive interaction). [Lean](#) [Lean](#) [Stmnt](#) Let $B \geq 0$ be a two-site operator whose overlapping placements commute: $(B \otimes \mathbb{1})(\mathbb{1} \otimes B) = (\mathbb{1} \otimes B)(B \otimes \mathbb{1})$. There are positive integers $d_{q,l}, d_{q,r}$, a one-site unitary U , and positive semidefinite operators $\eta_{q,h}$ on $\mathcal{H}_{q,r} \otimes \mathcal{H}_{h,l}$ such that

$$(U \otimes U)^* B (U \otimes U) = \bigoplus_{q,h} \mathbb{1}_{q,l} \otimes \eta_{q,h} \otimes \mathbb{1}_{h,r}.$$

This is the symmetric two-sided form obtained from [Bei12, Lemma 2.1 and Section III, equation (2)].

Proof. [Lean](#) [Proof](#) Apply the spatial decomposition to the two overlapping placements of B . In the resulting coordinates, matrix entries vanish unless both the left and right sector labels agree. For fixed sectors q, h , choose unit vectors $l_0 \in \mathcal{H}_{q,l}$ and $r_0 \in \mathcal{H}_{h,r}$ and define

$$\begin{aligned} \langle r, l | \eta_{q,h} | r', l' \rangle &= \langle l_0, r; l, r_0 | \widetilde{B} | l_0, r'; l', r_0 \rangle, \\ \widetilde{B} &= (U \otimes U)^* B (U \otimes U). \end{aligned}$$

The left block action makes the right outer factor an identity, while the right block action makes the left outer factor an identity. Hence the displayed matrix element is independent of the chosen outer coordinates and gives the stated direct sum. Each $\eta_{q,h}$ is a compression of the positive semidefinite matrix $\widetilde{B}$, so it is positive semidefinite. $\square$

Theorem 13.1.77 (Sector graph of a commuting parent interaction). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Suppose that the canonical length-two parent terms of A commute on a three-site chain. Then the complementary two-site interaction admits a finite sector graph as in Definition 13.1.75.

Proof. [Lean](#) [Proof](#) The canonical two-site parent interaction Q_A is an orthogonal projector. Thus $\mathbb{1} - Q_A$ is a positive semidefinite orthogonal projector. The overlapping placements of $\mathbb{1} - Q_A$ commute because those of Q_A commute. Apply Theorem 13.1.76 to obtain

$$(U \otimes U)^* (\mathbb{1} - Q_A) (U \otimes U) = \bigoplus_{q,h} \mathbb{1}_{q,l} \otimes \eta_{q,h} \otimes \mathbb{1}_{h,r}.$$

The left-hand side is idempotent, so every block on the right is idempotent. Evaluating the two outer identity factors at fixed unit vectors gives $\eta_{q,h}^2 = \eta_{q,h}$. Since $\eta_{q,h} \geq 0$, each $\eta_{q,h}$ is an orthogonal projector and therefore defines the required edge projection. $\square$

Theorem 13.1.78 (Ordered-cycle ground-space dimension formula). [Lean](#) [Stmnt](#) For every $N \geq 2$, the periodic ground-space dimension is

$$\dim \ker H_N = \sum_{(a_0, \dots, a_{N-1})} \prod_{n=0}^{N-1} \dim \text{ran } P_{a_n, a_{n+1}},$$

where the sum is over ordered cycles and the indices are read modulo N . This is equations (3)–(4) of [Bei12, Section III] for chains of length at least two. The one-site periodic interaction requires a separate convention.

Proof. ✓ Proof The translated parent interactions commute. Hence the ordered product of their complementary orthogonal projectors is the projector onto $\ker H_N$. In the cyclic sector coordinates this product is block diagonal, with the block indexed by $(a_0, \dots, a_{N-1})$ equal to the tensor product of the adjacent projectors $P_{a_n, a_{n+1}}$. Taking the trace gives the product of their ranks. A sequence containing a missing edge contributes zero, leaving exactly the displayed ordered-cycle sum. $\square$

Theorem 13.1.79 (Period-two ordered-cycle ground-space dimension formula). Lean ✓ Stmt *At period two, the ground-space dimension is*

$$\dim \ker H_2 = \sum_{\substack{a \rightarrow b \\ b \rightarrow a}} \dim \operatorname{ran} P_{a,b} \dim \operatorname{ran} P_{b,a}.$$

The sum is over ordered pairs: if $a \neq b$, then (a, b) and (b, a) are distinct summands. This is the length-two convention stated after equation (4) of [Bei12, Section III].

Proof. ✓ Proof Specialize the ordered-cycle formula to $N = 2$. Its two cyclic factors are $P_{a,b}$ and $P_{b,a}$, which gives the displayed product of ranks. $\square$

Theorem 13.1.80 (Project-local one-site zero convention). Lean Lean ✓ Stmt *For the convention that an L -site interaction is zero when $L > N$, the two-site parent Hamiltonian on a one-site periodic chain satisfies*

$$\begin{aligned} H_2^{(1)} &= 0, \\ \ker H_2^{(1)} &= \mathbb{C}^d. \end{aligned}$$

This convention is local to the present development. The two-site terms in [Bei12, Section III] are not assigned an action when both site labels denote the same site. The nearest-neighbor condition in [CPGSV16, Theorem 3.10(iii)], source lines 534–540, only concerns $N > 2$.

Proof. ✓ Proof Since $2 > 1$, the single translated interaction is zero by the stated convention. Its sum is therefore zero, and the kernel of the zero operator is the full one-site Hilbert space. $\square$

Theorem 13.1.81 (Classification of eventually constant weighted sector graphs). Lean Lean ✓ Stmt *Let G be a finite directed graph whose edge weights are non-negative integers, and let*

$$s_N = \sum_{(a_0, \dots, a_{N-1})} \prod_{n=0}^{N-1} w(a_n, a_{n+1})$$

be the weighted sum over ordered cyclic vertex sequences. Suppose that s_N is constant for all sufficiently large N . Then every edge that belongs to a directed cycle has weight one, every directed cycle is a repetition of a single loop, and, for every $N > 0$, $s_N = \#\{a : w(a, a) > 0\}$. This is the comparison in [Bei12, Section IV].

Proof. ✓ Proof Choose a sufficiently large N such that the sums at lengths N , $N+1$, and $N(N+1)$ have their common eventual value. Repeating each ordered N -cycle $N+1$ times gives an injection into the cycles of length $N(N+1)$, while repeating each ordered $(N+1)$ -cycle N times gives a second injection. Comparison of the three weighted sums shows first that every positive cyclic weight is one. Both injections must then be exhaustive. Hence a cycle at the common length has periods N and $N+1$; comparison of adjacent entries gives period one. Repeating an arbitrary shorter cycle to a sufficiently large length proves that it is constant. The only surviving ordered cycles at any positive length are therefore the constant words supported on positive loops, each of weight one. $\square$

Corollary 13.1.82 (Parent-ground-space dimension is the loop number). [Lean](#) [Lean](#) [✓ Stmt](#)
If the parent-ground-space dimension is eventually constant, then for every $N > 2$ it equals the number of positive loops in the sector graph. In particular, this conclusion holds under the BNT and all-chain nearest-neighbor parent-ground-space hypotheses of Theorem 13.1.71.

Proof. [✓ Proof](#) The ordered-cycle dimension formula identifies the eventual parent-ground-space dimension with the eventual weighted-cycle sum. The preceding theorem identifies that sum, at every positive length, with the loop number. Transport between coefficient and Euclidean coordinates preserves the dimension of the ground space. $\square$

Definition 13.1.83 (Tensor associated with a positive Beigi loop). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $a \rightarrow a$ be a positive loop and choose a nonzero vector $\varphi_a \in \text{ran } P_{a,a}$. In sector coordinates define the matrix product tensor $(C_a^{(r,l)})_{x,y} = \delta_{x,l} \varphi_a(r,y)$ and set it to zero outside sector a . The physical loop tensor is the image of C_a under the one-site unitary of the spatial decomposition. Its associated cyclic product pairs the right factor at site n with the left factor at site $n + 1$. The corresponding N -site product state is denoted by $\Psi_a^{(N)} \in (\mathbb{C}^d)^{\otimes N}$.

Definition 13.1.84 (Schmidt-support tensor associated with a positive Beigi loop). [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\Phi_a = (\varphi_a(r,l))_{r,l}$ and write $r_a = \text{rank } \Phi_a$. Choose a rank factorization

$$\begin{aligned}\Phi_a &= X_a Y_a, \\ X_a &\in M_{D_a^R, r_a}(\mathbb{C}), \\ Y_a &\in M_{r_a, D_a^L}(\mathbb{C}).\end{aligned}$$

The tensor on the minimal Schmidt support is

$$(\tilde{C}_a^{(r,l)})_{\alpha,\beta} = (Y_a)_{\alpha,l} (X_a)_{r,\beta},$$

again extended by zero outside sector a . Its physical tensor is its image under the same one-site unitary.

Definition 13.1.85 (Normalized minimal tensor associated with a positive Beigi loop). [Lean](#)
[Lean](#) [Lean](#) [✓ Stmt](#) Let $\nu_a = \|\varphi_a\|_2 > 0$, where the norm is the Hilbert-space norm of the bond vector, or equivalently the Frobenius norm of Φ_a . In either coordinate system, the normalized representative is $\hat{C}_a = \nu_a^{-1} \tilde{C}_a$. The raw coordinate and physical tensors are retained separately.

Lemma 13.1.86 (Positive Schmidt dimension of a Beigi loop). [Lean](#) [✓ Stmt](#) *The Schmidt rank r_a of the nonzero loop bond vector is positive.*

Proof. [✓ Proof](#) If $r_a = 0$, then the intermediate index set in the factorization $\Phi_a = X_a Y_a$ is empty. Every entry of $X_a Y_a$ is therefore zero, contrary to $\varphi_a \neq 0$. $\square$

Theorem 13.1.87 (Injectivity of the minimal Beigi loop tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The tensor $\tilde{C}_a$ on the Schmidt support is injective, as is its image under the one-site unitary. In particular, the physical tensor is normal.*

Proof. [✓ Proof](#) In a rank factorization $\Phi_a = X_a Y_a$ through $r_a = \text{rank } \Phi_a$, the columns of Y_a and the rows of X_a each span $\mathbb{C}^{r_a}$. Their pairwise outer products therefore span $M_{r_a}(\mathbb{C})$, and these outer products are precisely the loop-sector letters of $\tilde{C}_a$. A unitary change of physical coordinates preserves this span. One-site injectivity implies normality. $\square$

Proof. [✓ Proof](#) The two-site coordinate identity gives $(U^* \otimes U^*)(\mathbb{1} - h)(U \otimes U) = \tilde{Q}$ for one bond. Conjugation by $(U^*)^{\otimes N}$ carries every translated copy of this bond identity to the corresponding translated sector interaction. Since unitary conjugation distributes over an ordered product,

$$(U^*)^{\otimes N} \left(\prod_{n=0}^{N-1} Q_n \right) U^{\otimes N} = \prod_{n=0}^{N-1} ((U^*)^{\otimes N} Q_n U^{\otimes N}) = \prod_{n=0}^{N-1} \tilde{Q}_n = \tilde{Q}_N,$$

where Q_n and $\tilde{Q}_n$ are the translated physical and sector complementary interactions, respectively. $\square$

Theorem 13.1.91 (Fixed complementary-product vectors have zero parent energy). [Lean](#)

[✓ Stmt](#) Suppose that A is equipped with Beigi sector data. For $N \geq 2$, let $h_0, \dots, h_{N-1}$ be its translated length-two parent interactions and set $Q_N = \prod_{n=0}^{N-1} (\mathbb{1} - h_n)$. Every vector fixed by Q_N belongs to the kernel of the periodic parent Hamiltonian $H_N = \sum_n h_n$.

Proof. [✓ Proof](#) Under the Euclidean identification E , the matrix product of the complementary interactions is carried to

$$Q_N^E = E^{-1} Q_N E = \prod_{n=0}^{N-1} (\mathbb{1} - h_n^E).$$

The operators h_n^E are orthogonal projections. Beigi sector data supply pairwise commutativity, which Euclidean transport preserves. For all $0 \leq m, n < N$, one has $h_m^E h_n^E = h_n^E h_m^E$. For any commuting orthogonal projections $P_0, \dots, P_{N-1}$,

$$\prod_{n=0}^{N-1} (\mathbb{1} - P_n) = \text{proj}_{\bigcap_{n=0}^{N-1} \ker P_n}.$$

Applying this identity to the h_n^E gives

$$Q_N^E = \text{proj}_{\bigcap_{n=0}^{N-1} \ker h_n^E},$$

$$\bigcap_{n=0}^{N-1} \ker h_n^E = \ker \left(\sum_{n=0}^{N-1} h_n^E \right) = \ker H_N^E.$$

Thus $Q_N v = v$ implies $E^{-1} v \in \ker H_N^E$, and conjugating back by E gives $v \in \ker H_N$. $\square$

Theorem 13.1.92 (A positive Beigi loop gives a parent ground state). [Lean](#) [Lean](#) [✓ Stmt](#) Let $a \rightarrow a$ be a positive loop. For every $N \geq 2$, its physical cyclic product-of-pairs state belongs to the kernel of the length-two periodic parent Hamiltonian. Equivalently, its Euclidean representative belongs to the parent ground space. This is only a membership assertion; it does not assert that the loop states span the ground space.

Proof. [✓ Proof](#) In sector coordinates, the complete product $\tilde{Q}_N$ of the complementary two-site interactions is block diagonal over cyclic sector sequences. Write $\Phi_a(k; x)$ for the coefficient of the loop cyclic product in the block indexed by k . Its support lies in sector a , so for every $k \neq (a, \dots, a)$ and every edge-index tuple x , one has $\Phi_a(k; x) = 0$. Thus only the constant cyclic sector sequence contributes. On this block, $\tilde{Q}_N$ is the tensor product of the edge projectors.

Since $\varphi_a \in \text{ran } P_{a,a}$,

$$\begin{aligned}\widetilde{Q}_N \Phi_a &= P_{a,a}^{\otimes N} \Phi_a = \Phi_a, \\ \Phi_a(t_0, \dots, t_{N-1}) &= \prod_{n=0}^{N-1} \varphi_a(r_n, l_{n+1}).\end{aligned}$$

If U is the one-site spatial unitary and $\Psi_a = U^{\otimes N} \Phi_a$ is the physical loop state, unitary conjugation gives

$$Q_N \Psi_a = U^{\otimes N} \widetilde{Q}_N \Phi_a = U^{\otimes N} \Phi_a = \Psi_a.$$

Theorem 13.1.91 now yields $\Psi_a \in \ker H_N$. $\square$

Theorem 13.1.93 (Positive Beigi loops span the parent ground space). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the parent-ground-space dimension is eventually constant. For every $N > 2$, the positive-loop product states form a nonzero pairwise orthogonal family and

$$\text{span}\{\Psi_a^{(N)} : a \rightarrow a \text{ is a positive loop}\} = \ker H_N.$$

In particular, the same conclusion holds under the BNT and all-chain nearest-neighbor parent-ground-space hypotheses.

Proof. [✓ Proof](#) Write $\Phi_a^{(N)}$ for the cyclic product in sector coordinates and $\Psi_a^{(N)} = U^{\otimes N} \Phi_a^{(N)}$ for the physical state, where U is the one-site spatial unitary. Choose q with $\varphi_a(q) \neq 0$. On the constant edge-index configuration,

$$\begin{aligned}\Phi_a^{(N)}(q, \dots, q) &= \prod_{n=0}^{N-1} \varphi_a(q) \\ &\neq 0.\end{aligned}$$

Thus $\Phi_a^{(N)} \neq 0$, and unitarity gives $\Psi_a^{(N)} = U^{\otimes N} \Phi_a^{(N)} \neq 0$. If $a \neq b$, then every sector configuration s satisfies $\Phi_a^{(N)}(s) = 0$ or $\Phi_b^{(N)}(s) = 0$, and hence

$$\langle \Phi_a^{(N)}, \Phi_b^{(N)} \rangle = \sum_s \overline{\Phi_a^{(N)}(s)} \Phi_b^{(N)}(s) = 0.$$

Moreover,

$$\langle \Psi_a^{(N)}, \Psi_b^{(N)} \rangle = \langle U^{\otimes N} \Phi_a^{(N)}, U^{\otimes N} \Phi_b^{(N)} \rangle = \langle \Phi_a^{(N)}, \Phi_b^{(N)} \rangle = 0.$$

Thus the loop states are linearly independent. Their span is contained in $\ker H_N$ by Theorem 13.1.92. Its dimension is the number of positive loops, which equals $\dim \ker H_N$ by Corollary 13.1.82. The inclusion is consequently an equality. $\square$

Theorem 13.1.94 (All-chain nearest-neighbor commuting parent-Hamiltonian ground spaces imply a renormalization fixed point). [Lean](#) [✓ Stmt](#) Let B be normal and in left-canonical form. Suppose that the family A_j is a BNT for B , and that the nearest-neighbor parent Hamiltonian of B has the all-chain ground-space property of Definition 13.1.18: for every $N > 2$, the length-two translated parent interactions commute, $h_i(B, 2)h_j(B, 2) = h_j(B, 2)h_i(B, 2)$, the vector $V^{(N)}(B)$

has zero energy, and $\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}$. Then B is a renormalization fixed point.

This is the normal left-canonical representative case of the reverse implication in Theorem 3.10 of [CPGSV16]. The source theorem is stated for a tensor in canonical form; the unrestricted canonical-form statement is not asserted here.

Proof. ✓ Proof The commuting interaction determines Beigi's finite sector graph. Its positive loops span the parent ground space, and the normalized minimal loop tensors L_j are normal renormalization fixed points. For the normal representative B ,

$$\forall^\infty N, \quad V^{(N)}(B) \in \text{span}\{V^{(N)}(L_j) : j = 1, \dots, g\}.$$

Normal-tensor orthogonality applied to this eventual-span relation first identifies the bond dimensions and then gives an index j , an invertible matrix X , and a scalar ζ with $|\zeta| = 1$ such that $L_j^i = \zeta X B^i X^{-1}$. Transfer idempotence is preserved by these transformations. $\square$

Lemma 13.1.95 (Common physical unitaries preserve mixed transfer maps). Lean ✓ Stmt Let $A = (A_i)_i$ and $B = (B_i)_i$ be two tensors, possibly with different bond dimensions, and let u be unitary on the physical index. If

$$\tilde{A}_i = \sum_j u_{ij} A_j, \quad \tilde{B}_i = \sum_j u_{ij} B_j,$$

then

$$\mathcal{E}_{\tilde{A}, \tilde{B}} = \mathcal{E}_{A, B}.$$

Proof. ✓ Proof Expand both physical sums. The coefficient of $A_j X B_k^\dagger$ is $\sum_i u_{ij} \overline{u_{ik}} = \delta_{jk}$, so only the diagonal terms remain. $\square$

Lemma 13.1.96 (Gauge-phase transport of mixed-transfer vanishing). Lean Lean ✓ Stmt Let U, A and V, B be pairs of tensors with respective common bond dimensions. Suppose that $\zeta_A, \zeta_B \neq 0$ and that X_A, X_B are invertible matrices such that, for every physical index i ,

$$A^i = \zeta_A X_A U^i X_A^{-1}, \tag{13.5}$$

$$B^i = \zeta_B X_B V^i X_B^{-1}. \tag{13.6}$$

Then

$$\mathcal{E}_{A, B} = 0 \implies \mathcal{E}_{U, V} = 0. \tag{13.7}$$

Identifying equal bond dimensions on either tensor leg preserves this vanishing condition in both directions.

Proof. ✓ Proof Set $D_A = \zeta_A I$ and $D_B = \zeta_B I$. For every rectangular matrix Z , covariance of the mixed transfer map gives

$$\mathcal{E}_{A, B}(X_A Z X_B^\dagger) = X_A D_A \mathcal{E}_{U, V}(Z) (X_B D_B)^\dagger. \tag{13.8}$$

If $\mathcal{E}_{A, B} = 0$, the left-hand side vanishes. The two outer factors are invertible, so cancellation gives $\mathcal{E}_{U, V}(Z) = 0$ for every Z . $\square$

Lemma 13.1.97 (Distinct normalized Beigi loops are locally orthogonal). Lean ✓ Stmt If a and b are distinct positive loops in the sector graph, then the mixed transfer map of their normalized minimal loop tensors vanishes:

$$\mathcal{E}_{L_a, L_b} = 0.$$

Proof. ✓ Proof In sector coordinates, every physical letter of L_a is supported in sector a , while every physical letter of L_b is supported in sector b . Hence, for every rectangular matrix Z and every physical index i ,

$$L_a^i Z (L_b^i)^\dagger = 0, \quad (13.9)$$

$$\mathcal{E}_{L_a, L_b}(Z) = \sum_i L_a^i Z (L_b^i)^\dagger = 0. \quad (13.10)$$

The common one-site physical unitary preserves the mixed transfer map by Lemma 13.1.95. $\square$

Theorem 13.1.98 (Multi-sector NNCPH ground spaces imply a fixed point for the basis direct sum). Lean ✓ Stmt *Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and set*

$$B = \bigoplus_{j=1}^g A_j.$$

Suppose that, for every $N > 2$, the translated length-two parent interactions of B commute, every $V^{(N)}(A_j)$ has zero energy, and

$$\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}.$$

Then B is a renormalization fixed point.

This is the multiplicity-one, unit-weight representative case of the reverse implication in Theorem 3.10 of [CPGSV16]. It does not assert the printed theorem for repeated copies or arbitrary raw weights.

Proof. ✓ Proof Beigi's positive-loop product states span each parent ground space. Consequently every normal basis tensor A_j agrees, up to a unit-modulus scalar and an invertible virtual change of basis, with a normalized minimal loop tensor $L_{\ell(j)}$. Distinctness of the BNT basis makes $j \mapsto \ell(j)$ injective. Thus there are $\zeta_j \neq 0$ and invertible X_j such that

$$L_{\ell(j)}^i = \zeta_j X_j A_j^i X_j^{-1}. \quad (13.11)$$

Each $L_{\ell(j)}$ has an idempotent transfer map. If $j \neq k$, then $L_{\ell(j)}$ and $L_{\ell(k)}$ have disjoint sector support, so their mixed transfer map vanishes. Lemma 13.1.96 transports this vanishing through the two virtual changes of basis:

$$\mathcal{E}_{L_{\ell(j)}, L_{\ell(k)}} = 0 \implies \mathcal{E}_{A_j, A_k} = 0. \quad (13.12)$$

Thus the diagonal mixed transfer maps of the family A_j are idempotent and the off-diagonal ones vanish. The blockwise criterion is

$$\mathcal{E}_B \circ \mathcal{E}_B = \mathcal{E}_B \iff \mathcal{E}_{A_j, A_k} \circ \mathcal{E}_{A_j, A_k} = \mathcal{E}_{A_j, A_k} \quad (13.13)$$

for every pair j, k , and proves the claim. $\square$

Theorem 13.1.99 (Fixed points and commuting parent ground spaces for the basis direct sum). Lean ✓ Stmt *Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and set $A = \bigoplus_{j=1}^g A_j$. Then*

$$\mathcal{E}_A^2 = \mathcal{E}_A \iff A \text{ has all-chain commuting parent ground spaces spanned by } (A_j)_j. \quad (13.14)$$

This statement concerns the multiplicity-one, unit-weight direct sum; it does not include repeated copies or arbitrary raw weights.

Proof. ✓ Proof The forward implication is Theorem 13.1.33, and the reverse implication is Theorem 13.1.98. $\square$

13.2 Decorrelation and the idempotent-product parent-commuting condition

This section develops the idempotent-product formulation used for the decorrelation implication from Appendix D. It is not the full source condition. In [CPGSV16, Appendix D.2], the parent commuting Hamiltonian condition is stated for local projectors Q_{AX} and Q_{XB} . Writing hats for their lifts to the tripartite space, the condition is

$$[\widehat{Q}_{AX}, \widehat{Q}_{XB}] = 0, \\ K_{AXB} = (K_{AX} \otimes H_B) \cap (H_A \otimes K_{XB}).$$

Writing $P_{AX} = 1 - Q_{AX}$ and $P_{XB} = 1 - Q_{XB}$ gives, for the idempotent P_K used below, the full-space identity $\widehat{P}_{AX}\widehat{P}_{XB} = P_K$. The following definition records this idempotent-product consequence abstractly, not the tensor-product locality, orthogonal-projector, or ground-space-intersection hypotheses of Appendix D.2.

The following preparatory result is independent of the decorrelation condition. For a positive semidefinite operator on a bipartite space, the support projector of the retained left marginal, lifted to the full space, fixes the original operator. These are the support-projector identities used in [CPGSV16, Appendix D.2, lines 2225–2235].

Lemma 13.2.1 (Partial trace of the identity). Lean ✓ Stmt *For finite-dimensional spaces H_L and H_R , one has $\text{tr}_R(\mathbb{1}_{L \otimes R}) = (\dim H_R)\mathbb{1}_L$.*

Proof. ✓ Proof In product-basis coordinates,

$$(\text{tr}_R \mathbb{1}_{L \otimes R})_{ij} = \sum_k \delta_{ij} = (\dim H_R) \delta_{ij}.$$

□

Left and right absorption by the marginal support projector — for ρ positive semidefinite on $H_L \otimes H_R$ with P_L the orthogonal projector onto the support of $\text{tr}_R \rho$, the identities $(P_L \otimes \mathbb{1}_R)\rho = \rho$ and $\rho(P_L \otimes \mathbb{1}_R) = \rho$, which for $H_L = H_A \otimes H_X$ and $H_R = H_B$ are precisely the P_{AX} support-projector identities in [CPGSV16, Appendix D.2, lines 2228–2235] — are pure bipartite-operator facts with no tensor-network content, proved in the companion quantum-channel volume [LTC26, “Left absorption by the marginal support projector”, “Right absorption by the marginal support projector”, “Partial trace over the left factor”]. The partial trace over the left factor that they and the absorption theorems below are stated against is defined there as well.

Theorem 13.2.2 (Hermiticity of the left partial trace). Lean ✓ Stmt *If $\rho = \rho^\dagger$, then $\text{tr}_L \rho = (\text{tr}_L \rho)^\dagger$.*

Proof. ✓ Proof Entrywise Hermiticity gives

$$\overline{(\text{tr}_L \rho)_{r,s}} = \sum_k \overline{\rho_{(k,r),(k,s)}} = \sum_k \rho_{(k,s),(k,r)} = (\text{tr}_L \rho)_{s,r}.$$

□

Left and right absorption by the right marginal support — for $\rho \geq 0$ and P_R the support projector of $\text{tr}_L \rho$, the identities $(\mathbb{1}_L \otimes P_R)\rho = \rho$ and $\rho(\mathbb{1}_L \otimes P_R) = \rho$ — are the same kind of generic bipartite-operator fact, proved in the companion quantum-channel volume [LTC26, “Left

absorption by the right marginal support”, “Right absorption by the right marginal support”] alongside the left-marginal absorption theorems above.

We now use the explicit tripartite indexing $H_A \otimes (H_X \otimes H_B)$. Reassociation identifies it with $(H_A \otimes H_X) \otimes H_B$ whenever the AX marginal is taken.

Definition 13.2.3 (Tripartite local lifts). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Writing $\widehat{M}_R$ for the lift of an operator M_R supported on $R \in \{AX, XB, A, B\}$, the four lifts to $H_A \otimes (H_X \otimes H_B)$ are

$$\begin{aligned}\widehat{M}_{AX} &= M_{AX} \otimes \mathbb{1}_B, \\ \widehat{M}_{XB} &= \mathbb{1}_A \otimes M_{XB}, \\ \widehat{M}_A &= M_A \otimes \mathbb{1}_{XB}, \\ \widehat{M}_B &= \mathbb{1}_{AX} \otimes M_B.\end{aligned}$$

The first and fourth formulas use the canonical identification $(H_A \otimes H_X) \otimes H_B \cong H_A \otimes (H_X \otimes H_B)$.

Lemma 13.2.4 (Products and adjoints of tripartite lifts). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The lifts from AX and XB preserve multiplication,

$$\begin{aligned}\widehat{MN}_{AX} &= \widehat{M}_{AX} \widehat{N}_{AX}, \\ \widehat{MN}_{XB} &= \widehat{M}_{XB} \widehat{N}_{XB},\end{aligned}$$

and each of the four lifts from AX , XB , A , and B preserves adjoints:

$$(\widehat{M}_R)^\dagger = \widehat{M}_R^\dagger, \quad R \in \{AX, XB, A, B\}.$$

Proof. [✓ Proof](#) For the two-factor lifts,

$$\begin{aligned}(M_{AX} \otimes \mathbb{1}_B)(N_{AX} \otimes \mathbb{1}_B) &= (M_{AX} N_{AX}) \otimes \mathbb{1}_B, \\ (\mathbb{1}_A \otimes M_{XB})(\mathbb{1}_A \otimes N_{XB}) &= \mathbb{1}_A \otimes (M_{XB} N_{XB}).\end{aligned}$$

Moreover,

$$\begin{aligned}(M_{AX} \otimes \mathbb{1}_B)^\dagger &= M_{AX}^\dagger \otimes \mathbb{1}_B, \\ (\mathbb{1}_A \otimes M_{XB})^\dagger &= \mathbb{1}_A \otimes M_{XB}^\dagger,\end{aligned}$$

and similarly,

$$\begin{aligned}(M_A \otimes \mathbb{1}_{XB})^\dagger &= M_A^\dagger \otimes \mathbb{1}_{XB}, \\ (\mathbb{1}_{AX} \otimes M_B)^\dagger &= \mathbb{1}_{AX} \otimes M_B^\dagger,\end{aligned}$$

with the tensor factors reassociated to the tripartite space. $\square$

Definition 13.2.5 (Tripartite observable decorrelation). [Lean](#) [✓ Stmt](#) Regions A and B are decorrelated relative to P if, for all Hermitian observables $O_A = O_A^\dagger$ and $O_B = O_B^\dagger$, one has $PO_A(\mathbb{1} - P)O_BP = 0$. This is [CPGSV16, Definition D.1, lines 2187–2191].

Definition 13.2.6 (Complex-linear extension of decorrelation). [Lean](#) [✓ Stmt](#) The complex-linear extension requires $PO_A(\mathbb{1} - P)O_BP = 0$ for arbitrary complex matrices O_A and O_B .

Theorem 13.2.7 (Observable and complex-linear decorrelation are equivalent). [Lean](#) [✓ Stmt](#) Observable decorrelation holds if and only if its complex-linear extension holds.

Proof. [✓ Proof](#) Every complex matrix has the decomposition

$$\begin{aligned} O &= H + iK, \\ H &= \frac{O + O^\dagger}{2}, \\ K &= \frac{O - O^\dagger}{2i}, \end{aligned}$$

where $H = H^\dagger$ and $K = K^\dagger$. For analogous decompositions $O = H + iK$ and $O' = H' + iK'$,

$$\begin{aligned} PO(\mathbb{1} - P)O'P &= PH(\mathbb{1} - P)H'P + iPH(\mathbb{1} - P)K'P \\ &\quad + iPK(\mathbb{1} - P)H'P - PK(\mathbb{1} - P)K'P. \end{aligned}$$

Each summand vanishes by observable decorrelation applied to its two Hermitian components. $\square$

Theorem 13.2.8 (Reverse order of decorrelation). [Lean](#) [✓ Stmt](#) If $P = P^\dagger$ and A and B are decorrelated, then, for arbitrary complex matrices O_A and O_B , one has $PO_B(\mathbb{1} - P)O_AP = 0$.

Proof. [✓ Proof](#) Apply decorrelation to $O_A^\dagger$ and $O_B^\dagger$, and take adjoints. $\square$

Definition 13.2.9 (Tripartite reduced operators). [Lean](#) [Lean](#) [✓ Stmt](#) The two reduced operators are $\rho_{AX} = \text{tr}_B P$ and $\rho_{XB} = \text{tr}_A P$.

Theorem 13.2.10 (Matrix-unit expansions of the reduced operators). [Lean](#) [Lean](#) [✓ Stmt](#) For matrix units on A and B ,

$$\begin{aligned} \mathbb{1}_A \otimes \rho_{XB} &= \sum_{a,k} E_{ak}^A P E_{ka}^A, \\ \rho_{AX} \otimes \mathbb{1}_B &= \sum_{b,k} E_{bk}^B P E_{kb}^B. \end{aligned}$$

Proof. [✓ Proof](#) For the XB marginal, evaluation in the product basis gives

$$(E_{ak}^A P E_{ka}^A)_{(a_0,r),(a_1,s)} = \delta_{a,a_0} \delta_{a,a_1} P_{(k,r),(k,s)}.$$

Hence

$$\sum_{a,k} (E_{ak}^A P E_{ka}^A)_{(a_0,r),(a_1,s)} = \delta_{a_0,a_1} \sum_k P_{(k,r),(k,s)} = (\mathbb{1}_A \otimes \rho_{XB})_{(a_0,r),(a_1,s)}.$$

The AX identity follows symmetrically. $\square$

Theorem 13.2.11 (Decorrelation for the reduced operators). [Lean](#) [✓ Stmt](#) If A and B are decorrelated, then $\hat{\rho}_{XB}(\mathbb{1} - P)\hat{\rho}_{AX} = 0$.

Proof. [✓ Proof](#) Substitute the matrix-unit expansions. Every summand has the form

$$E_{ak}^A P E_{ka}^A (\mathbb{1} - P) E_{bl}^B P E_{lb}^B = E_{ak}^A [P E_{ka}^A (\mathbb{1} - P) E_{bl}^B P] E_{lb}^B = 0,$$

where the bracketed factor vanishes by decorrelation. $\square$

Theorem 13.2.12 (Reverse reduced-operator decorrelation). [Lean](#) [✓ Stmt](#) If $P = P^\dagger$ and A and B are decorrelated, then $\hat{\rho}_{AX}(\mathbb{1} - P)\hat{\rho}_{XB} = 0$.

Proof. ✓ Proof Substitute the matrix-unit expansions. Every summand has the form

$$E_{bl}^B P E_{lb}^B (\mathbb{1} - P) E_{ak}^A P E_{ka}^A = E_{bl}^B [P E_{lb}^B (\mathbb{1} - P) E_{ak}^A P] E_{ka}^A = 0,$$

where the bracketed factor vanishes by reverse-order decorrelation. □

Definition 13.2.13 (Tripartite marginal support projectors). Lean Lean ✓ Stmt For a Hermitian operator P , let P_{AX} and P_{XB} be the support projectors of $\text{tr}_B P$ and $\text{tr}_A P$, respectively, as in [CPGSV16, Appendix D.2, lines 2225–2229]. Write $\widehat{P}_{AX}$ and $\widehat{P}_{XB}$ for their lifts to the tripartite space.

Theorem 13.2.14 (Decorrelation for the two marginal supports). Lean Lean ✓ Stmt If $P = P^\dagger$ and A and B are decorrelated, then

$$\widehat{P}_{XB}(\mathbb{1} - P)\widehat{P}_{AX} = 0, \tag{13.15}$$

$$\widehat{P}_{AX}(\mathbb{1} - P)\widehat{P}_{XB} = 0. \tag{13.16}$$

Proof. ✓ Proof For $g(0) = 0$ and $g(x) = x^{-1}$ when $x \neq 0$, functional calculus gives

$$\begin{aligned} P_{AX} &= g(\rho_{AX})\rho_{AX} = \rho_{AX}g(\rho_{AX}), \\ P_{XB} &= g(\rho_{XB})\rho_{XB} = \rho_{XB}g(\rho_{XB}). \end{aligned}$$

Hence

$$\widehat{P}_{XB}(\mathbb{1} - P)\widehat{P}_{AX} = g(\widehat{\rho_{XB}})[\widehat{\rho_{XB}}(\mathbb{1} - P)\widehat{\rho_{AX}}]g(\widehat{\rho_{AX}}) = 0,$$

and, in the reverse order,

$$\widehat{P}_{AX}(\mathbb{1} - P)\widehat{P}_{XB} = g(\widehat{\rho_{AX}})[\widehat{\rho_{AX}}(\mathbb{1} - P)\widehat{\rho_{XB}}]g(\widehat{\rho_{XB}}) = 0.$$

□

Theorem 13.2.15 (Absorption by the lifted XB support projector). Lean Lean ✓ Stmt For $P \geq 0$, $\widehat{P}_{XB}P = P = P\widehat{P}_{XB}$.

Proof. ✓ Proof For the bipartition $A|XB$, the two support-absorption identities give

$$(\mathbb{1}_A \otimes P_{XB})P = P = P(\mathbb{1}_A \otimes P_{XB}).$$

□

Theorem 13.2.16 (Absorption by the lifted AX support projector). Lean Lean ✓ Stmt For $P \geq 0$, $\widehat{P}_{AX}P = P = P\widehat{P}_{AX}$.

Proof. ✓ Proof After reassociating the tripartite space as $AX|B$, the two support-absorption identities give

$$(P_{AX} \otimes \mathbb{1}_B)P = P = P(P_{AX} \otimes \mathbb{1}_B).$$

□

Theorem 13.2.17 (Product of the two marginal support projectors). Lean ✓ Stmt If $P = P^\dagger = P^2$ and A and B are decorrelated relative to P , then

$$\widehat{P}_{XB}\widehat{P}_{AX} = P, \tag{13.17}$$

$$\widehat{P}_{AX}\widehat{P}_{XB} = P. \tag{13.18}$$

Proof. ✓ Proof Inserting $\mathbb{1} = P + (\mathbb{1} - P)$ between the two support projectors gives

$$\widehat{P}_{XB}\widehat{P}_{AX} = \widehat{P}_{XB}P\widehat{P}_{AX} + \widehat{P}_{XB}(\mathbb{1} - P)\widehat{P}_{AX} = P + 0 = P.$$

Here the second equality uses the support-absorption identities and (13.15). The same calculation with the two factors reversed, using (13.16), gives (13.18). $\square$

Definition 13.2.18 (Parent commuting Hamiltonian on the tripartite space). Lean Lean Lean

✓ Stmt There are orthogonal projectors P_{AX} and P_{XB} on the two local factors whose lifts commute and whose ranges satisfy the source ground-space intersection condition $\text{ran } P = \text{ran } \widehat{P}_{AX} \cap \text{ran } \widehat{P}_{XB}$. Thus $Q_{AX} = \mathbb{1} - P_{AX}$ and $Q_{XB} = \mathbb{1} - P_{XB}$ are exactly the commuting parent terms of [CPGSV16, Definition D.2, lines 2205–2218].

Theorem 13.2.19 (Ground-space intersection and projector product). Lean Lean Lean Lean

✓ Stmt Let P , P_{AX} , and P_{XB} be orthogonal projectors, and write their lifts to the tripartite space as $\widehat{P}_{AX} = P_{AX} \otimes \mathbb{1}_B$ and $\widehat{P}_{XB} = \mathbb{1}_A \otimes P_{XB}$. Suppose that the lifted projectors commute, $\widehat{P}_{AX}\widehat{P}_{XB} = \widehat{P}_{XB}\widehat{P}_{AX}$. Then

$$\text{ran } P = \text{ran } \widehat{P}_{AX} \cap \text{ran } \widehat{P}_{XB} \iff \widehat{P}_{AX}\widehat{P}_{XB} = P. \quad (13.19)$$

This is the passage from [CPGSV16, Definition D.2, lines 2205–2218] to the projector identity at lines 2279–2289.

Proof. ✓ Proof A vector lies in the range of $\widehat{P}_{AX}\widehat{P}_{XB}$ precisely when it is fixed by both commuting idempotents. Hence $\text{ran}(\widehat{P}_{AX}\widehat{P}_{XB}) = \text{ran } \widehat{P}_{AX} \cap \text{ran } \widehat{P}_{XB}$. The product is again an orthogonal projector. Orthogonal projectors with equal ranges are equal, giving the forward implication in (13.19); the reverse implication follows by substituting the product identity. $\square$

Theorem 13.2.20 (Parent commuting Hamiltonian–complex-linear decorrelation equivalence).

Lean Lean Lean ✓ Stmt Let $P = P^\dagger = P^2$. Then P is the ground-space projector of a parent commuting Hamiltonian on AX and XB if and only if regions A and B satisfy complex-linear decorrelation.

Proof. ✓ Proof Starting from decorrelation, take the support projectors of ρ_{AX} and ρ_{XB} . The preceding product theorem proves their commutation and identifies their product with P .

Starting from commuting parent projectors, first use the ground-space intersection theorem to write $P = \widehat{P}_{AX}\widehat{P}_{XB} = \widehat{P}_{XB}\widehat{P}_{AX}$. Locality gives $[O_A, \widehat{P}_{XB}] = [O_B, \widehat{P}_{AX}] = 0$, while $\widehat{P}_{XB}(\mathbb{1} - P)\widehat{P}_{AX} = 0$. Therefore, using the two locality commutators,

$$PO_A(\mathbb{1} - P)O_BP = \widehat{P}_{AX}O_A[\widehat{P}_{XB}(\mathbb{1} - P)\widehat{P}_{AX}]O_B\widehat{P}_{XB} = 0.$$

$\square$

Theorem 13.2.21 (Parent commuting Hamiltonian–observable decorrelation equivalence).

Lean ✓ Stmt Let $P = P^\dagger = P^2$. Then P is the ground-space projector of a parent commuting Hamiltonian on AX and XB if and only if regions A and B are decorrelated when tested on Hermitian observables. This is [CPGSV16, Proposition D.3, lines 2221–2289].

Proof. ✓ Proof Combine the complex-linear parent equivalence with the decomposition $O = H + iK$ from Theorem 13.2.7. $\square$

Definition 13.2.22 (Decorrelation). Lean ✓ Stmt Given an endomorphism P_K and sets of observables $\mathcal{O}_A, \mathcal{O}_B$, regions A and B are *decorrelated* w.r.t. K when $P_K \circ O_A \circ (\mathbb{1} - P_K) \circ O_B \circ P_K = 0$ for all $O_A \in \mathcal{O}_A$ and $O_B \in \mathcal{O}_B$.

Definition 13.2.23 (Idempotent-product parent-commuting structure). [Lean](#) [✓ Stmt](#) The structure used below records only the idempotent-product consequence of the parent commuting Hamiltonian condition of [CPGSV16, Appendix D.2]. It consists of commuting idempotent endomorphisms P_{AX} and P_{XB} such that, for the idempotent P_K associated to K , $P_{AX} \circ P_{XB} = P_K$. In the source, this is the equation obtained from local projectors Q_{AX} and Q_{XB} satisfying $[Q_{AX}, Q_{XB}] = 0$ and $K_{AXB} = (K_{AX} \otimes H_B) \cap (H_A \otimes K_{XB})$.

Theorem 13.2.24 (Idempotence gives tautological idempotent-product data). [Lean](#) [✓ Stmt](#) If $P_K \circ P_K = P_K$, then P_K has idempotent-product data.

Proof. [✓ Proof](#) Take $P_{AX} = P_{XB} = P_K$. Their commutativity is immediate, and the product identity is exactly the assumed idempotence of P_K . $\square$

Lemma 13.2.25 (Cancellation for commuting idempotents). [Lean](#) [✓ Stmt](#) If P and Q are commuting idempotents, then $Q \circ (\mathbb{1} - P \circ Q) \circ P = 0$.

Proof. [✓ Proof](#) The cancellation is the computation

$$Q(\mathbb{1} - PQ)P = QP - QPQP = QP - QPPQ = QP - QP = 0,$$

using commutativity and idempotence. $\square$

Theorem 13.2.26 (Idempotence of the product projector). [Lean](#) [✓ Stmt](#) If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_K \circ P_K = P_K$.

Proof. [✓ Proof](#) Since P_{AX} and P_{XB} are idempotent and commute, their product $P_{AX} \circ P_{XB}$ is idempotent. Substituting $P_K = P_{AX} \circ P_{XB}$ gives the claim. $\square$

Theorem 13.2.27 (Absorption by the left projector). [Lean](#) [✓ Stmt](#) If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_{AX} \circ P_K = P_K$.

Proof. [✓ Proof](#) Using $P_K = P_{AX} \circ P_{XB}$ and the idempotence of P_{AX} ,

$$P_{AX} \circ P_K = P_{AX} \circ P_{AX} \circ P_{XB} = P_{AX} \circ P_{XB} = P_K.$$

$\square$

Theorem 13.2.28 (Absorption by the right projector). [Lean](#) [✓ Stmt](#) If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_K \circ P_{XB} = P_K$.

Proof. [✓ Proof](#) Using $P_K = P_{AX} \circ P_{XB}$ and the idempotence of P_{XB} ,

$$P_K \circ P_{XB} = P_{AX} \circ P_{XB} \circ P_{XB} = P_{AX} \circ P_{XB} = P_K.$$

$\square$

Theorem 13.2.29 (Cross-absorption by the right projector). [Lean](#) [✓ Stmt](#) If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_{XB} \circ P_K = P_K$.

Proof. [✓ Proof](#) By commutativity and the idempotence of P_{XB} ,

$$P_{XB} \circ P_K = P_{XB} \circ P_{AX} \circ P_{XB} = P_{AX} \circ P_{XB} \circ P_{XB} = P_{AX} \circ P_{XB} = P_K.$$

$\square$

Theorem 13.2.30 (Cross-absorption by the left projector). [Lean] [✓ Stmt] If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_K \circ P_{AX} = P_K$.

Proof. [✓ Proof] By commutativity and the idempotence of P_{AX} ,

$$P_K \circ P_{AX} = P_{AX} \circ P_{XB} \circ P_{AX} = P_{AX} \circ P_{AX} \circ P_{XB} = P_{AX} \circ P_{XB} = P_K.$$

□

Theorem 13.2.31 (Reverse product identity). [Lean] [✓ Stmt] If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then $P_{XB} \circ P_{AX} = P_K$.

Proof. [✓ Proof] This follows from commutativity of P_{AX} and P_{XB} together with the identity $P_{AX} \circ P_{XB} = P_K$. □

Theorem 13.2.32 (Commuting complementary projectors). [Lean] [✓ Stmt] If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data and $Q_{AX} = \mathbb{1} - P_{AX}$ and $Q_{XB} = \mathbb{1} - P_{XB}$, then $Q_{AX} \circ Q_{XB} = Q_{XB} \circ Q_{AX}$.

Proof. [✓ Proof] Expanding both products and using $P_{AX} \circ P_{XB} = P_{XB} \circ P_{AX}$ shows that the complementary projectors commute. □

Theorem 13.2.33 (Idempotent-product data imply decorrelation). [Lean] [✓ Stmt] If P_{AX} and P_{XB} are idempotent endomorphisms satisfying $P_{AX} \circ P_{XB} = P_K$ and $P_{AX} \circ P_{XB} = P_{XB} \circ P_{AX}$, and A -observables commute with P_{XB} while B -observables commute with P_{AX} , then regions A and B are decorrelated w.r.t. K . This is the idempotent-product form of the backward direction of [CPGSV16, Appendix D, Section D.2, Proposition D.3].

Proof. [✓ Proof] The key identity is $P_{XB} \circ (\mathbb{1} - P_{AX} \circ P_{XB}) \circ P_{AX} = 0$. Substituting $P_K = P_{AX} \circ P_{XB}$ and using $O_A \circ P_{XB} = P_{XB} \circ O_A$ and $O_B \circ P_{AX} = P_{AX} \circ O_B$ gives

$$P_K \circ O_A \circ (\mathbb{1} - P_K) \circ O_B \circ P_K = P_{AX} \circ O_A \circ [P_{XB} \circ (\mathbb{1} - P_{AX} \circ P_{XB}) \circ P_{AX}] \circ O_B \circ P_{XB} = 0.$$

□

Theorem 13.2.34 (Decorrelating idempotent-product data). [Lean] [✓ Stmt] Corollary of Theorem 13.2.33: if P_K has idempotent-product data, the A -observables commute with P_{XB} , and the B -observables commute with P_{AX} , then regions A and B are decorrelated w.r.t. K .

Proof. [✓ Proof] Apply Theorem 13.2.33 to the witnessing endomorphisms P_{AX} and P_{XB} supplied by the idempotent-product data. □

Lemma 13.2.35 (Frustration-free identity). [Lean] [✓ Stmt] For any endomorphisms P and Q ,

$$(\mathbb{1} - P) + (\mathbb{1} - Q) - (\mathbb{1} - P) \circ (\mathbb{1} - Q) = \mathbb{1} - P \circ Q. \quad (13.20)$$

Proof. [✓ Proof] Expanding the left-hand side of (13.20) and cancelling gives the result. □

Theorem 13.2.36 (Idempotent-product Hamiltonian identity). [Lean] [✓ Stmt] If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then

$$Q_{AX} + Q_{XB} - Q_{AX} \circ Q_{XB} = \mathbb{1} - P_K, \quad (13.21)$$

where $Q_{AX} = \mathbb{1} - P_{AX}$ and $Q_{XB} = \mathbb{1} - P_{XB}$ are the complementary projectors. This is the algebraic identity obtained from the complementary projectors in [CPGSV16, Appendix D, Section D.2]; it is not the full source parent commuting Hamiltonian condition.

Proof. ✓ Proof Apply Lemma 13.2.35 to P_{AX} and P_{XB} , and substitute $P_{AX} \circ P_{XB} = P_K$ to obtain (13.21). □

Theorem 13.2.37 (Ground-space membership). Lean ✓ Stmt *If $P_K = P_{AX} \circ P_{XB}$ is idempotent-product data, then*

$$P_K v = v \iff P_{AX} v = v \text{ and } P_{XB} v = v. \quad (13.22)$$

This is the algebraic consequence corresponding to equation (D.2) from [CPGSV16]: $K_{AXB} = (K_{AX} \otimes H_B) \cap (H_A \otimes K_{XB})$. The source equation itself also carries the tensor-product local-subspace hypotheses.

Proof. ✓ Proof ($\Rightarrow$) If $P_K v = v$, then

$$P_{AX}(P_K v) = (P_{AX} \circ P_K)v = P_K v = v$$

by left absorption, and similarly for P_{XB} .

($\Leftarrow$) If $P_{AX} v = v$ and $P_{XB} v = v$, then

$$P_K v = (P_{AX} \circ P_{XB})v = P_{AX}(P_{XB} v) = P_{AX} v = v.$$

This proves (13.22). □

13.3 Spectral gap

A frustration-free Hamiltonian is *gapped* if the smallest nonzero eigenvalue, equivalently the gap above the ground space, is bounded away from zero uniformly in the chain length N . For MPS parent Hamiltonians, the martingale method of Fannes–Nachtergaele–Werner and Nachtergaele [FNW92, Nac96], as recalled by Cirac–Perez-Garcia–Schuch–Verstraete [CPGSV21, Section IV.C] and in the later formulation of Kastoryano and Lucia [KL18], reduces a uniform gap to an anticommutator estimate for overlapping local terms. Principal angles between the corresponding local ground spaces give the geometric interpretation of this estimate. The results below state the anticommutator condition for the cyclic-window local terms and also record cyclic norm-compression estimates as sufficient stronger hypotheses.

Definition 13.3.1 (Parent ground space under the canonical ℓ^2 identification). Lean ✓ Stmt Denote by $\mathcal{G}_{N,L}^{\text{ES}}(A)$ the parent ground space after identifying the coefficient space with the Hilbert space $\ell^2(\{0, \dots, d-1\}^N)$.

Definition 13.3.2 (Parent Hamiltonian under the canonical ℓ^2 identification). Lean ✓ Stmt Denote by $H_{N,L}^{\text{ES}}(A)$ the conjugate of $H_N(A, L)$ by the canonical identification between the coefficient space and $\ell^2(\{0, \dots, d-1\}^N)$.

Theorem 13.3.3 (Kernel under the canonical ℓ^2 identification). Lean ✓ Stmt *Under the canonical ℓ^2 identification of the N -site state space, the parent ground space is exactly the kernel of the corresponding parent Hamiltonian.*

Proof. ✓ Proof This is an immediate unraveling of the definitions: both constructions are obtained from the same parent Hamiltonian by conjugating with the canonical identification between the site space and the ℓ^2 space, so the ground space is precisely the kernel after this identification. □

Lemma 13.3.4 (Membership criterion under the canonical ℓ^2 identification). Lean ✓ Stmt *A vector belongs to $\mathcal{G}_{N,L}^{\text{ES}}(A)$ exactly when the parent Hamiltonian annihilates it.*

Remark 13.3.5 (Local projectors under the canonical ℓ^2 identification). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let $P_L^{\text{ES}}(A)$ denote the orthogonal projector onto $(G_L^{\text{ES}}(A))^\perp$ on the canonical ℓ^2 L -site Hilbert space. For a cyclic window starting at i and an outside configuration τ , the map $R_{i,\tau}$ restricts an N -site vector to that length- L cyclic window with the outside configuration fixed by τ . The local term $h_{i,\text{ES}}$ is the corresponding N -site local projector, and $R_{i,\tau}^\dagger P_L^{\text{ES}}(A) R_{i,\tau}$ is the basic positive summand in its cyclic-averaging formula.

Lemma 13.3.6 (Positivity of the Hilbert-space local summands). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) *The Hilbert-space local projector $P_L^{\text{ES}}(A)$ is positive. Consequently, every summand $R_{i,\tau}^\dagger P_L^{\text{ES}}(A) R_{i,\tau}$ is positive, and so is the finite sum over τ .*

Proof. [Stmnt](#) Orthogonal projectors are positive. Conjugation by $R_{i,\tau}$ preserves positivity, and finite sums of positive operators remain positive. $\square$

Lemma 13.3.7 (Cyclic averaging for the Hilbert-space local terms). [Lean](#) [Lean](#) [Stmnt](#) *The Hilbert-space parent Hamiltonian is the sum of the Hilbert-space local terms, $H_{N,\text{ES}} = \sum_i h_{i,\text{ES}}$, and each local term admits the exact cyclic-averaging formula*

$$h_{i,\text{ES}} = d^{-L} \sum_{\tau} R_{i,\tau}^\dagger P_L^{\text{ES}}(A) R_{i,\tau}. \quad (13.23)$$

Proof. [Stmnt](#) Unfold the Hilbert-space local term and compare it configuration-by-configuration with the cyclic restrictions. The d^{-L} factor compensates for the d^L choices of the window coordinates in the ambient configuration parameter τ . $\square$

Lemma 13.3.8 (Positivity of the Hilbert-space Hamiltonian). [Lean](#) [Lean](#) [Stmnt](#) *Each Hilbert-space local term $h_{i,\text{ES}}$ is positive, hence the full Hilbert-space parent Hamiltonian $H_{N,\text{ES}}$ is positive.*

Proof. [Stmnt](#) Apply the averaging formula (13.23): $h_{i,\text{ES}}$ is a non-negative scalar multiple of a finite sum of positive conjugates of $P_L^{\text{ES}}(A)$, hence is itself positive. Summing over i yields positivity of $H_{N,\text{ES}}$. $\square$

Lemma 13.3.9 (Hilbert-space local terms are symmetric projections). [Lean](#) [Lean](#) [Stmnt](#) *The Hilbert-space local projector $P_L^{\text{ES}}(A)$ is a symmetric projection, and every Hilbert-space local term $h_{i,\text{ES}}$ on the N -site chain is a symmetric projection.*

Proof. [Stmnt](#) The local operator $P_L^{\text{ES}}(A)$ is an orthogonal projection onto $(G_L^{\text{ES}}(A))^\perp$. For $L \leq N$, restricting $h_{i,\text{ES}} v$ to a cyclic window gives $P_L^{\text{ES}}(A)$ applied to the corresponding restriction of v ; applying $h_{i,\text{ES}}$ a second time therefore applies $(P_L^{\text{ES}}(A))^2 = P_L^{\text{ES}}(A)$ on that window. The positivity result above supplies symmetry, and the case $L > N$ is the zero projection by definition. $\square$

Lemma 13.3.10 (Kernel of the Hilbert-space parent interaction). [Lean](#) [Stmnt](#) *The Hilbert-space local interaction $P_L^{\text{ES}}(A)$ annihilates exactly the Hilbert-space local ground space: $P_L^{\text{ES}}(A)v = 0$ if and only if $v \in G_L^{\text{ES}}(A)$.*

Proof. [Stmnt](#) Since $P_L^{\text{ES}}(A)$ is the orthogonal projector onto $(G_L^{\text{ES}}(A))^\perp$, its kernel is the original subspace $G_L^{\text{ES}}(A)$. $\square$

Lemma 13.3.11 (Local-term kernel via cyclic restrictions). [Lean](#) [Lean](#) [Stmnt](#) *For $L \leq N$, a Hilbert-space local term $h_{i,\text{ES}}$ annihilates a vector v if and only if every restriction of v to the cyclic length- L window starting at i , with the outside configuration fixed, lies in $G_L^{\text{ES}}(A)$.*

Proof. ✓ Proof Restricting $h_{i,\text{ES}}v$ to the same cyclic window gives $P_L^{\text{ES}}(A)$ applied to the corresponding restriction of v . Lemma 13.3.10 identifies the vanishing of this local projector with membership in $G_L^{\text{ES}}(A)$. $\square$

Theorem 13.3.12 (End-site and cyclic coordinate identifications). Lean Lean Lean ✓ Stmt Let ι_L denote the canonical identification of the L -site state space with its Hilbert-space realization, and let $R_{i,\tau}^{\text{cyc}}$ denote cyclic restriction to the length- L window beginning at i , with background configuration τ . For every $L \in \mathbb{N}$, every $(L+1)$ -site Hilbert-space vector v , and every background configuration τ ,

$$R_{\tau_L}^{\text{last}}(\iota_{L+1}^{-1}v) = \iota_L^{-1}R_{0,\tau}^{\text{cyc}}v. \quad (13.24)$$

On three sites, for every one-site prefix u and every state ψ ,

$$R_u^{\text{tail}}\psi = R_{u_0}^{\text{first}}\psi. \quad (13.25)$$

Finally, if $L > 0$, then for every $(L+1)$ -site Hilbert-space vector v and every background configuration τ ,

$$R_{\tau_0}^{\text{first}}(\iota_{L+1}^{-1}v) = \iota_L^{-1}R_{1,\tau}^{\text{cyc}}v. \quad (13.26)$$

No hypothesis on the physical or virtual dimension is required.

Proof. ✓ Proof Expand the restriction maps on a configuration. At site 0, cyclic order is ordinary order and fixing the complement fixes the final coordinate. A one-site prefix is its unique coordinate, which gives the middle identity. If $L > 0$, cyclic restriction beginning at site 1 fixes the first coordinate and preserves the order of the remaining L coordinates. $\square$

Theorem 13.3.13 (Adjacent three-site kernels and excitation projections). Lean Lean Lean Lean ✓ Stmt For a tensor B , let h_0 and h_1 be the two translated length-two parent interactions on the three-site Hilbert space. Then

$$\ker h_0 = \mathcal{U}_2(B), \quad \ker h_1 = \mathcal{V}_{1,2}(B). \quad (13.27)$$

Consequently, the orthogonal projectors onto the complementary excitation subspaces satisfy

$$P_{\mathcal{U}_2(B)^\perp} = h_0, \quad P_{\mathcal{V}_{1,2}(B)^\perp} = h_1. \quad (13.28)$$

These identities require neither injectivity nor a nonzero bond dimension.

Proof. ✓ Proof At site 0, fixing the final physical coordinate is the cyclic restriction defining h_0 . At site 1, fixing the one-site prefix is the first-site restriction and hence the cyclic restriction defining h_1 . The kernel characterization of a translated local parent interaction therefore gives the two displayed kernel identities. Each h_i is an orthogonal projector, so its range is the orthogonal complement of its kernel. Uniqueness of the orthogonal projector with a prescribed range gives the final two identities. $\square$

Theorem 13.3.14 (Zero energy gives the cyclic local constraints). Lean ✓ Stmt Assume $N > 0$ and $L \leq N$. If a vector ψ has zero energy for the finite periodic parent Hamiltonian, $H_N(A, L)\psi = 0$, then every cyclic length- L restriction of ψ lies in the local MPS ground space. Equivalently, $\ker H_N(A, L) \subseteq G_L^{(N)}(A)$.

Proof. [✓ Proof](#) Let e_N be the canonical ℓ^2 identification from the Hilbert-space model to the coefficient space. Since $H_N(A, L)$ is the e_N -conjugate of $H_{N,L}^{\text{ES}}(A)$, the hypothesis gives

$$H_{N,L}^{\text{ES}}(A)(e_N^{-1}\psi) = e_N^{-1}(H_N(A, L)\psi) = 0.$$

Put $P_i = h_{i,\text{ES}}$ and $H = \sum_i P_i$. If $Hv = 0$, then

$$0 = \text{Re}\langle Hv, v \rangle = \sum_i \text{Re}\langle P_i v, v \rangle = \sum_i \|P_i v\|^2.$$

Each summand is non-negative; hence $\|P_i v\|^2 = 0$ and therefore $P_i v = 0$ for every i . Applying this to $H_{N,L}^{\text{ES}}(A) = \sum_i h_{i,\text{ES}}$ gives the vanishing of every Hilbert-space local term. Lemma 13.3.11 identifies these equations with the conditions $R_{i,\tau}(e_N^{-1}\psi) \in G_L^{\text{ES}}(A)$ for all i and τ . Transporting these conditions back through e_N gives $R_{i,\tau}\psi \in G_L(A)$ for all i and τ , which are exactly the cyclic-window defining equations of $G_L^{(N)}(A)$. $\square$

Lemma 13.3.15 (Cyclic constraints give zero energy). [Lean](#) [✓ Stmt](#) Assume $N > 0$ and $L \leq N$. Then $G_L^{(N)}(A) \subseteq \ker H_N(A, L)$.

Proof. [✓ Proof](#) If $\psi \in G_L^{(N)}(A)$, then every local parent interaction annihilates ψ , for every site i : $h_i(A, L)\psi = 0$. Summing over all sites gives $H_N(A, L)\psi = \sum_i h_i(A, L)\psi = 0$. $\square$

Definition 13.3.16 (Hilbert-space cyclic constraint space). [Lean](#) [✓ Stmt](#) The Hilbert-space cyclic constraint space is the transport of $G_L^{(N)}(A)$ by the canonical ℓ^2 identification: $G_{L,\text{ES}}^{(N)}(A) = e_N^{-1}G_L^{(N)}(A)$.

Theorem 13.3.17 (The finite-volume kernel is the cyclic constraint space). [Lean](#) [Lean](#) [✓ Stmt](#) Assume $N > 0$ and $L \leq N$. Then $\ker H_N(A, L) = G_L^{(N)}(A)$. Equivalently, after the canonical Hilbert-space identification, $\mathcal{K}_N^{\text{ES}}(A, L) = e_N^{-1}G_L^{(N)}(A)$, where $\mathcal{K}_N^{\text{ES}}(A, L)$ is the zero-energy space of $H_{N,L}^{\text{ES}}(A)$.

Proof. [✓ Proof](#) The inclusion $\ker H_N(A, L) \subseteq G_L^{(N)}(A)$ is Theorem 13.3.14. For the converse, a vector in $G_L^{(N)}(A)$ satisfies all local equations $h_i(A, L)\psi = 0$. Therefore $H_N(A, L)\psi = \sum_i h_i(A, L)\psi = 0$. The Hilbert-space statement is the same equality transported by e_N^{-1} . $\square$

Theorem 13.3.18 (Intersection property for two adjacent local kernels). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If A is injective and $L \geq 2$, then on an $(L+1)$ -site chain the kernels of the two adjacent Hilbert-space L -site local terms have common kernel equal to the Hilbert-space $(L+1)$ -site MPS ground space:

$$h_{0,\text{ES}}v = 0 \text{ and } h_{1,\text{ES}}v = 0 \iff v \in G_{L+1}^{\text{ES}}(A).$$

Proof. [✓ Proof](#) Lemma 13.3.11 translates the two local kernel assumptions into the left and right L -site ground conditions. The open-chain intersection property, Theorem 12.4.4.1, then grows back the shared $(L-1)$ -site overlap to the $(L+1)$ -site ground space. The converse uses the forward restriction lemmas for ground-space vectors and the same kernel–restriction characterization. $\square$

Lemma 13.3.19 (Non-overlap positivity for Hilbert-space local terms). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $L \leq N$ and let $h_{i,\text{ES}}, h_{j,\text{ES}}$ be Hilbert-space local terms whose cyclic L -windows are site-disjoint: no site of the periodic chain has cyclic offset $< L$ from both i and j . Then the terms commute pointwise: for every vector v , $h_{i,\text{ES}}(h_{j,\text{ES}}v) = h_{j,\text{ES}}(h_{i,\text{ES}}v)$. The same commutation identity holds for the corresponding local terms before passing to the Hilbert-space model. Their ordered cross term is non-negative: $0 \leq \text{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle$.

Proof. ✓ Proof In coordinates, $h_{i, \text{ES}}$ applies $P_L^{\text{ES}}(A)$ only to the i -window variables and leaves the complementary variables fixed. Site disjointness gives two replacement identities: replacing the i -window does not change the extracted j -window, and the i - and j -window replacements commute. Therefore the two embedded local operators commute. The preceding projection-geometry identity for commuting symmetric projections gives the non-negative ordered cross term. $\square$

Lemma 13.3.20 (Quadratic form to norm bound). Lean ✓ Stmt *Let H be a positive semidefinite self-adjoint operator on a finite-dimensional complex Hilbert space, and suppose that, for all v , $\gamma \langle v, Hv \rangle \leq \langle Hv, Hv \rangle$, i.e., $H^2 \geq \gamma H$ as a quadratic form, for some $\gamma > 0$. Then, for every v in the orthogonal complement of $\ker H$, $\gamma \|v\| \leq \|Hv\|$. In particular, every positive eigenvalue of H is at least γ . The positive-semidefiniteness hypothesis is essential: without it, $H = -1$ with $\gamma = 2$ satisfies the quadratic-form inequality vacuously but violates the norm bound.*

Proof. ✓ Proof By the finite-dimensional spectral theorem, H diagonalises in an orthonormal eigenbasis with real eigenvalues λ_k . For a unit eigenvector ψ_k with eigenvalue λ_k , the hypothesis gives $\gamma \lambda_k \leq \lambda_k^2$, whence $\lambda_k \leq 0$ or $\lambda_k \geq \gamma$. Since H is positive semidefinite (its quadratic form is ≥ 0), the negative case is excluded and every nonzero eigenvalue satisfies $\lambda_k \geq \gamma$. Writing $v = \sum_k c_k \psi_k$ and projecting onto $(\ker H)^\perp$ keeps only terms with $\lambda_k \geq \gamma$, so

$$\|Hv\|^2 = \sum_k |c_k|^2 \lambda_k^2 \geq \gamma^2 \sum_k |c_k|^2 = \gamma^2 \|v\|^2.$$

$\square$

13.3.1 Martingale condition for a finite family of projections

Lemma 13.3.1.1 (Parent-Hamiltonian quadratic-form reduction). Lean Lean ✓ Stmt *Let $H_{N,L}^{\text{ES}}(A)$ be the Hilbert-space representative of the parent Hamiltonian. If a constant $\gamma > 0$ satisfies the uniform quadratic-form estimate*

$$\gamma \operatorname{Re} \langle H_{N,L}^{\text{ES}}(A)v, v \rangle \leq \operatorname{Re} \langle H_{N,L}^{\text{ES}}(A)v, H_{N,L}^{\text{ES}}(A)v \rangle \quad (13.29)$$

for every $N \geq 2L$ and every vector v , then $\gamma \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$ for every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$. In particular, with $L > 1$ and $\gamma = 1/(4L)$, the explicit gap-bound conclusion follows once this same quadratic-form estimate (13.29) is supplied.

Proof. ✓ Proof The Hilbert-space Hamiltonian is positive by Lemma 13.3.8, and its Hilbert-space ground space is its kernel by Theorem 13.3.3. Therefore the abstract spectral-theorem conversion of Lemma 13.3.20 applies directly. The explicit constant is positive because $L > 1$ implies $4L > 0$. $\square$

Definition 13.3.1.2 (Friedrichs overlap bound). Lean ✓ Stmt *Let U and V be subspaces of a finite-dimensional complex Hilbert space. Their Friedrichs overlap is at most η when*

$$|\langle x, y \rangle| \leq \eta \|x\| \|y\|$$

for every $x \in U \cap (U \cap V)^\perp$ and $y \in V \cap (U \cap V)^\perp$.

Theorem 13.3.1.3 (Projector defect bounds the Friedrichs overlap). Lean ✓ Stmt *Let P_U , P_V , and $P_{U \cap V}$ be the orthogonal projections onto U , V , and their intersection. If*

$$\|P_U P_V - P_{U \cap V}\| \leq \eta,$$

then the Friedrichs overlap of U and V is at most η .

Proof. ✓ Proof For $y \in V \cap (U \cap V)^\perp$, the defect applied to y is $P_U y$. Hence $\|P_U y\| \leq \eta \|y\|$. If also $x \in U \cap (U \cap V)^\perp$, self-adjointness of P_U gives $\langle x, y \rangle = \langle x, P_U y \rangle$, and Cauchy–Schwarz proves the claim. $\square$

Lemma 13.3.1.4 (Projection compression from a Friedrichs overlap bound). Lean ✓ Stmt Let P_U be the orthogonal projection onto U . If the Friedrichs overlap of U and V is at most η , where $\eta \geq 0$, then

$$\|P_U v\| \leq \eta \|v\|$$

for every $v \in V \cap (U \cap V)^\perp$.

Proof. ✓ Proof Put $x = P_U v$. Since the common intersection is contained in U , the projection preserves orthogonality to $U \cap V$. Thus $x \in U \cap (U \cap V)^\perp$. Moreover, $\langle x, v \rangle = \langle x, x \rangle = \|x\|^2$. The Friedrichs bound gives $\|x\|^2 \leq \eta \|x\| \|v\|$, and cancellation proves the result when $x \neq 0$; the case $x = 0$ is immediate. $\square$

Lemma 13.3.1.5 (Two-projection compression on the reduced space). Lean ✓ Stmt Let P_U and P_V be the orthogonal projections onto U and V . If the Friedrichs overlap of U and V is at most η , where $\eta \geq 0$, then every $v \in (U \cap V)^\perp$ satisfies

$$\|P_U P_V v\| \leq \eta \|P_V v\|.$$

Proof. ✓ Proof The vector $P_V v$ belongs to V and remains orthogonal to $U \cap V$. Apply Lemma 13.3.1.4. $\square$

Lemma 13.3.1.6 (Directional compression on a reduced projection range). Lean ✓ Stmt Under the hypotheses of Lemma 13.3.1.5, every $v \in U \cap (U \cap V)^\perp$ satisfies

$$\|P_U P_V v\| \leq \eta \|P_U v\|.$$

Proof. ✓ Proof Orthogonal projection onto V does not increase the norm, while $P_U v = v$. Hence $\|P_U P_V v\| \leq \eta \|P_V v\| \leq \eta \|v\| = \eta \|P_U v\|$. $\square$

Lemma 13.3.1.7 (Directional compression for kernel-complement projections). Lean ✓ Stmt Let P and Q be the orthogonal projections onto $K^\perp$ and $L^\perp$. If the Friedrichs overlap of these two ranges is at most $\eta \geq 0$, then every $v \in K^\perp \cap (K^\perp \cap L^\perp)^\perp$ satisfies

$$\|PQv\| \leq \eta \|Pv\|.$$

Proof. ✓ Proof Apply Lemma 13.3.1.6 to the two subspaces $K^\perp$ and $L^\perp$. $\square$

Remark 13.3.1.8 (Projection-geometry identities for row-sum arguments). Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt The projection-geometry reduction uses the following elementary Hilbert-space identities: the diagonal identity $\operatorname{Re}\langle Pv, Pv \rangle = \operatorname{Re}\langle Pv, v \rangle$ for symmetric projections; nonnegativity of this quadratic form; nonnegativity of $\operatorname{Re}\langle Pv, Qv \rangle$ for commuting symmetric projections P, Q ; the Cauchy–Schwarz conversion from $\|PQv\| \leq c\|Pv\|$ to $\operatorname{Re}\langle Pv, Qv \rangle \geq -c \operatorname{Re}\langle Pv, v \rangle$; the symmetric variant which, for $c \geq 0$, converts the operator-product bound $\|PQv\| \leq c\|Qv\|$ to the anticommutator estimate $\operatorname{Re}\langle Pv, Qv \rangle + \operatorname{Re}\langle Qv, Pv \rangle \geq -c(\operatorname{Re}\langle Pv, v \rangle + \operatorname{Re}\langle Qv, v \rangle)$; finite-sum expansions of $\operatorname{Re}\langle (\sum_i P_i)v, v \rangle$ and $\operatorname{Re}\langle (\sum_i P_i)v, (\sum_i P_i)v \rangle$; the conclusion that a vector annihilated by a finite sum of symmetric projections is annihilated by each projection; the diagonal/off-diagonal split of the ordered double sum; the aggregate off-diagonal estimate obtained from row sums $\sum_{j \neq i} c_{ij} \leq 1$; and the corresponding aggregate estimate for the anticommutator martingale condition.

Lemma 13.3.1.9 (Ordered projection row-sum quadratic-form reduction). [Lean](#) [Stmnt](#) Let P_i be a finite family of symmetric projections and let $H = \sum_i P_i$. If the ordered off-diagonal forms obey

$$\operatorname{Re}\langle P_i v, P_j v \rangle \geq -(1 - \gamma) c_{ij} \operatorname{Re}\langle P_i v, v \rangle \quad (i \neq j),$$

with row sums $\sum_{j \neq i} c_{ij} \leq 1$ and $\gamma \leq 1$, then $H^2 \geq \gamma H$ as a quadratic form.

Proof. [Proof](#) Expand $\operatorname{Re}\langle H v, H v \rangle$ as an ordered double sum. Symmetric idempotence identifies each diagonal term with $\operatorname{Re}\langle P_i v, v \rangle$, while the row-summable ordered cross-term bounds control the off-diagonal contribution by $-(1 - \gamma) \sum_i \operatorname{Re}\langle P_i v, v \rangle$. $\square$

Lemma 13.3.1.10 (Projection anticommutator row-sum reduction). [Lean](#) [Stmnt](#) Let P_i be a finite family of symmetric projections and let $H = \sum_i P_i$. Suppose the anticommutator forms obey

$$\operatorname{Re}\langle P_i v, P_j v \rangle + \operatorname{Re}\langle P_j v, P_i v \rangle \geq -(1 - \gamma) c_{ij} (\operatorname{Re}\langle P_i v, v \rangle + \operatorname{Re}\langle P_j v, v \rangle) \quad (i \neq j),$$

with row sums $\sum_{j \neq i} c_{ij} \leq 1$, column sums $\sum_{i \neq j} c_{ij} \leq 1$, and $\gamma \leq 1$. Then $H^2 \geq \gamma H$ as a quadratic form.

Proof. [Proof](#) Expand $\operatorname{Re}\langle H v, H v \rangle$ as an ordered double sum. The anticommutator bound controls the sum of each ordered term with its reversed term. The row and column estimates bound the coefficient-weighted diagonal contribution by twice $\sum_i \operatorname{Re}\langle P_i v, v \rangle$; after dividing by two, the same quadratic-form estimate follows. $\square$

Lemma 13.3.1.11 (Finite-overlap projection anticommutator reduction). [Lean](#) [Stmnt](#) Let P_i be a finite family of symmetric projections and let $\gamma \leq 1$. Suppose an interaction predicate $i \sim j$ has at most m off-diagonal neighbours in each row and in each column, where $m > 0$. If noninteracting pairs have non-negative anticommutator form and interacting pairs satisfy

$$\operatorname{Re}\langle P_i v, P_j v \rangle + \operatorname{Re}\langle P_j v, P_i v \rangle \geq -(1 - \gamma) \frac{1}{m} (\operatorname{Re}\langle P_i v, v \rangle + \operatorname{Re}\langle P_j v, v \rangle),$$

then $H = \sum_i P_i$ satisfies $H^2 \geq \gamma H$ as a quadratic form.

Proof. [Proof](#) Choose $c_{ij} = 1/m$ on interacting pairs and $c_{ij} = 0$ otherwise. The row and column cardinality bounds give the corresponding coefficient sums, and the noninteracting anticommutator non-negativity supplies the zero-coefficient estimates. Apply Lemma 13.3.1.10. $\square$

Lemma 13.3.1.12 (Finite-overlap projection row-sum reduction). [Lean](#) [Stmnt](#) Let P_i be a finite family of symmetric projections and let $\gamma \leq 1$. Suppose an interaction predicate $i \sim j$ has at most m off-diagonal neighbours in each row for some positive integer m . If noninteracting ordered cross terms are non-negative and, on each interacting pair,

$$\operatorname{Re}\langle P_i v, P_j v \rangle \geq -(1 - \gamma) \frac{1}{m} \operatorname{Re}\langle P_i v, v \rangle,$$

then $H = \sum_i P_i$ satisfies $H^2 \geq \gamma H$ as a quadratic form.

Proof. [Proof](#) Choose $c_{ij} = 1/m$ on interacting pairs and $c_{ij} = 0$ otherwise. The row cardinality bound gives $\sum_{j \neq i} c_{ij} \leq 1$; noninteracting nonnegativity supplies the zero-coefficient cross-term estimates. Apply Lemma 13.3.1.9. $\square$

Lemma 13.3.1.13 (Finite-overlap projection norm-compression reduction). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let P_i be a finite family of symmetric projections and let $\gamma \leq 1$. Suppose an interaction predicate has at most $m > 0$ off-diagonal neighbours in each row, noninteracting ordered cross terms are non-negative, and interacting pairs satisfy the norm-compression estimate

$$\|P_i P_j v\| \leq (1 - \gamma) \frac{1}{m} \|P_i v\|. \quad (13.30)$$

Then $H = \sum_i P_i$ satisfies $H^2 \geq \gamma H$ as a quadratic form. The same conclusion holds from a separate compression coefficient η whenever $\eta \leq (1 - \gamma)/m$. In particular, if $0 \leq \eta$ and $\eta m < 1$, then $H^2 \geq (1 - \eta m)H$ and $1 - \eta m > 0$.

Proof. [Stmnt](#) Cauchy–Schwarz and symmetric idempotence turn the norm-compression estimate into $\text{Re}\langle P_i v, P_j v \rangle \geq -(1 - \gamma)m^{-1} \text{Re}\langle P_i v, v \rangle$. If a separate coefficient η is used, the assumption $\eta \leq (1 - \gamma)/m$ first weakens that estimate to (13.30). The finite-overlap row-sum reduction then applies. For the strict form, take $\gamma = 1 - \eta m$. $\square$

Definition 13.3.1.14 (Cyclic window and complement indexing). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) If $L \leq N$ and $i \in \{0, \dots, N-1\}$, identify the sites of the periodic chain with the disjoint union of the cyclic window $i, i+1, \dots, i+L-1$ and its ordered complement $i+L, i+L+1, \dots, i+N-1$, with all indices taken modulo N .

Theorem 13.3.1.15 (Cyclic product factorization). [Lean](#) [Stmnt](#) Let M be a commutative monoid. If $L \leq N$, $i \in \{0, \dots, N-1\}$, and $f: \{0, \dots, N-1\} \rightarrow M$, then the product of f over the chain equals the product over the cyclic window beginning at i times the product over its ordered complement.

Proof. [Stmnt](#) Reindex the product by the cyclic window and complement identification of Definition 13.3.1.14, then split the product over the resulting disjoint union. $\square$

Theorem 13.3.1.16 (Three-site anticommutator transport to cyclic adjacent pairs). [Lean](#) [Lean](#) [Stmnt](#) Fix a tensor A with physical dimension $d > 0$ and a coefficient $c \in \mathbb{R}$. For every chain length N and site $i \in \{0, \dots, N-1\}$, set $h_i^{(N)} := h_{i, \text{ES}}^{(N,2)}(A)$, the Euclidean range-two parent term of this same tensor A at i . Suppose that, for every three-site vector w ,

$$-c(\text{Re}\langle h_0^{(3)} w, w \rangle + \text{Re}\langle h_1^{(3)} w, w \rangle) \leq \text{Re}\langle h_0^{(3)} w, h_1^{(3)} w \rangle + \text{Re}\langle h_1^{(3)} w, h_0^{(3)} w \rangle. \quad (13.31)$$

Then, for every $N \geq 3$, every site $i \in \{0, \dots, N-1\}$, and every N -site vector v , the forward pair satisfies

$$-c(\text{Re}\langle h_i^{(N)} v, v \rangle + \text{Re}\langle h_{i+1}^{(N)} v, v \rangle) \leq \text{Re}\langle h_i^{(N)} v, h_{i+1}^{(N)} v \rangle + \text{Re}\langle h_{i+1}^{(N)} v, h_i^{(N)} v \rangle, \quad (13.32)$$

and the backward pair satisfies

$$-c(\text{Re}\langle h_i^{(N)} v, v \rangle + \text{Re}\langle h_{i-1}^{(N)} v, v \rangle) \leq \text{Re}\langle h_i^{(N)} v, h_{i-1}^{(N)} v \rangle + \text{Re}\langle h_{i-1}^{(N)} v, h_i^{(N)} v \rangle. \quad (13.33)$$

All site indices are taken modulo N , so these conclusions include pairs and three-site windows crossing the periodic cut.

Proof. [Stmnt](#) For a cyclic three-site window beginning at i , write $\mathcal{C}_{N-3}$ for its ordered complementary configurations and $R_{i,\tau}^{(3)}$ for restriction with complement τ fixed. Definition 13.3.1.14

identifies every chain configuration uniquely with its window and complement, while the cyclic restriction operators of Remark 12.4.5.2 give the fiber decomposition

$$\operatorname{Re}\langle u, v \rangle = \sum_{\tau \in \mathcal{C}_{N-3}} \operatorname{Re} \langle R_{i,\tau}^{(3)} u, R_{i,\tau}^{(3)} v \rangle. \quad (13.34)$$

The restriction identities in Theorem 13.1.60 intertwine the two adjacent terms:

$$R_{i,\tau}^{(3)} h_i^{(N)} = h_0^{(3)} R_{i,\tau}^{(3)}, \quad R_{i,\tau}^{(3)} h_{i+1}^{(N)} = h_1^{(3)} R_{i,\tau}^{(3)}. \quad (13.35)$$

Applying (13.31) to $w_\tau = R_{i,\tau}^{(3)} v$ and summing over the complement gives

$$\sum_{\tau \in \mathcal{C}_{N-3}} -c(\operatorname{Re}\langle h_0^{(3)} w_\tau, w_\tau \rangle + \operatorname{Re}\langle h_1^{(3)} w_\tau, w_\tau \rangle) \leq \sum_{\tau \in \mathcal{C}_{N-3}} (\operatorname{Re}\langle h_0^{(3)} w_\tau, h_1^{(3)} w_\tau \rangle + \operatorname{Re}\langle h_1^{(3)} w_\tau, h_0^{(3)} w_\tau \rangle), \quad (13.36)$$

By the fiber decomposition (13.34) and restriction intertwining (13.35), this is (13.32); in particular, summation leaves c unchanged. For the backward pair, begin the forward window at $i-1$. Moving one step forward from that site returns i , including when $i=0$, and yields (13.33). $\square$

Lemma 13.3.1.17 (Quadratic-form estimate from the martingale condition). [Lean](#) [Stmt](#) For a fixed chain length N , suppose the Hilbert-space local terms $h_{i,\text{ES}}$ satisfy the ordered row-sum estimate

$$\operatorname{Re}\langle h_{i,\text{ES}} v, h_{j,\text{ES}} v \rangle \geq -(1-\gamma) c_{ij} \operatorname{Re}\langle h_{i,\text{ES}} v, v \rangle \quad (i \neq j), \quad (13.37)$$

with $\sum_{j \neq i} c_{ij} \leq 1$ and $\gamma \leq 1$. Then the Hilbert-space parent Hamiltonian satisfies

$$\gamma \operatorname{Re}\langle H_{N,L}^{\text{ES}} v, v \rangle \leq \operatorname{Re}\langle H_{N,L}^{\text{ES}} v, H_{N,L}^{\text{ES}} v \rangle$$

for every vector v .

Proof. [Lean](#) [Proof](#) Lemma 13.3.9 supplies the symmetric-projection hypothesis for the family $P_i = h_{i,\text{ES}}$. Apply Lemma 13.3.1.9 to this family and use $H_{N,L}^{\text{ES}} = \sum_i h_{i,\text{ES}}$. $\square$

Lemma 13.3.1.18 (Parent-Hamiltonian gap from ordered local row bounds). [Lean](#) [Stmt](#) Assume uniformly for every $N \geq 2L$ that the Hilbert-space local terms satisfy the ordered row-sum estimate (13.37) with $\gamma = 1/(4L)$. If $L > 1$, then $1/(4L) > 0$ and, for every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\frac{1}{4L} \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [Lean](#) [Proof](#) The fixed-chain row-sum reduction supplies the quadratic-form estimate with $\gamma = 1/(4L)$ for every admissible N . The positivity and kernel identification of the Hilbert-space Hamiltonian then allow Lemma 13.3.1.1 to convert this quadratic-form estimate into the norm lower bound on the orthogonal complement of the ground space. $\square$

Lemma 13.3.1.19 (Parent-Hamiltonian gap from the anticommutator martingale condition). [Lean](#) [Lean](#) [Stmt](#) Assume uniformly for every $N \geq 2L$ that the Hilbert-space local terms satisfy the anticommutator martingale estimate

$$\operatorname{Re}\langle h_{i,\text{ES}} v, h_{j,\text{ES}} v \rangle + \operatorname{Re}\langle h_{j,\text{ES}} v, h_{i,\text{ES}} v \rangle \geq -c_{ij} \left(1 - \frac{1}{4L}\right) (\operatorname{Re}\langle h_{i,\text{ES}} v, v \rangle + \operatorname{Re}\langle h_{j,\text{ES}} v, v \rangle)$$

for $i \neq j$, with row and column coefficient sums bounded by one. If $L > 1$, then $1/(4L) > 0$ and, for every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\frac{1}{4L} \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [✓ Proof](#) The fixed-chain anticommutator row-sum reduction supplies the quadratic-form estimate with $\gamma = 1/(4L)$ for every admissible N . The positivity and kernel identification of the Hilbert-space Hamiltonian then convert this quadratic-form estimate into the norm lower bound on the orthogonal complement of the ground space. $\square$

Lemma 13.3.1.20 (Parent-Hamiltonian finite-overlap anticommutator reduction). [Lean](#) [Lean](#)
[✓ Stmt](#) Fix $L > 1$ and set $m = 2(L - 1)$. Suppose uniformly for every $N \geq 2L$ that the overlap relation on local windows has at most m off-diagonal neighbours in each row and in each column. If non-overlapping local terms have non-negative anticommutator form, and overlapping local terms satisfy

$$\operatorname{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle + \operatorname{Re}\langle h_{j,\text{ES}}v, h_{i,\text{ES}}v \rangle \geq -\left(1 - \frac{1}{4L}\right) \frac{1}{m} (\operatorname{Re}\langle h_{i,\text{ES}}v, v \rangle + \operatorname{Re}\langle h_{j,\text{ES}}v, v \rangle),$$

then the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) Apply the finite-overlap anticommutator projection reduction to the Hilbert-space local terms. The resulting quadratic-form estimate has $\gamma = 1/(4L)$, and Lemma 13.3.1.1 converts it to the norm lower bound. $\square$

Lemma 13.3.1.21 (Parent-Hamiltonian finite-overlap martingale quadratic reduction). [Lean](#)
[✓ Stmt](#) For a fixed chain length N , let m be a positive integer and let $\gamma \leq 1$. Suppose an overlap predicate on local windows has at most m overlapping off-diagonal windows in each row. If non-overlapping local terms have non-negative ordered cross terms, and overlapping local terms satisfy the one-sided ordered cross-term estimate

$$\operatorname{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle \geq -(1 - \gamma) \frac{1}{m} \operatorname{Re}\langle h_{i,\text{ES}}v, v \rangle,$$

and the Hilbert-space local terms are symmetric projections, then the Hilbert-space Hamiltonian satisfies $H_{N,L}^{\text{ES}}{}^2 \geq \gamma H_{N,L}^{\text{ES}}$ as a quadratic form.

Proof. [✓ Proof](#) Apply the finite-overlap projection reduction with $P_i = h_{i,\text{ES}}$. $\square$

Lemma 13.3.1.22 (Finite-overlap norm-compression reduction). [Lean](#) [Lean](#) [✓ Stmt](#) For a fixed chain length N , let $m > 0$ and $\gamma \leq 1$. Suppose an overlap predicate on local windows has at most m overlapping off-diagonal windows in each row, non-overlapping local terms have non-negative ordered cross terms, and overlapping local terms satisfy

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq (1 - \gamma) \frac{1}{m} \|h_{i,\text{ES}}v\|.$$

Then $H_{N,L}^{\text{ES}}{}^2 \geq \gamma H_{N,L}^{\text{ES}}$ as a quadratic form. The same conclusion holds from a separate coefficient η satisfying $\eta \leq (1 - \gamma)/m$.

Proof. [✓ Proof](#) Apply the finite-overlap norm-compression projection reduction to the family $P_i = h_{i,\text{ES}}$. If a separate coefficient η is used, first weaken the compression estimate using $\eta \leq (1 - \gamma)/m$. $\square$

13.3.2 Cyclic-window overlap estimates

Definition 13.3.2.1 (Cyclic-window support). [Lean](#) [✓ Stmt](#) For a length- L cyclic window on $\{0, \dots, N-1\}$ starting at i , its support is $S_i = \{i, i+1, \dots, i+L-1\} \pmod{N}$, with repeated visits counted once.

Lemma 13.3.2.2 (Cyclic-window support offsets). [Lean](#) [✓ Stmt](#) If $L \leq N$, then a site k belongs to the cyclic support S_i exactly when its cyclic offset from i is below L : $k \in S_i \iff (k - i) \bmod N < L$.

Proof. [✓ Proof](#) Write $k = i + r \pmod{N}$ for a representative offset r . Membership in S_i gives an offset $r < L$, and conversely an offset below L gives the corresponding point of the support. $\square$

Definition 13.3.2.3 (Cyclic-window overlap). [Lean](#) [✓ Stmt](#) Two length- L cyclic windows overlap, written $i \sim_L j$, when their supports meet: $S_i \cap S_j \neq \emptyset$.

Lemma 13.3.2.4 (Cyclic-window overlap symmetry). [Lean](#) [✓ Stmt](#) For length- L cyclic windows, $i \sim_L j$ if and only if $j \sim_L i$.

Proof. [✓ Proof](#) Both assertions say that the two cyclic supports have a common site. $\square$

Lemma 13.3.2.5 (Concrete non-overlap positivity for cyclic supports). [Lean](#) [Lean](#) [✓ Stmt](#) If $L \leq N$ and the cyclic supports S_i and S_j do not meet, then the Hilbert-space local terms have non-negative ordered cross term: $0 \leq \text{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle$.

Proof. [✓ Proof](#) The support-offset characterization converts failure of $S_i \cap S_j \neq \emptyset$ into site-disjointness of the two cyclic windows. The disjoint-window positivity lemma then applies. $\square$

Lemma 13.3.2.6 (Cyclic-window overlap row cardinality). [Lean](#) [✓ Stmt](#) If $2L \leq N$ and $L > 1$, then, for every $i \in \{0, \dots, N-1\}$, $\#\{j \neq i : i \sim_L j\} \leq 2(L-1)$.

Proof. [✓ Proof](#) If S_i and S_j meet, choose offsets $a, b < L$ with $i + a \equiv j + b \pmod{N}$. Since $2L \leq N$, the offset from i to j is either the clockwise distance $a - b \in \{1, \dots, L-1\}$ or the counterclockwise distance $b - a \in \{1, \dots, L-1\}$; the zero distance is excluded by $j \neq i$. Thus every overlapping off-diagonal start lies among the $L-1$ clockwise starts or the $L-1$ counterclockwise starts. $\square$

Lemma 13.3.2.7 (Parent-Hamiltonian gap from a finite-overlap martingale estimate). [Lean](#) [✓ Stmt](#) Fix $L > 1$ and set $m = 2(L-1)$. Suppose uniformly for every $N \geq 2L$ that the hypotheses of the fixed-chain finite-overlap martingale reduction hold with $\gamma = 1/(4L)$ and this value of m . Then $1/(4L) > 0$, and the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) The fixed-chain finite-overlap reduction gives the required quadratic-form estimate for every admissible N . Lemma 13.3.1.1 converts that estimate to the norm lower bound. $\square$

Lemma 13.3.2.8 (Parent-Hamiltonian gap from a cyclic-window martingale estimate). [Lean](#) [✓ Stmt](#) Fix $L > 1$ and let $i \sim_L j$ mean that the cyclic supports of the length- L windows at i and j meet. Suppose uniformly for every $N \geq 2L$ that overlapping off-diagonal terms satisfy

$$\text{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle \geq -(1 - \frac{1}{4L}) \frac{1}{2(L-1)} \text{Re}\langle h_{i,\text{ES}}v, v \rangle. \quad (13.38)$$

Then $1/(4L) > 0$, and the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) Lemma 13.3.2.6 gives the row-cardinality bound $m = 2(L-1)$ for the overlap relation among length- L local windows, and Lemma 13.3.9 supplies the symmetric projection structure of each Hilbert-space local term. If two cyclic supports do not meet, the corresponding windows are site-disjoint, so Lemma 13.3.2.5 gives the required non-negative ordered cross term. Apply Lemma 13.3.2.7 with these facts and the overlapping-window estimate (13.38). $\square$

Lemma 13.3.2.9 (Parent-Hamiltonian gap from an overlapping cyclic martingale estimate).

[Lean](#) [✓ Stmt](#) Fix $L > 1$ and let $i \sim_L j$ mean that the cyclic supports of the length- L windows at i and j meet. Suppose uniformly for every $N \geq 2L$ that overlapping off-diagonal terms satisfy

$$\operatorname{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle \geq -\left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} \operatorname{Re}\langle h_{i,\text{ES}}v, v \rangle. \quad (13.39)$$

Then the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) Apply Lemma 13.3.2.8. Its non-overlap positivity and finite-family hypotheses are already internal to the preceding reduction; the overlapping estimate (13.39) supplies its remaining ordered cross-term condition. $\square$

Lemma 13.3.2.10 (Parent-Hamiltonian gap from a cyclic anticommutator estimate). [Lean](#)

[✓ Stmt](#) Fix $L > 1$ and let $i \sim_L j$ mean that the cyclic supports of the length- L windows at i and j meet. Suppose uniformly for every $N \geq 2L$ that overlapping off-diagonal terms satisfy

$$\operatorname{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle + \operatorname{Re}\langle h_{j,\text{ES}}v, h_{i,\text{ES}}v \rangle \geq -\left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} (\operatorname{Re}\langle h_{i,\text{ES}}v, v \rangle + \operatorname{Re}\langle h_{j,\text{ES}}v, v \rangle).$$

Then the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) The cyclic overlap relation is symmetric, and Lemma 13.3.2.6 gives at most $2(L-1)$ overlapping off-diagonal windows in each row and column. If two cyclic supports do not meet, Lemma 13.3.2.5 gives non-negative ordered cross terms in both orders. Apply the finite-overlap anticommutator reduction. $\square$

Lemma 13.3.2.11 (Parent-Hamiltonian gap from sufficient cyclic norm compression). [Lean](#)

[✓ Stmt](#) Fix $L > 1$. Suppose uniformly for every $N \geq 2L$ and every overlapping off-diagonal pair that

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} \|h_{i,\text{ES}}v\|.$$

Then the explicit norm lower bound with constant $1/(4L)$ follows for vectors in $(\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$.

Proof. [✓ Proof](#) For an overlapping pair, Cauchy-Schwarz and $h_{i,\text{ES}}^2 = h_{i,\text{ES}}$ give

$$\begin{aligned} \operatorname{Re}\langle h_{i,\text{ES}}v, h_{j,\text{ES}}v \rangle &= \operatorname{Re}\langle h_{i,\text{ES}}v, h_{i,\text{ES}}h_{j,\text{ES}}v \rangle \\ &\geq -\|h_{i,\text{ES}}v\| \|h_{i,\text{ES}}h_{j,\text{ES}}v\| \\ &\geq -\left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} \|h_{i,\text{ES}}v\|^2 \\ &= -\left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} \operatorname{Re}\langle h_{i,\text{ES}}v, v \rangle. \end{aligned}$$

This is the ordered lower bound used by the preceding overlapping-cyclic martingale reduction. $\square$

Lemma 13.3.2.12 (Parent-Hamiltonian gap from a cyclic overlap compression constant). [Lean](#)

[Lean](#) [✓ Stmt](#) Fix $L > 1$, and let $m = 2(L-1)$. Suppose uniformly for every $N \geq 2L$ and every overlapping off-diagonal pair that

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{i,\text{ES}}v\|.$$

If $\gamma > 0$, $\gamma \leq 1$, and $\eta \leq (1 - \gamma)/m$, then $0 < \gamma$ and, for every admissible N and every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\gamma\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. In particular, if $\eta \geq 0$ and $\eta m < 1$, then $0 < 1 - \eta m$ and, for every admissible N and every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $(1 - \eta m)\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. This strict all-vector compression form is a stronger conditional sufficient condition, not the source principal-angle estimate itself. It is a sufficient strengthening of the anticommutator martingale estimate in Cirac–Perez-Garcia–Schuch–Verstraete [CPGSV21, Section IV.C]. It does not prove the remaining MPS-specific estimate; it isolates the finite-row martingale argument once such a strengthening is supplied.

Proof. ✓ Proof The overlap-cardinality lemma gives at most $m = 2(L-1)$ overlapping off-diagonal neighbours in each row, while the non-overlap positivity lemma handles disjoint supports. The finite-overlap norm-compression reduction gives the quadratic-form estimate with the chosen γ , and the positive quadratic-form criterion converts it to the norm lower bound. The strict form is the same statement with $\gamma = 1 - \eta m$. □

Lemma 13.3.2.13 (Parent-Hamiltonian gap from a cyclic operator-product norm constant).

Lean Lean ✓ Stmt Fix $L > 1$, and let $m = 2(L-1)$. Suppose uniformly for every $N \geq 2L$ and every overlapping off-diagonal pair that

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{j,\text{ES}}v\|,$$

that is, a bound $\|h_{i,\text{ES}}h_{j,\text{ES}}\| \leq \eta$ on the operator-product norm. If $\gamma > 0$, $\gamma \leq 1$, and $\eta \leq (1 - \gamma)/m$, then $0 < \gamma$ and, for every admissible N and every $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\gamma\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. In particular, if $\eta \geq 0$ and $\eta m < 1$, then $0 < 1 - \eta m$ and the same norm bound holds with $\gamma = 1 - \eta m$.

This is the symmetric (operator-product) counterpart of Lemma 13.3.2.12, obtained from the anticommutator conversion recorded in Remark 13.3.1.8. Like that lemma it is a conditional reduction: the excitation projections satisfy $\|h_{i,\text{ES}}h_{j,\text{ES}}\| = 1$ whenever the overlapping ranges share a common nonzero vector, so the hypothesis is unsatisfiable for $\eta < 1$ in that case. It isolates the finite-row martingale argument once such a bound is supplied; the gap with the cited principal-angle estimates is documented in the paper-gap note [con26m].

Proof. ✓ Proof The overlap-cardinality lemma gives at most $m = 2(L-1)$ overlapping off-diagonal neighbours in each row, while the non-overlap positivity lemma handles disjoint supports. The symmetric anticommutator conversion of Remark 13.3.1.8 turns the operator-product bound into the per-pair anticommutator estimate with coefficient $(1 - \gamma)/m$, using the hypothesis $\eta \leq (1 - \gamma)/m$; the anticommutator finite-overlap reduction then gives the quadratic-form estimate with the chosen γ , and the positive quadratic-form criterion converts it to the norm lower bound. The strict form is the same statement with $\gamma = 1 - \eta m$, so the coefficient is then η . □

13.3.3 Spectator-indexed boundary maps

The tail and left boundary maps embed virtual-space data into a common ambient $(K + L)$ -site Hilbert space. This places the overlapping ground spaces as ranges of linear maps on the same ambient space, a necessary step before their projectors can be compared in the martingale condition. All statements are exact algebraic identities; no FNW contraction or norm estimate is derived here.

Definition 13.3.3.1 (Prefix restriction). Lean Lean ✓ Stmt For $K, L \in \mathbb{N}$ and a fixed suffix configuration $\tau: \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$, the *prefix restriction* is the $\mathbb{C}$ -linear map $P_\tau: \mathbb{C}^{\{0, \dots, d-1\}^{K+L}} \rightarrow \mathbb{C}^{\{0, \dots, d-1\}^K}$ defined by $(P_\tau\psi)(\sigma) = \psi(\sigma, \tau)$ for $\sigma: \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$. This is the left-sided analogue of the suffix restriction of Definition 12.4.5.5.

Lemma 13.3.3.2 (Prefix restriction of a ground-space vector). [Lean](#) [✓ Stmt](#) For every boundary matrix $X \in M_D(\mathbb{C})$ and every suffix τ ,

$$P_\tau(\Gamma_{K+L}(X)) = \Gamma_K(A^\tau X). \quad (13.40)$$

Proof. [✓ Proof](#) Expand $\Gamma_{K+L}(X)(\sigma, \tau) = \text{tr}(A^{\sigma, \tau} X)$, factor $A^{\sigma, \tau} = A^\sigma A^\tau$ by Lemma 2.1.4, and reassociate: $\text{tr}((A^\sigma A^\tau)X) = \text{tr}(A^\sigma(A^\tau X)) = \Gamma_K(A^\tau X)(\sigma)$. $\square$

Definition 13.3.3.3 (Tail boundary map). [Lean](#) [Lean](#) [✓ Stmt](#) For $K, L \in \mathbb{N}$, the *tail boundary map* is the $\mathbb{C}$ -linear map

$$\Gamma_{K,L}^{\text{tail}} : ((\{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}) \rightarrow M_D(\mathbb{C})) \rightarrow \mathbb{C}^{\{0, \dots, d-1\}^{K+L}} \quad (13.41)$$

defined for every family $Y : (\{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}) \rightarrow M_D(\mathbb{C})$ by

$$\Gamma_{K,L}^{\text{tail}}(Y)(u, \tau) = \text{tr}(A^\tau Y_u), \quad (13.42)$$

where $u : \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$ is the prefix and $\tau : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$ is the suffix. When $Y_u = XA^u$ the output equals the standard $\Gamma_{K+L}(X)$.

Definition 13.3.3.4 (Left boundary map). [Lean](#) [Lean](#) [✓ Stmt](#) For $K, L \in \mathbb{N}$, the *left boundary map* is the $\mathbb{C}$ -linear map

$$\Gamma_{K,L}^{\text{left}} : ((\{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}) \rightarrow M_D(\mathbb{C})) \rightarrow \mathbb{C}^{\{0, \dots, d-1\}^{K+L}} \quad (13.43)$$

defined for every family $Z : (\{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}) \rightarrow M_D(\mathbb{C})$ by

$$\Gamma_{K,L}^{\text{left}}(Z)(u, \tau) = \text{tr}(A^u Z_\tau), \quad (13.44)$$

where $u : \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$ is the prefix and $\tau : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$ is the suffix. When $Z_\tau = A^\tau X$ the output equals the standard $\Gamma_{K+L}(X)$.

Theorem 13.3.3.5 (Tail boundary map on a prefix-suffix pair). [Lean](#) [✓ Stmt](#) For every prefix $u : \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$ and suffix $\tau : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$,

$$\Gamma_{K,L}^{\text{tail}}(Y)(u, \tau) = \Gamma_L(Y_u)(\tau). \quad (13.45)$$

Proof. [✓ Proof](#) Unfold the definition of $\Gamma_{K,L}^{\text{tail}}$ on the pair $(u, \tau) = \{0, \dots, -1\} \text{append } u \tau$, use the decomposition $(\{0, \dots, -1\} \text{append } u \tau) \circ \{0, \dots, -1\} \text{natAdd } K = \tau$ and $(\{0, \dots, -1\} \text{append } u \tau) \circ \{0, \dots, -1\} \text{castAdd } L = u$, and apply $\Gamma_L(Y_u)(\tau) = \text{tr}(A^\tau Y_u)$. $\square$

Theorem 13.3.3.6 (Suffix restriction of the tail boundary map). [Lean](#) [✓ Stmt](#) Restricting the tail boundary map to the suffix for a fixed prefix $u \in \{0, \dots, d-1\}^K$ recovers $\Gamma_L(Y_u)$:

$$R_u^{\text{tail}}(\Gamma_{K,L}^{\text{tail}}(Y)) = \Gamma_L(Y_u). \quad (13.46)$$

Proof. [✓ Proof](#) This is Theorem 13.3.3.5 applied pointwise: for each τ , $(R_u^{\text{tail}}\psi)(\tau) = \psi(u, \tau)$. $\square$

Theorem 13.3.3.7 (Tail boundary map factorization). [Lean](#) [✓ Stmt](#) The standard length- $(K+L)$ boundary map factors through the tail boundary map:

$$\Gamma_{K+L}(X) = \Gamma_{K,L}^{\text{tail}}(u \mapsto XA^u) \quad (13.47)$$

for every $X \in M_D(\mathbb{C})$.

Proof. ✓ Proof Write a configuration $\sigma \in \{0, \dots, d-1\}^{K+L}$ as $\sigma = (u, \tau)$ with $u \in \{0, \dots, d-1\}^K$ and $\tau \in \{0, \dots, d-1\}^L$. Substitute $Y_u = XA^u$ into the definition of $\Gamma_{K,L}^{\text{tail}}$, apply the three-factor trace cycle $\text{tr}(A^\tau(XA^u)) = \text{tr}(A^u(A^\tau X))$, reassociate $\text{tr}(A^u(A^\tau X)) = \text{tr}((A^u A^\tau)X)$, factor $A^u A^\tau = A^{u,\tau}$ by multiplicativity of word evaluation (Lemma 2.1.4), and identify the result with $\Gamma_{K+L}(X)(u, \tau) = \text{tr}(A^{u,\tau} X)$. $\square$

Theorem 13.3.3.8 (Tail boundary map range). Lean ✓ Stmt *The range of the tail boundary map is exactly the tail ground space:*

$$\text{range}(\Gamma_{K,L}^{\text{tail}}) = \{\psi \in \mathbb{C}^{\{0, \dots, d-1\}^{K+L}} \mid \forall u \in \{0, \dots, d-1\}^K, R_u^{\text{tail}} \psi \in G_L(A)\}. \quad (13.48)$$

Proof. ✓ Proof For the forward inclusion, if $\psi = \Gamma_{K,L}^{\text{tail}}(Y)$ then $R_u^{\text{tail}} \psi = \Gamma_L(Y_u)$ for every prefix u , so the suffix restriction lies in $G_L(A)$. Conversely, given ψ with $R_u^{\text{tail}} \psi \in G_L(A)$ for every u , choose for each u a matrix $Y_u \in M_D(\mathbb{C})$ with $\Gamma_L(Y_u) = R_u^{\text{tail}} \psi$. Then $\Gamma_{K,L}^{\text{tail}}(Y)(u, \tau) = \text{tr}(A^\tau Y_u) = \Gamma_L(Y_u)(\tau) = (R_u^{\text{tail}} \psi)(\tau) = \psi(u, \tau)$, so ψ lies in the range. $\square$

Theorem 13.3.3.9 (Left boundary map on a prefix-suffix pair). Lean ✓ Stmt *For every prefix $u: \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$ and suffix $\tau: \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$,*

$$\Gamma_{K,L}^{\text{left}}(Z)(u, \tau) = \Gamma_K(Z_\tau)(u). \quad (13.49)$$

Proof. ✓ Proof Unfold the definition of $\Gamma_{K,L}^{\text{left}}$ on the pair (u, τ) , use $(\{0, \dots, -1\} \text{append } u \tau) \circ \{0, \dots, -1\} \text{castAdd } L = u$ and $(\{0, \dots, -1\} \text{append } u \tau) \circ \{0, \dots, -1\} \text{natAdd } K = \tau$, and apply $\Gamma_K(Z_\tau)(u) = \text{tr}(A^u Z_\tau)$. $\square$

Theorem 13.3.3.10 (Prefix restriction of the left boundary map). Lean ✓ Stmt *Restricting the left boundary map to the prefix for a fixed suffix $\tau \in \{0, \dots, d-1\}^L$ recovers $\Gamma_K(Z_\tau)$:*

$$P_\tau(\Gamma_{K,L}^{\text{left}}(Z)) = \Gamma_K(Z_\tau). \quad (13.50)$$

Proof. ✓ Proof Pointwise application of Theorem 13.3.3.9: for each σ , $(P_\tau \psi)(\sigma) = \psi(\sigma, \tau)$. $\square$

Theorem 13.3.3.11 (Left boundary map factorization). Lean ✓ Stmt *The standard length- $(K+L)$ boundary map factors through the left boundary map:*

$$\Gamma_{K+L}(X) = \Gamma_{K,L}^{\text{left}}(\tau \mapsto A^\tau X) \quad (13.51)$$

for every $X \in M_D(\mathbb{C})$.

Proof. ✓ Proof Substitute $Z_\tau = A^\tau X$ into the definition of $\Gamma_{K,L}^{\text{left}}$, giving $\text{tr}(A^u(A^\tau X)) = \text{tr}((A^u A^\tau)X)$. Factor $A^u A^\tau = A^{u,\tau}$ by Lemma 2.1.4 and identify the result with $\Gamma_{K+L}(X)(u, \tau)$. $\square$

Theorem 13.3.3.12 (Left boundary map range). Lean ✓ Stmt *The range of the left boundary map is exactly the left ground space:*

$$\text{range}(\Gamma_{K,L}^{\text{left}}) = \{\psi \in \mathbb{C}^{\{0, \dots, d-1\}^{K+L}} \mid \forall \tau \in \{0, \dots, d-1\}^L, P_\tau \psi \in G_K(A)\}. \quad (13.52)$$

Proof. ✓ Proof The proof is symmetric to Theorem 13.3.3.8. If $\psi = \Gamma_{K,L}^{\text{left}}(Z)$ then $P_\tau \psi = \Gamma_K(Z_\tau)$, so the prefix restriction lies in $G_K(A)$. Conversely, given ψ with $P_\tau \psi \in G_K(A)$ for every τ , choose Z_τ with $\Gamma_K(Z_\tau) = P_\tau \psi$; then $\Gamma_{K,L}^{\text{left}}(Z)(u, \tau) = \text{tr}(A^u Z_\tau) = \Gamma_K(Z_\tau)(u) = (P_\tau \psi)(u) = \psi(u, \tau)$. $\square$

Lemma 13.3.3.13 (Prefix restriction at $L = 1$). Lean ✓ Stmt *For $L = 1$, the prefix restriction P_τ coincides with the last-site restriction: $P_\tau \psi = R^{\text{last}}(\psi, \tau(0))$ for every $\tau: \{0, \dots, 1-1\} \rightarrow \{0, \dots, d-1\}$.*

Proof. [✓ Proof](#) Unfolding both sides: for $\sigma \in \{0, \dots, d-1\}^K$, $(P_\tau \psi)(\sigma) = \psi(\sigma, \tau)$ while $R^{\text{last}}(\psi, j)(\sigma) = \psi(\sigma_1, \dots, \sigma_K, j)$ with $j = \tau(0)$. These are equal because $\{0, \dots, -1\} \text{append } \sigma \tau$ puts τ after σ , matching the $\{0, \dots, -1\} \text{snoc}$ pattern of R^{last} . $\square$

Theorem 13.3.3.14 (Tail boundary map range: Euclidean-space transport). [Lean](#) [✓ Stmt](#) *For every $K, L \in \mathbb{N}$, transport of the tail boundary map range by the inverse canonical ℓ^2 isomorphism identifies it with the open-chain tail ground submodule:*

$$\iota_{K+L}(\text{range}(\Gamma_{K,L}^{\text{tail}})) = \mathcal{V}_{K,L}(A). \quad (13.53)$$

Concretely, a vector v in the $(K+L)$ -site Euclidean space belongs to $\mathcal{V}_{K,L}(A)$ exactly when the corresponding function satisfies the tail ground condition (Definition 12.4.5.6).

Proof. [✓ Proof](#) By Theorem 13.3.3.8, the range equals the intersection over prefixes u of the ground-space comaps $(R_u^{\text{tail}})^{-1}(G_L(A))$. Transporting this intersection through the canonical ℓ^2 identification and unfolding the definition of $\mathcal{V}_{K,L}(A)$ (Definition 12.4.5.12) gives the stated equality. $\square$

Theorem 13.3.3.15 (Left boundary map range: Euclidean-space transport ($L = 1$)). [Lean](#) [✓ Stmt](#) *At $L = 1$, transport of the left boundary map range by the inverse canonical ℓ^2 isomorphism identifies it with the open-chain left ground submodule:*

$$\iota_{K+1}(\text{range}(\Gamma_{K,1}^{\text{left}})) = \mathcal{U}_K(A). \quad (13.54)$$

Concretely, a vector v in the $(K+1)$ -site Euclidean space belongs to $\mathcal{U}_K(A)$ exactly when the corresponding function satisfies the left ground condition (every fixed-final-site restriction lies in $G_K(A)$; Definition 12.4.1.4).

Proof. [✓ Proof](#) By Theorem 13.3.3.12 with $L = 1$, the range equals the intersection over suffixes $\tau: \{0, \dots, 1-1\} \rightarrow \{0, \dots, d-1\}$ of the comaps $P_\tau^{-1}(G_K(A))$. For $L = 1$, each suffix τ is determined by a single index $j \in \{0, \dots, d-1\}$, and Lemma 13.3.3.13 identifies P_τ with $R^{\text{last}}(\cdot, j)$, the last-site restriction of Definition 12.4.1.2. Transporting the resulting intersection $\bigcap_{j \in \{0, \dots, d-1\}} (R^{\text{last}}(\cdot, j))^{-1}(G_K(A))$ through the canonical ℓ^2 identification and unfolding Definition 12.4.5.12 gives $\mathcal{U}_K(A)$. $\square$

Definition 13.3.3.16 (Euclidean spectator boundary maps). [Lean](#) [Lean](#) [✓ Stmt](#) Transport the tail and left boundary maps through the canonical Euclidean identifications on their virtual matrix families and physical coefficient spaces. Denote the resulting continuous maps by $\Gamma_{K,L}^{\text{tail}}$ and $\Gamma_{K,L}^{\text{left}}$.

Theorem 13.3.3.17 (Euclidean spectator boundary ranges and factorizations). [Lean](#) [Lean](#) [✓ Stmt](#) *Equip each virtual matrix fiber with the unweighted Hilbert–Schmidt inner product and flatten the finite family of fibers into a Euclidean space. The resulting continuous tail and left boundary maps have ranges $\mathcal{V}_{K,L}(A)$ and, at suffix length one, $\mathcal{U}_K(A)$. Moreover, if $J_K^{\text{tail}}(X)_u = XA^u$ and $J_L^{\text{left}}(X)_\tau = A^\tau X$, then*

$$\Gamma_{K,L}^{\text{tail}} J_K^{\text{tail}} = \Gamma_{K+L}, \quad \Gamma_{K,L}^{\text{left}} J_L^{\text{left}} = \Gamma_{K+L}. \quad (13.55)$$

Proof. [✓ Proof](#) Transport the algebraic range descriptions through the Euclidean identification. The two factorization identities follow by evaluating a virtual matrix X fiberwise and applying the corresponding algebraic tail or left factorization. $\square$

Theorem 13.3.3.18 (Injectivity of the left spectator boundary map). [Lean](#) [✓ Stmt](#) *If the length- K ground-space map is injective, then the left boundary map with any spectator length L is injective.*

Proof. [✓ Proof](#) Recover every spectator fiber by restricting to its fixed suffix and use injectivity of the length- K ground-space map. $\square$

Theorem 13.3.3.19 (Direct-sum spectator Gram operators). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The tail Gram is the direct sum over prefix configurations of copies of $\mathcal{K}_L$, and the left Gram is the direct sum over suffix configurations of copies of $\mathcal{K}_K$. Whenever the corresponding local boundary map is injective, the inverse Gram is the direct sum of the local inverse Grams. These identities include empty spectator index types and assert no norm estimate.*

Proof. [✓ Proof](#) Reindex the physical inner product by prefix-suffix configurations and evaluate each fiber. This gives the direct-sum Gram identities. Composing fiberwise with the local inverse Gram on either side gives the stated inverse formulas. $\square$

Theorem 13.3.3.20 (Norm of a fiberwise boundary operator). [Lean](#) [✓ Stmt](#) *Let S be a finite spectator index set, let V be the Euclidean virtual matrix space, and let $G : V \rightarrow V$ be linear. If $G^{\oplus S}$ acts on a boundary family $x = (x_s)_{s \in S}$ by $(G^{\oplus S}x)_s = Gx_s$, then*

$$\|G^{\oplus S}\| \leq \|G\|. \quad (13.56)$$

This estimate also holds when S is empty.

Proof. [✓ Proof](#) Square both sides of the pointwise estimate $\|Gx_s\| \leq \|G\|\|x_s\|$, sum over $s \in S$, and use the Euclidean direct-sum norm identity. $\square$

Theorem 13.3.3.21 (Exact C3 projector residual identity). [Lean](#) [✓ Stmt](#) *Fix $K, L_0 \in \mathbb{N}$. Suppose that the tail boundary map for $(K, L_0 + 1)$, the left boundary map for $(K + L_0, 1)$, and Γ_{K+L_0+1} are injective. The difference between the product of the first two inverse-Gram range projectors and the third inverse-Gram range projector is exactly the common-factorization residual obtained from the tail and left virtual maps. Their ranges are, respectively, $\mathcal{V}_{K, L_0+1}(A)$, $\mathcal{U}_{K+L_0}(A)$, and $\mathcal{G}_{K+L_0+1}(A)$. This identity asserts no numerical contraction estimate.*

Proof. [✓ Proof](#) Apply the common-factorization projector identity to the two spectator boundary maps, the full boundary map, and their tail and left virtual maps. The preceding range equalities identify the three physical ranges. $\square$

Theorem 13.3.3.22 (Arbitrary-increment mixed Gram of the overlapping spectator boundary maps). [Lean](#) [Lean](#) [✓ Stmt](#) *Let $q \in \mathbb{N}$, let $X = (X_u)_{u \in \{0, \dots, d-1\}^K}$, and let $Z = (Z_j)_{j \in \{0, \dots, d-1\}^q}$ be virtual boundary-matrix families. After the canonical identification of $K + (l + q)$ with $(K + l) + q$, the mixed Gram for the tail window of length $l + q$ and the left window with a q -site increment is*

$$\langle X, (\Gamma_{K, l+q}^{\text{tail}})^{\dagger} \Gamma_{K+l, q}^{\text{left}} Z \rangle = \sum_{u \in \{0, \dots, d-1\}^K} \sum_{j \in \{0, \dots, d-1\}^q} \langle U(A^j X_u), \mathcal{K}_l U(Z_j A^u) \rangle, \quad (13.57)$$

where U is Frobenius vectorization and $\mathcal{K}_l = \Gamma_l^{\dagger} \Gamma_l$. The multiplication order is fixed by $\text{tr}(A^{\tau, j} X_u) = \text{tr}(A^{\tau} (A^j X_u))$ and $\text{tr}(A^{u, \tau} Z_j) = \text{tr}(A^{\tau} (Z_j A^u))$. The one-site identity is the specialization $q = 1$. This is the exact algebraic whole-increment identity relevant to condition C3' in [Nac96, Equation (2.5)]; it does not prove the norm estimate, uniformity in q , or any numerical contraction required there.

Proof. [✓ Proof](#) Split a physical configuration into a prefix u , an overlap word τ of length l , and an increment configuration j of length q . The adjoint relation turns the mixed Gram into the physical inner product of the two boundary vectors. For each (u, τ, j) , word multiplicativity and trace cyclicity give

$$\mathrm{tr}(A^{\tau,j}X_u) = \mathrm{tr}(A^\tau(A^jX_u)), \quad \mathrm{tr}(A^{u,\tau}Z_j) = \mathrm{tr}(A^\tau(Z_jA^u)). \quad (13.58)$$

For fixed (u, j) , summing the resulting inner products over τ is exactly the boundary-map Gram pairing $\langle U(A^jX_u), \mathcal{K}_l U(Z_jA^u) \rangle$. Summing over u and j proves the arbitrary-increment identity; setting $q = 1$ gives the original formula. $\square$

Theorem 13.3.3.23 (Centered mixed Gram at the limiting metric). [Lean](#) [✓ Stmt](#) *Specialize the preceding theorem to $q = 1$. Let $X = (X_u)_{u \in \{0, \dots, d-1\}^\kappa}$ and $Z = (Z_j)_{j \in \{0, \dots, d-1\}^1}$ be the resulting boundary families. If $\rho \in M_D(\mathbb{C})$ has nonzero trace and $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$, then*

$$\langle X, (\Gamma_{K,l+1}^{\mathrm{tail}})^\dagger \Gamma_{K+l,1}^{\mathrm{left}} Z \rangle - \sum_{u,j} \langle U(A^jX_u), \mathcal{K}_\infty U(Z_jA^u) \rangle \quad (13.59)$$

$$= \sum_{u,j} \langle U(A^jX_u), (\mathcal{K}_l - \mathcal{K}_\infty) U(Z_jA^u) \rangle. \quad (13.60)$$

Here $\mathcal{K}_\infty U(Z_jA^u) = U((\mathrm{tr} \rho)^{-1} Z_j A^u \rho)$, so the limiting multiplication is on the right and is generally not the identity.

Proof. [✓ Proof](#) Substitute the exact mixed-Gram formula and the right-multiplication action of $\mathcal{K}_\infty$, then distribute subtraction through the two finite sums and the second argument of the inner product. $\square$

Theorem 13.3.3.24 (Limiting metric inverse and virtual-map adjoints). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *For arbitrary virtual dimension, the adjoints of the tail and left virtual maps are*

$$(J_K^{\mathrm{tail}})^\dagger (X_u)_u = U \left(\sum_u X_u (A^u)^\dagger \right), \quad (J_L^{\mathrm{left}})^\dagger (Z_\tau)_\tau = U \left(\sum_\tau (A^\tau)^\dagger Z_\tau \right). \quad (13.61)$$

If, in addition, $D > 0$ and $\rho \in M_D(\mathbb{C})$ is positive definite, set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and $S_\infty = \mathcal{K}_\infty^{-1}$. Then

$$\mathcal{K}_\infty U(X) = U((\mathrm{tr} \rho)^{-1} X \rho), \quad S_\infty U(X) = U((\mathrm{tr} \rho) X \rho^{-1}). \quad (13.62)$$

Proof. [✓ Proof](#) For the tail adjoint, test against an arbitrary matrix Y and use the Hilbert–Schmidt trace pairing fiberwise:

$$\sum_u \langle U(YA^u), U(X_u) \rangle = \left\langle U(Y), U \left(\sum_u X_u (A^u)^\dagger \right) \right\rangle. \quad (13.63)$$

The left formula follows in the same way from $\langle U(A^\tau Y), U(Z_\tau) \rangle = \langle U(Y), U((A^\tau)^\dagger Z_\tau) \rangle$.

Under the additional positive-definiteness hypothesis, the forward action of $\mathcal{K}_\infty$ is the exact limiting-metric formula and the operator is invertible. Applying the forward formula to $U((\mathrm{tr} \rho) X \rho^{-1})$ cancels both scalar factors and the two right matrix factors, which gives the displayed inverse action. $\square$

Theorem 13.3.3.25 (Exact limiting-metric telescope for the C3 residual). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\rho \in M_D(\mathbb{C})$ be positive definite and set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and $S_\infty = \mathcal{K}_\infty^{-1}$. Write J_t and J_l for the tail and left virtual maps, and write $S_n = \mathcal{K}_n^{-1}$ whenever the length- n boundary map is injective. The virtual residual in the exact tail-after-left C3 projector factorization has the telescoping decomposition

$$M_l - \mathcal{K}_{l+1}^{\oplus K} J_t S_{K+l+1} J_l^\dagger \mathcal{K}_{K+l}^{\oplus 1} \quad (13.64)$$

$$= (M_l - \mathcal{K}_\infty^{\oplus K} J_t S_\infty J_l^\dagger \mathcal{K}_\infty^{\oplus 1}) \quad (13.65)$$

$$- (\mathcal{K}_{l+1}^{\oplus K} - \mathcal{K}_\infty^{\oplus K}) J_t S_{K+l+1} J_l^\dagger \mathcal{K}_{K+l}^{\oplus 1} \quad (13.66)$$

$$- \mathcal{K}_\infty^{\oplus K} J_t (S_{K+l+1} - S_\infty) J_l^\dagger \mathcal{K}_{K+l}^{\oplus 1} \quad (13.67)$$

$$- \mathcal{K}_\infty^{\oplus K} J_t S_\infty J_l^\dagger (\mathcal{K}_{K+l}^{\oplus 1} - \mathcal{K}_\infty^{\oplus 1}), \quad (13.68)$$

where $M_l = (\Gamma_{K,l+1}^{\text{tail}})^\dagger \Gamma_{K+l,1}^{\text{left}}$ and the superscript $\oplus$ denotes the corresponding fiberwise spectator operator. Substituting this equality into the exact inverse-Gram factorization gives the same decomposition of the physical projector residual, sandwiched between the tail boundary map, the two local fiberwise inverse Grams, and the adjoint left boundary map.

The limiting product in the first parenthesis has, without an additional hypothesis, the explicit limiting mixed pairing

$$\langle X, \mathcal{K}_\infty^{\oplus K} J_t S_\infty J_l^\dagger \mathcal{K}_\infty^{\oplus 1} Z \rangle \quad (13.69)$$

$$= \sum_{u,j} \langle U(A^j X_u), \mathcal{K}_\infty U(Z_j A^u) \rangle. \quad (13.70)$$

Consequently, pairing the first parenthesis against boundary families is exactly the same finite sum with $\mathcal{K}_l - \mathcal{K}_\infty$ in place of $\mathcal{K}_l$.

These identities are reconstructed operator algebra. They assert neither the numerical FNW/Nachtergaele contraction nor any identification of a physical ground projector with P_ρ .

Proof. [✓ Proof](#) Add and subtract the product in which all three finite Gram factors are replaced by their limiting values, and telescope the resulting product from left to right. Substitute the direct-sum Gram and inverse-Gram formulas into the exact projector factorization.

It remains to identify the leading limiting product. In Frobenius coordinates the limiting metric and its inverse act by

$$\mathcal{K}_\infty U(X) = U((\text{tr } \rho)^{-1} X \rho), \quad S_\infty U(X) = U((\text{tr } \rho) X \rho^{-1}). \quad (13.71)$$

The adjoint left virtual map sends a family $W = (W_j)_j$ to $U(\sum_j (A^j)^\dagger W_j)$. For $W_j = (\text{tr } \rho)^{-1} Z_j \rho$, the right multiplication by ρ and the factor $(\text{tr } \rho)^{-1}$ supplied by the left limiting spectator metric cancel exactly through S_∞ . Applying the tail virtual map and the tail limiting spectator metric leaves, in the fiber indexed by u ,

$$U \left((\text{tr } \rho)^{-1} \sum_j (A^j)^\dagger Z_j A^u \rho \right). \quad (13.72)$$

Its inner product with X_u , followed by summation over u , is the displayed limiting mixed pairing by the Frobenius trace identity. No transfer fixed-point equation is needed for this zeroth-order cancellation. The centered mixed-Gram identity then turns the leading term into the pairing with $\mathcal{K}_l - \mathcal{K}_\infty$. $\square$

13.3.4 Martingale gap consequences

Theorem 13.3.4.1 (Martingale criterion for a finite family of projections). Lean ✓ Stmt Let P_i be a finite family of symmetric projections on a finite-dimensional complex Hilbert space, let $H = \sum_i P_i$, and let $0 < \gamma \leq 1$. Suppose that a real coefficient matrix (c_{ij}) has row and column sums at most one away from the diagonal, and that, for $i \neq j$ and every v ,

$$\operatorname{Re}\langle P_i v, P_j v \rangle + \operatorname{Re}\langle P_j v, P_i v \rangle \quad (13.73)$$

$$\geq -(1 - \gamma) c_{ij} (\operatorname{Re}\langle P_i v, v \rangle + \operatorname{Re}\langle P_j v, v \rangle). \quad (13.74)$$

Then

$$\gamma \operatorname{Re}\langle H v, v \rangle \leq \operatorname{Re}\langle H v, H v \rangle$$

for every v . Consequently, for every $v \in (\ker H)^\perp$,

$$\gamma \|v\| \leq \|H v\|.$$

Proof. ✓ Proof The row-and-column reduction gives the quadratic-form inequality. The sum of symmetric projections is positive, so the abstract martingale criterion converts that inequality into the norm lower bound on $(\ker H)^\perp$. $\square$

Theorem 13.3.4.2 (Reduced ground-space overlap bounds the excitation anticommutator).

Lean ✓ Stmt Let U and V be ground-space ranges in a finite-dimensional complex Hilbert space, let $W = U \cap V$, and let q_U and q_V be the orthogonal projectors onto U and V . Suppose that $\eta \geq 0$ and

$$|\langle x, y \rangle| \leq \eta \|x\| \|y\| \quad (13.75)$$

whenever $x \in U \cap W^\perp$ and $y \in V \cap W^\perp$. Define the complementary excitation projections by $h_U = \mathbb{1} - q_U$ and $h_V = \mathbb{1} - q_V$. Then, for every vector v ,

$$\operatorname{Re}\langle h_U v, h_V v \rangle + \operatorname{Re}\langle h_V v, h_U v \rangle \geq -\eta (\operatorname{Re}\langle h_U v, v \rangle + \operatorname{Re}\langle h_V v, v \rangle). \quad (13.76)$$

Proof. ✓ Proof Remove the component of v in the common intersection W ; both excitation projections are unchanged. If $\eta \leq 1$, apply (13.75) to the reduced ground-space components $q_U v$ and $q_V v$, and combine the resulting principal-angle estimate with the orthogonal decompositions $v = q_U v + h_U v = q_V v + h_V v$. If $\eta > 1$, the same conclusion follows from the universal overlap bound with coefficient 1 and the non-negativity of the two diagonal excitation forms. $\square$

13.3.5 The compatible open-chain Hamiltonian

For an interval, the finite-volume Hamiltonian contains only translates whose supports remain inside the interval. This is the compatible open chain of Nachtergaele [Nac96, Equation (3.12)], whose zero-energy space is identified in Equation (3.13). Equations (3.14)–(3.16) continue with comparison of compatible interactions, approximate commutation of ground-state projectors, and a uniform gap. Theorem 2.1(i) of the same source obtains a gap from its three stated hypotheses. Every compatible Hamiltonian below is positive because it is a sum of orthogonal projections. In addition to the compatible Hamiltonian and its zero-energy space, the statements below verify only the unit norm lower bound at the full-window volume $N = L$. They give no gap estimate for $N > L$ and no periodic coercivity.

Definition 13.3.5.1 (Compatible open-chain parent Hamiltonian). [Lean](#) [Lean](#) [✓ Stmt](#) For $L, N \in \mathbb{N}$, let

$$I_{L,N} = \{i \in \{0, \dots, N-1\} \mid i + L \leq N\}. \quad (13.77)$$

The compatible length- N open-chain parent Hamiltonian of interaction length L is

$$H_{N,L}^{\text{open}}(A) = \sum_{i \in I_{L,N}} h_{i,\text{ES}}(A, L). \quad (13.78)$$

Thus no summand crosses the two endpoints of the interval. For $L = l + 1$, the starting sites are $0, \dots, N - l - 1$, in agreement with [\[Nac96, Equation \(3.12\)\]](#).

Theorem 13.3.5.2 (Full-window compatible interaction). [Lean](#) [✓ Stmt](#) If $L > 0$, then

$$H_{L,L}^{\text{open}}(A) = P_L^{\text{ES}}(A). \quad (13.79)$$

Proof. [✓ Proof](#) The interaction has one window:

$$I_{L,L} = \{0\}. \quad (13.80)$$

The outside region is empty, so it has a unique configuration τ . Cyclic restriction to the sole window is the identity:

$$R_{0,\tau}v = v. \quad (13.81)$$

Likewise,

$$\sigma|_{[0,L-1]} = \sigma. \quad (13.82)$$

Therefore

$$H_{L,L}^{\text{open}}(A) = \sum_{i \in I_{L,L}} h_{i,\text{ES}}(A, L) = h_{0,\text{ES}}(A, L) = P_L^{\text{ES}}(A). \quad (13.83)$$

□

Theorem 13.3.5.3 (Unit gap at the full-window volume). [Lean](#) [Lean](#) [✓ Stmt](#) If $L > 0$ and $v \in (\ker H_{L,L}^{\text{open}}(A))^\perp$, then

$$\|H_{L,L}^{\text{open}}(A)v\| = \|v\|. \quad (13.84)$$

In particular,

$$\|v\| \leq \|H_{L,L}^{\text{open}}(A)v\|. \quad (13.85)$$

Thus $\gamma_{l+1} = 1$ at the full-window volume $N = L = l + 1$ in [\[Nac96, Theorem 2.1\(i\)\]](#). This neither asserts that the excited subspace is nonzero nor supplies a gap for any $N > L$.

Proof. [✓ Proof](#) The full-window identity and the local kernel formula give

$$\ker H_{L,L}^{\text{open}}(A) = \ker P_L^{\text{ES}}(A) = G_L^{\text{ES}}(A). \quad (13.86)$$

Hence v belongs to $(G_L^{\text{ES}}(A))^\perp$, where the orthogonal projector $P_L^{\text{ES}}(A)$ acts as the identity. Thus

$$\|H_{L,L}^{\text{open}}(A)v\| = \|P_L^{\text{ES}}(A)v\| = \|v\|. \quad (13.87)$$

Consequently,

$$\|v\| \leq \|H_{L,L}^{\text{open}}(A)v\|. \quad (13.88)$$

□

Lemma 13.3.5.4 (Positivity of the compatible open-chain Hamiltonian). [Lean](#) [✓ Stmt](#) For every A , L , and N , the operator $H_{N,L}^{\text{open}}(A)$ is positive.

Proof. [✓ Proof](#) Each $h_{i,\text{ES}}(A, L)$ is an orthogonal projection and hence is positive. Positivity is preserved by finite sums. $\square$

Theorem 13.3.5.5 (Zero energy forces every compatible local constraint). [Lean](#) [✓ Stmt](#) If $H_{N,L}^{\text{open}}(A)v = 0$, then, for every $i \in I_{L,N}$,

$$h_{i,\text{ES}}(A, L)v = 0. \quad (13.89)$$

Proof. [✓ Proof](#) Taking the inner product of the sum with v gives

$$\sum_{i \in I_{L,N}} \text{Re} \langle h_{i,\text{ES}}(A, L)v, v \rangle = \text{Re} \langle H_{N,L}^{\text{open}}(A)v, v \rangle = 0. \quad (13.90)$$

Each summand is the squared norm of the image of v under an orthogonal projection. Hence every local term annihilates v . $\square$

Theorem 13.3.5.6 (Restriction of an open-boundary MPS vector). [Lean](#) [✓ Stmt](#) Assume $s+L \leq N$. For every boundary matrix X and complementary configuration τ ,

$$\text{Res}_{s,L}^{\tau} \Gamma_N^A(X) \in G_L(A). \quad (13.91)$$

Proof. [✓ Proof](#) For the fixed words

$$w_L = (\tau_0, \dots, \tau_{s-1}), \quad (13.92)$$

and

$$w_R = (\tau_{s+L}, \dots, \tau_{N-1}), \quad (13.93)$$

the word of a configuration obtained by inserting the window word ω is

$$A^{w_L \omega w_R} = A^{w_L} A^{\omega} A^{w_R}. \quad (13.94)$$

Cyclicity of the trace gives

$$\text{tr}(A^{w_L} A^{\omega} A^{w_R} X) = \text{tr}(A^{\omega} A^{w_R} X A^{w_L}), \quad (13.95)$$

and hence

$$\text{Res}_{s,L}^{\tau} \Gamma_N^A(X) = \Gamma_L^A(A^{w_R} X A^{w_L}) \in G_L(A). \quad (13.96)$$

$\square$

Theorem 13.3.5.7 (Open-boundary vectors have zero compatible energy). [Lean](#) [✓ Stmt](#) For every A , L , and N ,

$$G_N^{\text{ES}}(A) \subseteq \ker H_{N,L}^{\text{open}}(A). \quad (13.97)$$

Proof. ✓ Proof Let $v = \iota_N(\Gamma_N^A(X)) \in G_N^{\text{ES}}(A)$. For every non-wrapping window $[s, s+L-1]$ and complementary configuration τ , Theorem 13.3.5.6 gives

$$\text{Res}_{s,L}^\tau \Gamma_N^A(X) \in G_L(A). \quad (13.98)$$

The non-wrapping identification and Lemma 12.3.2 then give

$$R_{s,\tau}v = \iota_L(\text{Res}_{s,L}^\tau \Gamma_N^A(X)) \in G_L^{\text{ES}}(A). \quad (13.99)$$

The local kernel criterion therefore gives

$$h_{s,\text{ES}}(A, L)v = 0. \quad (13.100)$$

Thus every compatible local term annihilates v . □

Theorem 13.3.5.8 (Compatible zero energy has open-boundary form). Lean ✓ Stmt Assume $D \geq 1$, $L_0 > 0$, and $N \geq L_0 + 1$. If A is L_0 -block-injective, then

$$\ker H_{N, L_0+1}^{\text{open}}(A) \subseteq G_N^{\text{ES}}(A). \quad (13.101)$$

Proof. ✓ Proof In the following display, s ranges over all starts satisfying $s + L_0 + 1 \leq N$, and τ ranges over all complementary configurations. The three ingredients give

$$H_{N, L_0+1}^{\text{open}}(A)v = 0 \implies (\forall s, h_{s,\text{ES}}(A, L_0 + 1)v = 0) \implies (\forall s, \tau, \text{Res}_{s, L_0+1}^\tau v \in G_{L_0+1}^{\text{ES}}(A)) \implies v \in G_N^{\text{ES}}(A). \quad (13.102)$$

The middle implication is the local kernel criterion, and the last is the open-chain intersection theorem. □

Theorem 13.3.5.9 (Ground-state property of the compatible open chain). Lean ✓ Stmt Assume $D \geq 1$, $L_0 > 0$, and $N \geq L_0 + 1$. If A is L_0 -block-injective, then

$$\ker H_{N, L_0+1}^{\text{open}}(A) = G_N^{\text{ES}}(A). \quad (13.103)$$

This is the canonical MPS instance of the ground-state property in [Nac96, Equation (3.13)]. The right-hand side is the open boundary-condition space and may therefore be degenerate.

Proof. ✓ Proof The two preceding inclusions identify each space as a subspace of the other, so antisymmetry gives the equality. □

Theorem 13.3.5.10 (Nachtergaele open-chain projector defect). Lean ✓ Stmt Assume $D \geq 1$ and let A be L_0 -block-injective with $L_0 > 0$. Set $n = K + L_0$ and $l = L_0$, and write

$$S = \mathcal{U}_{K+L_0}(A), \quad T = \mathcal{V}_{K, L_0+1}(A).$$

Thus S and T are the left and tail ground spaces in the Hilbert space on Λ_{n+1} . Let $q_S = P_{S^\perp}$ and $q_T = P_{T^\perp}$ be the complementary excitation projections. If

$$\|G_{\Lambda_{n+1} \setminus \Lambda_{n-l}} G_{\Lambda_n} - G_{\Lambda_{n+1}}\| \leq \eta,$$

then every vector v in the Hilbert space on Λ_{n+1} satisfies

$$-\eta(\text{Re}\langle q_T v, v \rangle + \text{Re}\langle q_S v, v \rangle) \leq \text{Re}\langle q_T v, q_S v \rangle + \text{Re}\langle q_S v, q_T v \rangle.$$

Proof. ✓ Proof The open-chain intersection theorem identifies $S \cap T$ with the ground space on Λ_{n+1} . The projector-defect adapter gives a Friedrichs overlap bound, and the reduced two-projection theorem gives the stated anticommutator inequality. $\square$

Theorem 13.3.5.11 (Uniform tail virtual-map bound). Lean Lean ✓ Stmt *Let A be a primitive MPS tensor with nonzero virtual dimension, and let $J_K = J_K^{\text{tail}}$ be its length- K tail virtual map. Then there is a constant $C > 0$ such that, for every $K \in \mathbb{N}$,*

$$\|J_K\| \leq C. \quad (13.104)$$

No positive-definiteness hypothesis on the fixed point is required.

Proof. ✓ Proof Reconstruct the norm infrastructure from the exact adjoint-self identity

$$J_K^\dagger J_K U(X) = U(X \mathcal{E}_A^K(\mathbf{1})). \quad (13.105)$$

The Hilbert–Schmidt product estimate then gives the fourth-power bound

$$\|J_K\|^4 \leq \sum_{i,j} |\mathcal{E}_A^K(\mathbf{1})_{ij}|^2. \quad (13.106)$$

For $K \geq 1$, decompose the transfer power into its fixed-point projection and the K -th power of the complementary map. The complementary powers are geometrically bounded by Theorem 12.3.20, while the entrywise mass is controlled by the L^2 operator norm with the virtual dimension factor from Theorem 12.3.15. The case $K = 0$ is bounded directly. This first gives a uniform bound for $\|J_K\|^4$ and hence a strictly positive uniform bound for $\|J_K\|$. No FNW or Nachtergaele contraction estimate enters. $\square$

Theorem 13.3.5.12 (Left-canonical normalization of fixed-length words). Lean ✓ Stmt *If A is left-canonical, then for every $Q \in \mathbb{N}$,*

$$\sum_{j \in \Omega_Q} (A^j)^\dagger A^j = \mathbf{1}, \quad (13.107)$$

where $\Omega_Q = \{0, \dots, (-1\}d^Q$ is the set of physical words of length Q .

Proof. ✓ Proof Induct on Q . Split a word of length $Q + 1$ into its first letter and its remaining suffix, apply the induction hypothesis to the suffix sum, and then use left-canonicity for the outer one-site sum. $\square$

Theorem 13.3.5.13 (Whole-increment left virtual norm). Lean Lean ✓ Stmt *If A is left-canonical, then the length- Q left virtual map J_Q^{left} preserves the Hilbert–Schmidt norm. In particular,*

$$\|J_Q^{\text{left}} X\| = \|X\|, \quad (13.108)$$

$$\|J_Q^{\text{left}}\| \leq 1. \quad (13.109)$$

Proof. ✓ Proof Expand the squared norm as a sum over $j \in \Omega_Q$. The identity from Theorem 13.3.5.12 reduces the sum to the Hilbert–Schmidt norm of X . The operator-norm bound follows immediately. $\square$

Definition 13.3.5.14 (Physical reassociation). Lean ✓ Stmt For $K, L, Q \in \mathbb{N}$, let

$$\mathcal{A}_{K,L,Q}: \ell^2(\Omega_{K+(L+Q)}) \longrightarrow \ell^2(\Omega_{(K+L)+Q}) \quad (13.110)$$

be the unitary coordinate map induced by associativity of addition.

Theorem 13.3.5.15 (Reassociation of the ground-space boundary map). [Lean](#) [✓ Stmt](#) For every $K, L, Q \in \mathbb{N}$,

$$\mathcal{A}_{K,L,Q} \Gamma_{K+(L+Q)}^{\text{ES}} = \Gamma_{K+L+Q}^{\text{ES}}. \quad (13.111)$$

Proof. [✓ Proof](#) Evaluate both sides at a physical word. Reassociation changes only the parenthesization of its coordinate set, so both coefficients are the trace pairing with the same word product. $\square$

Definition 13.3.5.16 (Reassociated tail boundary map). [Lean](#) [✓ Stmt](#) Define the tail boundary map in the common $(K + L + Q)$ -site coordinates by

$$\tilde{\Gamma}_{K;L,Q}^{\text{tail}} := \mathcal{A}_{K,L,Q} \Gamma_{K,L+Q}^{\text{tail}}. \quad (13.112)$$

Theorem 13.3.5.17 (Factorization of the reassociated tail map). [Lean](#) [✓ Stmt](#) The reassociated tail map satisfies

$$\tilde{\Gamma}_{K;L,Q}^{\text{tail}} J_K^{\text{tail}} = \Gamma_{K+L+Q}^{\text{ES}}. \quad (13.113)$$

Proof. [✓ Proof](#) Substitute the definition of the reassociated tail map, use the ordinary tail factorization, and then reassociate the full boundary map. $\square$

Theorem 13.3.5.18 (Whole-increment inverse-Gram projector residual). [Lean](#) [✓ Stmt](#) Let $T = \tilde{\Gamma}_{K;L,Q}^{\text{tail}}$, $S = \Gamma_{K+L,Q}^{\text{left}}$, and $C = \Gamma_{K+L+Q}^{\text{ES}}$. Assume separately that T , S , and C are injective. Since $T J_K^{\text{tail}} = C = S J_Q^{\text{left}}$, the difference $P_T P_S - P_C$ equals the exact inverse-Gram residual obtained from these two factorizations.

Proof. [✓ Proof](#) Apply the common-factorization range-projector identity to T , S , and C , using the two displayed factorizations. No canonicity or norm estimate is used. $\square$

Theorem 13.3.5.19 (Whole-increment mixed Gram after reassociation). [Lean](#) [✓ Stmt](#) With the notation of Theorem 13.3.3.22,

$$\langle X, (\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \Gamma_{K+L,Q}^{\text{left}} Z \rangle = \sum_{u \in \Omega_K} \sum_{j \in \Omega_Q} \langle U(A^j X_u), \mathcal{K}_L U(Z_j A^u) \rangle. \quad (13.114)$$

Proof. [✓ Proof](#) Expand the reassociated tail map. Its adjoint introduces $\mathcal{A}_{K,L,Q}^{-1}$, so the preceding coordinate formula applies. $\square$

Theorem 13.3.5.20 (Centered whole-increment mixed Gram). [Lean](#) [✓ Stmt](#) Let $\rho \in M_D(\mathbb{C})$ have nonzero trace and set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$. Then

$$\langle X, (\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \Gamma_{K+L,Q}^{\text{left}} Z \rangle - \sum_{u,j} \langle U(A^j X_u), \mathcal{K}_\infty U(Z_j A^u) \rangle \quad (13.115)$$

$$= \sum_{u,j} \langle U(A^j X_u), (\mathcal{K}_L - \mathcal{K}_\infty) U(Z_j A^u) \rangle. \quad (13.116)$$

Proof. [✓ Proof](#) Substitute the reassociated mixed-Gram formula and the action of the limiting metric, then distribute subtraction over the finite sums. $\square$

Definition 13.3.5.21 (Whole-increment left overlap factor). [Lean](#) [✓ Stmt](#) For $X = (X_u)_{u \in \Omega_K}$, define

$$(F_{K,Q} X)_{u,j} := A^j X_u. \quad (13.117)$$

Definition 13.3.5.22 (Whole-increment right overlap factor). [Lean](#) [✓ Stmt](#) For $Z = (Z_j)_{j \in \Omega_Q}$, define

$$(G_{K,Q}Z)_{u,j} := Z_j A^u. \quad (13.118)$$

Theorem 13.3.5.23 (Fiber formula for the whole-increment left factor). [Lean](#) [✓ Stmt](#) The matrix fiber of $F_{K,Q}X$ at (u, j) is $A^j X_u$.

Proof. [✓ Proof](#) Unfold the direct-sum factor and the Frobenius coordinate identification. $\square$

Theorem 13.3.5.24 (Fiber formula for the whole-increment right factor). [Lean](#) [✓ Stmt](#) The matrix fiber of $G_{K,Q}Z$ at (u, j) is $Z_j A^u$.

Proof. [✓ Proof](#) Unfold the direct-sum factor and the Frobenius coordinate identification. $\square$

Theorem 13.3.5.25 (Norm bounds for the whole-increment overlap factors). [Lean](#) [Lean](#) [✓ Stmt](#) The overlap factors satisfy

$$\|F_{K,Q}\| \leq \|J_Q^{\text{left}}\|, \quad (13.119)$$

$$\|G_{K,Q}\| \leq \|J_K^{\text{tail}}\|. \quad (13.120)$$

Proof. [✓ Proof](#) Expand the squared direct-sum norms fiberwise and apply the operator-norm bound to each fiber. Summing gives the two inequalities without a factor depending on the number of spectator words. $\square$

Definition 13.3.5.26 (Whole-increment centered residual). [Lean](#) [✓ Stmt](#) Assume $\text{tr}(\rho) \neq 0$ and let $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$. Define

$$R_{K,L,Q} := (\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \Gamma_{K+L,Q}^{\text{left}} - F_{K,Q}^\dagger \mathcal{K}_\infty^{\oplus(\Omega_K \times \Omega_Q)} G_{K,Q}. \quad (13.121)$$

Theorem 13.3.5.27 (Whole-increment centered factorization). [Lean](#) [✓ Stmt](#) The centered residual factors as

$$R_{K,L,Q} = F_{K,Q}^\dagger (\mathcal{K}_L - \mathcal{K}_\infty)^{\oplus(\Omega_K \times \Omega_Q)} G_{K,Q}. \quad (13.122)$$

Proof. [✓ Proof](#) Test both sides against arbitrary boundary families, use the two fiber formulas, and apply the centered mixed-Gram identity. $\square$

Theorem 13.3.5.28 (Whole-increment centered residual bound). [Lean](#) [✓ Stmt](#) The factorization gives

$$\|R_{K,L,Q}\| \leq \|J_Q^{\text{left}}\| \|\mathcal{K}_L - \mathcal{K}_\infty\| \|J_K^{\text{tail}}\|. \quad (13.123)$$

Proof. [✓ Proof](#) Apply submultiplicativity to the three factors and use the operator norm of a fiberwise direct sum. $\square$

Theorem 13.3.5.29 (Left-canonical whole-increment centered residual bound). [Lean](#) [✓ Stmt](#) If A is left-canonical, then

$$\|R_{K,L,Q}\| \leq \|\mathcal{K}_L - \mathcal{K}_\infty\| \|J_K^{\text{tail}}\|. \quad (13.124)$$

Proof. [✓ Proof](#) Substitute $\|J_Q^{\text{left}}\| \leq 1$ into the preceding estimate. $\square$

Theorem 13.3.5.30 (Primitive whole-increment centered residual bound). [Lean](#) [✓ Stmt](#) Let A be a primitive MPS tensor with positive-definite fixed point ρ . Then A satisfies the left-canonical centered residual estimate of Theorem 13.3.5.29.

Proof. [✓ Proof](#) Use the left-canonical normalization contained in the primitive-MPS hypotheses together with positive definiteness of ρ . $\square$

Theorem 13.3.5.31 (Limiting overlap product). [Lean](#) [✓ Stmt](#) Assume $D > 0$. For a positive-definite matrix ρ , the product $F_{K,Q}^\dagger \mathcal{K}_\infty^\oplus G_{K,Q}$ equals the corresponding virtual-word expression formed from J_K^{tail} , $(J_Q^{\text{left}})^\dagger$, $\mathcal{K}_\infty$, and $\mathcal{K}_\infty^{-1}$.

Proof. [✓ Proof](#) Compare matrix fibers, use the explicit limiting-metric action and its inverse, and evaluate the two virtual adjoints. $\square$

Theorem 13.3.5.32 (Gram of the reassociated tail map). [Lean](#) [✓ Stmt](#) Reassociation preserves the tail Gram:

$$(\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \tilde{\Gamma}_{K;L,Q}^{\text{tail}} = \mathcal{K}_{L+Q}^{\oplus \Omega_K}. \quad (13.125)$$

Proof. [✓ Proof](#) The reassociation is unitary, so its adjoint cancels it. The ordinary tail Gram is the fiberwise direct sum of $\mathcal{K}_{L+Q}$. $\square$

Theorem 13.3.5.33 (Inverse Gram of the reassociated tail map). [Lean](#) [✓ Stmt](#) If Γ_{L+Q}^{ES} is injective, then the inverse Gram of the reassociated tail map is the fiberwise direct sum of $(\mathcal{K}_{L+Q})^{-1}$.

Proof. [✓ Proof](#) Multiply the proposed fiberwise inverse on both sides by the Gram from Theorem 13.3.5.32 and use the inverse-Gram identities. $\square$

Theorem 13.3.5.34 (Whole-increment finite-Gram cancellations). [Lean](#) [Lean](#) [✓ Stmt](#) If Γ_{K+L}^{ES} and Γ_{L+Q}^{ES} are injective, respectively, then

$$(J_Q^{\text{left}})^\dagger \mathcal{K}_{K+L}^{\oplus \Omega_Q} (\mathcal{K}_{K+L}^{-1})^{\oplus \Omega_Q} (\Gamma_{K+L,Q}^{\text{left}})^\dagger = (\Gamma_{K+L+Q}^{\text{ES}})^\dagger. \quad (13.126)$$

Moreover,

$$(J_K^{\text{tail}})^\dagger \mathcal{K}_{L+Q}^{\oplus \Omega_K} (\mathcal{K}_{L+Q}^{-1})^{\oplus \Omega_K} (\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger = (\Gamma_{K+L+Q}^{\text{ES}})^\dagger. \quad (13.127)$$

Proof. [✓ Proof](#) The left identity is the preceding finite-Gram cancellation at length $K+L$. For the tail identity, cancel the fiberwise Gram with its inverse and take the adjoint of the reassociated tail factorization. $\square$

Definition 13.3.5.35 (Whole-increment tail finite-Gram correction). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and assume that Γ_{L+Q}^{ES} and $\Gamma_{K+L+Q}^{\text{ES}}$ are injective. The tail correction is the physical sandwich containing $\mathcal{K}_{L+Q} - \mathcal{K}_\infty$ between the reassociated tail pseudoinverse, J_K^{tail} , and the full boundary pseudoinverse.

Definition 13.3.5.36 (Whole-increment full inverse-Gram correction). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and assume that Γ_{L+Q}^{ES} and $\Gamma_{K+L+Q}^{\text{ES}}$ are injective. The full correction is the physical sandwich containing $(\mathcal{K}_{K+L+Q})^{-1} - \mathcal{K}_\infty^{-1}$.

Definition 13.3.5.37 (Whole-increment left finite-Gram correction). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and assume that Γ_{L+Q}^{ES} and Γ_{K+L}^{ES} are injective. The left correction is the physical sandwich containing $\mathcal{K}_{K+L} - \mathcal{K}_\infty$ and the left boundary pseudoinverse.

Theorem 13.3.5.38 (Reassociated tail pseudoinverse bound). [Lean](#) [✓ Stmt](#) If Γ_{L+Q}^{ES} is injective, then

$$\left\| \tilde{\Gamma}_{K;L,Q}^{\text{tail}} ((\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \tilde{\Gamma}_{K;L,Q}^{\text{tail}})^{-1} \right\| \leq \sqrt{\|\mathcal{K}_{L+Q}^{-1}\|}. \quad (13.128)$$

Proof. [✓ Proof](#) Use the square-root pseudoinverse identity, substitute the fiberwise inverse Gram, and bound the norm of the fiberwise map. $\square$

Theorem 13.3.5.39 (Whole-increment virtual correction telescope). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and assume that $\Gamma_{K+L+Q}^{\text{ES}}$ is injective. Write $K_t = \mathcal{K}_{L+Q}^{\oplus \Omega_K}$, $K_t^\infty = \mathcal{K}_\infty^{\oplus \Omega_K}$, $K_\ell = \mathcal{K}_{K+L}^{\oplus \Omega_Q}$, $K_\ell^\infty = \mathcal{K}_\infty^{\oplus \Omega_Q}$, $I_f = \mathcal{K}_{K+L+Q}^{-1}$, $I_\infty = \mathcal{K}_\infty^{-1}$, $J_t = J_K^{\text{tail}}$, and $J_\ell^\dagger = (J_Q^{\text{left}})^\dagger$. Then the exact virtual residual satisfies

$$(\tilde{\Gamma}_{K;L,Q}^{\text{tail}})^\dagger \Gamma_{K+L,Q}^{\text{left}} - K_t J_t I_f J_\ell^\dagger K_\ell \quad (13.129)$$

$$= R_{K,L,Q} - (K_t - K_t^\infty) J_t I_f J_\ell^\dagger K_\ell - K_t^\infty J_t (I_f - I_\infty) J_\ell^\dagger K_\ell \quad (13.130)$$

$$- K_t^\infty J_t I_\infty J_\ell^\dagger (K_\ell - K_\ell^\infty). \quad (13.131)$$

Proof. [✓ Proof](#) Substitute the centered residual and the limiting overlap product, distribute composition over subtraction, and telescope the product algebraically. $\square$

Definition 13.3.5.40 (Whole-increment centered projector sandwich). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and assume that Γ_{L+Q}^{ES} and Γ_{K+L}^{ES} are injective. The centered physical sandwich is obtained by placing $R_{K,L,Q}$ between the reassociated tail pseudoinverse and the left boundary pseudoinverse.

Theorem 13.3.5.41 (Norm bound for the whole-increment centered projector sandwich). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and suppose that Γ_{L+Q}^{ES} and Γ_{K+L}^{ES} are injective. Put

$$I_t = \mathcal{K}_{L+Q}^{-1}, \quad I_\ell = \mathcal{K}_{K+L}^{-1}. \quad (13.132)$$

Then the centered physical sandwich $\widehat{R}_{K,L,Q}$ satisfies

$$\|\widehat{R}_{K,L,Q}\| \leq \sqrt{\|I_t\|} \|R_{K,L,Q}\| \sqrt{\|I_\ell\|}. \quad (13.133)$$

Proof. [✓ Proof](#) Apply submultiplicativity to the centered residual between the two pseudoinverse factors. Bound the exterior factors by the square roots of the corresponding inverse-Gram norms. $\square$

Theorem 13.3.5.42 (Whole-increment physical correction telescope). [Lean](#) [✓ Stmt](#) Assume $D > 0$, let ρ be positive definite, and suppose that Γ_{L+Q}^{ES} , Γ_{K+L}^{ES} , and $\Gamma_{K+L+Q}^{\text{ES}}$ are injective. Then the whole-increment projector defect is the centered physical sandwich minus the tail Gram correction, the full inverse-Gram correction, and the left Gram correction.

Proof. [✓ Proof](#) Substitute the common-factorization residual, the two fiberwise Gram identities and inverse-Gram identities, and the virtual telescope. The left finite Gram then cancels its inverse, leaving the four physical sandwiches in the statement. $\square$

Theorem 13.3.5.43 (Whole-increment correction bounds). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Assume $D > 0$ and let ρ be positive definite. Put $I_t = \mathcal{K}_{L+Q}^{-1}$, $I_f = \mathcal{K}_{K+L+Q}^{-1}$, $I_\ell = \mathcal{K}_{K+L}^{-1}$, and $I_\infty = \mathcal{K}_\infty^{-1}$. Under the injectivity assumptions required to define the indicated inverse Grams, the tail, full, and left corrections C_t, C_f, C_ℓ satisfy

$$\|C_t\| \leq \sqrt{\|I_t\|} \|\mathcal{K}_{L+Q} - \mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \sqrt{\|I_f\|}. \quad (13.134)$$

The full correction satisfies

$$\|C_f\| \leq \sqrt{\|I_t\|} \|\mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \|I_\infty\| \|\mathcal{K}_{K+L+Q} - \mathcal{K}_\infty\| \sqrt{\|I_f\|}. \quad (13.135)$$

The left correction satisfies

$$\|C_\ell\| \leq \sqrt{\|I_t\|} \|\mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \|I_\infty\| \|J_Q^{\text{left}}\| \|\mathcal{K}_{K+L} - \mathcal{K}_\infty\| \sqrt{\|I_\ell\|}. \quad (13.136)$$

Proof. [✓ Proof](#) Unfold each correction and apply submultiplicativity. Use the reassociated tail pseudoinverse estimate and the fiberwise-map norm bound in all three cases. For the full correction, insert the inverse resolvent identity; for the left correction, use the left boundary pseudoinverse estimate. $\square$

Definition 13.3.5.44 (Common overlap factors and centered residual). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor, let Ω_K be the set of length- K words in the physical alphabet, and let $\mathcal{B}_K = \bigoplus_{u \in \Omega_K} M_D(\mathbb{C})$ with its Hilbert–Schmidt direct-sum norm. Define

$$F_K : \mathcal{B}_K \rightarrow \bigoplus_{(u,j) \in \Omega_K \times \Omega_1} M_D(\mathbb{C}) \quad (13.137)$$

by applying the one-site left virtual map independently to each prefix fiber. Define

$$G_K : \mathcal{B}_1 \rightarrow \bigoplus_{(u,j) \in \Omega_K \times \Omega_1} M_D(\mathbb{C}) \quad (13.138)$$

by applying the length- K tail virtual map independently to each final-site fiber. These definitions make sense without any nonemptiness assumption on the virtual dimension or on either configuration set.

When $D \geq 1$ and $\rho \in M_D(\mathbb{C})$ is positive definite, set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and define the centered virtual residual by

$$R_{K,l} = (\Gamma_{K,l+1}^{\text{tail}})^\dagger \Gamma_{K+l,1}^{\text{left}} - \mathcal{K}_\infty^{\oplus \Omega_K} J_K^{\text{tail}} \mathcal{K}_\infty^{-1} (J_1^{\text{left}})^\dagger \mathcal{K}_\infty^{\oplus \Omega_1}. \quad (13.139)$$

Theorem 13.3.5.45 (Centered overlap factorization and geometric bound). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every $K \in \mathbb{N}$, the common overlap factors have fiber formulas

$$(F_K X)_{u,j} = A^j X_u, \quad (G_K Z)_{u,j} = Z_j A^u, \quad (13.140)$$

and norm bounds

$$\|F_K\| \leq \|J_1^{\text{left}}\|, \quad \|G_K\| \leq \|J_K^{\text{tail}}\|. \quad (13.141)$$

The formulas and bounds remain valid for empty direct sums and incur no factor depending on the number of configurations.

Assume $D \geq 1$ and that $\rho \in M_D(\mathbb{C})$ is positive definite. The centered residual of Definition 13.3.5.44 factors exactly at the generally nonidentity limiting metric as

$$R_{K,l} = F_K^\dagger (\mathcal{K}_l - \mathcal{K}_\infty)^{\oplus(\Omega_K \times \Omega_l)} G_K, \quad (13.142)$$

and hence

$$\|R_{K,l}\| \leq \|J_1^{\text{left}}\| \|\mathcal{K}_l - \mathcal{K}_\infty\| \|J_K^{\text{tail}}\|. \quad (13.143)$$

If, in addition, A is primitive with fixed-point witness ρ , then there are constants $C > 0$ and $0 < r < 1$ such that, for every $K \in \mathbb{N}$ and every $l \geq 1$,

$$\|R_{K,l}\|^2 \leq D^3 (Cr^l)^2. \quad (13.144)$$

All formulas in this theorem are reconstructed algebra. They are not sourced formulas and assert neither a full projector-defect estimate nor the FNW–Nachtergaele rational coefficient.

Proof. ✓ Proof Evaluate the two factors on each common direct-sum fiber. The direct-sum norm identity gives the two bounds without a factor counting spectator configurations. Substitute these fiber formulas into the centered residual identity from Theorem 13.3.3.25 to obtain the exact factorization. Submultiplicativity and Theorem 13.3.3.20 give the product bound. Under primitivity, combine the uniform bound for J_K^{tail} with the squared geometric Gram estimate and absorb the fixed norm of J_1^{left} into the constant. $\square$

Theorem 13.3.5.46 (Fiberwise subtraction). Lean ✓ Stmt Let S be a finite set and let $G, H: V \rightarrow V$ be continuous linear endomorphisms of the virtual Euclidean space. Then

$$(G - H)^{\oplus S} = G^{\oplus S} - H^{\oplus S}. \quad (13.145)$$

Proof. ✓ Proof Evaluate both sides on each fiber indexed by $s \in S$. $\square$

Theorem 13.3.5.47 (Invertibility of the finite ground-space Gram). Lean ✓ Stmt If the ground-space map Γ_n^{ES} is injective, then its Gram endomorphism $\mathcal{K}_n = (\Gamma_n^{\text{ES}})^\dagger \Gamma_n^{\text{ES}}$ is invertible.

Proof. ✓ Proof The equality $\langle X, \mathcal{K}_n X \rangle = \|\Gamma_n^{\text{ES}} X\|^2$ shows that $\mathcal{K}_n$ is injective. An injective endomorphism of a finite-dimensional space is bijective and hence invertible. $\square$

Theorem 13.3.5.48 (Finite-to-limiting inverse resolvent identity). Lean ✓ Stmt Assume $D \geq 1$ and let $\rho \in M_D(\mathbb{C})$ be positive definite. Set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and $S_\infty = \mathcal{K}_\infty^{-1}$. If Γ_n^{ES} is injective and $S_n = \mathcal{K}_n^{-1}$, then

$$S_n - S_\infty = S_\infty (\mathcal{K}_\infty - \mathcal{K}_n) S_n. \quad (13.146)$$

Proof. ✓ Proof Both Gram endomorphisms are invertible. Apply the identity $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ with $A = \mathcal{K}_n$ and $B = \mathcal{K}_\infty$. $\square$

Theorem 13.3.5.49 (Left finite-Gram cancellation). Lean ✓ Stmt Suppose that Γ_K^{ES} is injective and write $S_K = \mathcal{K}_K^{-1}$. Then

$$(J_L^{\text{left}})^\dagger \mathcal{K}_K^{\oplus \Omega_L} S_K^{\oplus \Omega_L} (\Gamma_{K,L}^{\text{left}})^\dagger = (\Gamma_{K+L}^{\text{ES}})^\dagger. \quad (13.147)$$

Proof. ✓ Proof The direct-sum Gram and inverse Gram cancel fiberwise. The remaining composition is the adjoint of $\Gamma_{K,L}^{\text{left}} J_L^{\text{left}} = \Gamma_{K+L}^{\text{ES}}$. $\square$

Theorem 13.3.5.50 (Tail spectator pseudoinverse bound). [Lean](#) [✓ Stmt](#) If Γ_L^{ES} is injective, then

$$\left\| \Gamma_{K,L}^{\text{tail}} S_{\Gamma_{K,L}^{\text{tail}}} \right\| \leq \sqrt{\|S_L\|}. \quad (13.148)$$

Proof. [✓ Proof](#) The square-root pseudoinverse identity gives the left-hand side as the square root of the norm of the tail boundary inverse Gram. This inverse Gram is the fiberwise lift of S_L , whose norm is at most $\|S_L\|$. $\square$

Theorem 13.3.5.51 (Left spectator pseudoinverse bound). [Lean](#) [✓ Stmt](#) If Γ_K^{ES} is injective, then

$$\left\| S_{\Gamma_{K,L}^{\text{left}}} (\Gamma_{K,L}^{\text{left}})^\dagger \right\| \leq \sqrt{\|S_K\|}. \quad (13.149)$$

Proof. [✓ Proof](#) The adjoint-side square-root identity reduces the left-hand side to the norm of the left boundary inverse Gram. The direct-sum inverse-Gram formula and the fiberwise norm estimate give the result. $\square$

Definition 13.3.5.52 (Finite-Gram corrections in the C3 residual). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Assume $D \geq 1$ and let $\rho \in M_D(\mathbb{C})$ be positive definite. Set $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and $S_\infty = \mathcal{K}_\infty^{-1}$. Suppose that Γ_{l+1}^{ES} , Γ_{K+l}^{ES} , and $\Gamma_{K+l+1}^{\text{ES}}$ are injective, and write

$$T = \Gamma_{K,l+1}^{\text{tail}}, \quad (13.150)$$

$$L = \Gamma_{K+l,1}^{\text{left}}, \quad (13.151)$$

$$C = \Gamma_{K+l+1}^{\text{ES}}. \quad (13.152)$$

The three correction sandwiches in Theorem 13.3.3.25 are

$$Q_1 = T S_{l+1}^{\oplus \Omega_K} (\mathcal{K}_{l+1} - \mathcal{K}_\infty)^{\oplus \Omega_K} J_K^{\text{tail}} S_{K+l+1} (J_1^{\text{left}})^\dagger \mathcal{K}_{K+l}^{\oplus \Omega_1} S_{K+l}^{\oplus \Omega_1} L^\dagger, \quad (13.153)$$

$$Q_2 = T S_{l+1}^{\oplus \Omega_K} \mathcal{K}_\infty^{\oplus \Omega_K} J_K^{\text{tail}} (S_{K+l+1} - S_\infty) (J_1^{\text{left}})^\dagger \mathcal{K}_{K+l}^{\oplus \Omega_1} S_{K+l}^{\oplus \Omega_1} L^\dagger, \quad (13.154)$$

$$Q_3 = T S_{l+1}^{\oplus \Omega_K} \mathcal{K}_\infty^{\oplus \Omega_K} J_K^{\text{tail}} S_\infty (J_1^{\text{left}})^\dagger (\mathcal{K}_{K+l} - \mathcal{K}_\infty)^{\oplus \Omega_1} S_{K+l}^{\oplus \Omega_1} L^\dagger. \quad (13.155)$$

Theorem 13.3.5.53 (Exact factorizations of the C3 corrections). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Under the hypotheses of Definition 13.3.5.52, the three corrections have the exact factorizations

$$Q_1 = T S_{l+1}^{\oplus \Omega_K} (\mathcal{K}_{l+1} - \mathcal{K}_\infty)^{\oplus \Omega_K} J_K^{\text{tail}} S_{K+l+1} C^\dagger, \quad (13.156)$$

$$Q_2 = T S_{l+1}^{\oplus \Omega_K} \mathcal{K}_\infty^{\oplus \Omega_K} J_K^{\text{tail}} S_\infty (\mathcal{K}_\infty - \mathcal{K}_{K+l+1}) S_{K+l+1} C^\dagger, \quad (13.157)$$

$$Q_3 = T S_{l+1}^{\oplus \Omega_K} \mathcal{K}_\infty^{\oplus \Omega_K} J_K^{\text{tail}} S_\infty (J_1^{\text{left}})^\dagger (\mathcal{K}_{K+l} - \mathcal{K}_\infty)^{\oplus \Omega_1} S_{\Gamma_{K+l,1}^{\text{left}}} L^\dagger. \quad (13.158)$$

Proof. [✓ Proof](#) The direct-sum inverse-Gram formula identifies the initial factor with the pseudoinverse factor of T . Theorem 13.3.5.49 replaces the final product in Q_1 and Q_2 by $C^\dagger$. For Q_3 , apply the resolvent identity

$$S_{K+l+1} - S_\infty = S_\infty (\mathcal{K}_\infty - \mathcal{K}_{K+l+1}) S_{K+l+1}. \quad (13.159)$$

Fiberwise subtraction gives the displayed Gram differences. The factorization of Q_3 leaves the left pseudoinverse adjacent to $L^\dagger$. $\square$

Theorem 13.3.5.54 (Raw norm bounds for the C3 corrections). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
Under the hypotheses of Definition 13.3.5.52,

$$\|Q_1\| \leq \sqrt{\|S_{l+1}\|} \|\mathcal{K}_{l+1} - \mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \sqrt{\|S_{K+l+1}\|}, \quad (13.160)$$

$$\|Q_2\| \leq \sqrt{\|S_{l+1}\|} \|\mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \|S_\infty\| \|\mathcal{K}_{K+l+1} - \mathcal{K}_\infty\| \sqrt{\|S_{K+l+1}\|}, \quad (13.161)$$

$$\|Q_3\| \leq \sqrt{\|S_{l+1}\|} \|\mathcal{K}_\infty\| \|J_K^{\text{tail}}\| \|S_\infty\| \|J_1^{\text{left}}\| \|\mathcal{K}_{K+l} - \mathcal{K}_\infty\| \sqrt{\|S_{K+l}\|}. \quad (13.162)$$

Proof. [✓ Proof](#) Apply submultiplicativity to the factorizations in Theorem 13.3.5.53. The two spectator pseudoinverse estimates control the exterior factors, and Theorem 13.3.3.20 controls every fiberwise Gram factor without a term depending on $|\Omega_K|$. The square-root pseudoinverse identity controls $S_{K+l+1}C^\dagger$. $\square$

Definition 13.3.5.55 (Centered physical C3 sandwich). [Lean](#) [✓ Stmt](#) Let $R_{K,l}$ be the centered virtual residual and let S_n be the inverse Gram at length n . The centered physical sandwich is

$$\widehat{R}_{K,l} = \Gamma_{K,l+1}^{\text{tail}} S_{l+1}^{\oplus \Omega_K} R_{K,l} S_{K+l}^{\oplus \Omega_1} (\Gamma_{K+l,1}^{\text{left}})^\dagger. \quad (13.163)$$

Theorem 13.3.5.56 (Physical C3 assembly and centered bound). [Lean](#) [Lean](#) [✓ Stmt](#) Whenever the three finite Grams are invertible, the common-ambient defect is

$$P_{K,l+1}^{\text{tail}} P_{K+l}^{\text{left}} - P_{\mathcal{G}_{K+l+1}^{\text{ES}}(A)} = \widehat{R}_{K,l} - Q_1 - Q_2 - Q_3, \quad (13.164)$$

and

$$\|\widehat{R}_{K,l}\| \leq \sqrt{\|S_{l+1}\|} \|R_{K,l}\| \sqrt{\|S_{K+l}\|}. \quad (13.165)$$

Proof. [✓ Proof](#) Distribute the two exterior pseudoinverse factors across the exact four-term telescope. The three non-centred terms are precisely Q_1, Q_2, Q_3 . Submultiplicativity and the two square-root pseudoinverse estimates give the displayed norm bound for the remaining centered sandwich. $\square$

Theorem 13.3.5.57 (Spectator-uniform geometric C3 projector defect). [Lean](#) [✓ Stmt](#) [Discussion: #6134](#)
Let A be primitive with positive-definite fixed point ρ and nonzero bond dimension. There are constants $C, c, r > 0$, with $r < 1$, such that for every spectator length K and every $l \geq 1$ satisfying $cr^l < 1$,

$$\|P_{K,l+1}^{\text{tail}} P_{K+l}^{\text{left}} - P_{\mathcal{G}_{K+l+1}^{\text{ES}}(A)}\| \leq \frac{Cr^l}{1 - cr^l}.$$

The constants are independent of K .

Proof. [✓ Proof](#) Put $\mathcal{K}_\infty = \mathcal{R}(P_\rho)$ and $\mathcal{J}_\infty = \mathcal{K}_\infty^{-1}$. Choose $g, c, r > 0$ from the geometric Gram and inverse-Gram estimate, so that $c = \|\mathcal{J}_\infty\|g$, and choose $T > 0$ with $\|J_K^{\text{tail}}\| \leq T$ for every K . With

$$J = \|J_1^{\text{left}}\|, \quad M = \|\mathcal{K}_\infty\| \|\mathcal{J}_\infty\|, \quad (13.166)$$

take

$$C = 4\|\mathcal{J}_\infty\| Tg(J+1)(M+1). \quad (13.167)$$

The condition $cr^l < 1$ implies the same condition at $l + 1$, $K + l$, and $K + l + 1$. Hence every inverse Gram in the exact telescope is bounded by $(1 - cr^l)^{-1} \|\mathcal{J}_\infty\|$. Pairing the two square-root pseudoinverse factors gives this denominator only once. The centered term and each of the three correction terms are then at most

$$(1 - cr^l)^{-1} \frac{C}{4} r^l. \quad (13.168)$$

The triangle inequality gives the displayed estimate. Finally, the exact range identities replace the three inverse-Gram range projectors by the tail, left, and full physical ground-space projectors. The coefficient is the displayed reconstruction, not the optimized FNW rational numerator. $\square$

Theorem 13.3.5.58 (Prefix-uniform geometric whole-increment projector defect). [Lean](#) [✓ Stmt](#)
Let A be primitive with positive-definite fixed point ρ and nonzero bond dimension. There are constants $C, c, r > 0$, with $r < 1$, such that for all $K, L, Q \in \mathbb{N}$ with $L \geq 1$ and $cr^L < 1$,

$$\left\| P_{K;L,Q}^{\text{tail}} P_{K+L,Q}^{\text{left}} - P_{\mathcal{G}_{K+L+Q}^{\text{ES}}(A)} \right\| \leq \frac{Cr^L}{1 - cr^L}. \quad (13.169)$$

Here $P_{K;L,Q}^{\text{tail}}$ is the range projector of the reassocated tail boundary map and $P_{K+L,Q}^{\text{left}}$ is the range projector of the left boundary map. The constants are independent of both the prefix length K and the suffix length Q .

Proof. [✓ Proof](#) Bound the centered sandwich and the three finite-Gram corrections in the exact projector telescope. Geometric convergence of the finite Grams and the uniform tail-map bound make each term at most a fixed multiple of $r^L/(1 - cr^L)$, uniformly in K and Q . Sum the four estimates and absorb their constants into C . $\square$

Theorem 13.3.5.59 (Uniform whole-increment defect at the coefficient $7/16$). [Lean](#) [✓ Stmt](#)
Let A be primitive with positive-definite fixed point ρ and nonzero bond dimension. There is an integer $p > 0$, measured in sites of A , such that A is p -block injective and, for every $K \in \mathbb{N}$,

$$\left\| P_{K;p,p}^{\text{tail}} P_{K+p,p}^{\text{left}} - P_{\mathcal{G}_{K+2p}^{\text{ES}}(A)} \right\| \leq \frac{7}{16}. \quad (13.170)$$

Proof. [✓ Proof](#) Choose p above a positive block-injectivity length and sufficiently large that $cr^p < 1$ and $Cr^p/(1 - cr^p) < 7/16$. Block injectivity propagates to p , and the geometric whole-increment estimate applies with $L = Q = p$ for every prefix length K . $\square$

Theorem 13.3.5.60 (Three-block whole-increment defect at the coefficient $7/16$). [Lean](#) [✓ Stmt](#)
Under the same hypotheses, there is an integer $p > 0$, measured in sites of A , such that A is p -block injective and

$$\left\| P_{p;p,p}^{\text{tail}} P_{2p,p}^{\text{left}} - P_{\mathcal{G}_{3p}^{\text{ES}}(A)} \right\| \leq \frac{7}{16}. \quad (13.171)$$

Proof. [✓ Proof](#) Apply the prefix-uniform theorem at $K = p$. The three consecutive regions then each contain exactly p sites of the input tensor. $\square$

Theorem 13.3.5.61 (Uniform open-chain projector-defect threshold). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be a primitive MPS tensor with nonzero bond dimension and a positive-definite fixed point. There are an overlap length $l > 1$, measured in sites of A , and a number $\varepsilon_l \geq 0$ such that A is l -block injective, the projector defect is bounded uniformly in the left length K by ε_l , and*

$$\varepsilon_l < \frac{1}{\sqrt{l+1}}.$$

Nachtergaele's projector-defect identity following condition C3 [Nac96, Eq. (2.4)], together with the two-projection estimate, gives the corresponding anticommutator bound for every K . If A is already a blocked representative, one site of A is one blocked-chain filtration step.

Proof. ✓ Proof The geometric projector estimate supplies $C, c, r > 0$ with $r < 1$. Since

$$\sqrt{n+1} r^n \rightarrow 0, \quad (13.172)$$

continuity of inversion at 1 gives

$$\frac{Cr^n}{1-cr^n} \sqrt{n+1} \rightarrow 0. \quad (13.173)$$

Normality supplies a positive block-injectivity length. Choose l above that length and above 1 so that $cr^l < 1$ and the last display is less than 1. Block injectivity propagates to l , and the geometric estimate gives the uniform defect bound with $\varepsilon_l = (1-cr^l)^{-1}Cr^l$. The open-chain projector-defect theorem gives the anticommutator statement. $\square$

Lemma 13.3.5.62 (Blocked three-site projector and adjacent-pair transport). Lean ✓ Stmt Let A be an MPS tensor with nonzero physical dimension, let p be any blocking length, and set $B = \text{block}_p(A)$. Then

$$\|P_{1,2}^{\text{tail}}(B)P_2^{\text{left}}(B) - P_{\mathcal{G}_3^{\text{ES}}(B)}\| = \|P_{p;p,p}^{\text{tail}}P_{2p,p}^{\text{left}} - P_{\mathcal{G}_{3p}^{\text{ES}}(A)}\|. \quad (13.174)$$

If B is injective, then the bound on either side

$$\|P_{1,2}^{\text{tail}}(B)P_2^{\text{left}}(B) - P_{\mathcal{G}_3^{\text{ES}}(B)}\| \leq \frac{7}{16} \quad (13.175)$$

gives the adjacent three-site anticommutator inequality

$$h_0h_1 + h_1h_0 \geq -\frac{7}{16}(h_0 + h_1). \quad (13.176)$$

The same quadratic-form inequality holds for every forward or backward adjacent pair of cyclic range-two terms on every chain of length at least three, including pairs crossing the periodic cut.

Theorem 13.3.5.63 (Uniform gap for a blocked primitive MPS parent Hamiltonian). Lean ✓ Stmt Let A be a primitive MPS tensor with nonzero physical and bond dimensions, and suppose that its fixed point is positive definite. There is a blocking length $p > 0$ such that A is p -block injective and, with $B = \text{block}_p(A)$, for every $N \geq 4$ and every $v \in (\ker H_{N,2}(B))^\perp$,

$$\frac{1}{8}\|v\| \leq \|H_{N,2}(B)v\|. \quad (13.177)$$

Proof. ✓ Proof Choose p so that the three-block projector defect is at most $7/16$. Lemma 13.3.5.62 gives the same coefficient for every overlapping cyclic pair. For range two each row has at most two such pairs, so the martingale estimate gives $1 - 2(7/16) = 1/8$. $\square$

Theorem 13.3.5.64 (Anticommutator bound for two projections). Lean ✓ Stmt Let P and Q be orthogonal projections on a real or complex Hilbert space. For every vector v ,

$$\text{Re}\langle Pv, Qv \rangle + \text{Re}\langle Qv, Pv \rangle \leq \text{Re}\langle Pv, v \rangle + \text{Re}\langle Qv, v \rangle. \quad (13.178)$$

Proof. ✓ Proof Positivity of the inner product gives

$$0 \leq \operatorname{Re}\langle (P - Q)v, (P - Q)v \rangle, \quad (13.179)$$

$$\operatorname{Re}\langle (P - Q)v, (P - Q)v \rangle = \operatorname{Re}\langle Pv, v \rangle - \operatorname{Re}\langle Pv, Qv \rangle - \operatorname{Re}\langle Qv, Pv \rangle + \operatorname{Re}\langle Qv, v \rangle. \quad (13.180)$$

Symmetry and idempotence expand the left-hand side into the second line; rearranging it proves the claim. $\square$

Theorem 13.3.5.65 (Cyclic window double counting). Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let h_i , with $i \in \mathbb{Z}/N\mathbb{Z}$, be orthogonal projections. Define

$$H = \sum_i h_i, \quad (13.181)$$

$$A_s = \sum_{a=0}^{m-1} h_{s+a}, \quad (13.182)$$

$$Q_q = \sum_i (h_i h_{i+q} + h_i h_{i-q}), \quad (13.183)$$

$$C_q(v) = \sum_i (\operatorname{Re}\langle h_i v, h_{i+q} v \rangle + \operatorname{Re}\langle h_i v, h_{i-q} v \rangle) = \operatorname{Re}\langle Q_q v, v \rangle. \quad (13.184)$$

Then, for every vector v ,

$$\sum_s A_s^2 = mH + \sum_{q=1}^{m-1} (m-q)Q_q, \quad (13.185)$$

$$\sum_s \operatorname{Re}\langle A_s v, A_s v \rangle = m \operatorname{Re}\langle H v, v \rangle + \sum_{q=1}^{m-1} (m-q)C_q(v), \quad (13.186)$$

$$\sum_s \operatorname{Re}\langle A_s v, v \rangle = m \operatorname{Re}\langle H v, v \rangle. \quad (13.187)$$

Proof. ✓ Proof For fixed offsets $a, b \in \{0, \dots, m-1\}$, translation by a gives

$$\sum_s h_{s+a} h_{s+b} = \sum_i h_i h_{i+b-a}. \quad (13.188)$$

The case $a = b$ contributes mH , while each of the differences $b-a = q$ and $b-a = -q$ contributes $(m-q)Q_q$. Summing (13.188) over a and b gives (13.185). Taking the quadratic form of that operator identity at v turns each product $h_i h_{i\pm q}$ into the pair $\operatorname{Re}\langle h_i v, h_{i\pm q} v \rangle$, giving (13.186). Translation invariance also gives

$$\sum_s \operatorname{Re}\langle A_s v, v \rangle = m \operatorname{Re}\langle H v, v \rangle. \quad (13.189)$$

$\square$

Theorem 13.3.5.66 (Cyclic cross-term estimates). Lean Lean Lean Lean ✓ Stmt For Q_q as in Theorem 13.3.5.65,

$$\operatorname{Re}\langle Q_q v, v \rangle \leq 2 \operatorname{Re}\langle H v, v \rangle. \quad (13.190)$$

Suppose that h_i and h_{i+d} commute whenever $R \leq d \leq N-R$. Then the cross term is nonnegative for these offsets. If $1 \leq m$, $R \leq m$, and $2m \leq N$, it follows that

$$\operatorname{Re}\langle H v, v \rangle + \sum_{q=1}^{m-1} \operatorname{Re}\langle Q_q v, v \rangle \leq \operatorname{Re}\langle H v, H v \rangle. \quad (13.191)$$

Proof. ✓ Proof Translation of the reversed leg gives

$$\sum_i \operatorname{Re}\langle h_{i+q}v, h_i v \rangle = \sum_i \operatorname{Re}\langle h_i v, h_{i-q}v \rangle. \quad (13.192)$$

Summing Theorem 13.3.5.64 over i and using (13.192) proves the upper bound. If h_i and h_{i+d} commute, their symmetry and idempotence give

$$0 \leq \operatorname{Re}\langle h_i v, h_{i+d}v \rangle. \quad (13.193)$$

Partitioning the nonzero cyclic offsets into $1 \leq q < m$, their reflections $N-q$, and $m \leq k \leq N-m$ gives

$$\operatorname{Re}\langle Hv, Hv \rangle = \operatorname{Re}\langle Hv, v \rangle + \sum_{q=1}^{m-1} \operatorname{Re}\langle Q_q v, v \rangle + \sum_{k=m}^{N-m} \sum_i \operatorname{Re}\langle h_i v, h_{i+k}v \rangle. \quad (13.194)$$

The last sum in (13.194) is nonnegative by (13.193), which proves the lower bound. $\square$

Theorem 13.3.5.67 (Finite-range cyclic Knabe inequality). Lean ✓ Stmt Let $1 \leq R \leq m$, $2m \leq N$, and let h_i be cyclically indexed orthogonal projections such that h_i commutes with h_{i+d} whenever $R \leq d \leq N-R$. Suppose that, for every s and v ,

$$\gamma \operatorname{Re}\langle A_s v, v \rangle \leq \operatorname{Re}\langle A_s v, A_s v \rangle. \quad (13.195)$$

If $(R-1)^2 < m\gamma$, then, with

$$\delta = \frac{m\gamma - (R-1)^2}{m - R + 1} > 0, \quad (13.196)$$

one has

$$\delta \operatorname{Re}\langle Hv, v \rangle \leq \operatorname{Re}\langle Hv, Hv \rangle. \quad (13.197)$$

The coefficient for general R is derived here and is not attributed to the cited papers.

Proof. ✓ Proof Put $c = m - R + 1$. Summing the window gaps and applying Theorem 13.3.5.65 gives

$$m\gamma \operatorname{Re}\langle Hv, v \rangle \leq \sum_s \operatorname{Re}\langle A_s v, A_s v \rangle. \quad (13.198)$$

Decompose $m-q = c + (R-1-q)$ and expand (13.186): the weights $q \leq R-2$ carry the coefficient c plus the positive part $R-1-q$, while for $q \geq R$ the coefficient $R-1-q$ is nonpositive and the cross term is nonnegative, so those summands are bounded by $c C_q(v)$. For the positive parts, the anticommutator bound and the triangular identity

$$2 \sum_{q=1}^{R-1} (R-1-q) = (R-1)(R-2) \quad (13.199)$$

combine with the decomposition

$$m - c + (R-1)(R-2) = (R-1)^2, \quad c = m - R + 1, \quad (13.200)$$

so that

$$\begin{aligned} \sum_s \operatorname{Re}\langle A_s v, A_s v \rangle &\leq (m + (R-1)(R-2)) \operatorname{Re}\langle H v, v \rangle + c \sum_{q=1}^{m-1} C_q(v) \\ &\leq (m + (R-1)(R-2)) \operatorname{Re}\langle H v, v \rangle + c(\operatorname{Re}\langle H v, H v \rangle - \operatorname{Re}\langle H v, v \rangle). \end{aligned}$$

The cyclic cross-term estimate therefore gives

$$\sum_s \operatorname{Re}\langle A_s v, A_s v \rangle \leq c \operatorname{Re}\langle H v, H v \rangle + (R-1)^2 \operatorname{Re}\langle H v, v \rangle. \quad (13.201)$$

Rearranging and dividing by $c > 0$ proves the claim. $\square$

Theorem 13.3.5.68 (Nearest-neighbor cyclic Knabe inequality). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 13.3.5.67 with $R = 2$, if $1 < m\gamma$, then*

$$\frac{m\gamma - 1}{m - 1} \operatorname{Re}\langle H v, v \rangle \leq \operatorname{Re}\langle H v, H v \rangle. \quad (13.202)$$

Equivalently, the open-window threshold is $\gamma > 1/m$. This is Knabe's nearest-neighbor threshold [Kna88], quoted in [PGVWC07, lines 1483–1489] and [CPGSV21, lines 2194–2197].

Proof. [✓ Proof](#) Substitute $R = 2$ into the coefficient of Theorem 13.3.5.67. $\square$

Lemma 13.3.5.69 (Positive gap transfer under Loewner domination). [Lean](#) [Lean](#) [✓ Stmt](#) *Let P and Q be finite-dimensional complex endomorphisms, and let $\gamma \geq 0$. Suppose that P is positive and*

$$\gamma \|v\| \leq \|Pv\| \quad (13.203)$$

for every $v \in (\ker P)^\perp$. Then

$$\gamma \|v\|^2 \leq \operatorname{Re}\langle Pv, v \rangle \quad (13.204)$$

on $(\ker P)^\perp$. If also $P \leq Q$ and $\ker P = \ker Q$, then

$$\gamma \|v\| \leq \|Qv\| \quad (13.205)$$

for every $v \in (\ker Q)^\perp$.

Theorem 13.3.5.70 (Kernels of positively weighted sums of positive operators). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $(P_i)_{i \in I}$ be a finite family of positive operators on an inner product space. A positive operator P annihilates a vector v as soon as $\operatorname{Re}\langle Pv, v \rangle = 0$, and consequently, for real weights $w_i > 0$,*

$$\ker\left(\sum_{i \in I} w_i P_i\right) = \ker\left(\sum_{i \in I} P_i\right) = \bigcap_{i \in I} \ker P_i, \quad (13.206)$$

equivalently $(\sum_{i \in I} w_i P_i)v = 0$ exactly when $P_i v = 0$ for every $i \in I$. For the empty family both sums are zero and all three kernels are the full space. This is the analytic kernel step of the blocked-sum comparison in Lemma `block-objc` of Rozmán–Molnár–Schuch¹, in which the blocked parent Hamiltonian is a weighted sum $\sum_i \gamma_i h_i$ of positive local terms with $\gamma_i > 0$.

¹arXiv:2607.19078v1.

Proof. ✓ Proof If $(\sum_i w_i P_i)v = 0$, the real part of the inner product with v expands as

$$\operatorname{Re} \left\langle \left(\sum_i w_i P_i \right) v, v \right\rangle = \sum_i w_i \operatorname{Re} \langle P_i v, v \rangle = 0. \quad (13.207)$$

Each summand is nonnegative, so each vanishes, and $w_i > 0$ forces $\operatorname{Re} \langle P_i v, v \rangle = 0$. For a positive operator P , the quadratic form is nonnegative along the real affine line $t \mapsto v + t \cdot (Pv)$ and vanishes at $t = 0$; its linear coefficient is $2\|Pv\|^2$, so evaluating at small negative t forces $\|Pv\| = 0$, that is $Pv = 0$. The converse inclusion holds term by term, and the unweighted comparison is the case $w_i = 1$. $\square$

Theorem 13.3.5.71 (Loewner domination of positively weighted sums). Lean Lean ✓ Stmt Let $(P_i)_{i \in I}$ be a finite family of positive operators on an inner product space, with real weights $(w_i)_{i \in I}$. If $w_i \geq 0$ for every i , then $\sum_{i \in I} w_i P_i$ is positive. If $w_i \leq 1$ for every i , then

$$\sum_{i \in I} w_i P_i \leq \sum_{i \in I} P_i \quad (13.208)$$

in Loewner order. This is the comparison step of the blocked-sum argument in the source cited in Theorem 13.3.5.70, where the coefficients satisfy $0 < \gamma_i \leq 1$.

Proof. ✓ Proof Positivity of the weighted sum follows from positivity of each scalar multiple $w_i P_i$ and of finite sums of positive operators. For the domination, the difference is again a weighted sum,

$$\sum_{i \in I} P_i - \sum_{i \in I} w_i P_i = \sum_{i \in I} (1 - w_i) P_i, \quad (13.209)$$

now with nonnegative weights $1 - w_i \geq 0$, hence positive. $\square$

Theorem 13.3.5.72 (Gap transfer through a positively weighted sum). Lean ✓ Stmt Let $(P_i)_{i \in I}$ be a finite family of positive operators on a finite-dimensional complex inner product space, with real weights $0 < w_i \leq 1$, and let $\gamma \geq 0$. If

$$\gamma \|v\| \leq \left\| \left(\sum_{i \in I} w_i P_i \right) v \right\| \quad (13.210)$$

for every $v \in (\ker \sum_{i \in I} w_i P_i)^\perp$, then

$$\gamma \|v\| \leq \left\| \left(\sum_{i \in I} P_i \right) v \right\| \quad (13.211)$$

for every $v \in (\ker \sum_{i \in I} P_i)^\perp$.

Proof. ✓ Proof The weighted sum is positive and, by Theorem 13.3.5.71, dominated by the unweighted sum; by the kernel equality (13.206) of Theorem 13.3.5.70, the two sums have the same kernel. Lemma 13.3.5.69 transfers the norm gap from the smaller weighted sum to the larger unweighted sum. $\square$

Lemma 13.3.5.73 (Blocked local terms and the aligned sparse sum). Lean Lean Lean ✓ Stmt Assume $p > 0$ and $N \geq 2$. Write $h_j^{A,L}$ for the range- L local term of a tensor A at site j . Let

$B = \text{block}_p(A)$, and let U be the canonical configuration isometry from the blocked chain of length N to the original chain of length Np . For every blocked site i ,

$$Uh_i^{B,2}U^{-1} = h_{pi}^{A,2p}. \quad (13.212)$$

Hence the conjugated blocked Hamiltonian is the aligned sparse sum

$$UH_{N,2}(B)U^{-1} = \sum_{i=0}^{N-1} h_{pi}^{A,2p}, \quad (13.213)$$

and

$$UH_{N,2}(B)U^{-1} \leq H_{Np,2p}(A) \quad (13.214)$$

in Loewner order.

Theorem 13.3.5.74 (Original-site gap on block-divisible periodic chains). [Lean](#) [Stmt](#) Under the hypotheses of Theorem 13.3.5.63, one can choose the same $p > 0$ so that the original tensor has interaction range $2p$ and, for every $N \geq 4$ and every $v \in (\ker H_{Np,2p}(A))^\perp$,

$$\frac{1}{8}\|v\| \leq \|H_{Np,2p}(A)v\|. \quad (13.215)$$

Proof. [Proof](#) Let $B = \text{block}_p(A)$, and let U be the canonical configuration isometry. Lemma 13.3.5.73 gives, for every blocked site i ,

$$Uh_i^{B,2}U^{-1} = h_{pi}^{A,2p}, \quad (13.216)$$

and therefore

$$UH_{N,2}(B)U^{-1} = \sum_{i=0}^{N-1} h_{pi}^{A,2p} \leq \sum_{j=0}^{Np-1} h_j^{A,2p} = H_{Np,2p}(A). \quad (13.217)$$

Block injectivity identifies the two ground-state lines under U , hence

$$\ker(UH_{N,2}(B)U^{-1}) = \ker H_{Np,2p}(A). \quad (13.218)$$

Theorem 13.3.5.63 gives the norm gap $1/8$ for $H_{N,2}(B)$. Since U is isometric, the same bound holds for its conjugate. Lemma 13.3.5.69 then transfers this bound through the Loewner domination (13.214) and the kernel equality above. $\square$

Theorem 13.3.5.75 (Uniform gap for normal MPS parent Hamiltonians). [Not ready](#) [Discussion: #5926](#) Let A be a normal tensor. There exist an interaction range $L > 1$ and a constant $\gamma > 0$ such that, for every $N \geq 2L$ and every $v \in (\ker H_{N,L}(A))^\perp$,

$$\gamma\|v\| \leq \|H_{N,L}(A)v\|.$$

Theorem 13.3.5.74 proves the estimate with $L = 2p$ when the periodic chain length is divisible by p . The unrestricted statement still requires a remainder-length comparison for the finitely many congruence classes modulo p .

Proof. [Not ready](#) [Discussion: #5926](#) The divisible-length case is Theorem 13.3.5.74. It remains to control chains of length $Np + r$, $0 < r < p$, by a boundary/remainder argument that preserves the ground-space complement and a uniform positive constant. $\square$

Theorem 13.3.5.76 (Gap bound from cyclic overlap compression). [Lean](#) [✓ Stmt](#) Let A be a tensor and let $L > 1$. Suppose that, for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} \|h_{i,\text{ES}}v\|.$$

Then the parent Hamiltonian has the explicit gap constant $\gamma_L := 1/(4L) > 0$. Concretely, if $N \geq 2L$ and $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, then $\gamma_L \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [✓ Proof](#) The local averaging, positivity, and symmetric-projection statements are now established in Lemmas 13.3.7, 13.3.8, and 13.3.9. Lemma 13.3.1.1 reduces the present theorem to the remaining MPS-specific task: deriving the uniform quadratic-form estimate. Lemma 13.3.1.18 establishes the finite-sum algebra from ordered cross-term row bounds, while Lemma 13.3.1.19 records the source anticommutator martingale condition. Lemmas 13.3.1.21 and 13.3.2.7 state the common finite-overlap ordered-cross-term specialization with $m = 2(L-1)$. Lemma 13.3.19 supplies the non-overlap positivity statement for site-disjoint windows, and Lemma 13.3.2.5 translates this to the concrete support-disjoint cyclic windows. Lemma 13.3.2.6 gives the finite-overlap row-cardinality bound. Lemma 13.3.2.8 combines these results with the finite-overlap reduction, so the only additional input is the ordered anticommutator or cross-term estimate for overlapping cyclic windows. Lemma 13.3.2.9 states the same overlap-only formulation, and Lemma 13.3.2.11 states a sufficient norm-compression formulation. Equivalently, this is the constant-compression theorem below with $\eta = (1 - \frac{1}{4L}) \frac{1}{2(L-1)}$. Since $1 - \eta 2(L-1) = 1/(4L)$, the displayed hypothesis gives the stated lower bound with $\gamma_L = 1/(4L)$. $\square$

Theorem 13.3.5.77 (Gap bound from a cyclic anticommutator estimate). [Lean](#) [✓ Stmt](#) Let $L > 1$. Suppose that for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$h_{i,\text{ES}}h_{j,\text{ES}} + h_{j,\text{ES}}h_{i,\text{ES}} \geq -\left(1 - \frac{1}{4L}\right) \frac{1}{2(L-1)} (h_{i,\text{ES}} + h_{j,\text{ES}})$$

in quadratic form. Then the parent Hamiltonian has the explicit gap constant $\gamma_L := 1/(4L) > 0$. Concretely, if $N \geq 2L$ and $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, then $\gamma_L \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [✓ Proof](#) This is Lemma 13.3.2.10 stated as the public conditional gap theorem, with the anticommutator inequality retained as an explicit hypothesis. $\square$

Theorem 13.3.5.78 (Gap bound from a strict cyclic compression constant). [Lean](#) [✓ Stmt](#) Let $L > 1$. Put $m = 2(L-1)$. Suppose that there is a constant $\eta \geq 0$ such that $\eta m < 1$, and suppose that, for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{i,\text{ES}}v\|.$$

Then the parent Hamiltonian has the gap constant $1 - \eta m > 0$. Concretely, if $N \geq 2L$ and $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, then $(1 - \eta m) \|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [✓ Proof](#) Lemma 13.3.2.12, with $\gamma = 1 - \eta m$, applies to the displayed hypothesis

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{i,\text{ES}}v\|.$$

It gives $0 < 1 - \eta m$ and the stated norm lower bound. The preceding fixed-coefficient theorem is the specialization $\eta = (1 - 1/(4L))/(2(L-1))$. $\square$

Theorem 13.3.5.79 (Gap bound from a cyclic operator-product norm constant). [Lean](#) [Stmt](#)
Let $L > 1$. Put $m = 2(L-1)$. Suppose there is a constant $\eta \geq 0$ with $\eta m < 1$ and that, for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{j,\text{ES}}v\|,$$

that is, a bound $\|h_{i,\text{ES}}h_{j,\text{ES}}\| \leq \eta$ on the operator-product norm rather than the directional bound of Theorem 13.3.5.78. Then the parent Hamiltonian has the gap constant $1 - \eta m > 0$. Concretely, if $N \geq 2L$ and $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, then $(1 - \eta m)\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. This is a conditional reduction. The local terms $h_{i,\text{ES}} = p_i$ are the excitation projections. If their overlapping ranges have a common nonzero vector, then $\|h_{i,\text{ES}}h_{j,\text{ES}}\| = 1$, so the hypothesis is unsatisfiable for $\eta < 1$ in that case. In degenerate cases, such as a zero excitation projection, the operator-product norm can be smaller. In any case this is not the quantity controlled by the cited principal-angle estimates. The comparison is documented in [\[con26m\]](#).

Proof. [Proof](#) Lemma 13.3.2.13, with $\gamma = 1 - \eta m$, applies to the displayed operator-product hypothesis and gives $0 < 1 - \eta m$ together with the stated norm lower bound. $\square$

Theorem 13.3.5.80 (Conditional spectral gap for MPS parent Hamiltonians). [Lean](#) [Lean](#) [Stmt](#)
For a tensor A and a range $L > 1$, assume either the cyclic-window anticommutator estimate of Theorem 13.3.5.77 or the stronger overlapping cyclic-window norm-compression estimate of Theorem 13.3.5.76. Then there exists $\gamma > 0$ such that for every $N \geq 2L$ and every vector $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$ one has $\gamma\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. By Lemma 13.3.20, this implies the usual uniform lower bound on every nonzero eigenvalue of the parent Hamiltonian.

Proof. [Proof](#) This follows immediately from the corresponding explicit theorem by taking γ to be the constant produced there. $\square$

Theorem 13.3.5.81 (Conditional spectral gap from a strict cyclic compression constant). [Lean](#) [Stmt](#)
For a tensor A and a range $L > 1$, suppose there is a constant $\eta \geq 0$ such that $\eta 2(L-1) < 1$ and, for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{i,\text{ES}}v\|.$$

Then there exists $\gamma > 0$ such that, for every $N \geq 2L$ and every vector $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\gamma\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$.

Proof. [Proof](#) This follows from Theorem 13.3.5.78 by taking $\gamma = 1 - \eta 2(L-1)$. $\square$

Theorem 13.3.5.82 (Conditional spectral gap from an operator-product norm constant). [Lean](#) [Stmt](#)
For a tensor A and a range $L > 1$, suppose there is a constant $\eta \geq 0$ such that $\eta 2(L-1) < 1$ and, for every $N \geq 2L$ and every overlapping off-diagonal pair of cyclic length- L windows,

$$\|h_{i,\text{ES}}(h_{j,\text{ES}}v)\| \leq \eta \|h_{j,\text{ES}}v\|.$$

Then there exists $\gamma > 0$ such that, for every $N \geq 2L$ and every vector $v \in (\mathcal{G}_{N,L}^{\text{ES}}(A))^\perp$, $\gamma\|v\| \leq \|H_{N,L}^{\text{ES}}(A)v\|$. As in Theorem 13.3.5.79, this is a conditional reduction: the operator-product hypothesis is unsatisfiable for any overlapping pair whose excitation ranges have a common nonzero vector, since then $\|h_{i,\text{ES}}h_{j,\text{ES}}\| = 1$. The statement does not assert this norm identity in degenerate cases.

Proof. [Proof](#) This follows from Theorem 13.3.5.79 by taking $\gamma = 1 - \eta 2(L-1)$. $\square$

Chapter 14

Block-Injective Parent Hamiltonians and Degenerate Ground Spaces

For a block-injective tensor $A^i = \bigoplus_j \mu_j A_j^i$, the local ground spaces decompose according to the normal blocks. The first sections prove the open-segment intersection relations and the trace identities that separate distinct blocks.

The remaining sections analyze block-diagonal boundary conditions across the periodic cut. Under the stated length and closure hypotheses, they identify the periodic ground space with $\text{span}\{V^{(N)}(A_j)\}_j$, as in [CPGSV21, Theorem IV.16] and [PGVWC07, Theorem 12].

14.1 Block injectivity and degenerate ground spaces

For an MPS with several canonical blocks, the relevant local space is the linear sum of the block ranges. The formal argument below does not pass through an auxiliary “periodic ground space of a BNT” definition or through a commutant reconstruction of a boundary matrix. Instead it proves simultaneous word-span separation, the one-step restriction intersection, and the global change-of-cut boundary comparison directly. This route culminates in the source-normalized kernel theorem of Theorem 14.4.2.

Definition 14.1.1 (Vector space generated by a basis of normal tensors). Lean ✓ Stmt For a basis of normal tensors $A_1, \dots, A_g$, the vector space generated by their length- N matrix product vectors is $\text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}$.

Lemma 14.1.2 (BNT span as the sum of one-vector spaces). Lean ✓ Stmt For a family $A_0, \dots, A_{r-1}$ and a chain length N , let $M_j = \text{span}\{V^{(N)}(A_j)\}$. Then

$$\bigvee_{j=0}^{r-1} M_j = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, r-1\}.$$

Proof. ✓ Proof Both sides are the span of the same family of vectors. □

Lemma 14.1.3 (Inclusion of a block ground space). [Lean](#) [✓ Stmt](#) For each block index j with $\mu_j \neq 0$, and whenever $0 < N$ and $L \leq N$, the periodic-boundary ground space of the block A_j embeds into the corresponding ground space of the direct-sum tensor:

$$\mathcal{G}_{N,L}(A_j) \subseteq \mathcal{G}_{N,L}\left(\bigoplus_k \mu_k A_k\right).$$

Proof. [✓ Proof](#) The definition by length- L windows on the ring reduces the inclusion to the local block-embedding statement that each ground vector of A_j on the window lifts to the direct sum by the identity

$$\Gamma_L^{\oplus_k \mu_k A_k}(\iota_j(\mu_j^{-L} X)) = \Gamma_L^{A_j}(X).$$

Here ι_j inserts a matrix in the j -th diagonal block and has zero off that block. Thus $\iota_j(\mu_j^{-L} X)$ has exactly the local vector of the j -th block on that window. $\square$

Lemma 14.1.4 (Forward inclusion from the block ground spaces). [Lean](#) [✓ Stmt](#) Assume $\mu_j \neq 0$ for every block index j . Whenever $0 < N$ and $L \leq N$, the linear sum of the blockwise ground spaces with periodic boundary conditions is contained in the direct-sum tensor's ground space with periodic boundary conditions:

$$\sum_j \mathcal{G}_{N,L}(A_j) \subseteq \mathcal{G}_{N,L}\left(\bigoplus_k \mu_k A_k\right),$$

where j ranges over all block indices.

Proof. [✓ Proof](#) Take the linear sum of the inclusions $\mathcal{G}_{N,L}(A_j) \subseteq \mathcal{G}_{N,L}(\bigoplus_k \mu_k A_k)$ over all block indices j . $\square$

14.2 Block separation and intersection

This section establishes that block-separating words make the simultaneous block-word tuples $\omega \mapsto (A_1^\omega, \dots, A_g^\omega)$ span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$ at a single length, and that this single-length span, together with the normalization $\sum_a A_a^j A_a^{j\dagger} = I$, yields the one-step intersection identity $\bigvee_j G_{n+2}(A^j)$ used for the degenerate parent-Hamiltonian ground space.

Definition 14.2.1 (Pairwise block-separating equations). [Lean](#) [Lean](#) [✓ Stmt](#) Fix a finite target set T of block indices. A length- S block-separating equation for k against T is a choice of coefficients c_ω such that

$$\sum_{|\omega|=S} c_\omega A_\ell^\omega = \begin{cases} I, & \ell = k, \\ 0, & \ell \in T. \end{cases}$$

The pairwise version requires this equation for each ordered pair of distinct blocks, with $T = \{j\}$.

Definition 14.2.2 (Trace-dual pair product-span condition). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For two blocks A and B , the length- S pair word tuple is $w \mapsto (A^w, B^w)$. We also use the subalgebra of $M_{D_A}(\mathbb{C}) \times M_{D_B}(\mathbb{C})$ generated by the one-letter pairs (A_i, B_i) . The pair product-span condition says that the length- S tuples

Lemma 14.2.3 (Three-block trace reduction from a nonzero middle matrix). [Lean](#) [Lean](#) [Stmt](#)
Let A and B be tensors with virtual bond dimensions D_A and D_B . Assume that A is L -block injective. Let $\Delta_A \in M_{D_A}(\mathbb{C})$ with $\Delta_A \neq 0$, and let $\Delta_B \in M_{D_B}(\mathbb{C})$. Suppose that, for all words u, v, t of length L ,

Then for every $Z \in M_{D_A}(\mathbb{C})$ there is a matrix $W \in M_{D_B}(\mathbb{C})$ such that, for all words t of length L , $\text{tr}(ZA_t) + \text{tr}(WB_t) = 0$. This is the trace-dual algebraic step in the two-block case of the direct-sum lemma of [PGVWC07].

$$\text{tr}(A_u \Delta_A A_v A_t) + \text{tr}(B_u \Delta_B B_v B_t) = 0.$$

Lemma 14.2.4 (Finite-chain image inclusion from a three-block relation). [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) ✓ Stmt Under the hypotheses of Lemma 14.2.3, the length- L spaces $\mathcal{G}_L^A$ and $\mathcal{G}_L^B$ satisfy $\mathcal{G}_L^A \subseteq \mathcal{G}_L^B$. If the corresponding nonzero middle matrix exists on both sides, then $\mathcal{G}_L^A = \mathcal{G}_L^B$.

Lemma 14.2.5 (Dimension step for two-block local spaces). Lean Lean Lean Lean Lean
Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean
Lean Lean Lean ✓ Stmt If A and B are length- L block-injective, the map $Z \mapsto (t \mapsto \text{tr}(A_t Z))$ has image dimension D_A^2 , and similarly for B . Therefore an inclusion $\mathcal{G}_L^A \subseteq \mathcal{G}_L^B$, together with $D_B \leq D_A$, forces $D_A = D_B$ and $\mathcal{G}_L^A = \mathcal{G}_L^B$. In particular, the three-block trace relation from Lemma 14.2.4 gives this equality whenever the ordering $D_B \leq D_A$ is available. If the inclusion comes from the strictly larger block, $D_B < D_A$, then the same dimension step already contradicts the existence of the three-block trace relation. More generally, let H be a source hypothesis which excludes the simultaneous conclusion

Then the three-block intersection vanishes:

With injectivity at the three-block length, the strict-size branch gives

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One such source hypothesis is non-proportionality of the periodic MPS vectors at a sufficiently long chain length: $\mathbb{C} V^{(N)}(A) \neq \mathbb{C} V^{(N)}(B)$. Indeed,

$$\mathcal{G}_L^A = \mathcal{G}_L^B \implies \mathcal{G}_{N,L}(A) = \mathcal{G}_{N,L}(B) \implies \mathbb{C} V^{(N)}(A) = \mathbb{C} V^{(N)}(B),$$

by equality of the periodic-boundary constraint spaces, followed by injective uniqueness. For same-dimensional normalized BNT representatives that are not gauge-phase equivalent, the non-equivalence condition supplies this non-proportionality at arbitrarily large chain lengths; the direct-sum injectivity hypotheses remain explicit. Combining the equal-dimension and unequal-dimension branches over all ordered block pairs gives one common positive length S_0 with

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = S_0\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

For an ordered pair of blocks, the trace-separation conclusion has the form

$$[\forall |\omega| = S, \text{tr}(\Delta_A A^\omega) + \text{tr}(\Delta_B B^\omega) = 0] \implies \Delta_A = \Delta_B = 0.$$

Equivalently, the homogeneous pair span is full:

$$\text{span}\{(A^\omega, B^\omega) : |\omega| = S\} = M_{D_A}(\mathbb{C}) \times M_{D_B}(\mathbb{C}).$$

Concatenating words propagates this equality to positive multiples of S , and hence to a finite period window for every ordered pair. Combining that window with the conditional period-window homogenization theorem gives a single length S_0 such that

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = S_0\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

The same conclusion applies to the corresponding normal canonical-form hypotheses with a basis of normal tensors once one-site injectivity is supplied explicitly, since those hypotheses carry the irreducibility and normalization assumptions used in the direct-sum comparison.

Proof. ✓ Proof Block injectivity makes the boundary-to-chain map injective, so $\dim \mathcal{G}_L^A = D_A^2$ and $\dim \mathcal{G}_L^B = D_B^2$. Inclusion gives $D_A^2 \leq D_B^2$, while $D_B \leq D_A$ gives the reverse inequality. Hence $D_A = D_B$, and equality of dimensions inside the inclusion gives $\mathcal{G}_L^A = \mathcal{G}_L^B$. In the strict-size branch this contradicts $D_B < D_A$. For the conditional directness statement, suppose $\psi \in \mathcal{G}_{3L}^A \cap \mathcal{G}_{3L}^B$ has boundary representations $\psi = \Gamma_{3L}^A(X) = \Gamma_{3L}^B(Y)$. If $X = 0$, injectivity of the A boundary-to-chain map gives $\psi = 0$. Otherwise equality of the coefficient formulas gives $\forall |t| = 3L, \quad \text{tr}(XA^t) - \text{tr}(YB^t) = 0$, equivalently, $\forall |t| = 3L, \quad \text{tr}(XA^t) + \text{tr}((-Y)B^t) = 0$. The dimension step applied to this homogeneous three-block trace relation gives $D_A^2 \leq D_B^2$, contradicting $D_B < D_A$ in the strict-size branch. In the equal-size branch, equal local spaces impose the same length- L constraints on the N -site ring:

$$\mathcal{G}_L^A = \mathcal{G}_L^B \implies \mathcal{G}_{N,L}(A) = \mathcal{G}_{N,L}(B).$$

The injective uniqueness theorem identifies these periodic ground spaces with the one-dimensional MPS spans,

$$\begin{aligned} \mathcal{G}_{N,L}(A) &= \mathbb{C} V^{(N)}(A), \\ \mathcal{G}_{N,L}(B) &= \mathbb{C} V^{(N)}(B), \end{aligned}$$

and hence $\mathbb{C} V^{(N)}(A) = \mathbb{C} V^{(N)}(B)$, contradicting non-proportionality of the two MPS vectors. $\square$

Proof. ✓ Proof Let $M = (M_j)_j$ be a target tuple. Write

$$M_j = \sum_{|u|=L} a_u A_u^j,$$

$$I = \sum_{|v|=S} b_v A_v^j$$

simultaneously for every block j . Multiplying the two expansions gives

$$M_j = M_j I = \sum_{|u|=L} \sum_{|v|=S} a_u b_v A_u^j A_v^j = \sum_{|u|=L} \sum_{|v|=S} a_u b_v A_{uv}^j.$$

The words uv have length $L + S$, so the target tuple lies in the length- $L + S$ span. □

Lemma 14.2.12 (Period-window propagation for blockwise product spans). Lean Lean ✓ Stmt Let $p > 0$. Suppose the length- p simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$, and suppose that for every $0 \leq r < p$, the length- $s + r$ simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$. Then there is a length N such that, for every $n \geq N$, the length- n simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$. More concretely, one may take $N = s$: writing $n = s + r + qp$, with $0 \leq r < p$, the span at length $s + r$ propagates to the span at length n by multiplying q times by the period- p span.

Proof. ✓ Proof The Euclidean division of $n - s$ by p gives $n = s + r + qp$ with $0 \leq r < p$. The assumed residue-window span gives the full product algebra at length $s + r$. Applying homogeneous padding with the length- p span once gives the full product algebra at length $s + r + p$; iterating gives the full product algebra at length $s + r + qp$. □

Lemma 14.2.13 (Blockwise word span from block-selector equations). Lean ✓ Stmt Let J be a finite set of blocks, and let $(A_j)_{j \in J}$ be a family of tensors such that every A_j is L -block injective. Suppose that for each block k there are coefficients $c_\omega^{(k)}$, indexed by words of length S , such that

$$\sum_{|\omega|=S} c_\omega^{(k)} A_j^\omega = \begin{cases} I, & j = k, \\ 0, & j \neq k. \end{cases}$$

Then the simultaneous length- $(L + S)$ block-word evaluations span $\prod_{j \in J} M_{D_j}(\mathbb{C})$.

Proof. ✓ Proof For a target tuple $(M_j)_{j \in J}$, block injectivity gives coefficients $a_u^{(k)}$ such that $M_k = \sum_{|u|=L} a_u^{(k)} A_k^u$. Multiply this expansion by the selector equation for k and sum over k . For every block j this gives

$$M_j = \sum_{k \in J} \sum_{|u|=L} \sum_{|v|=S} a_u^{(k)} c_v^{(k)} A_j^u A_j^v.$$

Each product $A_j^u A_j^v$ is the evaluation of the concatenated word uv , which has length $L + S$. Hence the length- $(L + S)$ word tuples span the full product algebra. □

14.2.1 Trace identities for block intersections

Once the simultaneous block-word tuples span the full product algebra at a single length, non-degeneracy of the product trace pairing turns blockwise trace equalities into blockwise matrix identities $A_b^j C_a^j = A_b^j E^j A_a^j$. The lemmas here use those identities to place left-boundary trace decompositions in the block sum $\bigvee_j G_{n+2}(A^j)$ and to obtain the one-step block-intersection identity.

Lemma 14.2.1.1 (Arbitrary interpolation on one block of a pair). Lean ✓ Stmt Let $(A^\ell)_{\ell=1}^r$ be a finite family of tensors. Suppose that the length- S pair evaluations of blocks k and j span their product matrix algebra. For every $X \in M_{D_k}(\mathbb{C})$, there is a tuple $M = (M_\ell)_\ell$ in the span of the simultaneous length- S word evaluations such that $M_k = X$ and $M_j = 0$.

Proof. ✓ Proof Full pair span gives coefficients satisfying $(X, 0) = \sum_{|w|=S} c_w (A_w^k, A_w^j)$. Define $M_\ell = \sum_{|w|=S} c_w A_w^\ell$ for every block ℓ . Then M lies in the simultaneous word-tuple span, and its k - and j -components are X and 0, respectively. $\square$

Lemma 14.2.1.2 (Blockwise matrix equality from trace pairings). Lean ✓ Stmt Let $A^1, \dots, A^g$ be block tensors whose length- m simultaneous word tuples span the product algebra. If, for every word w of length m , two blockwise matrix families X_j and Y_j satisfy $\sum_j \text{tr}(X_j A_w^j) = \sum_j \text{tr}(Y_j A_w^j)$, then $X_j = Y_j$ for every block j .

Proof. ✓ Proof For every word w of length m ,

$$\sum_j \text{tr}((X_j - Y_j) A_w^j) = \sum_j \text{tr}(X_j A_w^j) - \sum_j \text{tr}(Y_j A_w^j) = 0.$$

Nondegeneracy of the product trace pairing, together with the spanning hypothesis, gives $X_j - Y_j = 0$ for every block j . $\square$

Lemma 14.2.1.3 (Boundary identities from block-sum membership). Lean ✓ Stmt Let $A^1, \dots, A^g$ be block tensors whose length- m simultaneous word tuples span the product algebra. Let $C_a^j \in M_{D_j}(\mathbb{C})$ be matrices indexed by blocks j and physical letters a . Suppose that, for every pair of boundary letters a, b and every word w of length m , a vector ψ has the left-boundary trace form $\psi(a, w, b) = \sum_j \text{tr}(A_b^j C_a^j A_w^j)$. If $\psi \in \bigvee_j G_{m+2}(A^j)$, then there are matrices E_j such that, for every block j and all a, b , $A_b^j C_a^j = A_b^j E_j A_a^j$.

Proof. ✓ Proof Write $\psi = \sum_j \phi_j$ with $\phi_j \in G_{m+2}(A^j)$. For each j , choose E_j whose image under the length- $m+2$ ground-space map for A^j is ϕ_j . Comparing the coefficient of the word awb in the two expressions for ψ gives

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(A_b^j E_j A_a^j A_w^j).$$

Lemma 14.2.1.2 gives the displayed blockwise identities. $\square$

Lemma 14.2.1.4 (Blockwise boundary compatibility from traces). Lean ✓ Stmt Let $A^1, \dots, A^g$ be block tensors with the following common word-span property: for a fixed m , the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$, $|w| = m$, span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Suppose that matrices $C_a^j, D_b^j \in M_{D_j}(\mathbb{C})$ satisfy, for all physical indices a, b and all words w of length m ,

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

Then $A_b^j C_a^j = D_b^j A_a^j$ for every block j and all a, b .

Proof. ✓ Proof For fixed a, b , set $\Delta_j = A_b^j C_a^j - D_b^j A_a^j$. The assumed trace equality says that $\sum_j \text{tr}(\Delta_j A_w^j) = 0$ for every word w of length m . Since the tuples $(A_w^1, \dots, A_w^g)$ span the product algebra, the product trace pairing is nondegenerate and hence every Δ_j is zero. $\square$

Lemma 14.2.1.5 (Word-valued boundary compatibility from traces). [Lean](#) [Lean](#) [✓ Stmt](#) Let $A^1, \dots, A^g$ be block tensors with the common word-span property at length m . Suppose that $C_\rho^j \in M_{D_j}(\mathbb{C})$ is indexed by a block j and a word ρ of some fixed length M , and that $D_\beta^j \in M_{D_j}(\mathbb{C})$ is indexed by a block j and a word β of some fixed length K . If, for every word ρ of length M , every word β of length K , and every middle word w of length m ,

$$\sum_j \operatorname{tr}(A_\beta^j C_\rho^j A_w^j) = \sum_j \operatorname{tr}(D_\beta^j A_\rho^j A_w^j),$$

then $A_\beta^j C_\rho^j = D_\beta^j A_\rho^j$ for every block j and all words β, ρ . In particular, taking $D_\beta^j = X_j A_\beta^j$ gives $A_\beta^j C_\rho^j = (X_j A_\beta^j) A_\rho^j$.

Proof. [✓ Proof](#) Fix β and ρ , and set $\Delta_j = A_\beta^j C_\rho^j - D_\beta^j A_\rho^j$. The trace equality says that $\sum_j \operatorname{tr}(\Delta_j A_w^j) = 0$ for every word w of length m . The same nondegenerate product trace-pairing argument as in Lemma 14.2.1.4 gives $\Delta_j = 0$ for every j . The specialization $D_\beta^j = X_j A_\beta^j$ is immediate. $\square$

Lemma 14.2.1.6 (Fixed complementary-word compatibility from traces). [Lean](#) [✓ Stmt](#) Let $A^1, \dots, A^g$ be block tensors with the common word-span property at length m . Fix a complementary word ρ of length M and matrices $C_\rho^j, X_j \in M_{D_j}(\mathbb{C})$. If, for every wrapped word β of length K and every middle word w of length m ,

$$\sum_j \operatorname{tr}(A_\beta^j C_\rho^j A_w^j) = \sum_j \operatorname{tr}((X_j A_\beta^j) A_\rho^j A_w^j),$$

then $A_\beta^j C_\rho^j = (X_j A_\beta^j) A_\rho^j$ for every block j and every wrapped word β .

Proof. [✓ Proof](#) Fix β and set $\Delta_j = A_\beta^j C_\rho^j - (X_j A_\beta^j) A_\rho^j$. The trace equality gives $\sum_j \operatorname{tr}(\Delta_j A_w^j) = 0$ for every word w of length m . The common word-span property and the product trace pairing imply $\Delta_j = 0$ for every j . $\square$

Lemma 14.2.1.7 (Normalized boundary matrix identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $(A_a)_{a \in I}$, $(C_a)_{a \in I}$, and $(D_b)_{b \in I}$ be matrices in $M_D(\mathbb{C})$, indexed by a finite alphabet. If

$$\begin{aligned} \sum_a A_a A_a^\dagger &= \mathbb{1}, \\ A_b C_a &= D_b A_a \end{aligned}$$

for all $a, b \in I$, and if $E = \sum_a C_a A_a^\dagger$, then

$$\begin{aligned} D_b &= A_b E, \\ A_b C_a &= A_b E A_a \end{aligned}$$

for all $a, b \in I$.

In particular, the same calculation applies when the alphabet is the finite set of words of a fixed length and A_β denotes $A_{\beta_1} \cdots A_{\beta_K}$.

The same argument also applies when the normalized family is indexed by a finite set J , while $(A_b)_{b \in I}$ and $(D_b)_{b \in I}$ are indexed by a set I . If $(B_a)_{a \in J}$ and $(C_a)_{a \in J}$ are indexed by J , and if, for all $a \in J$ and $b \in I$,

$$\begin{aligned} \sum_a B_a B_a^\dagger &= \mathbb{1}, \\ A_b C_a &= D_b B_a, \end{aligned}$$

then, with $E = \sum_a C_a B_a^\dagger$, one has

$$\begin{aligned} D_b &= A_b E, \\ A_b C_a &= A_b E B_a. \end{aligned}$$

This includes the case where I and J are word sets of different lengths.

Proof. ✓ Proof Multiply D_b by the normalization:

$$D_b = D_b \sum_a A_a A_a^\dagger = \sum_a D_b A_a A_a^\dagger = \sum_a A_b C_a A_a^\dagger = A_b E.$$

Substituting this identity into $A_b C_a = D_b A_a$ gives $A_b C_a = A_b E A_a$. For two alphabets, the corresponding step is

$$D_b = D_b \sum_a B_a B_a^\dagger = \sum_a D_b B_a B_a^\dagger = \sum_a A_b C_a B_a^\dagger = A_b E,$$

where $E = \sum_a C_a B_a^\dagger$. Substituting this identity into $A_b C_a = D_b B_a$ gives $A_b C_a = A_b E B_a$. □

Lemma 14.2.1.8 (Explicit C^j, D^j, E^j identities from trace decompositions). Lean ✓ Stmt Let $A^1, \dots, A^g$ be block tensors with the common word-span property at length m . Fix matrices X_j , and put $D_\beta^j = X_j A_\beta^j$. Suppose that C_ρ^j is indexed by a block j and a complementary word ρ of length M , and assume that, for every wrapped word β of length K , every complementary word ρ , and every middle word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_\rho^j A_w^j) = \sum_j \text{tr}(D_\beta^j A_\rho^j A_w^j).$$

If each block satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$, then, for every block j , there is a matrix $E^j = \sum_\rho C_\rho^j (A_\rho^j)^\dagger$ such that, for every wrapped word β and complementary word ρ ,

$$\begin{aligned} D_\beta^j &= A_\beta^j E^j, \\ A_\beta^j C_\rho^j &= A_\beta^j E^j A_\rho^j. \end{aligned}$$

This is the calculation in [PGVWC07, Theorem 12, proof lines 1446–1451], with the adjoint on A_ρ^j required by the displayed normalization.

Proof. ✓ Proof Lemma 14.2.1.5 gives $A_\beta^j C_\rho^j = D_\beta^j A_\rho^j$. The single-letter normalization gives the word normalization $\sum_\rho A_\rho^j (A_\rho^j)^\dagger = \mathbb{1}$. Lemma 14.2.1.7, applied to the two word alphabets, gives

$$\begin{aligned} E^j &= \sum_\rho C_\rho^j (A_\rho^j)^\dagger, \\ D_\beta^j &= A_\beta^j E^j, \\ A_\beta^j C_\rho^j &= A_\beta^j E^j A_\rho^j. \end{aligned}$$

□

Lemma 14.2.1.9 (Block-diagonal boundary C^j, D^j, E^j identities from trace decompositions).

Lean **✓ Stmt** Let $A^1, \dots, A^g$ be block tensors with the common word-span property at length m . Fix $N \in \mathbb{N}$, complex numbers μ_j , and block-diagonal boundary matrices X_j , and put $D_\beta^j = (\mu_j^N X_j) A_\beta^j$. Suppose that, for every wrapped word β of length K , every complementary word ρ of length M , and every middle word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_\rho^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

If each block satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$, then, for every block j , there is a matrix $E^j = \sum_\rho C_\rho^j (A_\rho^j)^\dagger$ such that, for every wrapped word β and complementary word ρ ,

$$\begin{aligned} (\mu_j^N X_j) A_\beta^j &= A_\beta^j E^j, \\ A_\beta^j C_\rho^j &= A_\beta^j E^j A_\rho^j. \end{aligned}$$

This is the boundary-matrix calculation in [PGVWC07, Theorem 12, proof lines 1446–1451], after substituting the block-diagonal boundary matrix $D_\beta^j = (\mu_j^N X_j) A_\beta^j$.

Proof. **✓ Proof** Apply Lemma 14.2.1.8 with $D_\beta^j = (\mu_j^N X_j) A_\beta^j$. □

Lemma 14.2.1.10 (Complementary-word boundary identities from a compatibility hypothesis).

Lean **✓ Stmt** Let A_β be the products over wrapped words of length K , let A_ρ be the products over complementary words of length M , and let $X \in M_D(\mathbb{C})$ be a matrix. Suppose that the complementary-word products satisfy $\sum_\rho A_\rho A_\rho^\dagger = \mathbb{1}$ and that there are matrices C_ρ , indexed by complementary words, such that, for every wrapped word β and complementary word ρ , $A_\beta C_\rho = (XA_\beta)A_\rho$. Then, for every complementary word ρ , with

$$E_\rho = \left(\sum_\gamma C_\gamma A_\gamma^\dagger \right) A_\rho,$$

one has, for every wrapped word β , $(XA_\beta)A_\rho = A_\beta E_\rho$.

Proof. **✓ Proof** Apply Lemma 14.2.1.7 with $D_\beta = XA_\beta$. It gives $A_\beta C_\rho = A_\beta E A_\rho$ for $E = \sum_\gamma C_\gamma A_\gamma^\dagger$. With $E_\rho = E A_\rho$, the compatibility hypothesis gives

$$(XA_\beta)A_\rho = A_\beta C_\rho = A_\beta E A_\rho = A_\beta E_\rho.$$

□

Lemma 14.2.1.11 (Complementary-word boundary identities from trace decompositions). **Lean**

✓ Stmt Let $A^1, \dots, A^g$ be block tensors with the common word-span property at length m . Fix matrices X_j . Suppose that C_ρ^j is indexed by a block j and a complementary word ρ of length M . Assume that, for every wrapped word β of length K , every complementary word ρ , and every middle word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_\rho^j A_w^j) = \sum_j \text{tr}(X_j A_\beta^j A_\rho^j A_w^j).$$

If each block satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$, then, for every block j and every complementary word ρ , there is a matrix $E_{j,\rho}$ such that, for every wrapped word β , $(X_j A_\beta^j) A_\rho^j = A_\beta^j E_{j,\rho}$.

Proof. [✓ Proof](#) Lemma 14.2.1.5, applied with $D_\beta^j = X_j A_\beta^j$, gives $A_\beta^j C_\rho^j = (X_j A_\beta^j) A_\rho^j$. The hypothesis $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ extends to words of length M : $\sum_\rho A_\rho^j (A_\rho^j)^\dagger = \mathbb{1}$. Lemma 14.2.1.10 gives the matrices $E_{j,\rho}$. $\square$

Lemma 14.2.1.12 (A left-boundary summand is a ground-space vector). [Lean](#) [✓ Stmt](#) *Let A be a block tensor, and let $C_a, E \in M_D(\mathbb{C})$ satisfy $A_b C_a = A_b E A_a$ for all physical indices a, b . For a word $\sigma = (\sigma_1, \dots, \sigma_{n+2})$, define*

$$\alpha(\sigma) = \text{tr}(A_{\sigma_{n+2}} C_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_{n+1}}).$$

Then $\alpha \in G_{n+2}(A)$.

Proof. [✓ Proof](#) The boundary identity gives

$$A_{\sigma_{n+2}} C_{\sigma_1} = A_{\sigma_{n+2}} E A_{\sigma_1}.$$

Hence, by cyclicity of the trace,

$$\alpha(\sigma) = \text{tr}(A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_{n+2}} E),$$

which is the ground-space parametrization of E . $\square$

Lemma 14.2.1.13 (Left-boundary summands are ground-space vectors). [Lean](#) [✓ Stmt](#) *Let $A^1, \dots, A^g$ be block tensors, and let $C_a^j, E^j \in M_{D_j}(\mathbb{C})$ satisfy $A_b^j C_a^j = A_b^j E^j A_a^j$ for every block j and all physical indices a, b . For a word $\sigma = (\sigma_1, \dots, \sigma_{n+2})$, define*

$$\alpha_j(\sigma) = \text{tr}(A_{\sigma_{n+2}}^j C_{\sigma_1}^j A_{\sigma_2}^j \cdots A_{\sigma_{n+1}}^j).$$

Then $\sum_j \alpha_j \in \bigvee_j G_{n+2}(A^j)$.

Proof. [✓ Proof](#) Fix j . By the assumed boundary identity and cyclicity of the trace,

$$\text{tr}(A_{\sigma_{n+2}}^j C_{\sigma_1}^j A_{\sigma_2}^j \cdots A_{\sigma_{n+1}}^j) = \text{tr}(A_{\sigma_1}^j A_{\sigma_2}^j \cdots A_{\sigma_{n+2}}^j E^j).$$

Thus α_j is the image of E^j under the parametrization of $G_{n+2}(A^j)$. Hence each α_j lies in $G_{n+2}(A^j)$, and the finite sum lies in the supremum of these subspaces. $\square$

Lemma 14.2.1.14 (Left-boundary trace decompositions lie in the block ground spaces). [Lean](#) [✓ Stmt](#) *Let $A^1, \dots, A^g$ be block tensors with common word-span separation at length n , and suppose that $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every block j . Suppose a vector $\psi \in (\mathbb{C}^d)^{\otimes(n+2)}$ has the left-boundary trace decomposition*

$$\psi = \sum_j \alpha_j,$$

$$\alpha_j(i_1, \dots, i_{n+2}) = \text{tr}(A_{i_{n+2}}^j C_{i_1}^j A_{i_2}^j \cdots A_{i_{n+1}}^j),$$

and suppose that the two trace decompositions agree for every middle word:

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

Then $\psi \in \bigvee_j G_{n+2}(A^j)$.

Proof. ✓ Proof The equality of the two boundary trace expressions and the common word-span hypothesis give $A_b^j C_a^j = D_b^j A_a^j$ for every j, a, b . By the normalization,

$$\begin{aligned} D_b^j &= D_b^j \sum_a A_a^j A_a^{j\dagger} = \sum_a A_b^j C_a^j A_a^{j\dagger} = A_b^j E^j, \\ E^j &:= \sum_a C_a^j A_a^{j\dagger}. \end{aligned}$$

Hence $A_b^j C_a^j = A_b^j E^j A_a^j$. Lemma 14.2.1.13 then gives $\psi = \sum_j \alpha_j \in \bigvee_j G_{n+2}(A^j)$. □

Lemma 14.2.1.15 (Trace decompositions from fixed-boundary restrictions). Lean ✓ Stmt Let $A^1, \dots, A^g$ be block tensors. If $\psi \in (\mathbb{C}^d)^{\otimes(n+2)}$ satisfies

$$\begin{aligned} \psi(a, -) &\in \bigvee_j G_{n+1}(A^j), \\ \psi(-, b) &\in \bigvee_j G_{n+1}(A^j) \end{aligned}$$

for every physical index a, b , then there are matrices C_a^j and D_b^j such that

$$\begin{aligned} \psi &= \sum_j \alpha_j, \\ \alpha_j(i_1, \dots, i_{n+2}) &= \text{tr}(A_{i_{n+2}}^j C_{i_1}^j A_{i_2}^j \cdots A_{i_{n+1}}^j), \end{aligned}$$

and, for every middle word w of length n ,

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

Proof. ✓ Proof Choose finite decompositions of the fixed-boundary vectors:

$$\begin{aligned} \psi(a, i_2, \dots, i_{n+2}) &= \sum_j \text{tr}(A_{i_2}^j \cdots A_{i_{n+2}}^j C_a^j), \\ \psi(i_1, \dots, i_{n+1}, b) &= \sum_j \text{tr}(A_{i_1}^j \cdots A_{i_{n+1}}^j D_b^j). \end{aligned}$$

For every middle word w ,

$$\psi(a, w, b) = \sum_j \text{tr}(A_w^j A_b^j C_a^j) = \sum_j \text{tr}(A_b^j C_a^j A_w^j),$$

and the second decomposition gives

$$\psi(a, w, b) = \sum_j \text{tr}(A_a^j A_w^j D_b^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

Hence

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

For an arbitrary word $(i_1, \dots, i_{n+2})$, the first decomposition gives

$$\psi(i_1, \dots, i_{n+2}) = \sum_j \text{tr}(A_{i_2}^j \cdots A_{i_{n+2}}^j C_{i_1}^j) = \sum_j \text{tr}(A_{i_{n+2}}^j C_{i_1}^j A_{i_2}^j \cdots A_{i_{n+1}}^j),$$

by cyclicity of the trace. □

Lemma 14.2.1.16 (One-step block intersection from block restrictions). [Lean](#) [Stmt](#) Let $A^1, \dots, A^g$ be block tensors with common word-span separation at length n , and suppose that $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every block j . If $\psi \in (\mathbb{C}^d)^{\otimes(n+2)}$ satisfies

$$\begin{aligned}\psi(a, -) &\in \bigvee_j G_{n+1}(A^j), \\ \psi(-, b) &\in \bigvee_j G_{n+1}(A^j)\end{aligned}$$

for every physical index a, b , then $\psi \in \bigvee_j G_{n+2}(A^j)$.

Proof. [Proof](#) Choose finite decompositions of the fixed-boundary vectors:

$$\begin{aligned}\psi(a, i_2, \dots, i_{n+2}) &= \sum_j \text{tr}(A_{i_2}^j \cdots A_{i_{n+2}}^j C_a^j), \\ \psi(i_1, \dots, i_{n+1}, b) &= \sum_j \text{tr}(A_{i_1}^j \cdots A_{i_{n+1}}^j D_b^j).\end{aligned}$$

Evaluating these two decompositions on (a, w, b) , where w has length n , gives

$$\sum_j \text{tr}(A_b^j C_a^j A_w^j) = \sum_j \text{tr}(D_b^j A_a^j A_w^j).$$

The first decomposition also gives

$$\begin{aligned}\psi &= \sum_j \alpha_j, \\ \alpha_j(i_1, \dots, i_{n+2}) &= \text{tr}(A_{i_{n+2}}^j C_{i_1}^j A_{i_2}^j \cdots A_{i_{n+1}}^j).\end{aligned}$$

Lemma 14.2.1.14 therefore gives $\psi \in \bigvee_j G_{n+2}(A^j)$. □

Lemma 14.2.1.17 (One-step block intersection characterization). [Lean](#) [Stmt](#) Under the hypotheses of Lemma 14.2.1.16, an $(n+2)$ -site vector satisfies $\psi \in \bigvee_j G_{n+2}(A^j)$ if and only if

$$\begin{aligned}\psi(a, -) &\in \bigvee_j G_{n+1}(A^j), \\ \psi(-, b) &\in \bigvee_j G_{n+1}(A^j)\end{aligned}$$

for every physical index a, b .

Proof. [Proof](#) The reverse implication is Lemma 14.2.1.16. For the forward implication, write

$$\begin{aligned}\psi &= \sum_j \gamma_j, \\ \gamma_j &\in G_{n+2}(A^j).\end{aligned}$$

Each γ_j has fixed-boundary restrictions in $G_{n+1}(A^j)$:

$$\begin{aligned}\gamma_j(a, -) &\in G_{n+1}(A^j), \\ \gamma_j(-, b) &\in G_{n+1}(A^j).\end{aligned}$$

Taking the finite sum over j gives the two restrictions of ψ in $\bigvee_j G_{n+1}(A^j)$. □

Lemma 14.2.1.18 (Subspace form of the one-step block intersection). [Lean](#) [✓ Stmt](#) Under the hypotheses of Lemma 14.2.1.16, set $S_n := \bigvee_j G_{n+1}(A^j)$. Then

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 14.2.1.17 gives the displayed equivalence for each vector ψ . Interpreting the two sides of that equivalence as subspaces gives the stated equality. $\square$

14.2.2 Eventual block-intersection consequences

Propagating the single-length product span through a period window gives the block-intersection identity for all sufficiently large n , both from a general period window and from the normalized BNT representative hypotheses. The full product span at length n also makes the sum of the local spaces $G_n(A^j)$ direct.

Lemma 14.2.2.1 (Period-window form of the block intersection identity). [Lean](#) [✓ Stmt](#) Let $p > 0$. Suppose the length- p simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$, and suppose that, for every $0 \leq r < p$, the length- $(s+r)$ simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$. Assume also the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every block j . Then there is a length N such that, for every $n \geq N$, if $S_n := \bigvee_j G_{n+1}(A^j)$, then

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 14.2.12 gives a length N such that the simultaneous block-word tuples span $\prod_j M_{D_j}(\mathbb{C})$ at every length $n \geq N$. For each such n , Lemma 14.2.1.18 applies and gives the displayed subspace equality. $\square$

Lemma 14.2.2.2 (BNT block separation gives one block-intersection step). [Lean](#) [✓ Stmt](#) Under the hypotheses of Lemma 9.3.3, assume in addition the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every block j . Let

$$n = (L_0 + 1) + (r - 1)3(L_0 + 1),$$

$$S_n = \bigvee_j G_{n+1}(A^j).$$

Then

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 9.3.3 gives $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C})$. Substituting this span into Lemma 14.2.1.18, together with the normalization, gives the stated subspace equality. $\square$

Lemma 14.2.2.3 (BNT block separation with a block-injectivity window). [Lean](#) [✓ Stmt](#) Let $A^1, \dots, A^r$ be normalized representatives of a basis of normal tensors: each representative is irreducible, the family is left-canonical, the self-overlaps are normalized, and no two distinct

same-dimensional representatives are gauge-phase equivalent. Assume $L > 1$, and assume that, for every block j ,

$$\begin{aligned}\text{span}\{A_a^j : a = 0, \dots, d-1\} &= M_{D_j}(\mathbb{C}), \\ \text{span}\{A_u^j : |u| = L\} &= M_{D_j}(\mathbb{C}), \\ \text{span}\{A_v^j : |v| = 3L\} &= M_{D_j}(\mathbb{C}).\end{aligned}$$

Set $S = 3L$ and $q = (r-1)S$. Suppose that $p > 0$ and let s_0 be a starting length. Assume that, for every block j , $\text{span}\{A_u^j : |u| = p\} = M_{D_j}(\mathbb{C})$, and that, for every $0 \leq s < p+q$ and every block j , $\text{span}\{A_v^j : |v| = s_0 + s\} = M_{D_j}(\mathbb{C})$. Assume also the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every block j . Then there is N such that, for every $n \geq N$, if $S_n := \bigvee_j G_{n+1}(A^j)$, then

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. ✓ Proof The direct-sum dimension step gives, for each ordered pair of distinct blocks $k \neq j$, coefficients $c_\omega^{k,j}$ of length S such that

$$\begin{aligned}\sum_{|\omega|=S} c_\omega^{k,j} A_\omega^k &= \mathbb{1}, \\ \sum_{|\omega|=S} c_\omega^{k,j} A_\omega^j &= 0.\end{aligned}$$

Multiplying these equations over the $r-1$ blocks $j \neq k$ gives coefficients $c_\omega^{(k)}$ of total length q satisfying

$$\sum_{|\omega|=q} c_\omega^{(k)} A_\omega^j = \begin{cases} \mathbb{1}, & j = k, \\ 0, & j \neq k. \end{cases}$$

Combining this block-separating equation with block injectivity at length p gives $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = p+q\} = \prod_j M_{D_j}(\mathbb{C})$. Combining the same separating equations with the assumed block-injectivity window gives the full product span at lengths $s_0 + q + s$, for $0 \leq s < p+q$. The period-window form of Lemma 14.2.2.1 then gives the displayed equality for all sufficiently large n . $\square$

Lemma 14.2.2.4 (Unitality propagates simultaneous product spans). Lean Lean ✓ Stmt Suppose that the length- q simultaneous products span the product algebra, $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = q\} = \prod_j M_{D_j}(\mathbb{C})$, and that every block satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Then the simultaneous products span the product algebra at every length $n \geq q$. This is the simultaneous-tuple form of the unital propagation in [PGVWC07, lines 893–898].

Proof. ✓ Proof Given matrices M_j , first represent the tuple $((A_a^j)^\dagger M_j)_j$ by words of length q . Prefixing the letter a and summing over a gives $(\sum_a A_a^j A_a^{j\dagger} M_j)_j = (M_j)_j$. This proves the step from q to $q+1$; induction gives every $n \geq q$. $\square$

Lemma 14.2.2.5 (Normalized block-separating equations give all large product spans). Lean Lean ✓ Stmt Let $A^1, \dots, A^r$ be blocks satisfying the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ and assume

that every block is injective at length L : $\text{span}\{A_u^j : |u| = L\} = M_{D_j}(\mathbb{C})$. If there are block-separating equations of length S , then, for every $n \geq L + (r-1)S$, $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C})$. In particular, for normalized BNT representatives satisfying the displayed unital normalization, if $L_0 > 0$ and $\Gamma_{L_0}^{A_j}$ is injective for every block j , the same conclusion holds with $S = 3(L_0 + 1)$ and $n \geq (L_0 + 1) + (r-1)3(L_0 + 1)$.

Proof. [✓ Proof](#) By Theorem 7.8.1, the normalization propagates the length- L span equality to every larger homogeneous length:

$$m \geq L \implies \text{span}\{A_u^j : |u| = m\} = M_{D_j}(\mathbb{C}).$$

For $n \geq L + (r-1)S$, take $m = n - (r-1)S$. Combining the length- m block span with the $r-1$ block-separating equations gives the displayed product span at length n . In the BNT case, the injectivity of $\Gamma_{L_0}^{A_j}$ gives the full span at length L_0 , and the normalization propagates it to the lengths $L_0 + 1$ and $3(L_0 + 1)$. Lemma 9.3.3 then supplies the block-separating equations with $S = 3(L_0 + 1)$. $\square$

Theorem 14.2.2.6 (Tensor irreducibility implies transfer-map irreducibility). [Lean](#) [✓ Stmt](#) *If an MPS tensor is irreducible, then its transfer map has no nontrivial invariant face.*

Proof. [✓ Proof](#) Tensor irreducibility is the absence of a nontrivial invariant matrix projection, which is precisely the corresponding irreducibility condition for the transfer map. $\square$

Lemma 14.2.2.7 (Positive square-root gauging preserves irreducibility). [Lean](#) [✓ Stmt](#) *Let A have positive bond dimension and irreducible transfer map. If $\Lambda > 0$, then the square-root-gauged tensor $\Lambda^{1/2} A \Lambda^{-1/2}$ is irreducible.*

Proof. [✓ Proof](#) The square-root gauge conjugates the transfer map by an invertible similarity. Invariant faces therefore correspond bijectively before and after gauging. $\square$

Lemma 14.2.2.8 (Gauge-phase equivalence is invariant under gauge replacement). [Lean](#) [✓ Stmt](#) *Let A, A', B, B' have the same bond dimension. If A is gauge equivalent to A' , A' is gauge-phase equivalent to B' , and B is gauge equivalent to B' , then A is gauge-phase equivalent to B .*

Proof. [✓ Proof](#) If X, Y, Z are the left gauge, gauge-phase intertwiner, and right gauge, respectively, then $Z^{-1}YX$ is the required intertwiner from A to B ; it carries the same unit-modulus phase as Y . $\square$

Lemma 14.2.2.9 (Gauge-phase equivalence transports across two gauges). [Lean](#) [✓ Stmt](#) *Gauge equivalent replacements on both sides preserve gauge-phase equivalence, including after identifying equal bond dimensions.*

Proof. [✓ Proof](#) Compose the left gauge, the gauge-phase intertwiner, and the inverse right gauge. After transporting across the bond-dimension identification, the composite is the required gauge-phase intertwiner. $\square$

Theorem 14.2.2.10 (Prepared-gauge simultaneous product span). [Lean](#) [✓ Stmt](#) *Let $r \geq 2$, and let the bond dimensions $D_1, \dots, D_r$ be positive. Suppose that every block A^j is irreducible and*

that no two distinct blocks of the same bond dimension are gauge-phase equivalent. Let $L_0 > 0$, and suppose that, for every j ,

$$\begin{aligned}\text{span}\{A_w^j : |w| = L_0\} &= M_{D_j}(\mathbb{C}), \\ \text{span}\{A_w^j : |w| = L_0 + 1\} &= M_{D_j}(\mathbb{C}), \\ \text{span}\{A_w^j : |w| = 3(L_0 + 1)\} &= M_{D_j}(\mathbb{C}).\end{aligned}$$

For every j , let Λ^j be positive definite and satisfy

$$\sum_a (A_a^j)^\dagger \Lambda^j A_a^j = \Lambda^j.$$

Then

$$\text{span}\{(A_w^1, \dots, A_w^r) : |w| = 3(r-1)(L_0 + 1)\} = \prod_j M_{D_j}(\mathbb{C}).$$

Diagonality of Λ^j is not needed.

Proof. ✓ Proof Define the prepared blocks by

$$B_a^j = (\Lambda^j)^{1/2} A_a^j (\Lambda^j)^{-1/2}.$$

Positive definiteness makes this an invertible gauge, while the dual fixed-point equation gives

$$\sum_a (B_a^j)^\dagger B_a^j = \mathbb{1}.$$

Gauge transport preserves irreducibility, the three fixed-length injectivity statements, and pairwise inequivalence up to gauge and phase. Thus every B^j is algebraically normal and left-canonical, so

$$O_{B^j B^j}(N) \longrightarrow 1.$$

The sharp separation theorem now gives

$$\text{span}\{(B_w^1, \dots, B_w^r) : |w| = 3(r-1)(L_0 + 1)\} = \prod_j M_{D_j}(\mathbb{C}).$$

Theorem 9.3.2 transports this equality back to the source representatives. □

Theorem 14.2.2.11 (Simultaneous product span at the source length). Lean ✓ Stmt Let $r \geq 2$, and let the bond dimensions $D_1, \dots, D_r$ be positive. Suppose that every block A^j is irreducible and that no two distinct blocks of the same bond dimension are gauge-phase equivalent. Let $L_0 > 0$, and suppose every block is injective at length L_0 :

$$\text{span}\{A_w^j : |w| = L_0\} = M_{D_j}(\mathbb{C}).$$

For every j , let Λ^j be positive definite and assume the canonical identities from [PGVWC07]:

$$\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}, \quad \sum_a (A_a^j)^\dagger \Lambda^j A_a^j = \Lambda^j.$$

For every $n \geq 3(r-1)(L_0 + 1)$,

$$\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C}).$$

No identity $\sum_a (A_a^j)^\dagger A_a^j = \mathbb{1}$ is assumed for the source blocks. This is the simultaneous spanning estimate used in [PGVWC07, Theorem 12, proof lines 1424–1456].

Proof. ✓ Proof Theorem 7.8.1 first gives

$$\begin{aligned} \text{span}\{A_w^j : |w| = L_0\} = M_{D_j}(\mathbb{C}) &\implies \text{span}\{A_w^j : |w| = L_0 + 1\} = M_{D_j}(\mathbb{C}) \\ &\implies \text{span}\{A_w^j : |w| = 3(L_0 + 1)\} = M_{D_j}(\mathbb{C}). \end{aligned}$$

Theorem 14.2.2.10 gives the full simultaneous span at $q = 3(r - 1)(L_0 + 1)$ on the source representatives. The source unital identity and Lemma 14.2.2.4 propagate this equality to every $n \geq q$. $\square$

Theorem 14.2.2.12 (Block-space independence at the source length). Lean ✓ Stmt *Under the hypotheses of Theorem 14.2.2.11, the spaces $G_n(A^j)$ form an internal sum for every $n \geq 3(r - 1)(L_0 + 1)$.*

Proof. ✓ Proof Write $\phi_j = \Gamma_n^{A^j}(X_j)$ and suppose that $\sum_j \phi_j = 0$. The simultaneous spanning equality gives

$$\sum_j \Gamma_n^{A^j}(X_j) = 0 \implies \left(\forall (M_j)_j, \sum_j \text{tr}(X_j M_j) = 0 \right) \implies \forall j, X_j = 0.$$

Hence every ϕ_j vanishes. $\square$

Theorem 14.2.2.13 (Direct-sum restriction intersection at the source length). Lean ✓ Stmt *Under the hypotheses of Theorem 14.2.2.11, let $m \geq 3(r - 1)(L_0 + 1) + 1$. Then*

$$\mathbb{C}^d \otimes \left(\sum_j G_m(A^j) \right) \cap \left(\sum_j G_m(A^j) \right) \otimes \mathbb{C}^d = \sum_j G_{m+1}(A^j).$$

Theorem 14.2.2.12 also shows that the sums at lengths m and $m + 1$ are internal. This is the direct-sum lemma in the proof of [PGVWC07, Theorem 12, proof lines 1346–1456].

Proof. ✓ Proof Let a vector belong to both restricted spaces. The two endpoint decompositions give boundary matrices C_a^j and D_b^j . For every pair of physical indices a, b and every middle word w of length $m - 1$, their coefficients obey

$$\sum_j \text{tr}((A_b^j C_a^j) A_w^j) = \sum_j \text{tr}((D_b^j A_a^j) A_w^j).$$

The simultaneous product span at length $m - 1$ and nondegeneracy of the trace pairing therefore give, block by block, $A_b^j C_a^j = D_b^j A_a^j$. Set $E^j = \sum_a C_a^j A_a^{j\dagger}$. The preceding identities and the right-unital normalization imply $A_b^j C_a^j = A_b^j E^j A_a^j$. Thus the component in block j lies in $G_{m+1}(A^j)$. The reverse inclusion follows by splitting a word of length $m + 1$ at either endpoint. $\square$

Theorem 14.2.2.14 (Doubly normalized simultaneous product span). Lean ✓ Stmt *Let $r \geq 2$, and let the bond dimensions be positive. Suppose that the blocks are irreducible, have normalized self-overlap, are pairwise inequivalent up to gauge and phase, and are injective at a common length $L_0 > 0$. Assume both normalization identities*

$$\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}, \quad \sum_a (A_a^j)^\dagger A_a^j = \mathbb{1}.$$

Then for every $n \geq 3(r - 1)(L_0 + 1)$,

$$\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C}).$$

Proof. ✓ Proof Propagate injectivity to the two longer lengths used by the sharp block separation theorem, obtain the simultaneous span at $3(r-1)(L_0+1)$, and propagate it using the right-unital identity. $\square$

Theorem 14.2.2.15 (Doubly normalized block-space independence). Lean ✓ Stmt *Under the hypotheses of Theorem 14.2.2.14, the spaces $G_n(A^j)$ form an internal sum for every $n \geq 3(r-1)(L_0+1)$.*

Proof. ✓ Proof Apply nondegeneracy of the trace pairing to the simultaneous product span. $\square$

Theorem 14.2.2.16 (Doubly normalized restriction intersection). Lean ✓ Stmt *Under the same hypotheses, if $m \geq 3(r-1)(L_0+1) + 1$, then*

$$\mathbb{C}^d \otimes \left(\sum_j G_m(A^j) \right) \cap \left(\sum_j G_m(A^j) \right) \otimes \mathbb{C}^d = \sum_j G_{m+1}(A^j).$$

Proof. ✓ Proof Use the simultaneous product span at length $m-1$ in the endpoint restriction calculation. $\square$

Lemma 14.2.2.17 (Product span makes the sum of local spaces direct). Lean Lean ✓ Stmt *Suppose $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C})$. Then the sum of the local spaces $G_n(A^j)$ is direct: if*

$$\begin{aligned} \phi_j &\in G_n(A^j), \\ \sum_j \phi_j &= 0, \end{aligned}$$

then $\phi_j = 0$ for every j . In particular, for the normalized BNT hypotheses above, this directness holds for every $n \geq (L_0+1) + (r-1)3(L_0+1)$.

Proof. ✓ Proof Write $\phi_j(\sigma) = \text{tr}(A_\sigma^j X_j)$. Evaluating $\sum_j \phi_j = 0$ on each word w of length n gives $\sum_j \text{tr}(A_w^j X_j) = 0$. Since $\text{tr}(A_w^j X_j) = \text{tr}(X_j A_w^j)$, the product-span equation gives, for every $(M_1, \dots, M_r) \in \prod_j M_{D_j}(\mathbb{C})$, $\sum_j \text{tr}(X_j M_j) = 0$. Taking only the j -th component nonzero and using nondegeneracy of the trace pairing yields $X_j = 0$, hence $\phi_j = 0$. The BNT statement follows by substituting Lemma 14.2.2.5. $\square$

14.3 Block-diagonal boundary conditions for periodic ground spaces

For a block-diagonal tensor $B = \bigoplus_{j=1}^g \mu_j A_j$ with $\mu_j \neq 0$, this section proves implications from explicit block-diagonal boundary identities to the component membership $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. These implications supply the boundary-condition step in the periodic ring used by [CPGSV21, Theorem IV.16].

Lemma 14.3.1 (Normalized BNT representatives give the large-length block intersection). Lean Lean ✓ Stmt *Let $A^1, \dots, A^r$ be normalized representatives of a basis of normal tensors: each representative is irreducible, the family is left-canonical, the self-overlaps are normalized, and no*

two distinct same-dimensional representatives are gauge-phase equivalent. Suppose $L_0 > 0$, and, for every block j , the map

$$\Gamma_{L_0}^{A^j} : M_{D_j}(\mathbb{C}) \longrightarrow (\mathbb{C}^d)^{\otimes L_0}$$

is injective. Assume also the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Set $N = (L_0 + 1) + (r - 1)3(L_0 + 1)$. For every $n \geq N$, set $S_n := \bigvee_j G_{n+1}(A^j)$. The two consecutive sums are internal direct sums:

$$\begin{aligned} \bigvee_j G_{n+1}(A^j) &= \bigoplus_j G_{n+1}(A^j), \\ \bigvee_j G_{n+2}(A^j) &= \bigoplus_j G_{n+2}(A^j). \end{aligned}$$

With this notation,

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 14.2.2.5 gives $\text{span}\{(A_w^1, \dots, A_w^r) : |w| = n\} = \prod_j M_{D_j}(\mathbb{C})$ for every $n \geq N$. Substituting this product-span equation into Lemma 14.2.1.18, together with the normalization equation, gives the displayed intersection identity. Lemma 14.2.2.17 at lengths $n + 1$ and $n + 2$ gives the two directness statements. $\square$

Lemma 14.3.2 (Normalized BNT representatives give an eventual block intersection). [Lean](#) [✓ Stmt](#) Let $A^1, \dots, A^r$ be normalized representatives of a basis of normal tensors: each representative is irreducible, the family is left-canonical, the self-overlaps are normalized, and no two distinct same-dimensional representatives are gauge-phase equivalent. Suppose $L_0 > 0$, and, for every block j , the map

$$\Gamma_{L_0}^{A^j} : M_{D_j}(\mathbb{C}) \longrightarrow (\mathbb{C}^d)^{\otimes L_0}$$

is injective. Assume also the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Then there is a length N such that, for every $n \geq N$, with $S_n := \bigvee_j G_{n+1}(A^j)$, one has

$$\{\psi : \forall b, \psi(-, b) \in S_n\} \cap \{\psi : \forall a, \psi(a, -) \in S_n\} = \bigvee_j G_{n+2}(A^j).$$

Proof. [✓ Proof](#) Apply Lemma 14.3.1 and take $N = (L_0 + 1) + (r - 1)3(L_0 + 1)$. $\square$

Lemma 14.3.3 (Block-diagonal boundary-condition formula). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$. For block-diagonal boundary conditions X_j ,

$$\Gamma_L^B \left(\bigoplus_{j=1}^g X_j \right) = \sum_{j=1}^g \Gamma_L^{A_j}(\mu_j^L X_j).$$

Proof. [✓ Proof](#) This is the diagonal-block trace formula specialized to a block-diagonal boundary matrix. The j -th diagonal block of $\bigoplus_k X_k$ is X_j , giving the displayed sum. $\square$

Lemma 14.3.4 (Boundary matrices for a sum of open-boundary block spaces). [Lean](#) [✓ Stmt](#)
Let $B = \bigoplus_{j=1}^g \mu_j A_j$, where $\mu_j \neq 0$ for every j . For every length L and every $\psi \in \bigvee_{j=1}^g G_L(A_j)$, there are boundary matrices X_j such that

$$\psi = \Gamma_L^B \left(\bigoplus_{j=1}^g X_j \right).$$

For every j , one also has $\Gamma_L^{A_j}(\mu_j^L X_j) \in G_L(A_j)$.

Proof. [✓ Proof](#) Choose $\phi_j \in G_L(A_j)$ such that

$$\psi = \sum_{j=1}^g \phi_j.$$

Choose Y_j with $\phi_j = \Gamma_L^{A_j}(Y_j)$, and set $X_j = \mu_j^{-L} Y_j$. Since $\mu_j \neq 0$, one has, for every j ,

$$\Gamma_L^{A_j}(\mu_j^L X_j) = \phi_j.$$

Lemma 14.3.3 now gives

$$\Gamma_L^B \left(\bigoplus_{j=1}^g X_j \right) = \sum_{j=1}^g \Gamma_L^{A_j}(\mu_j^L X_j) = \sum_{j=1}^g \phi_j = \psi.$$

□

Lemma 14.3.5 (One block lies in the local parent space of the direct-sum tensor). [Lean](#) [✓ Stmt](#)
Let $B = \bigoplus_{k=1}^g \mu_k A_k$. If $\mu_j \neq 0$, then, for every length L , one has $G_L(A_j) \subseteq G_L(B)$.

Proof. [✓ Proof](#) A vector in $G_L(A_j)$ has the form $\Gamma_L^{A_j}(X)$. Put $\mu_j^{-L} X$ in the j -th diagonal block of a boundary matrix for B and put zero in every other block. If $\iota_j(Z)$ denotes this block insertion, Lemma 14.3.3 gives

$$\Gamma_L^B(\iota_j(\mu_j^{-L} X)) = \sum_{k=1}^g \Gamma_L^{A_k}(\mu_k^L (\iota_j(\mu_j^{-L} X))_{kk}) = \Gamma_L^{A_j}(X).$$

Hence $\Gamma_L^{A_j}(X) \in G_L(B)$.

□

Lemma 14.3.6 (Local ground space of a block-diagonal tensor). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$, where $\mu_j \neq 0$ for $1 \leq j \leq g$. Then, for every length L , $G_L(B) = \bigvee_{j=1}^g G_L(A_j)$.

Proof. [✓ Proof](#) A boundary matrix X for B has diagonal blocks X_j , and block-diagonality gives $\Gamma_L^B(X) = \sum_{j=1}^g \Gamma_L^{A_j}(\mu_j^L X_j)$. This proves $G_L(B) \subseteq \bigvee_j G_L(A_j)$. The reverse inclusion is Lemma 14.3.5 for each block j .

□

Lemma 14.3.7 (Periodic constraints for a block-diagonal tensor). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$, where $\mu_j \neq 0$ for $1 \leq j \leq g$, and set $S_M := \bigvee_{j=1}^g G_M(A_j)$. If, for every $M \geq L$, $\mathbb{C}^d \otimes S_M \cap S_M \otimes \mathbb{C}^d = S_{M+1}$, then $\mathcal{G}_{N,L}(B) \subseteq S_N$.

Proof. [✓ Proof](#) Lemma 14.3.6 gives $G_L(B) = S_L$. Apply Lemma 12.4.5.18 to the sequence S_M . $\square$

Lemma 14.3.8 (Periodic constraints lie in the block local-space sum). [Lean](#) [✓ Stmt](#) *Let $A_1, \dots, A_g$ be normalized representatives of a basis of normal tensors: each representative is irreducible, the family is left-canonical, the self-overlaps are normalized, and no two distinct same-dimensional representatives are gauge-phase equivalent. Suppose $L_0 > 0$, and, for every block j , the map*

$$\Gamma_{L_0}^{A_j} : M_{D_j}(\mathbb{C}) \longrightarrow (\mathbb{C}^d)^{\otimes L_0}$$

is injective. Assume also the normalization $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. For nonzero scalars μ_j , set

$$B = \bigoplus_{j=1}^g \mu_j A_j,$$

$$S_M := \bigvee_{j=1}^g G_M(A_j).$$

If $L \geq (L_0 + 1) + (g - 1)3(L_0 + 1) + 1$, then, for every $N \geq L$, $\mathcal{G}_{N,L}(B) \subseteq S_N$.

Proof. [✓ Proof](#) Put $T = (L_0 + 1) + (g - 1)3(L_0 + 1)$. For $M \geq L$, one has $M - 1 \geq T$. Applying Lemma 14.3.1 at $n = M - 1$ gives $\mathbb{C}^d \otimes S_M \cap S_M \otimes \mathbb{C}^d = S_{M+1}$ for every $M \geq L$. Lemma 14.3.7 then propagates the local constraint from L to N . $\square$

Lemma 14.3.9 (Periodic constraints and internality of the block local-space sum). [Lean](#) [✓ Stmt](#) *With the hypotheses and notation of Lemma 14.3.8, set $S_N := \bigvee_{j=1}^g G_N(A_j)$. Then, for every $N \geq L$, $\mathcal{G}_{N,L}(B) \subseteq S_N$, and the sum defining S_N is internal: if*

$$\phi_j \in G_N(A_j),$$

$$\sum_{j=1}^g \phi_j = 0,$$

then $\phi_j = 0$ for every j .

Proof. [✓ Proof](#) The inclusion is Lemma 14.3.8. Since $N \geq L$ and $L \geq (L_0 + 1) + (g - 1)3(L_0 + 1) + 1$, the product-span directness range of Lemma 14.2.2.17 applies at length N :

$$\text{span}\{(A_w^1, \dots, A_w^g) : |w| = N\} = \prod_j M_{D_j}(\mathbb{C})$$

$$\implies \ker \left(\bigoplus_{j=1}^g G_N(A_j) \longrightarrow (\mathbb{C}^d)^{\otimes N}, \quad (\phi_j)_j \mapsto \sum_j \phi_j \right) = 0.$$

$\square$

Lemma 14.3.10 (Periodic parent-space vectors decompose in the local block spaces). [Lean](#) [✓ Stmt](#) *With the hypotheses and notation of Lemma 14.3.8, every vector $\psi \in \mathcal{G}_{N,L}(\bigoplus_{j=1}^g \mu_j A_j)$ has a unique family $(\psi_j)_{j=1}^g$ such that*

$$\psi = \sum_{j=1}^g \psi_j,$$

$$\psi_j \in G_N(A_j).$$

Proof. [✓ Proof](#) Lemma 14.3.9 gives

$$\mathcal{G}_{N,L} \left(\bigoplus_{j=1}^g \mu_j A_j \right) \subseteq \bigvee_{j=1}^g G_N(A_j),$$

and the spaces $G_N(A_j)$ are independent: for every j and every $x_j \in G_N(A_j)$, $\sum_j x_j = 0 \implies x_j = 0$. Thus every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ has a unique decomposition $\psi = \sum_j \psi_j$ with $\psi_j \in G_N(A_j)$. $\square$

Lemma 14.3.11 (Block-diagonal boundary conditions for periodic parent-space vectors). [Lean](#)
[✓ Stmt](#) With the hypotheses and notation of Lemma 14.3.8, every vector $\psi \in \mathcal{G}_{N,L}(\bigoplus_{j=1}^g \mu_j A_j)$ admits block-diagonal boundary conditions X_j such that

$$\begin{aligned} \psi &= \Gamma_N^B \left(\bigoplus_{j=1}^g X_j \right), \\ \Gamma_N^{A_j}(\mu_j^N X_j) &\in G_N(A_j) \end{aligned}$$

for every j .

Proof. [✓ Proof](#) Lemma 14.3.8 places ψ in $\bigvee_{j=1}^g G_N(A_j)$. Applying Lemma 14.3.4 gives boundary matrices X_j for which, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in G_N(A_j)$, and

$$\psi = \Gamma_N^B \left(\bigoplus_{j=1}^g X_j \right).$$

$\square$

Lemma 14.3.12 (Block-diagonal boundary conditions at the source length). [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Let $g \geq 2$. Let A_j be irreducible blocks, pairwise inequivalent up to gauge and phase, and injective at the common length $L_0 > 0$. Assume the canonical normalization of [PGVWC07]:

$$\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}, \quad \sum_a A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j, \quad \Lambda_j > 0.$$

Let every weight μ_j be nonzero. If $0 < L \leq N$ and $L \geq 3(g-1)(L_0+1)+1$, then

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) \subseteq \bigoplus_j G_N(A_j).$$

In particular, every vector ψ in the left-hand side admits matrices X_j such that

$$\begin{aligned} \psi &= \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right), \\ \Gamma_N^{A_j}(\mu_j^N X_j) &\in G_N(A_j). \end{aligned}$$

This is the open-boundary decomposition used in [PGVWC07, Theorem 12, proof lines 1424–1456].

Proof. ✓ **Proof** Put $S_M = \sum_j G_M(A_j)$. For every M with $L \leq M < N$, Theorem 14.2.2.13 gives

$$\left(\bigcap_b \text{Res}_{-,b}^{-1} S_M \right) \cap \left(\bigcap_a \text{Res}_{a,-}^{-1} S_M \right) = S_{M+1}.$$

Induction through the cyclic local constraints therefore gives

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) \subseteq S_N.$$

Since $N \geq L \geq 3(g-1)(L_0+1)+1$, Theorem 14.2.2.12 shows that the spaces $G_N(A_j)$ form an internal sum. Lemma 14.3.4 then supplies matrices X_j such that $\Gamma_N^{A_j}(\mu_j^N X_j) \in G_N(A_j)$ for every j and

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right).$$

□

Theorem 14.3.13 (Doubly normalized block-diagonal boundary conditions at the source length).

Lean Lean Lean ✓ **Stmt** Let $g \geq 2$, assume the physical alphabet is nonempty, and let A_j be irreducible blocks with positive bond dimensions, pairwise inequivalent up to gauge and phase, and injective at the common length $L_0 > 0$. Assume normalized self-overlap and

$$\begin{aligned} \sum_a (A_a^j)^\dagger A_a^j &= \mathbb{1}, \\ \sum_a A_a^j (A_a^j)^\dagger &= \mathbb{1}. \end{aligned}$$

Let every weight μ_j be nonzero. If $0 < L \leq N$ and $L \geq 3(g-1)(L_0+1)+1$, then

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) \subseteq \bigoplus_j G_N(A_j).$$

In particular, every vector ψ in the left-hand side admits matrices X_j such that

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in G_N(A_j)$$

for every j .

Proof. ✓ **Proof** Put $S_M = \sum_j G_M(A_j)$. For every M with $L \leq M < N$, Theorem 14.2.2.16 gives

$$\left(\bigcap_b \text{Res}_{-,b}^{-1} S_M \right) \cap \left(\bigcap_a \text{Res}_{a,-}^{-1} S_M \right) = S_{M+1}.$$

Induction through the cyclic local constraints therefore yields

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) \subseteq S_N.$$

At the stated length, the block-space independence theorem makes this sum internal. Lemma 14.3.4 then supplies the matrices X_j . □

The theorem gives $\Gamma_N^{A_j}(\mu_j^N X_j) \in G_N(A_j)$. This is not enough for periodic-boundary membership when a cyclic window crosses the chosen cut. The boundary-cut comparison of Lemma 14.3.1.24 concerns the separate assertion $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

14.3.1 Cyclic windows at the boundary cut

A cyclic length- L window either stays inside the chosen linear interval ($i + L \leq N$) or crosses the boundary cut ($N < i + L$). The non-wrapping case follows from trace cyclicity, while the crossing case requires a boundary-matrix identity $\mu_j^N X_j A_\beta^j = A_\beta^j Y$. The lemmas below establish the single-block periodic membership $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ in both cases.

Lemma 14.3.1.1 (Cyclic restrictions of block-diagonal boundary vectors). [Lean](#) [Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$, and fix block-diagonal boundary conditions X_j . For every cyclic restriction $R_{i,\tau}$ from N sites to L sites,

$$R_{i,\tau} \left(\Gamma_N^B \left(\bigoplus_{j=1}^g X_j \right) \right) = \sum_{j=1}^g R_{i,\tau} \left(\Gamma_N^{A_j}(\mu_j^N X_j) \right).$$

Proof. [Proof](#) Apply Lemma 14.3.3 at length N , then use linearity of the cyclic restriction $R_{i,\tau}$. $\square$

Lemma 14.3.1.2 (Local direct-sum constraint from block-diagonal boundary conditions). [Lean](#) [Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$, with $\mu_j \neq 0$, and suppose

$$\begin{aligned} \psi &= \Gamma_N^B \left(\bigoplus_{j=1}^g X_j \right), \\ \psi &\in \mathcal{G}_{N,L}(B). \end{aligned}$$

Then every local constraint of ψ gives

$$\sum_{j=1}^g R_{i,\tau} \left(\Gamma_N^{A_j}(\mu_j^N X_j) \right) \in \bigvee_{j=1}^g G_L(A_j).$$

Proof. [Proof](#) The hypothesis $\psi \in \mathcal{G}_{N,L}(B)$ says that the left-hand side before decomposition lies in $G_L(B)$. Lemma 14.3.1.1 rewrites this local vector as the displayed sum, and Lemma 14.3.6 identifies $G_L(B)$ with $\bigvee_j G_L(A_j)$. $\square$

Lemma 14.3.1.3 (Cyclic windows not crossing the cut). [Lean](#) [Stmt](#) Fix block-diagonal boundary conditions X_j . If the cyclic window beginning at i stays inside the chosen linear interval, $i + L \leq N$, then for every block j ,

$$R_{i,\tau} \left(\Gamma_N^{A_j}(\mu_j^N X_j) \right) \in G_L(A_j).$$

Proof. [Proof](#) When the window stays inside the chosen linear interval, the boundary matrix remains outside the window. Write $A_{\text{left}} = A_{\tau_0}^j \cdots A_{\tau_{i-1}}^j$ and $A_{\text{right}} = A_{\tau_{i+L}}^j \cdots A_{\tau_{N-1}}^j$. Trace cyclicity gives

$$R_{i,\tau} \left(\Gamma_N^{A_j}(\mu_j^N X_j) \right) = \Gamma_L^{A_j} (A_{\text{right}} \mu_j^N X_j A_{\text{left}}).$$

Hence the restricted vector lies in $G_L(A_j)$. $\square$

Lemma 14.3.1.4 (Boundary-crossing coefficient formula). [Lean](#) [✓ Stmt](#) Assume $0 < N, L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Write $a = i + L - N$. For a local word σ of length L , set

$$\begin{aligned}\beta &= \sigma_{N-i} \cdots \sigma_{L-1}, \\ \rho &= \tau_a \cdots \tau_{i-1}, \\ \alpha &= \sigma_0 \cdots \sigma_{N-i-1}.\end{aligned}$$

Then the corresponding coefficient of the j -th block component is

$$R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) (\sigma) = \text{tr} \left(A_\beta^j A_\rho^j A_\alpha^j \mu_j^N X_j \right).$$

Proof. [✓ Proof](#) Let ω be the N -site word obtained by inserting σ in the cyclic interval beginning at i and by using τ outside that interval. Since the interval crosses the cut, $\omega = \beta\rho\alpha$. Hence

$$\Gamma_N^{A_j} (\mu_j^N X_j) (\omega) = \text{tr} \left(A_\beta^j A_\rho^j A_\alpha^j \mu_j^N X_j \right).$$

This is exactly $R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) (\sigma)$. □

Lemma 14.3.1.5 (Boundary-crossing trace decomposition from the local constraint). [Lean](#) [✓ Stmt](#) Assume $0 < N, L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Put $a = i + L - N$. For a local word σ of length L , write

$$\begin{aligned}\beta &= \sigma_{N-i} \cdots \sigma_{L-1}, \\ \rho &= \tau_a \cdots \tau_{i-1}, \\ \alpha &= \sigma_0 \cdots \sigma_{N-i-1}.\end{aligned}$$

These are the word coordinates obtained by opening the periodic boundary at the chosen cut, in the comparison with the source expressions involving $C_{i_1}^j$ and $D_{i_{m+1}}^j$. If

$$\sum_j R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in \bigvee_j G_L(A_j),$$

then there are matrices $C_{i,\tau}^j$ such that, for every σ ,

$$\sum_j \text{tr} \left(A_\beta^j C_{i,\tau}^j A_\alpha^j \right) = \sum_j \text{tr} \left(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_\alpha^j \right).$$

Proof. [✓ Proof](#) Decompose the local vector in $\bigvee_j G_L(A_j)$ as a sum of vectors $\Gamma_L^{A_j}(C_{i,\tau}^j)$. Evaluating at σ gives $\sum_j \text{tr} \left(A_{\alpha\beta}^j C_{i,\tau}^j \right)$. Trace cyclicity rewrites this as $\sum_j \text{tr} \left(A_\beta^j C_{i,\tau}^j A_\alpha^j \right)$. On the other hand, Lemma 14.3.1.4 gives the second trace expression for the same local coefficient sum. □

Lemma 14.3.1.6 (Boundary representation gives fixed-tail boundary traces). [Lean](#) [✓ Stmt](#) Assume that the physical dimension d is positive, $0 < N, L \leq N$, and $\mu_j \neq 0$ for every block j . Let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Let

$$\begin{aligned}\psi &\in \mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right), \\ \psi &= \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right).\end{aligned}$$

For every outside word ρ of length $N - L$, there are matrices $C_{i,\rho}^j$ such that, for every word β of length $i + L - N$ before the cut and every wrapped tail word α of length $N - i$,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_\alpha^j) = \sum_j \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_\alpha^j).$$

Proof. ✓ Proof Since $d > 0$, extend the outside word ρ to a full outside configuration τ whose entries between the end of the interval and the cut are the word ρ . By the nonzero-weight hypothesis, Lemma 14.3.1.2 applies and puts the sum of the corresponding block restrictions in $\bigvee_j G_L(A_j)$. Applying Lemma 14.3.1.5 gives the displayed trace equality after rewriting the opened cyclic word as $\beta\rho\alpha$. □

Lemma 14.3.1.7 (Boundary representation gives all fixed-tail boundary traces). Lean ✓ Stmt Assume that the physical dimension d is positive, $0 < N$, $L \leq N$, and $\mu_j \neq 0$ for every block j . Let

$$\begin{aligned} \psi &\in \mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right), \\ \psi &= \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right). \end{aligned}$$

Then there are matrices $C_{i,\rho}^j$, indexed by the boundary-crossing start i and by an outside word ρ of length $N - L$, such that whenever $N < i + L$,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_\alpha^j) = \sum_j \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_\alpha^j)$$

for every word β of length $i + L - N$ before the cut and every wrapped tail word α of length $N - i$.

Proof. ✓ Proof Apply Lemma 14.3.1.6 to each boundary-crossing start i and each outside word ρ , and choose the resulting matrices $C_{i,\rho}^j$. □

Theorem 14.3.1.8 (Fixed boundary-crossing C^j, D^j comparison from the local block-sum constraint). Lean ✓ Stmt Assume $0 < N$, $L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Put $a = i + L - N$ and $\rho = \tau_a \cdots \tau_{i-1}$. Suppose also that the block tensors have the common word-span property at length $N - i$. If

$$\sum_j R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in \bigvee_j G_L(A_j),$$

then there are matrices $C_{i,\tau}^j$ such that, for every block j and every word β of length a ,

$$A_\beta^j C_{i,\tau}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j. \tag{14.1}$$

Proof. ✓ Proof Lemma 14.3.1.5 gives matrices $C_{i,\tau}^j$ with

$$\sum_j \text{tr}(A_\beta^j C_{i,\tau}^j A_\alpha^j) = \sum_j \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_\alpha^j)$$

for every word α of length $N - i$ and every word β of length a . The fixed complementary-word trace comparison, applied with X_j replaced by $\mu_j^N X_j$, gives (14.1). □

Theorem 14.3.1.9 (C^j, D^j comparison for a block-diagonal boundary representation). [Lean](#)

[✓ Stmt](#) Let

$$\begin{aligned}\psi &\in \mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right), \\ \psi &= \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right).\end{aligned}$$

Assume that the physical alphabet is nonempty, that every μ_j is nonzero, that $0 < N$, and that $L \leq N$. For every boundary-crossing interval beginning at i , assume that the simultaneous products

$$\begin{aligned}w &\mapsto (A_w^1, \dots, A_w^r), \\ |w| &= N - i,\end{aligned}$$

span the product algebra $\prod_j M_{D_j}(\mathbb{C})$. Then there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every block j , and every word β before the cut, $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$.

Proof. [✓ Proof](#) Lemma 14.3.1.7 gives matrices $C_{i,\rho}^j$ such that, for every boundary-crossing start i , outside word ρ of length $N - L$, word β of length $i + L - N$ before the cut, and wrapped tail word α of length $N - i$,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_\alpha^j) = \sum_j \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_\alpha^j).$$

Apply Lemma 14.2.1.6 with the stated product-algebra span at length $N - i$. Equality of the traces for all α then gives $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$ for every block j . $\square$

Theorem 14.3.1.10 (Fixed boundary-crossing component in the local block space). [Lean](#)

[✓ Stmt](#) Assume $0 < N$, $L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Suppose that the block tensors have the common word-span property at length $N - i$ and that

$$\sum_j R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in \bigvee_j G_L(A_j).$$

Then each block component of the fixed boundary-crossing restriction lies in its local block space:

$$R_{i,\tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in G_L(A_j).$$

Proof. [✓ Proof](#) Theorem 14.3.1.8 gives matrices $C_{i,\tau}^j$ satisfying (14.1). Thus $E = C_{i,\tau}^j$ satisfies the boundary matrix identity required in Lemma 14.3.1.11, and the displayed membership follows for every block j . $\square$

Lemma 14.3.1.11 (Boundary-crossing cyclic intervals from the boundary matrix identity).

[Lean](#) [✓ Stmt](#) Assume $0 < N$, $L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Put $a = i + L - N$. Suppose that there is a matrix E such that for every word β of length a ,

$$\mu_j^N X_j A_\beta^j A_{\tau_a}^j \cdots A_{\tau_{i-1}}^j = A_\beta^j E. \quad (14.2)$$

Then

$$R_{i,\tau}\left(\Gamma_N^{A_j}(\mu_j^N X_j)\right) \in G_L(A_j).$$

Proof. ✓ Proof Write a length- L word in the boundary-crossing interval as $\alpha\beta$, where α is the segment inside $\{i, \dots, N-1\}$ and β is the segment after the interval wraps past the cut, inside $\{0, \dots, a-1\}$. The full N -site word appearing in the restriction is then $\beta\tau_a \cdots \tau_{i-1}\alpha$. Trace cyclicity gives

$$\text{tr}\left(A_{\beta}^j A_{\tau_a}^j \cdots A_{\tau_{i-1}}^j A_{\alpha}^j \mu_j^N X_j\right) = \text{tr}\left(A_{\alpha}^j \mu_j^N X_j A_{\beta}^j A_{\tau_a}^j \cdots A_{\tau_{i-1}}^j\right) = \text{tr}\left(A_{\alpha}^j A_{\beta}^j E\right),$$

where the second equality follows from (14.2). The last expression is the coefficient of $\Gamma_L^{A_j}(E)$ at the word $\alpha\beta$. Hence the restricted vector lies in $G_L(A_j)$. $\square$

Lemma 14.3.1.12 (Last crossing-window component trace form). Lean ✓ Stmt Write $N = M+1$ and let the local length be $m+2 \leq M+1$. Consider the cyclic interval beginning at the last site M . For an outside configuration τ , put

$$\begin{aligned} \rho &= \tau_{m+1} \cdots \tau_{M-1}, \\ C_a^j &= A_{\rho}^j A_a^j (\mu_j^{M+1} X_j). \end{aligned}$$

Then the j -th component of the last crossing-window restriction has the left-boundary trace form

$$\begin{aligned} R_{M,\tau}\left(\Gamma_{M+1}^{A_j}(\mu_j^{M+1} X_j)\right) &= \Phi^j, \\ \Phi^j(a, w, b) &= \text{tr}\left(A_b^j C_a^j A_w^j\right). \end{aligned}$$

Proof. ✓ Proof The cyclic interval beginning at M reads the last site first and then wraps to the sites $0, \dots, m$. Thus, for the local word awb , the full word in the trace is $wb\rho a$. Hence the j -th coefficient is

$$\text{tr}\left(A_w^j A_b^j A_{\rho}^j A_a^j (\mu_j^{M+1} X_j)\right) = \text{tr}\left(A_b^j A_{\rho}^j A_a^j (\mu_j^{M+1} X_j) A_w^j\right),$$

where the equality follows from trace cyclicity. This is the displayed left-boundary trace form. $\square$

Lemma 14.3.1.13 (Last crossing-window trace form). Lean ✓ Stmt Write $N = M+1$ and let the local length be $m+2 \leq M+1$. Consider the cyclic interval beginning at the last site M . For an outside configuration τ , put

$$\begin{aligned} \rho &= \tau_{m+1} \cdots \tau_{M-1}, \\ C_a^j &= A_{\rho}^j A_a^j (\mu_j^{M+1} X_j). \end{aligned}$$

Then the block sum of the last crossing-window restrictions has the left-boundary trace form

$$\begin{aligned} \sum_j R_{M,\tau}\left(\Gamma_{M+1}^{A_j}(\mu_j^{M+1} X_j)\right) &= \sum_j \Phi^j, \\ \Phi^j(a, w, b) &= \text{tr}\left(A_b^j C_a^j A_w^j\right). \end{aligned}$$

Proof. ✓ Proof Sum Lemma 14.3.1.12 over the block index. $\square$

Lemma 14.3.1.14 (Last crossing-window coefficient comparison). [Lean](#) [✓ Stmt](#) In the setting of Lemma 14.3.1.13, assume that the length- m simultaneous word tuples span the product algebra and that

$$\sum_j R_{M,\tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in \bigvee_j G_{m+2}(A^j).$$

Then there are matrices E_j such that, for every block j and all boundary letters a, b ,

$$A_b^j (A_\rho^j A_a^j (\mu_j^{M+1} X_j)) = A_b^j E_j A_a^j. \quad (14.3)$$

Proof. [✓ Proof](#) Lemma 14.3.1.13 writes the local block sum in the form

$$\begin{aligned} \Phi(a, w, b) &= \sum_j \text{tr} \left(A_b^j C_a^j A_w^j \right), \\ C_a^j &= A_\rho^j A_a^j (\mu_j^{M+1} X_j). \end{aligned}$$

Applying Lemma 14.2.1.3 gives (14.3). □

Lemma 14.3.1.15 (Last crossing-window component membership). [Lean](#) [✓ Stmt](#) In the setting of Lemma 14.3.1.13, assume that the length- m simultaneous word tuples span the product algebra and that

$$\sum_j R_{M,\tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in \bigvee_j G_{m+2}(A^j).$$

Then, for every block j ,

$$R_{M,\tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in G_{m+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 14.3.1.14 gives matrices E_j satisfying (14.3). By Lemma 14.3.1.12, the j -th cyclic restriction is the left-boundary component determined by C_a^j . The displayed identity makes that component equal to $\Gamma_{m+2}^{A_j}(E_j)$, hence it lies in $G_{m+2}(A^j)$. □

Lemma 14.3.1.16 (Last crossing-window coefficients from a block-diagonal chain vector). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_j \mu_j A^j$, and assume $\mu_j \neq 0$ for every block j . Suppose

$$\begin{aligned} \psi &= \Gamma_{M+1}^B \left(\bigoplus_j X_j \right), \\ \psi &\in \mathcal{G}_{M+1, m+2}(B). \end{aligned}$$

Assume also that the length- m simultaneous word tuples span the product algebra. Then, for every outside configuration τ , with $\rho = \tau_{m+1} \cdots \tau_{M-1}$, there are matrices E_j such that $A_b^j (A_\rho^j A_a^j (\mu_j^{M+1} X_j)) = A_b^j E_j A_a^j$ for every block j and all boundary letters a, b .

Proof. [✓ Proof](#) The local constraint of ψ at the cyclic interval beginning at M gives

$$\sum_j R_{M,\tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in \bigvee_j G_{m+2}(A^j)$$

by Lemma 14.3.1.2. Applying Lemma 14.3.1.14 gives the displayed equations. □

Lemma 14.3.1.17 (Last crossing-window component membership from a chain vector). [Lean](#)

[✓ Stmt](#) Let $B = \bigoplus_j \mu_j A^j$, and assume $\mu_j \neq 0$ for every block j . Suppose

$$\begin{aligned}\psi &= \Gamma_{M+1}^B \left(\bigoplus_j X_j \right), \\ \psi &\in \mathcal{G}_{M+1, m+2}(B).\end{aligned}$$

If the length- m simultaneous word tuples span the product algebra, then for every outside configuration τ and every block j ,

$$R_{M, \tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in G_{m+2}(A^j).$$

Proof. [✓ Proof](#) Lemma 14.3.1.2, applied to the window beginning at the last site, gives

$$\sum_j R_{M, \tau} \left(\Gamma_{M+1}^{A_j} (\mu_j^{M+1} X_j) \right) \in \bigvee_j G_{m+2}(A^j).$$

Lemma 14.3.1.15 then applies to each block component. □

Lemma 14.3.1.18 (Boundary-crossing intervals from a right boundary-matrix identity). [Lean](#)

[✓ Stmt](#) Assume $0 < N$ and $L \leq N$, and let the cyclic interval beginning at i cross the boundary cut, so $N < i + L$. Put $a = i + L - N$. Suppose that there is a matrix Y such that, for every word β of length a , $\mu_j^N X_j A_\beta^j = A_\beta^j Y$. Then

$$R_{i, \tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in G_L(A_j).$$

Proof. [✓ Proof](#) Let $E := Y A_{\tau_a}^j \cdots A_{\tau_{i-1}}^j$. Then, for every word β on the segment $0, \dots, a-1$,

$$\mu_j^N X_j A_\beta^j A_{\tau_a}^j \cdots A_{\tau_{i-1}}^j = A_\beta^j E.$$

Lemma 14.3.1.11 gives the stated local constraint. □

Lemma 14.3.1.19 (Boundary-crossing cyclic intervals suffice). [Lean](#) [✓ Stmt](#) Assume $0 < N$ and $L \leq N$. Fix block-diagonal boundary conditions X_j . Suppose that, for every block j , every cyclic interval beginning at i with $N < i + L$, and every configuration τ ,

$$R_{i, \tau} \left(\Gamma_N^{A_j} (\mu_j^N X_j) \right) \in G_L(A_j).$$

Then, for every block j , $\Gamma_N^{A_j} (\mu_j^N X_j) \in \mathcal{G}_{N, L}(A_j)$.

Proof. [✓ Proof](#) By definition, membership in $\mathcal{G}_{N, L}(A_j)$ is the collection of all cyclic length- L local constraints. For an interval beginning at i , either $i + L \leq N$ or $N < i + L$. In the first case, use Lemma 14.3.1.3. In the second case, use the displayed hypothesis. □

Lemma 14.3.1.20 (Right boundary-matrix identities give single-block periodic-boundary constraints). [Lean](#) [✓ Stmt](#) Assume $0 < N$ and $L \leq N$. Fix block-diagonal boundary conditions X_j . Suppose that, for every block j , every boundary-crossing interval beginning at i , and every configuration τ , there is a matrix $Y_{j, i, \tau}$ such that, for every word β of length $a = i + L - N$, $\mu_j^N X_j A_\beta^j = A_\beta^j Y_{j, i, \tau}$. Then, for every block j , $\Gamma_N^{A_j} (\mu_j^N X_j) \in \mathcal{G}_{N, L}(A_j)$.

Proof. [✓ Proof](#) Lemma 14.3.1.18 turns the displayed boundary identity into the local constraint for each boundary-crossing interval. Lemma 14.3.1.19 then yields $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$, handling both boundary-crossing intervals $N < i + L$ and non-crossing intervals $i + L \leq N$. $\square$

Lemma 14.3.1.21 (Spanning matrix identities give single-block constraints). [Lean](#) [✓ Stmt](#) Assume $0 < N$ and $L \leq N$, and assume that, for every block j , the length- $N - L$ word products of A_j span the full matrix algebra. Suppose that, for every block j , every boundary-crossing interval beginning at i , and every outside word ρ of length $N - L$, there is a matrix $E_{j,i,\rho}$ such that, for every word β before the cut of length $i + L - N$, $\mu_j^N X_j A_\beta^j A_\rho^j = A_\beta^j E_{j,i,\rho}$. Then, for every block j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. [✓ Proof](#) For fixed j, i , apply Theorem 12.8.2 to the family

$$\begin{aligned} Z_\beta &= \mu_j^N X_j A_\beta^j, \\ F_\beta &= A_\beta^j. \end{aligned}$$

Since the length- $N - L$ outside-word products span the full matrix algebra, there is a single matrix $Y_{j,i}$ such that, for every β , $\mu_j^N X_j A_\beta^j = A_\beta^j Y_{j,i}$. Lemma 14.3.1.20 then gives the periodic-boundary constraints for each block. $\square$

Lemma 14.3.1.22 (Matrix identities under block injectivity give single-block constraints). [Lean](#) [✓ Stmt](#) Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies the normalization equation $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Suppose also that the boundary-crossing matrix identities of Lemma 14.3.1.21 hold for every outside word ρ of length $N - L$. Then, for every block j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. [✓ Proof](#) Since $L + L_0 \leq N$, the complementary length satisfies $L_0 \leq N - L$. Theorem 7.8.1 therefore gives, for every block j , $S_{N-L}(A_j) = M_{D_j}(\mathbb{C})$. Apply Lemma 14.3.1.21. $\square$

Lemma 14.3.1.23 (Normalization for words). [Lean](#) [✓ Stmt](#)

If $\sum_a A_a A_a^\dagger = \mathbb{1}$, then, for every length M , $\sum_\rho A_\rho A_\rho^\dagger = \mathbb{1}$, where ρ ranges over all words of length M .

Proof. [✓ Proof](#)

The case $M = 0$ is the identity matrix. For the induction step, write a word of length $M + 1$ as $a\rho$. Since $A_{a\rho} = A_a A_\rho$,

$$\sum_{a,\rho} A_a A_\rho A_\rho^\dagger A_a^\dagger = \sum_a A_a \left(\sum_\rho A_\rho A_\rho^\dagger \right) A_a^\dagger = \sum_a A_a A_a^\dagger = \mathbb{1}.$$

$\square$

Lemma 14.3.1.24 (Compatibility hypotheses under block injectivity give single-block constraints). [Lean](#) [✓ Stmt](#) Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Fix block-diagonal boundary conditions X_j . For every boundary-crossing interval beginning at i , suppose there are matrices $C_{i,\rho}^j$, indexed by outside words ρ of length $N - L$, such that, for every word β before the cut of length $i + L - N$,

$A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. After opening the periodic boundary at the chosen cut, this is the source comparison

$$\begin{aligned} A_{i_{m+1}}^j C_{i_1}^j &= D_{i_{m+1}}^j A_{i_1}^j, \\ D_\beta^j &= (\mu_j^N X_j) A_\beta^j. \end{aligned}$$

Then, for every block j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. ✓ Proof The normalization equation gives, by induction on word length, $\sum_\rho A_\rho^j A_\rho^{j\dagger} = \mathbb{1}$ for words ρ of length $N - L$. Applying Lemma 14.2.1.10 with $X = \mu_j^N X_j$ gives, for every outside word ρ , a matrix $E_{j,i,\rho}$ such that, for every word β before the cut, $((\mu_j^N X_j) A_\beta^j) A_\rho^j = A_\beta^j E_{j,i,\rho}$. Lemma 14.3.1.22 then gives the periodic-boundary constraint for each block. $\square$

Lemma 14.3.1.25 (Boundary representation and crossing constraints give periodic components). Lean ✓ Stmt Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Let

$$\begin{aligned} \psi &\in \mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right), \\ \psi &= \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right). \end{aligned}$$

If every boundary-crossing interval has the corresponding tail-word spanning property, then there are block-diagonal boundary conditions X_j with the displayed representation of ψ and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. ✓ Proof Theorem 14.3.1.9 supplies matrices $C_{i,\rho}^j$ satisfying $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. Lemma 14.3.1.24 then gives the single-block periodic constraints. $\square$

Lemma 14.3.1.26 (Boundary comparison implies trace comparison). Lean ✓ Stmt Fix block-diagonal boundary matrices X_j . Suppose the opened-boundary matrices $C_{i,\rho}^j$ satisfy, for every boundary-crossing interval i , every outside word ρ , and every word β before the cut, $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. Then, for every middle word w ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_w^j).$$

Proof. ✓ Proof Substituting the displayed comparison identity in each summand gives

$$\text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \text{tr}(((\mu_j^N X_j) A_\beta^j) A_\rho^j A_w^j).$$

Summing over j gives the conclusion. $\square$

Lemma 14.3.1.27 (Boundary trace comparisons give single-block constraints). Lean ✓ Stmt Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Assume also that the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$ at the middle-word length m span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Fix complex numbers μ_j and block-diagonal boundary conditions X_j . For every boundary-crossing interval beginning at i , suppose there are

matrices $C_{i,\rho}^j$, indexed by outside words ρ of length $N - L$, such that, for every word β before the cut, outside word ρ , and middle word w ,

$$\sum_j \operatorname{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \operatorname{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then, for every block j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. ✓ **Proof** Applying Lemma 14.2.1.9 with the same scalars μ_j and boundary matrices X_j , for every block j and every boundary-crossing interval i , gives a matrix E_i^j such that, for every word β before the cut, $(\mu_j^N X_j) A_\beta^j = A_\beta^j E_i^j$. Hence, for every outside word ρ , taking $E_{j,i,\rho} = E_i^j A_\rho^j$ gives $((\mu_j^N X_j) A_\beta^j) A_\rho^j = A_\beta^j E_{j,i,\rho}$. These are precisely the block-diagonal boundary-condition equations of Lemma 14.3.1.22, which gives the periodic-boundary constraint for each block. $\square$

Lemma 14.3.1.28 (Boundary trace comparisons with a block-diagonal representation). Lean
✓ **Stmt** Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Assume also that the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$ at the middle-word length m span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Let

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right)$$

for some block-diagonal boundary conditions X_j . Suppose that whenever ψ has such a representation, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every word β of length $i + L - N$ before the cut, and every word w of length m ,

$$\sum_j \operatorname{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \operatorname{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then there are block-diagonal boundary conditions X_j with

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right)$$

and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$.

Proof. ✓ **Proof** Choose a block-diagonal boundary representation of ψ . For that representation, the boundary trace comparison supplies the matrices $C_{i,\rho}^j$. Applying Lemma 14.3.1.27 gives the desired containment $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every block j . $\square$

Lemma 14.3.1.29 (C^j boundary comparison with a block-diagonal representation). Lean
✓ **Stmt** Assume $0 < N$, $L \leq N$, and $L + L_0 \leq N$. Suppose each block A_j is L_0 -block-injective and satisfies $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$. Assume also that the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$ at the middle-word length m span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Let $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$ for some block-diagonal boundary conditions X_j . Suppose that whenever ψ has such a representation, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, and every word β of length $i + L - N$ before the cut, $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. Then

there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ **Proof** Choose a block-diagonal boundary representation of ψ . For that representation, Lemma 14.3.1.26 gives, for every boundary-crossing interval i , outside word ρ , word β before the cut, and middle word w ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Applying Lemma 14.3.1.28 to these trace comparisons gives $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . $\square$

Lemma 14.3.1.30 (Boundary trace comparisons give periodic-boundary single-block states).

Lean ✓ **Stmt** With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$. Assume also that, for some middle-word length m , the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$, where w ranges over words of length m in $\{0, \dots, d-1\}$, span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Suppose that whenever $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every word β before the cut, and every word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ **Proof** By Lemma 14.3.11, there are matrices X_j with $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$. Lemma 14.3.1.28 applies to this representation and the assumed equality of boundary trace expressions. Thus, $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . $\square$

Lemma 14.3.1.31 (Basis-of-normal-tensors product span gives boundary-trace periodic components). Lean ✓ **Stmt** With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$ and $L \geq (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1))) + 1$. Put $m = (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1)))$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Suppose that whenever $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every word β before the cut, and every word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ Proof Lemma 14.2.2.5 gives $\text{span}\{(A_w^1, \dots, A_w^g) : |w| = m\} = \prod_j M_{D_j}(\mathbb{C})$. Applying Lemma 14.3.1.30 with this middle-word span gives the stated block-diagonal boundary conditions and the periodic-boundary constraint in each block. $\square$

Lemma 14.3.1.32 (C^j boundary comparison gives periodic single-block states). Lean ✓ Stmt With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Suppose that whenever $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$, there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, and every word β before the cut, $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ Proof By Lemma 14.3.11, there are matrices X_j with $\psi = \Gamma_N^{\oplus_j \mu_j A_j}(\bigoplus_j X_j)$. For these boundary conditions, the assumed C^j boundary comparison supplies the matrices $C_{i,\rho}^j$. Lemma 14.3.1.24 gives $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . $\square$

Theorem 14.3.1.33 (Crossing-tail spans give periodic single-block states). Lean ✓ Stmt With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$ and that every boundary-crossing interval beginning at i has the simultaneous tail-word spanning property: the products

$$w \mapsto (A_w^1, \dots, A_w^g),$$

$$|w| = N - i,$$

span the product algebra $\prod_j M_{D_j}(\mathbb{C})$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ Proof Lemma 14.3.11 supplies block-diagonal boundary conditions X_j , and Lemma 14.3.1.25, applied with the assumed tail-word spans, places each $\Gamma_N^{A_j}(\mu_j^N X_j)$ in $\mathcal{G}_{N,L}(A_j)$. $\square$

Theorem 14.3.1.34 (Short crossing-tail spans give periodic single-block states). [Lean](#) [✓ Stmt](#) With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$. Assume also that, for every boundary-crossing interval beginning at i whose tail length lies below the block-separation bound, $N - i < (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1)))$, the simultaneous products

$$w \mapsto (A_w^1, \dots, A_w^g),$$

$$|w| = N - i,$$

span the product algebra $\prod_j M_{D_j}(\mathbb{C})$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

This derives the boundary-condition comparison of [PGVWC07, Theorem 12, proof lines 1446–1456] at every crossing tail whose length reaches the displayed block-separation bound.

Proof. [✓ Proof](#) For a boundary-crossing interval beginning at i , either $N - i < (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1)))$ or $N - i \geq (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1)))$. In the first case, the assumed short-tail span applies; in the second case, Lemma 14.2.2.5 gives the simultaneous product span at length $N - i$. Hence every boundary-crossing interval has the tail-word spanning property, and Theorem 14.3.1.33 gives the block-diagonal boundary conditions and $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . $\square$

Definition 14.3.1.35 (Cyclic translation of configurations). [Lean](#) [✓ Stmt](#) For a periodic configuration σ on N sites and $s \in \mathbb{Z}/N\mathbb{Z}$, define $(T_s \sigma)(k) = \sigma(k + s)$.

Definition 14.3.1.36 (Cyclic translation of states). [Lean](#) [✓ Stmt](#) For a state ψ on a periodic chain, define $(T_s \psi)(\sigma) = \psi(T_s \sigma)$.

Theorem 14.3.1.37 (Cyclic translations compose). [Lean](#) [✓ Stmt](#) For offsets $s, t \in \mathbb{Z}/N\mathbb{Z}$ and a state ψ on the periodic chain,

$$T_s(T_t \psi) = T_{s+t} \psi.$$

Proof. [✓ Proof](#) For every configuration σ , the two sides evaluate ψ on the same configuration because $T_t(T_s \sigma) = T_{s+t} \sigma$. $\square$

Lemma 14.3.1.38 (Cyclic translation preserves the periodic chain space). [Lean](#) [✓ Stmt](#) If $\psi \in \mathcal{G}_{N,L}(A)$, then $T_s \psi \in \mathcal{G}_{N,L}(A)$ for every cyclic translation s .

Proof. [✓ Proof](#) A length- L interval of $T_s \psi$ beginning at i is the corresponding interval of ψ beginning at $i - s$, with the outside configuration translated by s . $\square$

Lemma 14.3.1.39 (Translation exchanges the two words at a cut). [Lean](#) [✓ Stmt](#) If β has length K and α has positive length S , then $T_K(\beta\alpha) = \alpha\beta$ as configurations on the periodic chain of length $K + S$.

Proof. ✓ Proof This is addition modulo $K + S$, separated according to whether the site lies before or after the cut. $\square$

Lemma 14.3.1.40 (Comparison across a global cut). Lean ✓ Stmt Let $X_j, Y_j \in M_{D_j}(\mathbb{C})$, and suppose

$$T_K \left(\sum_j \Gamma_{K+S}^{A_j}(X_j) \right) = \sum_j \Gamma_{K+S}^{A_j}(Y_j).$$

If the simultaneous block words of length K span $\prod_j M_{D_j}(\mathbb{C})$, then, for every block j and every word α of length S , $X_j A_\alpha^j = A_\alpha^j Y_j$.

Proof. ✓ Proof Evaluate the translated equality on the word $\beta\alpha$, where $|\beta| = K$. Cyclicity of the trace gives

$$\sum_j \text{tr}(X_j A_\alpha^j A_\beta^j) = \sum_j \text{tr}(A_\alpha^j Y_j A_\beta^j).$$

The simultaneous spanning hypothesis separates the blocks and gives the asserted matrix identities. $\square$

Theorem 14.3.1.41 (A global change of cut gives periodic single-block states). Lean ✓ Stmt With the hypotheses and notation of Lemma 14.3.12, assume in addition that $L + L_0 \leq N$. Thus

$$\begin{aligned} g &\geq 2, \\ L &\geq 3(g-1)(L_0+1)+1, \\ N &\geq L + L_0. \end{aligned}$$

Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:

$$\begin{aligned} \psi &= \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right), \\ \Gamma_N^{A_j}(\mu_j^N X_j) &\in \mathcal{G}_{N,L}(A_j). \end{aligned}$$

No simultaneous spanning assumption is imposed at the short crossing tails.

Proof. ✓ Proof Lemma 14.3.12 gives boundary matrices X_j . Translate the periodic state so that the cut moves past the final L_0 sites. By Lemma 14.3.1.38, the translated state remains in the periodic chain space, so applying Lemma 14.3.12 again yields boundary matrices Y_j for the translated state. For a word α of length L_0 , comparison of the two decompositions against every complementary word w of length $N - L_0$ gives

$$\sum_j \text{tr}((\mu_j^N X_j) A_\alpha^j A_w^j) = \sum_j \text{tr}(A_\alpha^j (\mu_j^N Y_j) A_w^j).$$

The length bound and Theorem 14.2.2.11 imply that the simultaneous products of length $N - L_0$ span the product algebra. Therefore, Lemma 14.3.1.40 gives, block by block, $(\mu_j^N X_j) A_\alpha^j =$

$A_\alpha^j(\mu_j^N Y_j)$. Since the words of length L_0 span each block matrix algebra, evaluation at the identity gives

$$\mu_j^N X_j = \mu_j^N Y_j. \quad (14.4)$$

Thus (14.4) says that $\mu_j^N X_j$ commutes with every word of length L_0 , and consequently with every one-site matrix. The boundary condition may then be moved across every interval crossing the cut, which places each block component in its periodic chain space. This is the change-of-cut argument of [PGVWC07, Theorem 12, proof lines 1276–1289 and 1454–1456]. $\square$

Theorem 14.3.1.42 (Doubly normalized global-cut boundary closing). Lean ✓ Stmt *Let $g \geq 2$, assume the physical alphabet is nonempty, and let A_j be irreducible blocks with positive bond dimensions, pairwise inequivalent up to gauge and phase, and injective at a common length $L_0 > 0$. Assume normalized self-overlap and the two normalization identities*

$$\begin{aligned} \sum_a (A_a^j)^\dagger A_a^j &= \mathbb{1}, \\ \sum_a A_a^j (A_a^j)^\dagger &= \mathbb{1}. \end{aligned}$$

Let every μ_j be nonzero, and suppose

$$\begin{aligned} L &\geq 3(g-1)(L_0+1)+1, \\ N &\geq L+L_0. \end{aligned}$$

If $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$, then there are matrices X_j such that, for every j ,

$$\begin{aligned} \psi &= \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right), \\ \Gamma_N^{A_j}(\mu_j^N X_j) &\in \mathcal{G}_{N,L}(A_j). \end{aligned}$$

This theorem is restricted to the case in which the positive dual fixed matrices of the source normalization are all the identity, with normalized self-overlap imposed separately.

Proof. ✓ Proof Obtain block-diagonal boundary matrices at the chosen cut. Move the cut past the final L_0 sites and compare the two boundary representations. The doubly normalized simultaneous product-span theorem separates the blocks and gives

$$(\mu_j^N X_j) A_\alpha^j = A_\alpha^j (\mu_j^N Y_j).$$

for every word α of length L_0 . Injectivity identifies the two boundary matrices and permits the cut to move across every local interval, proving the periodic single-block constraints. $\square$

Theorem 14.3.1.43 (Periodic block decomposition from a global cut). Lean ✓ Stmt *Let $g \geq 2$. Let A_j be irreducible blocks, pairwise inequivalent up to gauge and phase, and injective at a common length $L_0 > 0$. Assume $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ and, for positive-definite Λ_j , $\sum_a A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j$. Let $\mu_j \neq 0$ for every j , and assume*

$$\begin{aligned} L &\geq 3(g-1)(L_0+1)+1, \\ N &\geq L+L_0. \end{aligned}$$

Then

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \sum_j \mathcal{G}_{N,L}(A_j),$$

as asserted in [PGVWC07, Theorem 12, proof lines 1424–1456].

Proof. ✓ Proof This is the equality component of Theorem 14.3.1.44. □

Theorem 14.3.1.44 (Periodic block decomposition and block-space independence). Lean

✓ Stmt Under the same hypotheses,

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \sum_j \mathcal{G}_{N,L}(A_j).$$

The open-boundary spaces $G_N(A_j)$ also form an internal sum.

Proof. ✓ Proof Theorem 14.3.1.41 gives the forward inclusion in the displayed equality. Every periodic single-block vector is a periodic vector for the block-diagonal tensor, which gives the reverse inclusion. For independence, write $\phi_j = \Gamma_N^{A_j}(X_j)$ and suppose that $\sum_j \phi_j = 0$. Theorem 14.2.2.12 and nondegeneracy of the trace pairing give

$$\sum_j \Gamma_N^{A_j}(X_j) = 0 \implies \left(\forall (M_j)_j, \sum_j \text{tr}(X_j M_j) = 0 \right) \implies \forall j, X_j = 0.$$

The last implication follows by choosing a tuple supported in one block. Thus every ϕ_j vanishes, and the open-boundary spaces form an internal sum. □

Theorem 14.3.1.45 (Doubly normalized periodic block decomposition and independence).

Lean ✓ Stmt Under the hypotheses of Theorem 14.3.1.42,

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \sum_j \mathcal{G}_{N,L}(A_j),$$

and the open-boundary spaces $G_N(A_j)$ form an internal sum.

Proof. ✓ Proof Apply Theorem 14.3.1.44 with $\Lambda_j = \mathbb{1}$. Left canonicity gives the required dual fixed-point identity, while the remaining hypotheses are unchanged. □

Theorem 14.3.1.46 (Doubly normalized periodic block decomposition). Lean ✓ Stmt Let $g \geq 2$, assume the physical alphabet is nonempty, and let A_j be irreducible blocks with positive bond dimensions, pairwise inequivalent up to gauge and phase, and injective at a common length $L_0 > 0$. Assume normalized self-overlap and the two normalization identities

$$\begin{aligned} \sum_a (A_a^j)^\dagger A_a^j &= \mathbb{1}, \\ \sum_a A_a^j (A_a^j)^\dagger &= \mathbb{1}. \end{aligned}$$

Let every μ_j be nonzero, and suppose

$$\begin{aligned} L &\geq 3(g-1)(L_0+1)+1, \\ N &\geq L+L_0. \end{aligned}$$

Then

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \sum_j \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ **Proof** This is the equality component of Theorem 14.3.1.45. □

Theorem 14.3.1.47 (Unique decomposition into open-boundary block spaces). Lean ✓ Stmt
 Let $g \geq 2$, assume the physical alphabet is nonempty, and let A_j be irreducible blocks with positive bond dimensions, pairwise inequivalent up to gauge and phase, and injective at a common length $L_0 > 0$. Assume the source normalization

$$\begin{aligned} \sum_a A_a^j (A_a^j)^\dagger &= \mathbb{1}, \\ \sum_a (A_a^j)^\dagger \Lambda_j A_a^j &= \Lambda_j, \\ \Lambda_j &> 0. \end{aligned}$$

Let every μ_j be nonzero. If $0 < L \leq N$ and $L \geq 3(g-1)(L_0+1)+1$, then every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ has a unique decomposition

$$\begin{aligned} \psi &= \sum_j \phi_j, \\ \phi_j &\in G_N(A_j) \quad \text{for every } j. \end{aligned}$$

Proof. ✓ **Proof** The source-length open-boundary theorem places ψ in $\sum_j G_N(A_j)$ and proves that the spaces $G_N(A_j)$ form an internal sum. Existence follows from membership in the sum, and internality makes the components unique. □

Corollary 14.3.1.48 (Kernel containment from the global cut). Lean ✓ Stmt Under the same hypotheses, suppose additionally that every periodic single-block chain space is contained in its matrix-product-vector line. Then

$$\ker H_L^{(N)}\left(\bigoplus_j \mu_j A_j\right) \subseteq \text{span}\{V^{(N)}(A_j) : 1 \leq j \leq g\}.$$

Proof. ✓ **Proof** The parent-Hamiltonian kernel is the periodic chain space. Apply the preceding block decomposition and then the assumed single-block containments. □

Theorem 14.3.1.49 (Doubly normalized kernel containment from a global cut). Lean ✓ Stmt
 Under the hypotheses of Theorem 14.3.1.42, suppose in addition that, for every j ,

$$\mathcal{G}_{N,L}(A_j) \subseteq \text{span}\{|V^{(N)}(A_j)\rangle\}.$$

Then

$$\ker H_L^{(N)}\left(\bigoplus_j \mu_j A_j\right) \subseteq \text{span}\{|V^{(N)}(A_j)\rangle : 1 \leq j \leq g\}.$$

This theorem is restricted to the doubly normalized case in which every positive dual fixed matrix is the identity and normalized self-overlap is assumed.

Proof. ✓ Proof Theorem 14.3.1.46 identifies the periodic chain space of the block sum with the sum of the single-block chain spaces. Apply the assumed containment to each summand, and use the identification of the parent-Hamiltonian kernel with the periodic chain space. $\square$

Lemma 14.3.1.50 (Matrix identities give periodic single-block states). Lean ✓ Stmt *With the hypotheses and notation of Lemma 14.3.11, assume in addition that $L + L_0 \leq N$. Let $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$. Suppose that whenever $\psi = \Gamma_N^{\bigoplus_j \mu_j A_j}(\bigoplus_j X_j)$, the corresponding boundary conditions satisfy the block-diagonal boundary-condition equations: for every boundary-crossing interval i , every word β before the cut, and every outside word ρ of length $N - L$, $\mu_j^N X_j A_\beta^j A_\rho^j = A_\beta^j E_{j,i,\rho}$. Then there are block-diagonal boundary conditions X_j such that the displayed representation holds and, for every j , the displayed component lies in the corresponding single-block ground space:*

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

$$\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ Proof By Lemma 14.3.11, there are matrices X_j with $\psi = \Gamma_N^{\bigoplus_j \mu_j A_j}(\bigoplus_j X_j)$. These X_j satisfy the assumed identities, so Lemma 14.3.1.22 gives $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . $\square$

14.3.2 Periodic-boundary ground space for a block-diagonal tensor

The periodic ground-space theorem states that, for sufficiently long periodic systems, the periodic-boundary ground space of a parent Hamiltonian is generated by a basis of normal tensors of the initial tensor A [CPGSV21, Theorem IV.16]. The lemmas in this subsection are conditional reductions toward that statement. Their hypotheses explicitly include a boundary inclusion, a block-diagonal boundary representation, or periodic-boundary membership for the single-block states; in the source theorem these conditions are obtained from the boundary-condition comparison in the periodic ring. The source proof [PGVWC07] writes this comparison with $C_{i_1}^j, D_{i_{m+1}}^j$, and the normalized matrix E^j ; the symbols β and ρ below are the words obtained after opening the periodic boundary at the chosen cut. The source theorem writes the number of blocks as b and assumes $b \geq 2$; here the same number is denoted by g . In these coordinates the source comparison input becomes $A_\beta^j C_{i,\rho}^j = (\mu_j^N X_j A_\beta^j) A_\rho^j$, which is the form obtained after opening the boundary from $A_{i_{m+1}}^j C_{i_1}^j = D_{i_{m+1}}^j A_{i_1}^j$ with $D_\beta^j = (\mu_j^N X_j) A_\beta^j$. Once this compatibility is known, the normalized E^j calculation is the matrix identity of Lemma 14.2.1.9; the local boundary form used after opening the periodic boundary is Lemma 14.2.1.10.

Lemma 14.3.2.1 (The block-diagonal periodic-boundary space lies between two block sums).

Lean ✓ Stmt *With the hypotheses and notation of Lemma 14.3.8, one has*

$$\bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j) \subseteq \mathcal{G}_{N,L} \left(\bigoplus_{j=1}^g \mu_j A_j \right) \subseteq \bigvee_{j=1}^g G_N(A_j),$$

and the sum defining the right-hand side is internal.

Proof. ✓ Proof The first inclusion is Lemma 14.1.4. The second inclusion and internal directness are Lemma 14.3.9. $\square$

Lemma 14.3.2.2 (Boundary inclusion gives the block-diagonal periodic-boundary equality). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$, where $\mu_j \neq 0$ for $1 \leq j \leq g$. If the periodic-boundary comparison with block-diagonal boundary conditions gives

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j),$$

then $\mathcal{G}_{N,L}(B) = \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j)$.

Proof. [✓ Proof](#) The reverse inclusion is Lemma 14.1.4; combine the two inclusions. $\square$

Lemma 14.3.2.3 (Componentwise periodic-boundary decomposition gives the reverse inclusion). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$. Suppose that every vector $\psi \in \mathcal{G}_{N,L}(B)$ can be written as

$$\begin{aligned} \psi &= \sum_{j=1}^g \psi_j, \\ \psi_j &\in \mathcal{G}_{N,L}(A_j). \end{aligned}$$

Then

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j).$$

Proof. [✓ Proof](#) For $\psi \in \mathcal{G}_{N,L}(B)$, choose the displayed decomposition. Each summand lies in the corresponding subspace $\mathcal{G}_{N,L}(A_j)$, hence their finite sum lies in $\bigvee_j \mathcal{G}_{N,L}(A_j)$. $\square$

Lemma 14.3.2.4 (Block-diagonal boundary vectors in the periodic-boundary block sum). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$. Fix block-diagonal boundary conditions X_j . Suppose that, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. Then

$$\Gamma_N^B\left(\bigoplus_{j=1}^g X_j\right) \in \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j).$$

Proof. [✓ Proof](#) The block-diagonal boundary formula gives

$$\Gamma_N^B\left(\bigoplus_{j=1}^g X_j\right) = \sum_{j=1}^g \Gamma_N^{A_j}(\mu_j^N X_j).$$

Each summand belongs to the corresponding block periodic-boundary space by assumption, so the finite sum lies in the linear span of these spaces. $\square$

Lemma 14.3.2.5 (Block-diagonal boundary representation gives the reverse inclusion). [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_{j=1}^g \mu_j A_j$. Suppose every $\psi \in \mathcal{G}_{N,L}(B)$ admits block-diagonal boundary conditions X_j such that $\psi = \Gamma_N^B\left(\bigoplus_{j=1}^g X_j\right)$, and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. Then

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j).$$

Proof. ✓ Proof Apply Lemma 14.3.2.4 to the block-diagonal boundary representation of each $\psi \in \mathcal{G}_{N,L}(B)$. □

Lemma 14.3.2.6 (Block-diagonal boundary representation reduces to periodic-boundary equality). Lean ✓ Stmt *With the hypotheses and notation of Lemma 14.3.8, suppose every vector $\psi \in \mathcal{G}_{N,L}(B)$ has block-diagonal boundary conditions X_j such that $\psi = \Gamma_N^B(\bigoplus_{j=1}^g X_j)$, and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. Then $\mathcal{G}_{N,L}(B) = \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j)$, and the sum defining $\bigvee_j G_N(A_j)$ is internal.*

Proof. ✓ Proof The block-diagonal boundary representation gives

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j)$$

by Lemma 14.3.2.5. The equality follows from Lemma 14.3.2.2; the internality statement is Lemma 14.3.9. □

Lemma 14.3.2.7 (Complementary-word matrix identities reduce to periodic-boundary equality). Lean ✓ Stmt *With the hypotheses and notation of Lemma 14.3.2.6, assume in addition that $L + L_0 \leq N$. Assume further that, for every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ and every block-diagonal boundary representation*

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

the complementary-word matrix identities hold: for every block j , boundary-crossing interval i , word β before the cut, and outside word ρ of length $N - L$, there is a matrix $E_{j,i,\rho}$ such that $\mu_j^N X_j A_\beta^j A_\rho^j = A_\beta^j E_{j,i,\rho}$. Then

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the N -site ground spaces $G_N(A_j) = \mathcal{G}_{N,N}(A_j)$ have internal sum.

Proof. ✓ Proof For each ψ , Lemma 14.3.1.50 gives matrices X_j with

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right)$$

and, for every j , the membership $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ holds. Lemma 14.3.2.6 then gives

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the internality of the sum of the $G_N(A_j)$. □

Lemma 14.3.2.8 (C^j, D^j boundary-condition comparisons reduce to periodic-boundary equality). [Lean](#) [✓ Stmt](#) With the hypotheses and notation of Lemma 14.3.2.6, assume in addition that $L + L_0 \leq N$. Assume further that, for every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ and every block-diagonal boundary representation

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, and every word β before the cut, $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. This is the C^j, D^j compatibility input; the subsequent normalized E^j identities follow from Lemma 14.2.1.10. Then

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the N -site ground spaces $G_N(A_j) = \mathcal{G}_{N,N}(A_j)$ have internal sum.

Proof. [✓ Proof](#) For each ψ , Lemma 14.3.1.32 gives matrices X_j with

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right)$$

and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. Lemma 14.3.2.6 then gives

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the internality of the sum of the $G_N(A_j)$. □

Lemma 14.3.2.9 (C^j, D^j comparisons give the periodic block ground-space equality). [Lean](#) [✓ Stmt](#) With the hypotheses and notation of Lemma 14.3.2.8, one has

$$\mathcal{G}_{N,L} \left(\bigoplus_j \mu_j A_j \right) = \bigvee_j \mathcal{G}_{N,L}(A_j).$$

This records only the ground-space equality conclusion obtained after the C^j, D^j boundary-condition comparison input in [PGVWC07, Theorem 12, proof lines 1446–1456]; the local normalized E^j calculation is Lemma 14.2.1.10. The independence of the N -site spaces is a separate conclusion of Lemma 14.3.2.8. The statement is still in the length- L_0 injectivity range displayed in that lemma, not the shorter range in [PGVWC07, Theorem 12].

Proof. [✓ Proof](#) The claim is the equality part of Lemma 14.3.2.8. □

Lemma 14.3.2.10 (C^j, D^j comparisons at boundary-crossing intervals reduce to periodic-boundary equality). [Lean](#) [✓ Stmt](#) With the hypotheses and notation of Lemma 14.3.2.6, assume in addition that $L + L_0 \leq N$. Assume also that, for every boundary-crossing interval beginning at i , the simultaneous products

$$\begin{aligned} w &\mapsto (A_w^1, \dots, A_w^g), \\ |w| &= N - i, \end{aligned}$$

span the product algebra $\prod_j M_{D_j}(\mathbb{C})$. Then

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the N -site ground spaces $G_N(A_j) = \mathcal{G}_{N,N}(A_j)$ have internal sum. The present statement is the span-dependent consequence, after opening the boundary, of the C^j, D^j compatibility input in [PGVWC07, Theorem 12, proof lines 1436–1451] in the length- L_0 injectivity range $L \geq (L_0 + 1) + 3(g-1)(L_0 + 1) + 1$. The source theorem has the shorter bound $L \geq 3(g-1)(L_0 + 1) + 1$.

Proof. ✓ Proof Lemma 14.3.2.6 reduces the claim to producing, for each ψ , block-diagonal boundary conditions X_j whose single-block components lie in $\mathcal{G}_{N,L}(A_j)$. The crossing comparison theorem supplies this: it first uses the boundary-comparison-free BNT representation to obtain X_j , and then uses the stated tail-word spans and Theorem 14.3.1.9 to produce matrices $C_{i,\rho}^j$ satisfying $A_\beta^j C_{i,\rho}^j = ((\mu_j^N X_j) A_\beta^j) A_\rho^j$. Lemma 14.3.1.24 then gives $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$ for every j . Applying Lemma 14.3.2.6 to these boundary conditions gives the displayed equality and internality. $\square$

Lemma 14.3.2.11 (Boundary-crossing comparisons give the periodic block ground-space equality). Lean ✓ Stmt With the hypotheses and notation of Lemma 14.3.2.10, one has

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \bigvee_j \mathcal{G}_{N,L}(A_j).$$

This records the ground-space equality conclusion of the conditional crossing-span version above. It is the corresponding part of [PGVWC07, Theorem 12], but only under the visible tail-word span hypothesis and in the longer stated range. The independence of the N -site spaces is a separate conclusion of Lemma 14.3.2.10. This intermediate statement remains conditional on the short tail-word spans. The source-range conclusion without them is supplied by the global change-of-cut theorem Theorem 14.3.1.41 under the unrestricted source normalization, with positive definite dual fixed points satisfying

$$\sum_a A_j^{a\dagger} \Lambda_j A_j^a = \Lambda_j.$$

The doubly normalized theorem with $\Lambda_j = \mathbb{1}$ remains available as a specialization of that result.

Proof. ✓ Proof The claim is the equality part of Lemma 14.3.2.10. $\square$

Lemma 14.3.2.12 (Boundary trace comparisons reduce to periodic-boundary equality). Lean

✓ Stmt With the hypotheses and notation of Lemma 14.3.2.6, assume in addition that $L + L_0 \leq N$. Assume also that, for some middle-word length m , the simultaneous tuples $w \mapsto (A_w^1, \dots, A_w^g)$, where w ranges over words of length m in $\{0, \dots, d-1\}$, span the full product algebra $\prod_j M_{D_j}(\mathbb{C})$. Assume further that, for every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ and every block-diagonal boundary representation

$$\psi = \Gamma_N^{\bigoplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every word β before the cut, and every word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the N -site ground spaces $G_N(A_j) = \mathcal{G}_{N,N}(A_j)$ have internal sum.

Proof. ✓ Proof For each ψ , Lemma 14.3.1.30 gives matrices X_j with

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right)$$

and, for every j , $\Gamma_N^{A_j}(\mu_j^N X_j) \in \mathcal{G}_{N,L}(A_j)$. Lemma 14.3.2.6 then gives

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the internality of the sum of the $G_N(A_j)$. □

Lemma 14.3.2.13 (Basis-of-normal-tensors product span gives boundary-trace ground-space equality). Lean ✓ Stmt With the hypotheses and notation of Lemma 14.3.2.6, assume in addition that $L + L_0 \leq N$ and $L \geq (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1))) + 1$. Put $m = (L_0 + 1) + (g - 1)((L_0 + 1) + ((L_0 + 1) + (L_0 + 1)))$. Assume that, for every $\psi \in \mathcal{G}_{N,L}(\bigoplus_j \mu_j A_j)$ and every block-diagonal boundary representation

$$\psi = \Gamma_N^{\oplus_j \mu_j A_j} \left(\bigoplus_j X_j \right),$$

there are matrices $C_{i,\rho}^j$ such that, for every boundary-crossing interval i , every outside word ρ of length $N - L$, every word β before the cut, and every word w of length m ,

$$\sum_j \text{tr}(A_\beta^j C_{i,\rho}^j A_w^j) = \sum_j \text{tr}((\mu_j^N X_j) A_\beta^j A_\rho^j A_w^j).$$

Then

$$\mathcal{G}_{N,L}\left(\bigoplus_j \mu_j A_j\right) = \bigvee_j \mathcal{G}_{N,L}(A_j),$$

and the N -site ground spaces $G_N(A_j) = \mathcal{G}_{N,N}(A_j)$ have internal sum.

Proof. ✓ Proof Lemma 14.2.2.5 gives $\text{span}\{(A_w^1, \dots, A_w^g) : |w| = m\} = \prod_j M_{D_j}(\mathbb{C})$. Applying Lemma 14.3.2.12 with this middle-word span gives the equality of periodic-boundary ground spaces and the internality of the sum of the N -site spaces. □

Lemma 14.3.2.14 (Boundary decomposition reduces to periodic-boundary equality). [Lean](#)

[✓ Stmt](#) Let $A_1, \dots, A_g$ be normalized representatives of a basis of normal tensors: each representative is irreducible, the family is left-canonical, the self-overlaps are normalized, and no two distinct same-dimensional representatives are gauge-phase equivalent, and let $B = \bigoplus_{j=1}^g \mu_j A_j$, where $\mu_j \neq 0$. Assume each block is injective at length L_0 , assume $\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}$ for every j , and suppose

$$\begin{aligned} L &\geq (L_0 + 1) + (g - 1)3(L_0 + 1) + 1, \\ N &\geq L. \end{aligned}$$

If every vector $\psi \in \mathcal{G}_{N,L}(B)$ can be written as

$$\begin{aligned} \psi &= \sum_{j=1}^g \psi_j, \\ \psi_j &\in \mathcal{G}_{N,L}(A_j), \end{aligned}$$

then $\mathcal{G}_{N,L}(B) = \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j)$, and the sum $\bigvee_j \mathcal{G}_{N,L}(A_j)$ is internal.

Proof. [✓ Proof](#) Lemma 14.3.2.3 gives

$$\mathcal{G}_{N,L}(B) \subseteq \bigvee_{j=1}^g \mathcal{G}_{N,L}(A_j).$$

The blockwise inclusion gives the reverse containment, hence equality. Internality follows from the directness conclusion of Lemma 14.3.9. $\square$

14.4 Periodic-boundary ground space generated by a basis of normal tensors

Let $A_0, \dots, A_{g-1}$ be irreducible canonical blocks and let every weight μ_j be nonzero. The preceding sections prove the simultaneous word-span, one-step intersection, global boundary-closing, and periodic chain-space statements at the source length of [PGVWC07, Theorem 12]. They yield the exact kernel equality under the source normalization and the stronger assumption that distinct equal-dimensional blocks are inequivalent up to gauge and phase. The source instead states pairwise distinctness of the finite-ring component vectors; no implication from that condition to gauge-phase inequivalence is used here. For each block, let Λ_j be positive definite and satisfy

$$\sum_a A_a^j A_a^{j\dagger} = \mathbb{1}, \quad \sum_a A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j.$$

Setting $\Lambda_j = \mathbb{1}$ gives the two displayed normalization identities. The doubly-normalized theorem below also assumes normalized self-overlap, which is not a premise of the general dual-fixed-point theorem. It is therefore a stronger corollary with $\Lambda_j = \mathbb{1}$, not an exact hypothesis-for-hypothesis specialization.

Theorem 14.4.1 (Doubly-normalized PGVWC07 source-length kernel equality). [Lean](#) [✓ Stmt](#)

Let $d \geq 1$ and $g \geq 2$. For $0 \leq j < g$, let $D_j \geq 1$ and let $A_j = (A_a^j)_{a=0}^{d-1}$ be an irreducible left-canonical block with normalized self-overlap. Assume that no two distinct blocks are equivalent up

to gauge and phase. For every $0 \leq j < g$, let $\mu_j \neq 0$. Suppose there is a common $L_0 > 0$ such that the words of length L_0 in each block A_j span $M_{D_j}(\mathbb{C})$. In this doubly-normalized specialization, assume for every $0 \leq j < g$ that

$$\sum_{a=0}^{d-1} A_a^j A_a^{j\dagger} = \mathbb{1},$$

$$\sum_{a=0}^{d-1} A_a^{j\dagger} A_a^j = \mathbb{1}.$$

The first identity is the normalization used in the source. In the general source canonical form, the second identity is replaced by a positive definite matrix Λ_j satisfying

$$\sum_{a=0}^{d-1} A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j.$$

The relation $\Lambda_j = \mathbb{1}$ therefore recovers the displayed dual-fixed-point equation. This theorem additionally assumes normalized self-overlap, so it is a stronger corollary of the general statement below rather than an exact specialization of its hypotheses. If

$$L \geq 3(g-1)(L_0 + 1) + 1,$$

$$N \geq L + L_0,$$

then the parent-Hamiltonian kernel is exactly the span of the BNT component vectors:

$$\ker H_L^{(N)} \left(\bigoplus_j \mu_j A_j \right) = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, g-1\}. \quad (14.5)$$

Proof. ✓ Proof Apply the general-canonical-normalization theorem below with $\Lambda_j = \mathbb{1}$. Positive definiteness is immediate, and the assumed left-canonical identity is exactly the required dual fixed-point equation at $\Lambda_j = \mathbb{1}$. The normalized self-overlap premise is retained as an additional hypothesis of this corollary. $\square$

Theorem 14.4.2 (PGVWC07 source-length kernel equality at general canonical normalization). Lean ✓ Stmt Retain the same nonzero dimensions and weights, irreducibility, gauge-phase inequivalence, common block-injectivity length, and source-length bounds, but do not assume left-canonicity. For every $0 \leq j < g$, assume there is a positive definite matrix Λ_j such that

$$\sum_{a=0}^{d-1} A_a^j A_a^{j\dagger} = \mathbb{1},$$

$$\sum_{a=0}^{d-1} A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j.$$

Then

$$\ker H_L^{(N)} \left(\bigoplus_j \mu_j A_j \right) = \text{span}\{V^{(N)}(A_j) : j = 0, \dots, g-1\}.$$

Proof. ✓ Proof The source length bounds imply $L_0 < L$, $L \leq N$, and $L_0 + 1 < N$. Theorem 12.9.2.3 identifies each single-block periodic chain space with its matrix-product-vector line. Corollary 14.3.1.48 therefore gives the containment of the parent-Hamiltonian kernel in the BNT span. Conversely, Lemma 13.1.21 gives the reverse containment, and antisymmetry yields equality. $\square$

Chapter 15

Exponential Decay of Correlations

This chapter records an auxiliary-space transfer-map interface for thermodynamic-limit correlation functions and the resulting exponential decay estimates. The key mechanism is spectral: after removing the leading eigenvalue contribution, the connected part is governed by subleading eigenvalues of the transfer map. Passing from physical local observables to the auxiliary insertions used here requires the corresponding observable-inserted transfer map and is not supplied by the present definitions.

When the MPS tensor generates a parent Hamiltonian (Chapter 12), these auxiliary formulas control spatial ground-state correlations once physical observable insertions have been identified with the corresponding transfer-map insertions. The exponential decay rate is then set by the spectral properties of the transfer map, i.e. by the modulus of its subleading eigenvalues.

The spectral analysis of the transfer map and the mixed transfer operator, including the eigenvalue bound $\rho(F_{AB}) \leq 1$, the strict transfer-operator gap $\rho(F_{AB}) < 1$ for non-equivalent blocks, and the resulting overlap decay, is developed in Chapter 6. This chapter focuses on the auxiliary *correlator* interface: one-point and two-point functionals, their spectral decomposition, and quantitative bounds on decay rates.

15.1 Connected correlators from the transfer map

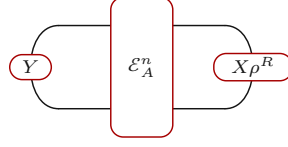
Definition 15.1.1 (Auxiliary-space one-point functional). [Lean](#) [✓ Stmt](#) Fix an MPS tensor and a right fixed point of its transfer map. Write the tensor as A and the fixed point as ρ^R . Under the hypotheses of Theorem 5.3.2, that fixed point exists and is unique up to scaling; we normalize it so that $\text{tr}(\rho^R) = 1$. For an auxiliary-space insertion $X \in M_D(\mathbb{C})$, define

$$\langle X_0 \rangle = \text{tr}(X \rho^R).$$

Definition 15.1.2 (Auxiliary-space two-point functional at distance n). [Lean](#) [✓ Stmt](#) For auxiliary-space insertions $X, Y \in M_D(\mathbb{C})$ and $n \geq 0$, define

$$\langle X_0 Y_n \rangle = \text{tr}(Y \mathcal{E}_A^n(X \rho^R)).$$

Equivalently, insert X at site 0, propagate by $\mathcal{E}_A^n$, then insert Y and take the trace.



Definition 15.1.3 (Auxiliary-space connected correlator). [Lean](#) [✓ Stmt](#) The connected two-point function is

$$C(X, Y; n) = \langle X_0 Y_n \rangle - \langle X_0 \rangle \langle Y_0 \rangle.$$

Definition 15.1.4 (Fixed-point-centered auxiliary insertion). [Lean](#) [✓ Stmt](#) For a matrix $\rho \in M_D(\mathbb{C})$ and an auxiliary insertion $X \in M_D(\mathbb{C})$, define

$$Z_\rho(X) = X\rho - \text{tr}(X\rho)\rho. \quad (15.1)$$

When $\text{tr}(\rho) = 1$, this removes the trace component of $X\rho$.

Lemma 15.1.5 (Trace of the centered insertion). [Lean](#) [✓ Stmt](#) If $\text{tr}(\rho) = 1$, then

$$\text{tr}(Z_\rho(X)) = 0. \quad (15.2)$$

Proof. [✓ Proof](#) By linearity of the trace,

$$\text{tr}(Z_\rho(X)) = \text{tr}(X\rho) - \text{tr}(X\rho)\text{tr}(\rho) = 0. \quad (15.3)$$

□

Theorem 15.1.6 (Fixed-point centering identity for connected correlators). [Lean](#) [✓ Stmt](#) Suppose that ρ is a right fixed point of the transfer map, $\mathcal{E}_A(\rho) = \rho$. Then, for every $n \geq 0$,

$$C(X, Y; n) = \text{tr}(Y \mathcal{E}_A^n(Z_\rho(X))). \quad (15.4)$$

When moreover $\text{tr}(\rho) = 1$, the preceding lemma shows that $Z_\rho(X)$ is traceless. The identity isolates the fixed-point contribution before the remaining transfer iterates are analysed.

Proof. [✓ Proof](#) Since $\mathcal{E}_A(\rho) = \rho$, one has $\mathcal{E}_A^n(\rho) = \rho$ for every n . Linearity therefore gives

$$\text{tr}(Y \mathcal{E}_A^n(Z_\rho(X))) = \text{tr}(Y \mathcal{E}_A^n(X\rho)) - \text{tr}(X\rho)\text{tr}(Y\rho), \quad (15.5)$$

which is the definition of $C(X, Y; n)$. □

15.2 Spectral expansion and exponential decay

Theorem 15.2.1 (Sum of exponentials (spectral expansion interface)). [Lean](#) [✓ Stmt](#) If coefficients $c_j \in \mathbb{C}$ and eigenvalues $\lambda_j \in \mathbb{C}$, for $j = 1, \dots, D^2 - 1$, satisfy, for all $n \geq 0$,

$$C(X, Y; n) = \sum_{j=1}^{D^2-1} c_j \lambda_j^n, \quad (15.6)$$

then the connected correlator equals the stated sum of exponentials. The source [CPGSV21, Section II.B.3] states that the connected correlator is a sum of $D^2 - 1$ pure exponentials, $C(X, Y; n) = \sum_{j \geq 2}^{D^2} c_{XY}(j) \lambda_j^n$. This pure-exponential representation follows when the relevant subleading spectral data of the transfer map are diagonalizable. For a non-diagonalizable transfer map, nontrivial Jordan blocks can instead produce terms of the form $n^k \lambda^n$. Accordingly, the pure-exponential formula is retained here as an explicit interface hypothesis rather than asserted for every normal MPS.

Proof. ✓ Proof Given the expansion hypothesis (15.6), the equality is immediate. Under the corresponding diagonalizability hypothesis, the expansion follows from the spectral decomposition of $\mathcal{E}_A^n$ restricted to the complement of the fixed-point eigenspace. The leading eigenvalue $\lambda_1 = 1$ contributes $\langle X_0 \rangle \langle Y_0 \rangle$, which is subtracted in the connected correlator, leaving the sum over subleading eigenvalues. This is the spectral mechanism described in [CPGSV21, Section II.B.3]. $\square$

Theorem 15.2.2 (Exponential decay bound (interface)). Lean ✓ Stmt If a constant $C_{XY} \in \mathbb{R}$ and $\lambda_2 \in \mathbb{C}$ satisfy, for all $n \geq 0$,

$$|C(X, Y; n)| \leq C_{XY} |\lambda_2|^n, \quad (15.7)$$

then the connected correlator satisfies that exponential decay bound. The source [CPGSV21, Section II.B.3] obtains this by combining the sum-of-exponentials expansion with the triangle inequality, while under the primitive or injective hypotheses stated in Chapter 6, the transfer-map gap results ensure that the subleading eigenvalues lie strictly inside the unit disk.

Proof. ✓ Proof Given the bound hypothesis (15.7), the estimate is immediate. For a pure-exponential expansion, the source [CPGSV21, Section II.B.3] obtains such a bound by applying the triangle inequality and using $|\lambda_j| \leq |\lambda_2|$ for all subleading eigenvalues; when the transfer-map gap gives $|\lambda_2| < 1$, this is a decaying bound. $\square$

Remark 15.2.3 (Transfer-map spectral hypothesis). The condition $|\lambda_2| < 1$ turns the bound in Theorem 15.2.2 into exponential decay. It is a transfer-map gap condition: all non-leading eigenvalues of $\mathcal{E}_A$ lie strictly inside the unit disk. It is distinct from the Hamiltonian spectral gap of Chapter 13. The complementary transfer-map gap $\rho(\mathcal{E}_A - P) < 1$ is established for primitive channels (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]). The mixed-transfer gap needed for separation of non-equivalent canonical blocks is supplied for irreducible trace-preserving tensors (Theorem 6.6.2), and quantitative single-tensor transfer-map bounds for injective tensors are recorded in Theorem 15.3.3.

These transfer-map gap results give exponential convergence of the complementary transfer map at every rate strictly larger than its spectral radius. Deriving a connected-correlator bound still requires extracting the spectral coefficients and, in the presence of nontrivial Jordan blocks, the generalized spectral terms from $\mathcal{E}_A$. This derivation is not included here.

Definition 15.2.4 (Correlation length). Lean ✓ Stmt For every $\lambda_2 \in \mathbb{C}$, define the associated correlation-length quantity by

$$\xi = -\frac{1}{\log |\lambda_2|}.$$

For a nonzero subleading eigenvalue in the physical range $0 < |\lambda_2| < 1$, this quantity is positive. If all subleading spectral values vanish and correlations therefore vanish after finitely many transfer steps, the total definition gives the limiting value $\xi = 0$.

Lemma 15.2.5 (Correlation length is positive). Lean ✓ Stmt If $|\lambda_2| < 1$ and $\lambda_2 \neq 0$, then $\xi > 0$.

Proof. ✓ Proof Since $0 < |\lambda_2| < 1$, we have $\log |\lambda_2| < 0$, so $\xi = -1/\log |\lambda_2| > 0$. $\square$

15.3 Quantitative transfer-map gap bounds

The companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”] establishes $\rho(\mathcal{E}_A - P) < 1$ for primitive channels qualitatively. This section gives quantitative refinements in terms of geometric constants and rates for the single-tensor transfer map $\mathcal{E}_A$.

Theorem 15.3.1 (Exponential convergence of injective primitive channels). [Lean] [✓ Stmt] Fix an injective, trace-preserving MPS tensor A with positive definite fixed point ρ . Let

$$P(X) = \frac{\text{tr}(X)}{\text{tr}(\rho)} \rho \quad (15.8)$$

be the corresponding fixed-point projection. Then there exist $C > 0$ and $0 < \delta \leq 1$ such that, for all $n \geq 0$ and $X \in M_D(\mathbb{C})$,

$$\|\mathcal{E}_A^n(X) - P(X)\| \leq C(1 - \delta)^n \|X\|. \quad (15.9)$$

The transfer-map gap δ exists by primitivity (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”], the complementary transfer-map gap).

Proof. [✓ Proof] Injectivity implies primitivity of the transfer map. The complementary spectral-radius bound $\rho(\mathcal{E}_A - P) < 1$ then follows from the primitive transfer-map gap theorem. Choose r with $\rho(\mathcal{E}_A - P) < r < 1$. The Gelfand formula yields $\|(\mathcal{E}_A - P)^n\| \leq Cr^n$ for a suitable constant $C > 0$. Since P is the fixed-point projection, $\mathcal{E}_A P = P \mathcal{E}_A = P$, and hence $\mathcal{E}_A^n - P = (\mathcal{E}_A - P)^n$ for $n \geq 1$. The case $n = 0$ is absorbed by increasing C if necessary. Setting $\delta = 1 - r$ gives (15.9). $\square$

Theorem 15.3.2 (Correlation length bound). [Lean] [✓ Stmt] For an injective trace-preserving MPS tensor, there exist $C > 0$ and $\xi > 0$ such that, for all $n \geq 0$ and all traceless $X \in M_D(\mathbb{C})$, i.e. $\text{tr}(X) = 0$,

$$\|\mathcal{E}_A^n(X)\| \leq C e^{-n/\xi} \|X\|. \quad (15.10)$$

Traceless matrices lie in $\ker P$, where P is the fixed-point projection (15.8). Since $\mathcal{E}_A - P$ has spectral radius strictly less than 1, choose any r with $\rho(\mathcal{E}_A - P) < r < 1$. The Gelfand formula then gives exponential decay with $\xi = -1/\log r$. Choosing a rate strictly above the spectral radius is necessary in general because a Jordan block at the spectral radius can prevent a uniform bound proportional to $\rho(\mathcal{E}_A - P)^n$.

Proof. [✓ Proof] For traceless X , we have $P(X) = 0$, so $\mathcal{E}_A^n(X) = (\mathcal{E}_A - P)^n(X)$. Applying the Gelfand-formula estimate with $\rho(\mathcal{E}_A - P) < r < 1$ gives $\|\mathcal{E}_A^n(X)\| \leq Cr^n \|X\|$. With $\xi = -1/\log r > 0$, the identity $r^n = e^{-n/\xi}$ yields (15.10). $\square$

Theorem 15.3.3 (Transfer-map gap from injectivity). [Lean] [✓ Stmt] For an injective trace-preserving MPS tensor, there exists $\delta > 0$ such that every eigenvalue $\mu \neq 1$ of the transfer map satisfies $|\mu| \leq 1 - \delta$.

Proof. [✓ Proof] Injectivity implies eventual full Kraus rank. This yields primitivity, hence a complementary transfer-map gap. Since the transfer map has only finitely many eigenvalues, one may choose a uniform $\delta > 0$ separating 1 from the remaining spectrum. $\square$

15.4 Relation to parent Hamiltonians

Remark 15.4.1. Suppose that the MPS tensor A generates a parent Hamiltonian $H_N(A, L)$ (Definition 12.3.58) whose ground state is the matrix product vector (Lemma 12.3.62), and that physical observable insertions reduce to the auxiliary functionals defined above. If the hypotheses of Theorem 15.2.2 hold, then the corresponding ground-state correlations decay with finite correlation length ξ .

Chapter 16

Concrete Examples

This chapter collects concrete MPS tensors from [CPGSV21, Section III]. Each example is specified by its physical dimension d , bond dimension D , and explicit matrix family $\{A^i\}$. Key structural properties (injectivity, RFP status, symmetry) are stated for each tensor.

16.1 The GHZ state

The Greenberger–Horne–Zeilinger (GHZ) state is the prototypical non-injective, renormalization-fixed-point MPS.

Definition 16.1.1 (GHZ tensor). [Lean](#) [✓ Stmt](#) Set $d = D = 2$. The GHZ tensor is the family $\{A^0, A^1\} \subset M_2(\mathbb{C})$ defined by

$$A^0 = |0\rangle\langle 0| = \text{diag}(1, 0), \quad A^1 = |1\rangle\langle 1| = \text{diag}(0, 1).$$

For a system of N sites, the resulting MPV is the GHZ state

$$|\text{GHZ}_N\rangle = |0\rangle^{\otimes N} + |1\rangle^{\otimes N}.$$

Definition 16.1.2 (Two-site computational vector). [Lean](#) [✓ Stmt](#) For $a, b \in \{0, 1\}$, the vector $|ab\rangle \in (\{0, 1\}^2 \rightarrow \mathbb{C})$ is the indicator of the configuration $(\sigma_1, \sigma_2) = (a, b)$.

Definition 16.1.3 (Constant product vector). [Lean](#) [✓ Stmt](#) For $a \in \{0, 1\}$ and $N \geq 0$,

$$|a\rangle^{\otimes N}(\sigma_1, \dots, \sigma_N) = \begin{cases} 1, & (\sigma_1, \dots, \sigma_N) = (a, \dots, a), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 16.1.4 (GHZ transfer map is idempotent). [Lean](#) [✓ Stmt](#) The transfer map of the GHZ tensor satisfies $\mathcal{E}_A^2 = \mathcal{E}_A$.

Proof. [✓ Proof](#) The transfer map acts by $\mathcal{E}_A(X)_{ij} = \delta_{ij}X_{ij}$ (extraction of the diagonal). This operation is idempotent. $\square$

Theorem 16.1.5 (GHZ tensor is an RFP). [Lean](#) [✓ Stmt](#) The GHZ tensor is a renormalization fixed point.

Proof. [✓ Proof](#) The claim follows from Theorem 16.1.4. $\square$

Theorem 16.1.6 (GHZ tensor is not injective). [Lean](#) [✓ Stmt](#) *The GHZ tensor is not injective: $\text{span}_{\mathbb{C}}\{A^0, A^1\}$ is the two-dimensional space of diagonal matrices in $M_2(\mathbb{C})$, which does not equal $M_2(\mathbb{C})$.*

Proof. [✓ Proof](#) Every element of $\text{span}_{\mathbb{C}}\{A^0, A^1\}$ has zero off-diagonal entries, so the matrix with all entries equal to 1 is not in the span. $\square$

Theorem 16.1.7 ($\mathbb{Z}_2$ on-site symmetry of the GHZ tensor). [Lean](#) [✓ Stmt](#) *Let $\mathbb{Z}_2 = \{1, \sigma_x\}$ act on $\mathbb{C}^2$ by the Pauli- X matrix σ_x . The GHZ tensor is on-site symmetric under this action: for each $g \in \mathbb{Z}_2$, the twisted tensor $\tilde{A}_g$ defines the same MPV family as A , i.e. $\mathcal{V}(A) = \mathcal{V}(\tilde{A}_g)$.*

Proof. [✓ Proof](#) The identity element is trivial. For the generator $g = \sigma_x$, the twisted tensor satisfies $\tilde{A}_g^i = A^{1-i}$, i.e. the two matrices are swapped. Conjugation by σ_x implements the same swap, giving gauge equivalence and hence the same MPV family. $\square$

Definition 16.1.8 (Virtual $\mathbb{Z}_2$ representation of the GHZ tensor). [Lean](#) [✓ Stmt](#) *The virtual $\mathbb{Z}_2$ representation on the bond space is the homomorphism $\mathbb{Z}_2 \rightarrow M_2(\mathbb{C})$ sending the identity to I and the generator to $\sigma_z = \text{diag}(1, -1)$.*

Theorem 16.1.9 (Virtual $\mathbb{Z}_2$ symmetry of the GHZ tensor). [Lean](#) [✓ Stmt](#) *The matrix σ_z commutes with every GHZ matrix A^i . A non-scalar matrix commuting with all A^i is the hallmark of a non-injective ($\mathbb{Z}_2$ -injective) tensor: the on-site symmetry of Theorem 16.1.7 swaps the two one-dimensional injective blocks $A^0 \leftrightarrow A^1$, while σ_z acts on each block by a phase.*

Proof. [✓ Proof](#) Both A^0 and A^1 are diagonal, so each commutes with the diagonal matrix σ_z . $\square$

Theorem 16.1.10 (GHZ order parameter is a transfer-map fixed point). [Lean](#) [✓ Stmt](#) *The order-parameter operator σ_z is a fixed point of the GHZ transfer map, $\mathcal{E}_A(\sigma_z) = \sigma_z$.*

Proof. [✓ Proof](#) Since $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ and both A^i are diagonal, the claim is the entrywise computation $\mathcal{E}_A(\sigma_z) = \sigma_z$. $\square$

Remark 16.1.11. A primitive (injective) tensor has a one-dimensional transfer-map fixed-point space, spanned by a positive matrix that equals the identity only in a normalized gauge. That the GHZ transfer map fixes the additional non-scalar operator σ_z is the transfer-matrix signature of the extra non-decaying mode behind its long-range ferromagnetic correlations.

Theorem 16.1.12 (GHZ zero correlation length). [Lean](#) [✓ Stmt](#) *The GHZ tensor has zero correlation length: its idempotent transfer map (Theorem 16.1.5) gives correlations independent of the separation. This contrasts with the AKLT state (Theorem 16.2.1.4), whose subleading eigenvalue $-\frac{1}{3}$ produces a finite correlation length $1/\log 3$.*

Proof. [✓ Proof](#) The claim follows from the idempotence of the GHZ transfer map and the equivalence of zero correlation length with transfer-map idempotence (Theorem 18.5.21). $\square$

16.2 The AKLT state

The Affleck–Kennedy–Lieb–Tasaki (AKLT) state is the canonical example of a normal (block-injective) MPS with non-trivial symmetry-protected topological (SPT) order.

Definition 16.2.1 (AKLT tensor). [Lean](#) [✓ Stmt](#) Set $d = 3$ (spin-1) and $D = 2$. The AKLT tensor is the family $\{A^0, A^1, A^2\} \subset M_2(\mathbb{C})$ defined via the Pauli matrices by

$$A^0 = \frac{1}{\sqrt{3}}\sigma_z, \quad A^1 = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle\langle 1|, \quad A^2 = -\frac{\sqrt{2}}{\sqrt{3}}|1\rangle\langle 0|.$$

Here $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, so equivalently $A^1 = \frac{1}{\sqrt{3}}\sigma_+$ and $A^2 = -\frac{1}{\sqrt{3}}\sigma_-$ for $\sigma_+ = \sqrt{2}|0\rangle\langle 1|$ and $\sigma_- = \sqrt{2}|1\rangle\langle 0|$. Equivalently, the matrices are the Clebsch–Gordan coefficients coupling spin-1 and spin- $\frac{1}{2}$ to spin- $\frac{1}{2}$.

Theorem 16.2.2 (AKLT tensor is not injective). [Lean](#) [✓ Stmt](#) *The AKLT tensor is not injective at a single site: $\text{span}_{\mathbb{C}}\{A^0, A^1, A^2\}$ is the three-dimensional space of traceless matrices $\mathfrak{sl}(2, \mathbb{C})$, which is a proper subspace of $M_2(\mathbb{C})$.*

Theorem 16.2.3 (AKLT tensor is normal). [Lean](#) [✓ Stmt](#) *The transfer map of the AKLT tensor has a unique eigenvalue of maximal modulus (a simple spectral-radius eigenvalue); in particular the AKLT tensor is normal. Concretely, the length-2 blocked tensor is injective.*

Theorem 16.2.4 ($\mathbb{Z}_2 \times \mathbb{Z}_2$ on-site symmetry of the AKLT tensor). [Lean](#) [✓ Stmt](#) *The AKLT tensor is on-site symmetric under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\text{SO}(3)$ generated by two commuting π -rotations about orthogonal axes, acting on the spin-1 physical index.*

Theorem 16.2.5 (Anticommuting virtual gauges of the AKLT symmetry). [Lean](#) [✓ Stmt](#) *The two virtual matrices that implement the symmetry of Theorem 16.2.4, namely $i\sigma_y$ and σ_z , anti-commute.*

Definition 16.2.6 (AKLT factor system). [Lean](#) [✓ Stmt](#) The AKLT factor system on $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\omega(g, h) = (-1)^{(g_1+g_2)h_1}$ (the value -1 when $g_1 + g_2 = 1$ and $h_1 = 1$, and 1 otherwise). It is the factor system of the explicit projective representation with virtual gauges $i\sigma_y$ and σ_z ; the diagonal term $g_1 h_1$ relative to the cluster case reflects $(i\sigma_y)^2 = -\mathbb{1}$.

Lemma 16.2.7 (AKLT commutator phase). [Lean](#) [✓ Stmt](#) *The commutator phase of the AKLT factor system on the two generators is -1 .*

Theorem 16.2.8 (The AKLT factor system is a 2-cocycle). [Lean](#) [✓ Stmt](#) *The AKLT factor system is a genuine 2-cocycle, being the factor system of an explicit projective representation, so it represents an element of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1))$.*

Theorem 16.2.9 (The AKLT $\mathbb{Z}_2 \times \mathbb{Z}_2$ phase is non-trivial). [Lean](#) [✓ Stmt](#) *The AKLT factor system (Theorem 16.2.8), built from the anticommuting gauges $i\sigma_y$ and σ_z (Theorem 16.2.5), is not a coboundary: its commutator phase on the two generators is -1 . Its class in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1)) \cong \mathbb{Z}_2$ is therefore non-trivial, so the AKLT chain realizes a non-trivial SPT phase rather than a trivial product state.*

Remark 16.2.10. This $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the smallest subgroup of $\text{SO}(3)$ that still carries a non-trivial 2-cocycle. That minimality, and the full continuous classification over $\text{SO}(3)$, are not used in this discrete-subgroup example (Remark 16.2.20).

Definition 16.2.11 (Blocked AKLT tensor). [Lean](#) [✓ Stmt](#) The length-2 blocked AKLT tensor groups two physical sites into one, with blocked physical dimension $9 = 3^2$ and the same bond dimension $D = 2$.

Definition 16.2.12 (Blocked $\mathbb{Z}_2 \times \mathbb{Z}_2$ on-site action). [Lean](#) [✓ Stmt](#) The length-2 blocked on-site action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the 9-dimensional blocked physical space is the Kronecker square $g \mapsto P_g \otimes P_g$ of the single-site spin-1 π -rotation representation P_g that implements the symmetry of Theorem 16.2.4. Identifying the blocked physical index with the pair of single-site indices, the entry of $P_g \otimes P_g$ at $(i_1 i_2, j_1 j_2)$ is the product $(P_g)_{i_1 j_1} (P_g)_{i_2 j_2}$ over the two grouped sites. This is the on-site action that makes the blocked twist agree with blocking the single-site twist.

Theorem 16.2.13 (The blocked AKLT tensor is injective). [Lean](#) [✓ Stmt](#) *The length-2 blocked AKLT tensor is injective: its nine matrices span $M_2(\mathbb{C})$.*

Proof. [✓ Proof](#) This is the blocked-tensor form of the normality of the single-site tensor (Theorem 16.2.3): the single-site tensor is 2-block injective, which is exactly injectivity of its length-2 blocking. $\square$

Theorem 16.2.14 ($\mathbb{Z}_2 \times \mathbb{Z}_2$ on-site symmetry of the blocked AKLT tensor). [Lean](#) [✓ Stmt](#) *The length-2 blocked AKLT tensor is on-site symmetric under $\mathbb{Z}_2 \times \mathbb{Z}_2$, with each group element acting by the Kronecker square $P_g \otimes P_g$ of the corresponding single-site spin-1 π -rotation P_g (Definition 16.2.12).*

Proof. [✓ Proof](#) Writing A for the single-site tensor and $A^{[2]}$ for its length-2 blocking, twisting the blocked tensor by $P_g \otimes P_g$ equals blocking the twisted single-site tensor, $(A^{[2]})^{P_g \otimes P_g} = (A^{P_g})^{[2]}$, where the superscript denotes the twist. Since the single-site tensor is symmetric, A^{P_g} has the same matrix product vector family as A (Theorem 16.2.4), and blocking preserves the matrix product vector family, so $(A^{[2]})^{P_g \otimes P_g}$ has the same matrix product vector family as $A^{[2]}$. $\square$

Theorem 16.2.15 (The single-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ action is unitary). [Lean](#) [✓ Stmt](#) *Each single-site spin-1 π -rotation matrix P_g of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action is unitary: $P_g P_g^\dagger = \mathbb{1}$.*

Theorem 16.2.16 (The AKLT state has string order under its $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry). [Lean](#) [✓ Stmt](#) *For every element g of $\mathbb{Z}_2 \times \mathbb{Z}_2$, the length-2 blocked AKLT tensor has string order in the sense of Definition 11.3.11 for the twist by g , with the maximally mixed boundary state $\Lambda = \frac{1}{2}\mathbb{1}$ as its stationary boundary state.*

Proof. [✓ Proof](#) The blocked tensor is injective (Theorem 16.2.13) and on-site symmetric under $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Theorem 16.2.14), and each group element acts by a real symmetric involutive matrix, hence unitarily. The maximally mixed state $\Lambda = \frac{1}{2}\mathbb{1}$ is positive definite and has trace 1. The single-site AKLT matrices satisfy $\sum_i A^i (A^i)^\dagger = \mathbb{1}$ and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. These identities propagate to the length-2 blocked tensor $A^{[2]}$, whose matrices are the products $A^{i_1} A^{i_2}$. In particular, the blocked transfer map is unital:

$$\mathcal{E}_{A^{[2]}}(\mathbb{1}) = \sum_{i_1, i_2} A^{i_1} A^{i_2} (A^{i_2})^\dagger (A^{i_1})^\dagger = \mathbb{1}.$$

Moreover, $\Lambda = \frac{1}{2}\mathbb{1}$ is a fixed point of the adjoint transfer map:

$$\mathcal{E}_{A^{[2]}}^\dagger(\Lambda) = \sum_{i_1, i_2} (A^{i_2})^\dagger (A^{i_1})^\dagger \Lambda A^{i_1} A^{i_2} = \Lambda.$$

The first identity sums the right-canonical relation $\sum_i A^i (A^i)^\dagger = \mathbb{1}$ over the inner letter and then the outer; the second sums the left-canonical relation $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ over the outer letter and then the inner, using $\Lambda = \frac{1}{2}\mathbb{1}$. These are exactly the hypotheses of Theorem 11.4.4, which then yields string order for every g . Like the cluster state, the AKLT state is an explicit witness of that criterion, sharing the same virtual-boundary caveat on the string order parameter. $\square$

Definition 16.2.17 (Cartesian AKLT tensor). [Lean](#) [✓ Stmt](#) The Cartesian presentation of the AKLT tensor has $d = 3$, $D = 2$, and $A^x = \sigma_x$, $A^y = \sigma_y$, $A^z = \sigma_z$, the three Pauli matrices. This is the rotationally natural form used in [CPGSV21] (around line 1159); it is the AKLT state in the Cartesian basis of the spin-1 site, the counterpart of the $|m\rangle$ -basis form of Definition 16.2.1.

Theorem 16.2.18 (The spin- $\frac{1}{2}$ cover is surjective onto $\text{SO}(3)$). [Lean](#) [✓ Stmt](#) Every rotation $R \in \text{SO}(3)$ is realized by conjugating the Pauli vector by some $U \in \text{SU}(2)$: there is $U \in \text{SU}(2)$ with $(\frac{1}{2} \text{tr}(\sigma_i U \sigma_j U^{-1}))_{ij} = R$. Equivalently, the adjoint double cover $\text{SU}(2) \rightarrow \text{SO}(3)$ is surjective.

Proof. [✓ Proof](#) Factor $R = R_z(\alpha)R_x(\beta)R_z(\gamma)$ by its ZXZ Euler decomposition and map each coordinate rotation to an explicit $\text{SU}(2)$ generator:

$$\begin{aligned} \text{diag}(e^{-i\theta/2}, e^{i\theta/2}) &\mapsto R_z(\theta), \\ \begin{pmatrix} \cos(\beta/2) & -i \sin(\beta/2) \\ -i \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} &\mapsto R_x(\beta). \end{aligned}$$

□

Theorem 16.2.19 ($\text{SO}(3)$ on-site symmetry of the AKLT tensor). [Lean](#) [✓ Stmt](#) The Cartesian AKLT tensor is on-site symmetric under the spin-1 (defining) representation of $\text{SO}(3)$ on the physical index.

Remark 16.2.20. The bond index carries the projective spin- $\frac{1}{2}$ representation: the gauge for a rotation is either of its two $\text{SU}(2)$ preimages, so the assignment is a representation only up to sign and factors through the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$. Its class is the non-trivial element of $H^2(\text{SO}(3), \text{U}(1)) \cong \mathbb{Z}_2$, witnessing the SPT phase.

16.2.1 Correlation length

The AKLT state is the canonical example of a finite correlation length. Its transfer map has leading eigenvalue 1 and a single subleading eigenvalue $-\frac{1}{3}$ of multiplicity three; the three Pauli matrices are the subleading eigenvectors and the identity is the leading one. The subleading modulus $\frac{1}{3}$ gives the correlation length $\xi = 1/\log 3$, and connected correlations decay as $(\frac{1}{3})^n$.

Lemma 16.2.1.1 (Complex square of a real square root). [Lean](#) [✓ Stmt](#) If $x \geq 0$, then the complex coercion of its real square root satisfies $((\sqrt{x} : \mathbb{R}) : \mathbb{C})^2 = x$.

Proof. [✓ Proof](#) Apply the real identity $(\sqrt{x})^2 = x$ and commute the real embedding with powers.

□

Theorem 16.2.1.2 (Pauli eigenvectors of the AKLT transfer map). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Each of the three Pauli matrices σ_x , σ_y , σ_z is an eigenvector of the AKLT transfer map with eigenvalue $-\frac{1}{3}$:

$$\mathcal{E}_A(\sigma_k) = -\frac{1}{3} \sigma_k, \quad k \in \{x, y, z\}.$$

Proof. [✓ Proof](#) Each identity is an entrywise computation from the AKLT matrices, using $(\sqrt{3})^2 = 3$ and $(\sqrt{2})^2 = 2$ to clear the prefactors.

□

Theorem 16.2.1.3 (Subleading eigenvalue of the AKLT transfer map). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The value 1 is the leading eigenvalue of the AKLT transfer map, with the identity $\mathbb{1}$ as eigenvector, and $-\frac{1}{3}$ is a subleading eigenvalue, with σ_z as eigenvector. The subleading modulus is therefore $\frac{1}{3}$.

Theorem 16.2.1.4 (AKLT correlation length). [Lean](#) [Lean](#) [✓ Stmt](#) *The AKLT correlation length, computed from the subleading eigenvalue $\lambda_2 = -\frac{1}{3}$, equals*

$$\xi = -\frac{1}{\log |\lambda_2|} = \frac{1}{\log 3}.$$

It is finite and positive.

Proof. [✓ Proof](#) The subleading modulus is $|\lambda_2| = \frac{1}{3}$, so rewriting the correlation length at $\lambda_2 = -\frac{1}{3}$ gives $\xi = -1/\log(\frac{1}{3}) = 1/\log 3$. Since $0 < \frac{1}{3} < 1$, positivity is the specialization of Lemma 15.2.5 to $\lambda_2 = -\frac{1}{3}$. $\square$

16.3 The Majumdar–Ghosh state

The Majumdar–Ghosh state is the exact dimer ground state of the spin- $\frac{1}{2}$ chain $H = \sum S_i \cdot S_{i+1} + \frac{1}{2} \sum S_i \cdot S_{i+2}$. On an even periodic chain it is the symmetric superposition of the two nearest-neighbour singlet coverings $(1, 2)(3, 4) \dots$ and $(2, 3)(4, 5) \dots (N, 1)$, the one-dimensional analog of a resonating valence-bond state.

Definition 16.3.1 (Majumdar–Ghosh tensor). [Lean](#) [✓ Stmt](#) Set $d = 2$ (spin- $\frac{1}{2}$) and $D = 3$. The Majumdar–Ghosh tensor is the family $\{A^0, A^1\} \subset M_3(\mathbb{C})$ defined by

$$A^0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The three bond levels are the two spin- $\frac{1}{2}$ values of an open singlet partner together with the closed inter-dimer link; the relative sign $-\frac{1}{\sqrt{2}}$ is the antisymmetry of the singlet. The equality between the matrix product vectors of these matrices and the even-ring dimer superposition, with normalization and cyclic convention fixed, is a separate verification, not part of this definition.

Theorem 16.3.2 (Majumdar–Ghosh tensor is in left-canonical form). [Lean](#) [✓ Stmt](#) *The Majumdar–Ghosh tensor is isometric: $(A^0)^\dagger A^0 + (A^1)^\dagger A^1 = \mathbb{1}$.*

Proof. [✓ Proof](#) Direct multiplication of the two displayed matrices gives $(A^0)^\dagger A^0 = \text{diag}(\frac{1}{2}, 1, 0)$ and $(A^1)^\dagger A^1 = \text{diag}(\frac{1}{2}, 0, 1)$. $\square$

Theorem 16.3.3 (Transfer map of the Majumdar–Ghosh tensor). [Lean](#) [✓ Stmt](#) *The transfer map of the Majumdar–Ghosh tensor sends the identity to the diagonal matrix $\mathcal{E}_A(\mathbb{1}) = \text{diag}(2, \frac{1}{2}, \frac{1}{2})$.*

Proof. [✓ Proof](#) Substitute the two matrices in $\mathcal{E}_A(\mathbb{1}) = A^0(A^0)^\dagger + A^1(A^1)^\dagger$ and multiply. $\square$

Theorem 16.3.4 (Majumdar–Ghosh tensor is not injective). [Lean](#) [✓ Stmt](#) *The Majumdar–Ghosh tensor is not injective: every element of $\text{span}_{\mathbb{C}}\{A^0, A^1\}$ has vanishing $(2, 2)$ entry, so the span is a proper subspace of $M_3(\mathbb{C})$.*

Proof. [✓ Proof](#) The matrix unit E_{22} has nonzero $(2, 2)$ entry, while every linear combination of A^0 and A^1 has zero $(2, 2)$ entry. $\square$

Theorem 16.3.5 (Majumdar–Ghosh tensor is not normal). [Lean](#) [✓ Stmt](#) *The Majumdar–Ghosh tensor is not normal: no blocking length makes the products of that length span the full matrix algebra $M_3(\mathbb{C})$. This non-normality is the algebraic signature expected from the two-periodic dimer-covering structure.*

Proof. ✓ Proof The bond carries a two-periodic grading with complementary support sets

$$\begin{aligned} &\{(0, 1), (0, 2), (1, 0), (2, 0)\}, \\ &\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2)\}. \end{aligned}$$

Left multiplication with a generator exchanges these sets. Let E_{ij} denote the (i, j) matrix unit. Then, for every blocking length N ,

$$\begin{aligned} N \text{ odd} &\Rightarrow (\forall w, |w| = N \Rightarrow (A^w)_{22} = 0) \Rightarrow E_{22} \notin \text{span}_{\mathbb{C}}\{A^w : |w| = N\}, \\ N \text{ even} &\Rightarrow (\forall w, |w| = N \Rightarrow (A^w)_{10} = 0) \Rightarrow E_{10} \notin \text{span}_{\mathbb{C}}\{A^w : |w| = N\}. \end{aligned}$$

Hence no fixed-length word span is all of $M_3(\mathbb{C})$. □

Remark 16.3.6. The review writes the tensor in the valence-bond shorthand $A = [|0\rangle((02| + (20|) + |1\rangle((12| + (21|)) \otimes Y$ with the singlet $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Read literally as $(ab| = |a\rangle\langle b|$ on a separate three-level factor tensored with the two-level singlet space, that shorthand would describe a six-dimensional bond. Its two-site contractions do not have the singlet-pair support pattern: same-letter traces can be nonzero while opposite-letter traces vanish. The shorthand is therefore treated here as a schematic of the valence-bond construction: contracting the singlet Y leaves a matrix product operator on a three-dimensional bond, with the antisymmetry of Y supplying the relative sign. The contracted tensor is the $D = 3$ representative in Definition 16.3.1. The source-notation gap and the missing matrix-product-vector identification are recorded in [con26v].

The remaining target for the Majumdar–Ghosh example is the ground-space description of its parent Hamiltonian. Because the tensor is non-normal (Theorem 16.3.5) with a two-periodic bond grading, its periodic ground space is governed by the block-diagonal boundary-condition machinery for non-normal tensors (Section 14.3) rather than by the injective single-state result. Thus the relevant target is the following range-three periodic ground-space statement.

Theorem 16.3.7 (Two-fold dimer degeneracy of the Majumdar–Ghosh ground space). Not ready

On an even periodic chain of length $N = 2m \geq 4$, let $H_{N, \text{MG}}^{(3)}$ be the range-three parent Hamiltonian of the Majumdar–Ghosh tensor, the periodic sum of translates of the local projector with kernel $\mathcal{G}_3(A_{\text{MG}})$. Then $H_{N, \text{MG}}^{(3)}$ has a two-dimensional ground space, spanned by the two nearest-neighbour singlet coverings $(1, 2)(3, 4) \cdots (N-1, N)$ and $(2, 3)(4, 5) \cdots (N, 1)$. This two-fold degeneracy reflects the spontaneous breaking of the one-site translation symmetry of the spin- $\frac{1}{2}$ chain $H = \sum S_i \cdot S_{i+1} + \frac{1}{2} \sum S_i \cdot S_{i+2}$ of [CPGSV21].

16.4 The W state

The W state on N sites is the single-excitation symmetric state

$$|W_N\rangle = |10 \cdots 0\rangle + |010 \cdots 0\rangle + \cdots + |0 \cdots 01\rangle,$$

the equal-weight superposition of all configurations with exactly one excitation. Unlike the preceding examples it has no translation-invariant representation with bond dimension bounded independently of N ; it is presented with open boundary conditions, where the matrix product is closed by a left and a right boundary vector rather than by a trace.

Definition 16.4.1 (Open-boundary contraction). Lean ✓ Stmt For a tensor A with bond dimension D , a left boundary covector $v_L \in \mathbb{C}^D$, a right boundary vector $v_R \in \mathbb{C}^D$, and a word $w = (i_1, \dots, i_k)$, the open-boundary contraction is the bilinear pairing

$$(v_L | A^{i_1} A^{i_2} \cdots A^{i_k} | v_R) = v_L^\top (A^{i_1} \cdots A^{i_k}) v_R.$$

Definition 16.4.2 (Open-boundary state). [Lean](#) [✓ Stmt](#) The open-boundary state on N sites assigns to each configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ the amplitude $(v_L | A^{\sigma_1} \dots A^{\sigma_N} | v_R)$.

Definition 16.4.3 (W tensor and boundary vectors). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Set $d = D = 2$. The W tensor is the site-independent family

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with left boundary covector $\langle l| = \langle 0|$ and right boundary vector $|r\rangle = |1\rangle$. Here A^1 is the single raising operator: it sends $|1\rangle \mapsto |0\rangle$ and annihilates $|0\rangle$, and squares to zero.

Definition 16.4.4 (W-state indicator). [Lean](#) [✓ Stmt](#) For $N \geq 0$, the W-state amplitude is the indicator of the single-excitation configurations:

$$|W_N\rangle(\sigma_1, \dots, \sigma_N) = \begin{cases} 1, & \text{exactly one } \sigma_j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 16.4.5 (The open-boundary contraction is the W state). [Lean](#) [✓ Stmt](#) For every N , the open-boundary state of the W tensor with boundary vectors $\langle 0|$ and $|1\rangle$ equals the W state:

$$|W_N\rangle = \sum_{\sigma} \langle 0 | A^{\sigma_1} \dots A^{\sigma_N} | 1 \rangle |\sigma\rangle.$$

Proof. [✓ Proof](#) Because A^0 is the identity and A^1 is the raising operator with $A^1 A^1 = 0$, the ordered product $A^{\sigma_1} \dots A^{\sigma_N}$ depends only on the number of excitations of σ : it is the identity with no excitation, the raising operator with one excitation, and zero with two or more. Reading off the $(0, 1)$ matrix element against the boundary vectors therefore gives 1 on the single-excitation configurations and 0 elsewhere, which is the W-state amplitude. $\square$

Theorem 16.4.6 (Three-site single excitations). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) On three sites, each of the three single-excitation configurations $|100\rangle$, $|010\rangle$, $|001\rangle$ receives W-state amplitude 1, while the vacuum $|000\rangle$ and the doubly-excited $|110\rangle$ receive amplitude 0.

Proof. [✓ Proof](#) Direct instances of Theorem 16.4.5: the excitation count is one in the first three cases and zero or two in the others. $\square$

Remark 16.4.7 (No fixed-bond translation-invariant family). The open-boundary representation above is not translation invariant: the chain is closed by boundary vectors, not by a trace. The review observes that there is no translation-invariant representation whose bond dimension is bounded independently of N : any translation-invariant matrix product state equal to $|W_N\rangle$ must have bond dimension growing with N , satisfying a bound of the form $D^3 \log D = \Omega(N)$. The review obtains this by combining the matrix product state representation theory of [PGVWC07] with the quantitative quantum Wielandt inequality. That asymptotic lower bound is not used in the present example.

16.5 The cluster state

The cluster state is a universal resource for measurement-based quantum computation and provides another example of a non-trivial SPT phase.

Definition 16.5.1 (Cluster-state tensor). [Lean](#) [✓ Stmt](#) Set $d = D = 2$. The cluster-state tensor is the family $\{A^0, A^1\} \subset M_2(\mathbb{C})$ defined by

$$A^0 = |+\rangle\langle 0| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A^1 = |-\rangle\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The review [\[CPGSV21\]](#) writes the reflected convention $A^0 = |0\rangle\langle +|$, $A^1 = |1\rangle\langle -|$; the transpose taken here is the same state read in the opposite direction and carries the same SPT order.

Theorem 16.5.2 (Cluster-state tensor is not injective). [Lean](#) [✓ Stmt](#) *The cluster-state tensor is not injective: $\text{span}_{\mathbb{C}}\{A^0, A^1\} = \text{span}_{\mathbb{C}}\{|+\rangle\langle 0|, |-\rangle\langle 1|\}$ has dimension 2, not $\dim M_2(\mathbb{C}) = 4$.*

Proof. [✓ Proof](#) Both A^0 and A^1 are rank-one matrices with linearly independent column spaces, so they are linearly independent. Their span is two-dimensional, which is strictly less than $\dim M_2(\mathbb{C}) = 4$. $\square$

Theorem 16.5.3 (Cluster-state tensor is normal). [Lean](#) [✓ Stmt](#) *Although the cluster-state tensor is not injective at a single site, its length-2 blocking is injective: the four length-2 products span $M_2(\mathbb{C})$. In particular, the cluster-state tensor is normal.*

Remark 16.5.4. Although the cluster-state tensor is not injective at length 1, the length-2 blocked tensor is injective. Using $\langle 0|\pm\rangle = \frac{1}{\sqrt{2}}$, $\langle 1|+\rangle = \frac{1}{\sqrt{2}}$, and $\langle 1|-\rangle = -\frac{1}{\sqrt{2}}$, the four products $A^{ij} = A^i A^j$ are

$$A^{00} = \frac{1}{\sqrt{2}}|+\rangle\langle 0|, \quad A^{01} = \frac{1}{\sqrt{2}}|+\rangle\langle 1|, \quad A^{10} = \frac{1}{\sqrt{2}}|-\rangle\langle 0|, \quad A^{11} = -\frac{1}{\sqrt{2}}|-\rangle\langle 1|. \quad (16.1)$$

Since $\{|+\rangle, |-\rangle\}$ and $\{|0\rangle, |1\rangle\}$ are bases, these four rank-one matrices are linearly independent and span $M_2(\mathbb{C})$.

Theorem 16.5.5 ($\mathbb{Z}_2 \times \mathbb{Z}_2$ on-site symmetry of the blocked cluster-state tensor). [Lean](#) [✓ Stmt](#) *The length-2 blocked cluster-state tensor is on-site symmetric under $\mathbb{Z}_2 \times \mathbb{Z}_2$. The two generators act on the 4-dimensional blocked physical space $(\mathbb{C}^2)^{\otimes 2}$ by $\sigma_x \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_x$, which commute and each square to the identity, defining a linear representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 16.5.6 (Anticommuting virtual gauges of the cluster-state symmetry). [Lean](#) [✓ Stmt](#) *The virtual matrices implementing the two generators of Theorem 16.5.5, namely $V_1 = \sigma_z$ and $V_2 = \sigma_x$, anticommute ($V_1 V_2 = -V_2 V_1$).*

Definition 16.5.7 (Cluster-state factor system). [Lean](#) [✓ Stmt](#) The cluster-state factor system on $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\omega(g, h) = (-1)^{g_2 h_1}$ (the value -1 when $g_2 = 1$ and $h_1 = 1$, and 1 otherwise). It is the factor system of the explicit projective representation with virtual gauges σ_z and σ_x , both squaring to the identity, so there is no diagonal term.

Lemma 16.5.8 (Cluster-state commutator phase). [Lean](#) [✓ Stmt](#) *The commutator phase of the cluster-state factor system on the two generators is -1 .*

Theorem 16.5.9 (The cluster-state factor system is a 2-cocycle). [Lean](#) [✓ Stmt](#) *The cluster-state factor system is a genuine 2-cocycle, being the factor system of an explicit projective representation, so it represents an element of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1))$.*

Theorem 16.5.10 (The cluster-state $\mathbb{Z}_2 \times \mathbb{Z}_2$ phase is non-trivial). [Lean](#) [✓ Stmt](#) *The cluster-state factor system (Theorem 16.5.9), built from the anticommuting gauges σ_z and σ_x (Theorem 16.5.6), is not a coboundary: its commutator phase on the two generators is -1 . Its class in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1)) \cong \mathbb{Z}_2$ is therefore non-trivial, so the cluster state realizes a non-trivial SPT phase.*

Theorem 16.5.11 (The cluster state has string order under its $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry). [Lean](#) [✓ Stmt](#) For every element g of $\mathbb{Z}_2 \times \mathbb{Z}_2$, the length-2 blocked cluster-state tensor has string order in the sense of Definition 11.3.11 for the twist by g , with the maximally mixed boundary state $\Lambda = \frac{1}{2}\mathbb{1}$ as its stationary boundary state.

Proof. [✓ Proof](#) The blocked tensor is injective (Remark 16.5.4) and on-site symmetric under $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Theorem 16.5.5), and each group element acts by a real symmetric involutive permutation matrix, hence unitarily. The maximally mixed state $\Lambda = \frac{1}{2}\mathbb{1}$ is positive definite and has trace 1. The four blocked matrices in (16.1) satisfy $\sum_i A^i (A^i)^\dagger = \mathbb{1}$ and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, so the transfer map is unital and Λ is a fixed point of the adjoint transfer map. These are exactly the hypotheses of Theorem 11.4.4, which then yields string order for every g . The cluster state is thus the first explicit witness of that criterion. $\square$

Theorem 16.5.12 (Blocked cluster transfer map is idempotent). [Lean](#) [Lean](#) [✓ Stmt](#) The length-2 blocked cluster transfer map satisfies $\mathcal{E}_A^2 = \mathcal{E}_A$. In closed form, it sends every X to the scalar $(X_{00} + X_{11})/2$ times the identity, so its image is one-dimensional.

Proof. [✓ Proof](#) The single-site cluster transfer map is not idempotent, but the four blocked matrices give the closed form $\mathcal{E}_A(X) = \frac{1}{2}(X_{00} + X_{11})\mathbb{1}$, which is manifestly idempotent. $\square$

Theorem 16.5.13 (Cluster zero correlation length). [Lean](#) [✓ Stmt](#) The length-2 blocked cluster tensor has zero correlation length: its idempotent transfer map (Theorem 16.5.12) gives correlations independent of the separation. Like the GHZ state, and unlike the AKLT state, the cluster state is a renormalization fixed point with zero correlation length.

Proof. [✓ Proof](#) The claim follows from the idempotence of the blocked cluster transfer map and the equivalence of zero correlation length with transfer-map idempotence (Theorem 18.5.21). $\square$

16.6 Correlation independence without a renormalization fixed point

The following example is [CPGSV16, Example 3.4]. It separates physical correlation independence from the renormalization fixed-point condition. It does not assert local orthogonality or zero correlation length in the stronger basis-of-normal-tensors sense.

Definition 16.6.1 (The tensor of CPSV16 Example 3.4). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Set $d = D = 2$ and

$$A^0 = \text{diag}\left(1, \frac{1}{\sqrt{2}}\right), \quad A^1 = \text{diag}\left(0, \frac{1}{\sqrt{2}}\right).$$

The first and second virtual diagonal entries define the bond-dimension-one product tensors for $|0\rangle$ and $|+\rangle$, respectively.

Theorem 16.6.2 (Positive-length amplitudes). [Lean](#) [✓ Stmt](#) For $N > 0$ and $i = (i_1, \dots, i_N) \in \{0, 1\}^N$,

$$\text{Tr}(A^{i_1} \dots A^{i_N}) = \mathbf{1}_{\{i=(0,\dots,0)\}} + \left(\frac{1}{\sqrt{2}}\right)^N.$$

Hence the generated MPV is $|0\rangle^{\otimes N} + |+\rangle^{\otimes N}$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

Proof. ✓ Proof Both matrices are diagonal. The first diagonal entry contributes 1 only to the all-zero word, while the second contributes $(1/\sqrt{2})^N$ to every word. $\square$

Theorem 16.6.3 (Physical correlations are independent of distance). Lean ✓ Stmt *The tensor in Definition 16.6.1 satisfies physical correlation independence for arbitrary disjoint physical regions, including adjacent regions.*

Proof. ✓ Proof Every word matrix is diagonal. Consequently every observable-inserted transfer operator and the ordinary transfer operator act entrywise on bond matrices, so they commute. The two-region expectation therefore depends on the two complementary gap lengths only through their sum. $\square$

Theorem 16.6.4 (Failure of the renormalization fixed-point equation). Lean Lean ✓ Stmt *The transfer map is not idempotent, and the tensor has no physical blocking isometry. Thus it does not satisfy the pure-state renormalization fixed-point equation.*

Proof. ✓ Proof Let $E_{01} = |0\rangle\langle 1|$. The transfer map obeys

$$\mathcal{E}_A(E_{01}) = \frac{1}{\sqrt{2}}E_{01}, \quad \mathcal{E}_A^2(E_{01}) = \frac{1}{2}E_{01}.$$

Hence $\mathcal{E}_A^2 \neq \mathcal{E}_A$. The equivalence between transfer idempotence and a physical blocking isometry gives the second conclusion. $\square$

Theorem 16.6.5 (Blocked product-state overlap). Lean ✓ Stmt *Let A_0 and A_+ be the bond-dimension-one product tensors obtained from the first and second virtual diagonal entries of A . Their formal MPV overlap at length n is*

$$O_{A_0 A_+}(n) = \left(\frac{1}{\sqrt{2}} \right)^n.$$

Proof. ✓ Proof Only the all-zero configuration contributes, with amplitude $(1/\sqrt{2})^n$. $\square$

16.7 Parent Hamiltonian statements for the examples

The parent-Hamiltonian construction of [CPGSV21, lines 1988–2000], reviewed in Chapter 12, is recorded here in the form

$$\mathcal{G}_L(A) = \left\{ \sum_{i_1, \dots, i_L} \text{tr}(A^{i_1} \dots A^{i_L} X) |i_1 \dots i_L\rangle : X \in M_D(\mathbb{C}) \right\}.$$

Thus

$$\begin{aligned} \ker h_L &= \mathcal{G}_L(A), \\ H_N^{(L)} &= \sum_i (h_L)_i. \end{aligned}$$

We write $\text{GS}(H) = \ker(H - E_0(H)\mathbb{1})$ for the ground eigenspace.

Theorem 16.7.1 (GHZ two-site parent space). Lean ✓ Stmt *For the GHZ tensor A of Definition 16.1.1,*

$$\mathcal{G}_2(A) = \text{span}_{\mathbb{C}}\{|00\rangle, |11\rangle\}.$$

Proof. ✓ Proof By definition, $\Gamma_2(X)(\sigma_1, \sigma_2) = \text{tr}(A^{\sigma_1} A^{\sigma_2} X)$. For the GHZ matrices $A^0 = \text{diag}(1, 0)$ and $A^1 = \text{diag}(0, 1)$, whenever $\sigma_1 \neq \sigma_2$, one has $A^{\sigma_1} A^{\sigma_2} = 0$. Hence $\Gamma_2(X) = X_{00}|00\rangle + X_{11}|11\rangle$, and the image of Γ_2 is $\text{span}_{\mathbb{C}}\{|00\rangle, |11\rangle\}$. $\square$

Theorem 16.7.2 (Constant GHZ product vectors satisfy the cyclic constraints). Lean ✓ Stmt For $N \geq 2$ and $a \in \{0, 1\}$, one has $|a\rangle^{\otimes N} \in \mathcal{G}_{N,2}(A_{\text{GHZ}})$.

Proof. ✓ Proof For a cyclic nearest-neighbour window and outside assignment τ , set $R_{i,\tau} := |a\rangle^{\otimes N}|_{[i,i+1],\tau}$. For $b, c \in \{0, 1\}$ with $b \neq c$, the completed configuration is not constant, hence $R_{i,\tau}(b, c) = 0$. Therefore

$$R_{i,\tau} \in \{v : v(0, 1) = v(1, 0) = 0\} = \text{span}_{\mathbb{C}}\{|00\rangle, |11\rangle\} = \mathcal{G}_2(A_{\text{GHZ}}),$$

so $|a\rangle^{\otimes N} \in \mathcal{G}_{N,2}(A_{\text{GHZ}})$. $\square$

Theorem 16.7.3 (GHZ periodic nearest-neighbour ground space). Lean ✓ Stmt For $N \geq 2$, the common nearest-neighbour cyclic ground space of the GHZ tensor satisfies

$$\mathcal{G}_{N,2}(A_{\text{GHZ}}) = \text{span}_{\mathbb{C}}\{|0\rangle^{\otimes N}, |1\rangle^{\otimes N}\}.$$

This is the cyclic-window form of the two-fold degeneracy in [CPGSV21, line 2205].

Proof. ✓ Proof Let $\psi \in \mathcal{G}_{N,2}(A_{\text{GHZ}})$. For every cyclic nearest-neighbour window and outside assignment τ , one has

$$\psi|_{[i,i+1],\tau} \in \mathcal{G}_2(A_{\text{GHZ}}) = \text{span}_{\mathbb{C}}\{|00\rangle, |11\rangle\}.$$

Taking the outside assignment from the word σ gives $\psi|_{[i,i+1],\sigma}(\sigma_i, \sigma_{i+1}) = \psi(\sigma)$. Hence a mixed nearest-neighbour word has zero coefficient: $\sigma_i \neq \sigma_{i+1} \implies \psi(\sigma) = 0$. A nonconstant binary cyclic word has a mixed edge: $\sigma \notin \{(0, \dots, 0), (1, \dots, 1)\} \implies \exists i, \sigma_i \neq \sigma_{i+1}$. Therefore

$$\psi = \psi(0, \dots, 0)|0\rangle^{\otimes N} + \psi(1, \dots, 1)|1\rangle^{\otimes N}.$$

Conversely, Theorem 16.7.2 gives $|0\rangle^{\otimes N}, |1\rangle^{\otimes N} \in \mathcal{G}_{N,2}(A_{\text{GHZ}})$, and hence their span lies in $\mathcal{G}_{N,2}(A_{\text{GHZ}})$. $\square$

Remark 16.7.4 (GHZ parent interaction). For the GHZ tensor of Definition 16.1.1,

$$\begin{aligned} \mathcal{G}_2(A) &= \text{span}\{|00\rangle, |11\rangle\}, \\ h_{\text{GHZ}} &= |01\rangle\langle 01| + |10\rangle\langle 10| = \mathbb{1} - |00\rangle\langle 00| - |11\rangle\langle 11|. \end{aligned}$$

The periodic nearest-neighbour parent Hamiltonian therefore satisfies

$$\begin{aligned} H_{\text{GHZ},N} &= \sum_i (h_{\text{GHZ}})_i, \\ \ker H_{\text{GHZ},N} &= \text{span}\{|0\rangle^{\otimes N}, |1\rangle^{\otimes N}\}, \end{aligned}$$

using the cyclic-window equality $\mathcal{G}_{N,2}(A_{\text{GHZ}}) = \text{span}\{|0\rangle^{\otimes N}, |1\rangle^{\otimes N}\}$ from Theorem 16.7.3. This is the ferromagnetic Ising parent Hamiltonian [CPGSV21, lines 1194–1196 and line 2205]; see also [FGSW⁺15, Section 3] for the displayed two-site space and interaction.

Remark 16.7.5 (Cluster-state stabilizer convention). The one-dimensional cluster tensor in Definition 16.5.1 is the reflected convention of the tensor displayed in [CPGSV21, lines 2364–2369]. The sourced stabilizer is $K_i^{ZZX} = \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z$. The corresponding positive parent interaction is

$$\begin{aligned} h_i^{ZZX} &= \frac{1}{2}(\mathbb{1} - K_i^{ZZX}), \\ H_{\text{cl},N}^{ZZX} &= \sum_i h_i^{ZZX}, \\ \dim \ker H_{\text{cl},N}^{ZZX} &= 1 \end{aligned}$$

[PGVWC07, local TeX lines 374–387]. Thus an XZX formula should be used for this tensor only after establishing an equivalence such as

$$\begin{aligned} U H_{\text{cl},N}^{ZZX} U^\dagger &= H_{\text{cl},N}^{XZX}, \\ T H_{\text{cl},N}^{ZZX} T^{-1} &= H_{\text{cl},N}^{XZX}. \end{aligned}$$

Remark 16.7.6 (AKLT parent Hamiltonian and edge modes). For the AKLT tensor of Definition 16.2.1, let $\mathcal{G}_{[a,b]}$ denote the local ground space on sites $a, \dots, b$. Then $L_0 = 2$, and the three-site intersection property is

$$(\mathcal{G}_{[1,2]} \otimes \mathcal{H}_3) \cap (\mathcal{H}_1 \otimes \mathcal{G}_{[2,3]}) = \mathcal{G}_{[1,3]}.$$

This is the dimensionally explicit form of the identity in [CPGSV21, line 2095]. The displayed local spin-chain Hamiltonian is

$$H_{\text{AKLT},N} = \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right),$$

up to an additive constant [PGVWC07, local TeX lines 340–349]; [SPGC11, lines 2126–2130]. The spin-2 projector form is related to the polynomial Hamiltonian by the spin-addition identity

$$P_{i,i+1}^{(2)} = \frac{1}{2} \left(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{2}{3}\mathbb{1} \right)$$

for two spin-1 particles. For open boundary conditions and $N \geq 2$,

$$\begin{aligned} \dim \text{GS}(H_{\text{AKLT},N}^{\text{open}}) &= 4, \\ \dim(\text{GS}(H_{\text{AKLT},N}^{\text{open}}) \cap \mathcal{H}_{S=0}) &= 1 \end{aligned}$$

[CPGSV21, lines 1169–1174].

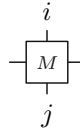
Chapter 17

Matrix Product Operators and Density Operators

Fix a physical dimension d and a bond dimension D . An MPO tensor carries two physical indices, one ket index and one bra index, together with the virtual left and right indices. The renormalization fixed-point conditions and their algebraic consequences are developed in Chapters 20 and 21.

17.1 MPO tensors

Definition 17.1.1 (MPO tensor). [Lean](#) [✓ Stmt](#) An *MPO tensor* is a family of matrices $\{M^{ij}\}_{i,j=0}^{d-1}$ with $M^{ij} \in M_D(\mathbb{C})$, indexed by a ket index i and a bra index j . Diagrammatically,



Definition 17.1.2 (Doubled-index MPS view). [Lean](#) [✓ Stmt](#) By combining the pair (i, j) into a single physical index in $\{0, \dots, d^2 - 1\}$, one obtains an MPS tensor with physical dimension d^2 and bond dimension D , with $M_{(i,j)}^{\text{MPS}} = M^{ij}$, identifying $\{0, \dots, d-1\} \times \{0, \dots, d-1\}$ with $\{0, \dots, d^2 - 1\}$ via the product encoding.

Definition 17.1.3 (Word evaluation). [Lean](#) [✓ Stmt](#) For words $u = (i_1, \dots, i_N)$ and $v = (j_1, \dots, j_N)$ in $\{0, \dots, d-1\}^N$, the *word evaluation* is

$$M^{u,v} := M^{i_1 j_1} M^{i_2 j_2} \dots M^{i_N j_N} \in M_D(\mathbb{C}). \quad (17.1)$$

The empty pair of words evaluates to the identity, $M^{\emptyset, \emptyset} = \mathbb{1}_D$, and word pairs of different lengths evaluate to 0.

Lemma 17.1.4 (Word-evaluation factorization). [Lean](#) [✓ Stmt](#) For words u_1, v_1 of equal length and arbitrary words u_2, v_2 , word evaluation factors as $M^{u_1 u_2, v_1 v_2} = M^{u_1, v_1} M^{u_2, v_2}$.

Proof. [✓ Proof](#) We argue by induction on the common length of u_1 and v_1 . For the empty prefix, $M^{u_2, v_2} = \mathbb{1}_D M^{u_2, v_2}$. If $u_1 = iu$ and $v_1 = jv$, then

$$M^{iuu_2, jvv_2} = M^{ij} M^{uu_2, vv_2} = M^{ij} M^{u, v} M^{u_2, v_2} = M^{iu, jv} M^{u_2, v_2},$$

using associativity of matrix multiplication. $\square$

Lemma 17.1.5 (Trace cyclicity of the closed MPO word). [Lean](#) [✓ Stmt](#) Let $a, b \in \{0, \dots, d-1\}$, and let u, v be words of equal length. Writing au, bv for adding the letters at the beginning and ua, vb for adding them at the end, one has $\text{tr}(M^{au, bv}) = \text{tr}(M^{ua, vb})$.

Proof. [✓ Proof](#) Lemma 17.1.4 gives $M^{ua, vb} = M^{u, v} M^{ab}$ and $M^{au, bv} = M^{ab} M^{u, v}$. Hence, by cyclicity of the trace,

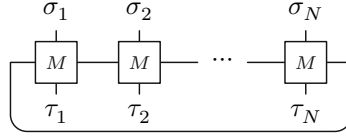
$$\text{tr}(M^{ua, vb}) = \text{tr}(M^{u, v} M^{ab}) = \text{tr}(M^{ab} M^{u, v}) = \text{tr}(M^{au, bv}).$$

$\square$

Definition 17.1.6 (MPO operator family). [Lean](#) [✓ Stmt](#) The operator generated by M on N sites is the matrix $\rho^{(N)}(M)$ indexed by configurations $\sigma, \tau \in \{0, \dots, d-1\}^N$ and given by

$$\rho^{(N)}(M)_{\sigma, \tau} = \text{tr}(M^{\sigma, \tau}). \quad (17.2)$$

Diagrammatically,



Definition 17.1.7 (Cyclic-shift reindexing). [Lean](#) [✓ Stmt](#) The *cyclic shift* of configurations on N sites is the bijection that relabels each site by one position, $\sigma \mapsto \sigma \circ r$, where r is the cyclic rotation $i \mapsto i + 1$ of $\{0, \dots, N-1\}$.

Theorem 17.1.8 (Translation invariance of the periodic MPDO). [Lean](#) [✓ Stmt](#) The operator $\rho^{(N)}(M)$ is invariant under the simultaneous cyclic shift of its bra and ket configurations: $\rho^{(N)}(M)_{r\sigma, r\tau} = \rho^{(N)}(M)_{\sigma, \tau}$.

Proof. [✓ Proof](#) The entry is a closed bond trace, $\rho^{(N)}(M)_{\sigma, \tau} = \text{tr}(M^{\sigma_1 \tau_1} \dots M^{\sigma_N \tau_N})$. Shifting both configurations by r rotates the matrix product by one factor, and the trace is invariant under that cyclic rotation:

$$\text{tr}(M^{\sigma_2 \tau_2} \dots M^{\sigma_N \tau_N} M^{\sigma_1 \tau_1}) = \text{tr}(M^{\sigma_1 \tau_1} \dots M^{\sigma_N \tau_N}),$$

so $\rho^{(N)}(M)_{r\sigma, r\tau} = \rho^{(N)}(M)_{\sigma, \tau}$. $\square$

Definition 17.1.9 (Hermitian MPO tensor). [Lean](#) [✓ Stmt](#) An MPO tensor is *Hermitian* if $M^{ij} = (M^{ji})^\dagger$ for all $i, j \in \{0, \dots, d-1\}$.

Definition 17.1.10 (Adjoint MPO tensor). [Lean](#) [Lean](#) [✓ Stmt](#) The *adjoint tensor* $M^\dagger$ of an MPO tensor M exchanges the ket and bra physical indices and conjugate-transposes each virtual matrix: $(M^\dagger)^{ij} = (M^{ji})^\dagger$. A tensor equals its adjoint tensor, $M^\dagger = M$, precisely when it is Hermitian.

Theorem 17.1.11 (Reflection identity for the adjoint tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
The adjoint tensor generates the adjoint operator family up to spatial reflection: entrywise,

$$\rho^{(N)}(M^\dagger)_{\sigma,\tau} = \overline{\rho^{(N)}(M)_{\tau \circ \text{rev}, \sigma \circ \text{rev}}}, \quad (17.3)$$

where rev reverses the site order. For an MPDO and $N \geq 1$, the conjugation disappears:

$$\rho^{(N)}(M^\dagger)_{\sigma,\tau} = \rho^{(N)}(M)_{\sigma \circ \text{rev}, \tau \circ \text{rev}}, \quad (17.4)$$

and the adjoint tensor is again an MPDO.

Proof. [✓ Proof](#) At the level of word evaluations, the conjugate transpose of a product reverses the factors, so

$$(M^\dagger)^{\sigma,\tau} = (M^{\tau \circ \text{rev}, \sigma \circ \text{rev}})^\dagger.$$

Taking traces gives the entrywise identity

$$\rho^{(N)}(M^\dagger)_{\sigma,\tau} = \text{tr}((M^{\tau \circ \text{rev}, \sigma \circ \text{rev}})^\dagger) = \overline{\rho^{(N)}(M)_{\tau \circ \text{rev}, \sigma \circ \text{rev}}}.$$

For an MPDO, $\rho^{(N)}(M)$ is Hermitian, so

$$\overline{\rho^{(N)}(M)_{\tau \circ \text{rev}, \sigma \circ \text{rev}}} = \rho^{(N)}(M)_{\sigma \circ \text{rev}, \tau \circ \text{rev}};$$

positive semidefiniteness is preserved under the simultaneous relabeling of rows and columns by the reflection. $\square$

17.2 Transfer maps

Definition 17.2.1 (MPO transfer map). [Lean](#) [✓ Stmt](#) The *transfer map* associated to M is the linear map $\mathcal{E}_M : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ defined by

$$\mathcal{E}_M(X) = \sum_{i,j=0}^{d-1} M^{ij} X (M^{ij})^\dagger. \quad (17.5)$$

Lemma 17.2.2 (Identification with the doubled-index MPS transfer map). [Lean](#) [✓ Stmt](#) The *transfer map of an MPO tensor agrees with the transfer map of its doubled-index MPS view*: $\mathcal{E}_M = \mathcal{E}_{M^{\text{MPS}}}$.

Proof. [✓ Proof](#) After identifying the single physical index of the doubled tensor with a pair (i, j) , for every $X \in M_D(\mathbb{C})$,

$$\mathcal{E}_{M^{\text{MPS}}}(X) = \sum_{(i,j)} M_{(i,j)}^{\text{MPS}} X (M_{(i,j)}^{\text{MPS}})^\dagger = \sum_{i,j=0}^{d-1} M^{ij} X (M^{ij})^\dagger = \mathcal{E}_M(X).$$

$\square$

Theorem 17.2.3 (Positivity of the MPO transfer map). [Lean](#) [✓ Stmt](#) If $X \geq 0$, then $\mathcal{E}_M(X) \geq 0$.

Proof. [✓ Proof](#) By Lemma 17.2.2, $\mathcal{E}_M$ is the transfer map of the doubled-index MPS tensor. The corresponding MPS transfer map is positive, so $\mathcal{E}_M(X) \geq 0$ whenever $X \geq 0$. $\square$

17.3 MPDOs and LPDOs

Definition 17.3.1 (MPDO). [Lean](#) [✓ Stmt](#) An MPO tensor is an *MPDO* if, for every $N \geq 1$,

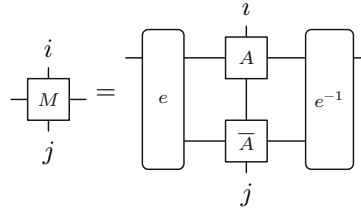
$$\rho^{(N)}(M) \geq 0. \quad (17.6)$$

This is the global positivity condition of [CPGSV16, Section 4].

Definition 17.3.2 (LPDO). [Lean](#) [✓ Stmt](#) An MPO tensor is an *LPDO* if there exist a Kraus dimension d_K , an inner bond dimension D' , a bond-space identification $e : \{0, \dots, D-1\} \simeq \{0, \dots, D'-1\} \times \{0, \dots, D'-1\}$ (equivalently, D and $(D')^2$ have the same cardinality), and matrices $A^{(i,k)} \in M_{D'}(\mathbb{C})$ such that, for all physical indices i, j ,

$$M^{ij} = \left(\sum_{k=0}^{d_K-1} A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_{e,e}, \quad (17.7)$$

where $\otimes$ denotes the Kronecker product, $\overline{(\cdot)}$ denotes entrywise complex conjugation, and $(\cdot)_{e,e}$ denotes reindexing along e back to $\{0, \dots, D-1\}$. See [CPGSV16, Section 4.3]. The local contraction is



Theorem 17.3.3 (LPDOs are MPDOs). [Lean](#) [✓ Stmt](#) Every LPDO tensor is an MPDO.

Proof. [✓ Proof](#) By the mixed-product property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, the N -site product decomposes as a sum of Kronecker products after transporting along the bond-space identification e :

$$M^{\sigma_1 \tau_1} \dots M^{\sigma_N \tau_N} = \left(\sum_{\kappa} P_{\kappa} \otimes \overline{Q_{\kappa}} \right) \Big|_e,$$

where $P_{\kappa} = A^{(\sigma_1, \kappa_1)} \dots A^{(\sigma_N, \kappa_N)}$ and $Q_{\kappa} = A^{(\tau_1, \kappa_1)} \dots A^{(\tau_N, \kappa_N)}$. Since the trace is invariant under the reindexing by e , we apply $\text{tr}(X \otimes Y) = \text{tr}(X) \text{tr}(Y)$ to obtain

$$\rho^{(N)}(M)_{\sigma, \tau} = \sum_{\kappa} \text{tr}(P_{\kappa}) \overline{\text{tr}(Q_{\kappa})} = \sum_{\kappa} \psi_{\kappa}(\sigma) \overline{\psi_{\kappa}(\tau)},$$

where $\psi_{\kappa}(\sigma) = \text{tr}(P_{\kappa})$. Hence $\rho^{(N)}(M) = \sum_{\kappa} |\psi_{\kappa}\rangle \langle \psi_{\kappa}| \geq 0$. □

17.4 Horizontal and Vertical Canonical Forms

Definition 17.4.1 (Normalized BNT-refined horizontal form for an MPO). [Lean](#) [✓ Stmt](#) An MPO tensor is in *normalized BNT-refined horizontal form* if its doubled-index MPS tensor is a BNT sector decomposition with an independent gauge on every repeated copy:

$$M^{ab} = \bigoplus_{j=0}^{g-1} \bigoplus_{q=0}^{r_j-1} \mu_{j,q} X_{j,q} A_j^{ab} X_{j,q}^{-1}. \quad (17.8)$$

Here the A_j are the distinct minimal representatives, $r_j > 0$, and every weight $\mu_{j,q}$ is nonzero, $|\mu_{j,q}| \leq 1$, and at least one weight has modulus one. The BNT hypotheses include irreducibility, left-canonicity, normalized self-overlap, eventual linear independence of the representative MPV families, and the exclusion of similarity up to phase between distinct representatives. Repeated gauge-equivalent copies are retained over the same representative rather than separated as distinct blocks. The total bond dimension of the displayed direct sum is exactly D . Equivalently, with the block-diagonal matrix $X = \bigoplus_{j,q} X_{j,q}$,

$$M^{ab} = X \left[\bigoplus_{j,q} \mu_{j,q} A_j^{ab} \right] X^{-1},$$

$$X = \bigoplus_{j,q} X_{j,q}.$$

This is a strengthened representative-grouped form of the canonical decomposition in [CPGSV16, Section 2.3]. In contrast with Definition 8.6.7, the retained direct sum has the full bond dimension, every copy weight is nonzero and normalized, and gauge-equivalent normal blocks are grouped over a BNT representative with a separate gauge for each copy. It is therefore not the literal CPSV canonical-form hypothesis.

Definition 17.4.2 (MPV representation associated with horizontal canonical form). [Lean](#)

[✓ Stmt](#) An MPO tensor has a *horizontal-canonical-form MPV representation* if its doubled-index MPS tensor generates the same positive-length MPV family as a BNT canonical form $\bigoplus_{j=0}^{g-1} \bigoplus_{q=0}^{r_j-1} \mu_{j,q} A_j$. Its length- N matrix product vector is

$$\sum_{j=0}^{g-1} \left(\sum_{q=0}^{r_j-1} \mu_{j,q}^N \right) V^{(N)}(A_j). \quad (17.9)$$

This is the MPV-level decomposition in [CPGSV16, Section 2.3, lines 298–301].

Theorem 17.4.3 (BNT-refined horizontal form gives the BNT MPV representation). [Lean](#)

[✓ Stmt](#) Every MPO tensor in normalized BNT-refined horizontal form has the associated positive-length BNT matrix-product-vector representation.

Proof. [✓ Proof](#) Each copy similarity cancels inside every positive-length closed matrix-product trace. $\square$

$$K = X \left(\bigoplus_j \mathcal{K}_j \right) X^{-1}$$

Figure 17.1: Horizontal canonical form of the doubled tensor: the gauge X , the direct sum of basis tensors $\mathcal{K}_j$ carrying the ket and bra indices, and X^{-1} . In the source specialized simple-biCF/ZCL display the weights μ_j are independent of the copy index and absorbed into the basis tensors, so the basis tensors depend only on the block index j while the gauge factors $X_{j,q}$ depend on both block indices. This is [CPGSV16, Appendix C.2, lines 1660–1665].

Definition 17.4.4 (Vertically viewed tensor of an MPO). [Lean](#) [Lean](#) [✓ Stmt](#) The *vertical direction* of an MPO tensor exchanges the notion of physical and virtual indices [CPGSV16, line 943]: the vertically viewed tensor $\widetilde{M}$ has its letters indexed by the horizontal bond pair (a, b) , and each letter is the operator on the physical space with matrix elements $(\widetilde{M}_{ab})_{ij} = M_{ab}^{ij}$. Concatenating $\widetilde{M}$ vertically generates the matrix product vectors of the vertical direction.

Lemma 17.4.5 (Vertical reflection identity for the adjoint tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Adjunction exchanges the two oriented horizontal bond indices in the vertical view:*

$$\widetilde{M}_{a,b}^\dagger = (\widetilde{M}_{b,a})^\dagger. \quad (17.10)$$

Proof. [✓ Proof](#) This follows entrywise from the exchange of the ket and bra indices and conjugate transposition of the virtual matrix. $\square$

Definition 17.4.6 (Vertical canonical form for an MPO). [Lean](#) [✓ Stmt](#) An MPO tensor is in *vertical canonical form* if there are a basis of normal tensors $\{M_\alpha\}_\alpha$ for its vertically viewed tensor $\widetilde{M}$, positive diagonal matrices $\mu_\alpha = \text{diag}(\omega_{\alpha,0}, \dots, \omega_{\alpha,r_\alpha-1})$ with $r_\alpha \geq 1$, and a matrix U from the physical space onto the retained nonzero sector space such that

$$\begin{aligned} U\widetilde{M}U^\dagger &= B, \\ \widetilde{M} &= U^\dagger B U, \\ B &= \bigoplus_\alpha \mu_\alpha \otimes M_\alpha. \end{aligned} \quad (17.11)$$

This is the conclusion of [CPGSV16, Proposition 4.13, lines 1863–1870]. In this orientation $UU^\dagger = I$ on the retained space, while $U^\dagger U$ is its support projection in the original physical space. The reconstruction identity says that every letter of $\widetilde{M}$ is supported on this projection, so the discarded orthogonal complement and the off-diagonal corners vanish. The orthogonal complement may be a zero sector: the source’s general canonical-form construction explicitly permits the sum of the nonzero block dimensions to be strictly smaller than the original dimension [CPGSV16, lines 214–225]. Since each μ_α is diagonal, the summand $\mu_\alpha \otimes M_\alpha$ is the direct sum $\bigoplus_{k=0}^{r_\alpha-1} \omega_{\alpha,k} M_\alpha$ of weighted copies of M_α , so the right-hand side is a block-diagonal tensor over the pairs (α, k) in which the multiplicities r_α and the diagonal entries $\omega_{\alpha,k}$ stay explicit; the coefficients $m_\alpha = \text{tr}(\mu_\alpha) = \sum_{k=0}^{r_\alpha-1} \omega_{\alpha,k}$ appearing in the renormalization fixed-point characterization are recovered from them.

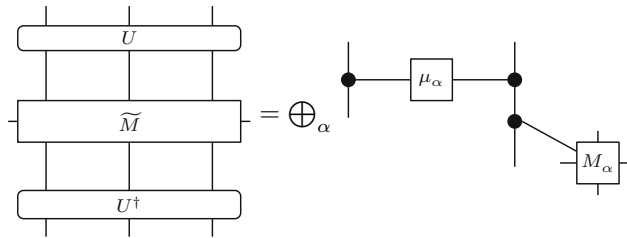


Figure 17.2: Vertical direct-sum form of the tensor: after the physical isometry U , each block factorizes as the positive diagonal weight μ_α on the multiplicity factor times the basis tensor M_α , and only M_α carries the virtual legs. This is the decomposition $U\widetilde{M}U^\dagger = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha$ of [CPGSV16, Proposition 4.13, lines 1864–1870].

Remark 17.4.7 (Finite witnesses for block-injective canonical form). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) Let $(A_k)_k$ be a finite family of block tensors of physical dimension d , and fix a length L ; write A^w_k for the length- L word evaluation of A_k along a word $w \in \{0, \dots, d-1\}^L$. The finite-length spanning hypothesis is that the tuple-valued evaluations span the full product algebra of block matrices,

$$\text{span}_{\mathbb{C}} \{ (A_k^w)_k : w \in \{0, \dots, d-1\}^L \} = \prod_k M_{D_k}(\mathbb{C}). \quad (17.12)$$

The corresponding finite-length separation property is that some length L makes the trace pairing against block words nondegenerate: there is a length L such that, for every tuple $(\Delta_k)_k$ of block matrices,

$$\left(\forall w \in \{0, \dots, d-1\}^L, \sum_k \text{tr}(\Delta_k A_k^w) = 0 \right) \implies \forall k, \Delta_k = 0. \quad (17.13)$$

Spanning at length L gives separation at the same length: a trace functional that vanishes on every tuple $(A_k^w)_k$ vanishes on their span, hence on the whole product algebra, so nondegeneracy of the product trace pairing forces every $\Delta_k = 0$. Combined with injectivity, left-canonicity, and nonzero block weights, this finite-length separation supplies the older per-copy trace-separation hypotheses for the block-diagonal tensor.

The spanning hypothesis has an equivalent entrywise form. For a block index k and matrix entry (a, b) , let $w \mapsto (A_k^w)_{a,b}$ be the corresponding scalar word family, indexed jointly by the triple (k, a, b) . Linear independence of this scalar family already implies the spanning condition, hence the separation property, and hence the older per-copy trace-separation hypotheses directly.

The block-injectivity proposition of [CPGSV16, lines 340–345] gives a concrete sufficient criterion for the spanning condition: a common length L with every block L -block-injective, together with a length S at which, for every k , there is a coefficient family $(c_w)_{w \in \{0, \dots, d-1\}^S}$ with

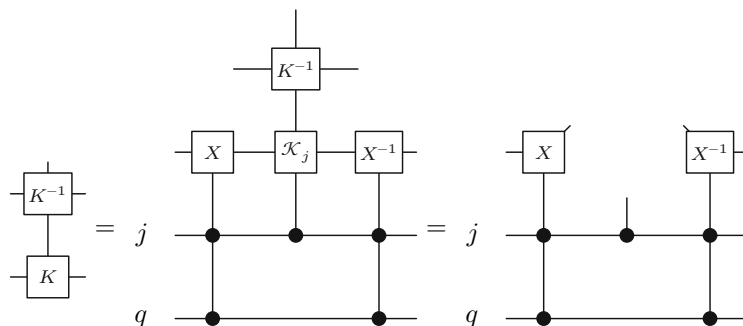
$$\begin{aligned} \sum_w c_w A_k^w &= \mathbb{1}, \\ \sum_w c_w A_j^w &= 0 \quad (j \neq k). \end{aligned}$$

Concatenating the injective prefix with this finite selector family produces the spanning condition at length $L+S$, hence the separation property and the older per-copy trace-separation hypotheses.

The inverse tensor of a block-injective canonical form satisfies

$$\sum_n (\mathcal{K}_j^p)_{\alpha,\beta} (K_{j'}^{-1,p})_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \delta_{j,j'}. \quad (17.14)$$

Writing $K = X(\bigoplus_j \mathcal{K}_j)X^{-1}$ in this contraction places the gauges on the virtual legs of K and identifies the block index of $\mathcal{K}_j$ with the label j :



Consequently, $K^{-1}K = \bigoplus_j \mathbb{1}_j$. This is the inverse identity in [CPGSV16, Appendix C.2, lines 1666–1670].

Theorem 17.4.8 (Finite block-separation criterion). Lean ✓ Stmt *If a block family admits the finite-length data described in the block-injectivity proposition of [CPGSV16, lines 340–345], then it is in block-injective canonical form.*

Proof. ✓ Proof If the common injective prefix has length L and the selector has length S , concatenation gives simultaneous words of length $L + S$. For every tuple $(X_j)_j$, the resulting coefficients satisfy $\sum_{|w|=L+S} c_w A_j^w = X_j$. Hence the simultaneous word tuples span the product algebra, and nondegeneracy of the trace pairing gives finite-length trace separation. $\square$

Lemma 17.4.9 (Single-block word-tuple span). Lean ✓ Stmt *Let $(A_j)_{j=0}^{g-1}$ be a family with $g = 1$. If its unique block is L -block injective, then the simultaneous length- L word evaluations span the full one-block product algebra.*

Proof. ✓ Proof For every matrix X , block injectivity gives coefficients such that $X = \sum_{|w|=L} c_w A_0^w$. Since the family has only the block A_0 , this is precisely the required simultaneous word-tuple spanning identity. $\square$

Definition 17.4.10 (Representative word-tuple separation). Lean Lean ✓ Stmt *Let $(A_j)_{j=0}^{g-1}$ be the basis representatives of a sector decomposition. They have representative word-tuple separation at length L when*

$$\text{span}_{\mathbb{C}}\{(A_j)_{j=0}^{g-1} : w \in \{0, \dots, d-1\}^L\} = \prod_{j=0}^{g-1} M_{D_j}(\mathbb{C}). \quad (17.15)$$

They have *eventual representative word-tuple separation* when this equality holds at every sufficiently large length. Notice that the family is indexed by the minimal representatives j , and not by the repeated copies lying over each j .

Theorem 17.4.11 (Representative-grouped Lemma L). Lean Lean Lean ✓ Stmt *Let A_{tot} be the tensor of a sector decomposition with basis representatives $(A_j)_{j=0}^{g-1}$, multiplicities $r_j > 0$, nonzero weights $\mu_{j,q}$, and grouped coefficients $c_j^{(N)} = \sum_{q=0}^{r_j-1} \mu_{j,q}^N$. Suppose that the first-site actions of $Y, Z \in M_d(\mathbb{C})$ agree on the MPV family of A_{tot} . If representative word-tuple separation holds at length L and $c_{j_0}^{(L+1)} \neq 0$, then $Y A_{j_0} = Z A_{j_0}$. Consequently, under eventual representative word-tuple separation, $Y A_j = Z A_j$ for every representative j . The same conclusion holds if the first-site equality is initially given for any tensor with the same positive-length MPV family as A_{tot} .*

The representative grouping is the argument of [CPGSV16, Appendix C.3, lines 1835–1858].

In the first auxiliary assertion, eventual representative word-tuple separation is an explicit assumption. The later post-blocked theorem derives the needed finite representative separation from the full BNT canonical-form hypotheses. The sharp $3D^5$ representative-separation bound is Theorem 9.6.1.22; its detailed source comparison is recorded in [con26u].

Proof. ✓ Proof At a fixed tail length L , expand the first-site equality at system size $L + 1$ and group all copies lying over the same representative. For every length- L word w this gives

$$\sum_{j=0}^{g-1} \text{tr}(c_j^{(L+1)}(Y A_j - Z A_j) A_j^w) = 0.$$

Representative word-tuple separation therefore forces each matrix $c_j^{(L+1)}(YA_j - ZA_j)$ to vanish. A nonzero coefficient may then be cancelled.

For the eventual assertion, choose a threshold L_0 and put $M = L_0 + 1$. For a fixed representative j , apply Corollary 9.5.2 to the nonzero numbers $\mu_{j,q}^M$. It yields $r \geq 1$ such that $c_j^{(Mr)} = \sum_q (\mu_{j,q}^M)^r \neq 0$. Since $Mr - 1 \geq L_0$, the fixed-length argument applies with $L = Mr - 1$. Finally, equality of the positive-length MPV families gives the first-site hypothesis for the sector-decomposition tensor. $\square$

Definition 17.4.12 (Direct sum with one marked letter). [Lean](#) [✓ Stmt](#) Let

$$A_{\text{tot}}^i = \bigoplus_{j=0}^{g-1} \bigoplus_{q=0}^{r_j-1} \mu_{j,q} A_j^i, \quad (17.16)$$

be the tensor of a sector decomposition. Let s range over a second finite alphabet, and for each representative A_j let C_j^s be a matrix on its bond space. The corresponding direct sum with one marked letter is

$$C_{\text{tot}}^s = \bigoplus_{j=0}^{g-1} \bigoplus_{q=0}^{r_j-1} \mu_{j,q} C_j^s. \quad (17.17)$$

The marked alphabet need not coincide with the alphabet of the unmarked tensor. This is the algebraic form of the marked matrices in the two diagrams of [CPGSV16, Proposition 4.13, lines 1909–1919].

Lemma 17.4.13 (Trace expansion with one marked letter). [Lean](#) [✓ Stmt](#) For every marked index s and every unmarked word w ,

$$\begin{aligned} \text{tr}(C_{\text{tot}}^s A_{\text{tot}}^w) &= \sum_{j=0}^{g-1} c_j^{(|w|+1)} \text{tr}(C_j^s A_j^w), \\ c_j^{(N)} &= \sum_{q=0}^{r_j-1} \mu_{j,q}^N. \end{aligned} \quad (17.18)$$

The exponent $|w| + 1$ includes the weight at the marked site.

Proof. [✓ Proof](#) Expand both matrices as direct sums. Their product is the direct sum of $\mu_{j,q}^{|w|+1} C_j^s A_j^w$. Taking the trace and summing first over q gives the stated coefficient $c_j^{(|w|+1)}$. $\square$

Theorem 17.4.14 (Algebraic marked-letter extension of representative separation). [Lean](#) [Lean](#) [✓ Stmt](#) Let (C_j^s) and (E_j^s) be two marked families over the same sector decomposition. Fix a tail length L , and suppose that, for every marked index s and every unmarked word w of length L ,

$$\text{tr}(C_{\text{tot}}^s A_{\text{tot}}^w) = \text{tr}(E_{\text{tot}}^s A_{\text{tot}}^w).$$

If representative word-tuple separation holds at length L and $c_{j_0}^{(L+1)} \neq 0$, then $C_{j_0} = E_{j_0}$. Under eventual representative word-tuple separation, agreement for every marked index and every positive-length unmarked word implies $C_j = E_j$ for every j .

Lemma L is stated for marks induced by physical operators on the first site. After its closed-chain identity has been expanded, the separation calculation uses only the matrices in the marked slot. The theorem is the resulting algebraic extension of [CPGSV16, Appendix C.3, lines 1848–1858]; it is not attributed to the paper as a separately stated arbitrary-mark theorem.

Proof. [✓ Proof](#) Fix s and a tail length L , and put $\Delta_j = c_j^{(L+1)}(C_j^s - E_j^s)$. Lemma 17.4.13 gives $\sum_{j=0}^{g-1} \text{tr}(\Delta_j A_j^w) = 0$ for every length- L word w . Representative word-tuple separation implies $\Delta_j = 0$ for every j . If $c_{j_0}^{(L+1)} \neq 0$, cancellation gives $C_{j_0}^s = E_{j_0}^s$. Since s was arbitrary, $C_{j_0} = E_{j_0}$.

Under eventual separation, apply Corollary 9.5.2 to the nonzero weights above each fixed representative, exactly as in Theorem 17.4.11. This selects a sufficiently large L with $c_j^{(L+1)} \neq 0$ and proves $C_j = E_j$. $\square$

Theorem 17.4.15 (Full-coordinate marked trace identity). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let T_{grp} be the full grouped tensor of Theorem 9.6.1.9, and let P_{rep} be its representative sector decomposition. For a marked family $(C_j^s)_j$, define the marked tensor $\widehat{C}_{\text{grp}}$ on all grouped listed coordinates as follows. The copy (j, q) carries the marked block $\lambda_{j,q} C_j^s$, where $\lambda_{j,q} = \nu_{k(j,q)} \zeta_{k(j,q)}$.

For every marked letter s and every unmarked word w , one has

$$\begin{aligned} \text{tr}(\widehat{C}_{\text{grp}}^s T_{\text{grp}}^w) &= \text{tr}(C_{\text{rep}}^s P_{\text{rep}}^w) \\ &= \sum_j \left(\sum_q \lambda_{j,q}^{|w|+1} \right) \text{tr}(C_j^s C_j^w). \end{aligned} \quad (17.19)$$

The exponent $|w| + 1$ includes the factor at the marked site.

Proof. [✓ Proof](#) Reindex the finite sum over listed coordinates by the class-copy pairs (j, q) and use the bond-dimension identifications to replace each block by its unit-phase representative. The marked block supplies one factor $\lambda_{j,q}$ and the word supplies $|w|$ further factors. Summing first over q gives (17.19). $\square$

Theorem 17.4.16 (Retained-coordinate representative-grouped Lemma L). [Lean](#) [Lean](#) [✓ Stmt](#) Let $(C_j^s)_j$ and $(E_j^s)_j$ be two marked families on the representatives. Form their full-coordinate marked tensors. If

$$\text{tr}(\widehat{C}_{\text{grp}}^s T_{\text{grp}}^w) = \text{tr}(\widehat{E}_{\text{grp}}^s T_{\text{grp}}^w) \quad (17.20)$$

for every s and every word w of positive length, then $C_j = E_j$ for every representative j .

In particular, let $Y, Z \in M_d(\mathbb{C})$. If their first-site actions agree on the positive-length matrix product vectors of T_{grp} , then

$$(YC_j)^i = (ZC_j)^i \quad (17.21)$$

for every representative j and every physical letter i .

Proof. [✓ Proof](#) Theorem 17.4.15 rewrites (17.20) into marked-trace agreement for P_{rep} . Its representatives have eventual word-tuple separation, and every copy weight is nonzero. Theorem 17.4.14 therefore gives $C_j = E_j$.

First-site agreement for T_{grp} also holds for P_{rep} because their matrix product vectors agree at every positive length. Applying Theorem 17.4.11 gives (17.21), namely

$$(YC_j)^i = (ZC_j)^i. \quad (17.22)$$

$\square$

The first-site actions of Y, Z agree at every length if and only if

$$\mathrm{tr}((YA)^i A^w) = \mathrm{tr}((ZA)^i A^w), \quad (17.28)$$

for every physical index i and every finite word w .

Conversely, suppose that A is L -block injective. If the two induced insertion tensors on $A^{[L+1]}$ are equal, then $YA = ZA$.

Proof. ✓ Proof For the forward implication, split the first block into its first letter and its remaining L letters. The insertion identity (17.27) reduces the blocked MPV equality to the original first-site equality, with the remaining letters of the first block prepended to the flattened tail.

For the converse, put $\Delta^i = (YA)^i - (ZA)^i$. Equality after blocking gives $\Delta^i A^w = 0$ for every word w of length L . Since the length- L words span $M_D(\mathbb{C})$, the same equality holds with A^w replaced by the identity, and therefore $\Delta^i = 0$ for every physical index i . $\square$

Theorem 17.4.20 (Representative separation after injective blocking). Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let P be a BNT canonical form with basis representatives $(A_j)_{j=0}^{g-1}$, and suppose that the representatives have already been blocked so that every A_j is injective. Then representative word-tuple separation holds at every length $L \geq 6(g-1) + 1$. Consequently, if the first-site actions of $Y, Z \in M_d(\mathbb{C})$ agree on the positive-length MPV family of the tensor represented by P , then $YA_j = ZA_j$ for $0 \leq j < g$. The same conclusion holds when the first-site equality is given for any tensor with the same positive-length MPV family as the tensor represented by P .

More generally, let P be a BNT canonical form before physical blocking, let $p > 0$, and suppose that every p -blocked representative $A_j^{[p]}$ is injective. Then the blocked sector decomposition $P^{[p]}$ has representative word-tuple separation at every blocked length $L \geq 6(g-1) + 1$.

Without changing the physical alphabet, a BNT canonical form also has eventual representative word-tuple separation. If $p > 0$ is a common block-injectivity length, then one may take pairwise separating words of length $6p$, form selectors of length $6p(g-1)$, and append homogeneous words of any length at least p .

Finally, the BNT canonical-form hypotheses imply that there is a common $L > 0$ for which every unblocked representative A_j is L -block injective. Indeed, irreducibility and left-canonicity give a finite peripheral period m_j . The self-overlap tends to m_j along multiples of m_j , but by normalization it also tends to 1; hence $m_j = 1$. Thus every representative is normal, and finiteness gives a common positive block-injectivity length. If the first-site actions of Y, Z agree on the tensor represented by P , then $YA_j = ZA_j$ for $0 \leq j < g$. The same conclusion holds when the first-site equality is given for any tensor with the same positive-length MPV family. This composes the blocking conclusion of [CPGSV16, lines 318–345] with Appendix C.3, Lemma L [CPGSV16, lines 1835–1858].

No additional common-length hypothesis is required.

Proof. ✓ Proof Injectivity of A_j implies that its homogeneous words span $M_{D_j}(\mathbb{C})$ at every positive length. Apply the three-block separation argument with source length 1. For each ordered pair of distinct representatives it gives a pairwise block-separating equation at length 6. Multiplying the equations for the $g-1$ representatives different from A_j gives coefficients $c_w^{(j)}$ satisfying

$$\sum_{|w|=6(g-1)} c_w^{(j)} A_k^w = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \quad (17.29)$$

If $L \geq 6(g-1) + 1$, put $m = L - 6(g-1) > 0$. The length- m words of each A_j span $M_{D_j}(\mathbb{C})$. Concatenating these words with the selectors in (17.29) shows that the simultaneous length- L

word tuples span $\prod_j M_{D_j}(\mathbb{C})$. This proves eventual representative word-tuple separation. The representative-grouped Lemma L gives the two first-site conclusions.

For the unblocked canonical form, propagate p -block injectivity to the lengths $2p - 1$, $2p$, and $6p$. Applying the direct-sum argument to the resulting injective tensors gives pair trace separation at original length $6p$. The bijection between blocked words of length 6 and original words of length $6p$ gives pair trace separation for the p -blocked representatives. The preceding selector argument then applies verbatim to $P^{[p]}$.

The same pairwise equations at original length $6p$ give selectors for the unblocked representatives directly. For every $L \geq 6p(g - 1) + p$, the remaining prefix has length at least p and hence spans each full representative matrix algebra. Concatenating the prefix with the selectors proves eventual representative word-tuple separation before blocking.

For the final assertion, first consider one representative A_j . Irreducibility and left-canonicity give a positive peripheral period m_j . The periodic self-overlap formula and the normalized self-overlap hypothesis give, along the same subsequence,

$$\begin{aligned} \lim_{k \rightarrow \infty} O_{A_j A_j}(m_j k) &= m_j, \\ \lim_{k \rightarrow \infty} O_{A_j A_j}(m_j k) &= 1. \end{aligned}$$

Hence $m_j = 1$. Primitivity then implies normality. Taking a common multiple of the finitely many normality lengths gives a positive length L at which every representative is block injective. Block $L + 1$ sites. Injectivity propagates from length L to length $L + 1$, so the blocked representatives are injective. Act on the first letter of each physical block and apply the blocked representative separation followed by the representative-grouped Lemma L. This gives equality of the two induced insertions on every blocked representative. The remaining L letters of the first block form a spanning family, so Lemma 17.4.19 recovers $Y A_j = Z A_j$ before blocking. $\square$

Corollary 17.4.21 (Marked-letter separation for a BNT canonical form). [Lean](#) [✓ Stmt](#) *Let (C_j^s) and (E_j^s) be two marked families over a BNT canonical form. If their closed marked chains agree for every marked index s and every unmarked word w of positive length, then $C_j^s = E_j^s$ for every representative j and every marked index s .*

Proof. [✓ Proof](#) The BNT canonical-form hypotheses give eventual representative word-tuple separation. Apply Theorem 17.4.14. $\square$

Corollary 17.4.22 (Physical-letter marked separation in BNT-refined horizontal form). [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let M be an MPO tensor in normalized BNT-refined horizontal form. For two coefficient families $f_{u,z}$ and $g_{u,z}$, put $C^u = \sum_z f_{u,z} M^z$ and $E^u = \sum_z g_{u,z} M^z$. If $\text{tr}(C^u M^w) = \text{tr}(E^u M^w)$ for every marked index u and every word w of positive length, then $C^u = E^u$ for every u .*

The marks are required to lie in the span of the physical letters. No assertion is made for arbitrary bond-space matrices, since off-diagonal matrices between repeated sectors are invisible to closed chains. The corollary is the physical-insertion consequence of [CPGSV16, Appendix C.3, lines 1835–1858] used in the contractions of [CPGSV16, Proposition 4.13, lines 1909–1919].

Proof. [✓ Proof](#) Pass through the block-diagonal similarity in the normalized BNT-refined horizontal form. Linear combinations of physical letters are covariant under this similarity, and cyclicity of the trace removes it from each closed marked chain. On the representative-indexed

sector decomposition one has

$$\begin{aligned} C_{\text{tot}}^u &= \bigoplus_{j,q} \mu_{j,q} \sum_z f_{u,z} A_j^z, \\ E_{\text{tot}}^u &= \bigoplus_{j,q} \mu_{j,q} \sum_z g_{u,z} A_j^z. \end{aligned}$$

Corollary 17.4.21 identifies the two marks on every minimal representative. Taking their weighted direct sums and restoring the common block-diagonal similarity gives $C^u = E^u$. $\square$

Theorem 17.4.23 (Representative and literal reduction from first-site contractions). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M be in normalized BNT-refined horizontal form. For doubled-index matrices Y, Z , suppose that $Y_1 V^{(N)}(M) = Z_1 V^{(N)}(M)$ for every $N \geq 1$. Then the corresponding inserted tensors agree on the original bond space, $YM = ZM$.

Now suppose in addition that M generates an MPDO, $Q = Q^\dagger$, and that $Q\widetilde{M} = Q\widetilde{M}Q$ on the vertically viewed tensor. Then, on every representative, the insertions corresponding to $\widetilde{M}Q$ and $Q\widetilde{M}Q$ agree: $Y_R(Q)A_j = Y_C(Q)A_j$ for $0 \leq j < g$. This equality extends through every repeated weighted sector and the direct sum of the copy gauges, so that the original MPO tensor satisfies $\widetilde{M}Q = Q\widetilde{M}Q$. The specialized assertion follows from the two entrywise identities $Q_1 H^{(N)} = Q_1 H^{(N)} Q_1$ and $Q_1 H^{(N)} = H^{(N)} Q_1$ at every positive length.

The corresponding literal-CF assertion also holds. Let A be in literal CPSV canonical form. If the first-site actions of $Y, Z \in M_d(\mathbb{C})$ agree on every positive-length matrix product vector of A , then

$$YA = ZA \quad (17.30)$$

on the original bond space. In particular, suppose that M generates matrix product density operators, $P = P^\dagger$, and $P\widetilde{M} = P\widetilde{M}P$. Positivity gives agreement of the first-site actions associated with $\widetilde{M}P$ and $P\widetilde{M}P$ at every positive length. Hence (17.30) gives the opposite one-sided identity on the original bond space.

This is the representative-indexed conclusion of the invariant-projection step in [CPGSV16, Proposition 4.13, lines 1873–1887]. Copies with a common normal representative are grouped before separation, exactly as in Appendix C.3, Lemma L. The source assumes that Q is an orthogonal projector; this implication only uses its Hermiticity.

Proof. ✓ Proof For the normalized BNT-refined assertion, positive-length MPV equality gives the first-site identity for the BNT sector decomposition. The representative-grouped Lemma L gives $YA_j = ZA_j$ for every representative. First-site insertion commutes with the weighted direct sum, and conjugating each copy by its gauge gives $YM = ZM$ on the original bond space.

For the specialized assertion, positivity and Hermiticity give $Q_1 H^{(N)} = Q_1 H^{(N)} Q_1 = H^{(N)} Q_1$. These two identities give the corresponding first-site equalities for the BNT sector decomposition. Apply the representative-grouped Lemma L to identify first $Y_L(Q) A_j$ with $Y_C(Q) A_j$ and then with $Y_R(Q) A_j$. Transitivity gives the stated equality for every representative, and the same direct-sum and copy-gauge argument gives the literal equality on $\widetilde{M}$.

For literal CPSV canonical form, first-site agreement is a marked-trace identity on the original tensor. Theorem 17.4.17 needs only the identities with a nonempty unmarked tail; it identifies the two linear marks and gives (17.30). In the invariant-projection case, positivity first yields $P_1 H^{(N)} = P_1 H^{(N)} P_1 = H^{(N)} P_1$; its matrix coefficients are exactly the required first-site agreement. $\square$

Lemma 17.4.24 (Blockwise equality from agreement on the MPV family). Lean ✓ Stmt Let $\bigoplus_k \mu_k A_k$ be a block-diagonal tensor whose blocks are injective, left-canonical, weighted by nonzero scalars, and jointly block-injective in the sense that for some blocking length L the tuple of trace pairings against length- L block words is faithful across all blocks simultaneously. For $Y \in M_d(\mathbb{C})$, write

$$(YA_k)^i := \sum_{j=0}^{d-1} Y_{i,j} A_k^j$$

for the physical-index insertion of Y into block A_k , and for a configuration $\sigma = (i_1, \dots, i_N)$ write

$$(Y_1 V^{(N)}(A))_{(i_1, \dots, i_N)} := \sum_{j=0}^{d-1} Y_{i_1, j} V^{(N)}(A)_{(j, i_2, \dots, i_N)}$$

for Y acting on the first physical index of the MPV. If $Y, Z \in M_d(\mathbb{C})$ satisfy $Y_1 V^{(N)}(A) = Z_1 V^{(N)}(A)$ for every $N \geq 1$, then $YA_k = ZA_k$ for every block k .

Scope restriction (per-block separation). The source’s Lemma L separates blockwise contributions at the level of a minimal basis of normal tensors, grouping gauge-equivalent copies of one representative before the trace-separation step; the block-injective hypothesis stated here instead demands trace separation simultaneously across every block, including gauge-equivalent copies, and so excludes decompositions with repeated normal tensors. Documented in [con26u].

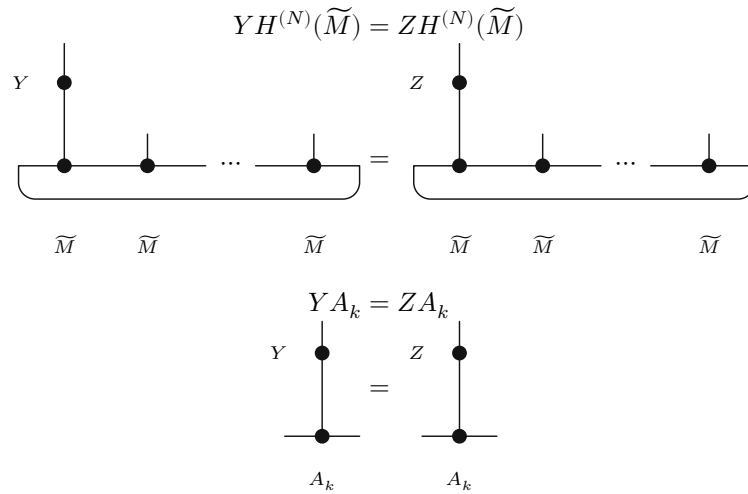


Figure 17.3: Blockwise equality from agreement on the MPV family. Top: the hypothesis $Y_1 V^{(N)}(A) = Z_1 V^{(N)}(A)$ for every positive chain length $N \geq 1$, with the operator inserted on the physical leg of the first site. Bottom: the conclusion that the two insertions agree on every block A_k of the canonical form. These are the two displays of Lemma L, [CPGSV16, Appendix C.3, lines 1835–1846].

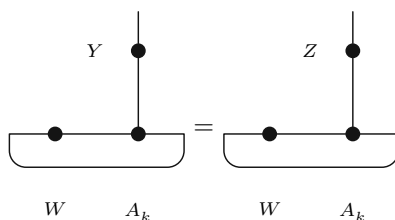
Proof. ✓ Proof This is the per-block-separated form of Lemma L of [CPGSV16, Appendix C.3, lines 1835–1846], using the finite criterion from Theorem 17.4.8 applied directly to the full block family rather than to a minimal basis of representatives. Expanding the hypothesis over a

word of length $L + 1$ beginning with the fixed site and folding each block contribution into a trace pairing against the corresponding first-site insertion tensor yields, upon taking the Y - Z difference, a tuple $(\Delta_k)_k$ of block matrices. For every word w of length L ,

$$\sum_k \text{tr}(\mu_k^{L+1} \Delta_k A_k^w) = 0.$$

The finite block-separation hypothesis applied to the tuple $(\mu_k^{L+1} \Delta_k)_k$ forces each weighted block difference to be zero, and the nonzero weights give the blockwise equality of the two first-site insertions. $\square$

The trace-pairing step used above is the following local identity, $\text{tr}(WY A_k) = \text{tr}(WZ A_k)$:



For every W in the span of the length- N word evaluations of the canonical form, it states $\text{tr}(W A^Y) = \text{tr}(W A^Z)$, where A^Y is obtained by applying Y to the physical index. Compare the trace-pairing identities in the proof of Lemma L, [CPGSV16, Appendix C.3, lines 1848–1858].

Theorem 17.4.25 (One-sided invariant corners of Hermitian operators). Lean ✓ Stmt Let H and P be Hermitian matrices. If $PH = PHP$, then $(\mathbb{1} - P)HP = 0$.

Proof. ✓ Proof Taking adjoints gives $HP = PHP$, and the conclusion follows by subtraction. $\square$

Theorem 17.4.26 (One-sided invariant corners of positive operators). Lean ✓ Stmt Let H be a nonempty-chain density operator generated by an MPDO, and let $P = P^\dagger$. If $PH = PHP$, then $(\mathbb{1} - P)HP = 0$. This is the operator identity used in the proof of [CPGSV16, Proposition 4.13].

Proof. ✓ Proof Since H and P are Hermitian, taking adjoints gives $HP = (PH)^\dagger = (PHP)^\dagger = PHP$. Therefore $(\mathbb{1} - P)HP = HP - PHP = 0$. $\square$

Theorem 17.4.27 (First-site contraction reduction). Lean ✓ Stmt Let $\bigoplus_{j,q} \mu_{j,q} A_j$ be a BNT canonical form and let $Y_L, Y_C, Y_R \in M_d(\mathbb{C})$, with $Y_1 V^{(N)}(A)$ denoting Y acting on the first physical index of the MPV. If, for every $N \geq 1$,

$$\begin{aligned} (Y_L)_1 V^{(N)}(A) &= (Y_C)_1 V^{(N)}(A), \\ (Y_L)_1 V^{(N)}(A) &= (Y_R)_1 V^{(N)}(A), \end{aligned}$$

then $Y_R A_j = Y_C A_j$ for every minimal representative A_j . This is the representative-grouped application of Lemma L and anticipates the concrete instantiation with Y_L, Y_C, Y_R the physical-index insertions of P, PHP, HP in Theorem 17.4.31.

Proof. ✓ Proof Apply the representative-grouped Lemma L first to the actions Y_L, Y_R and then to Y_L, Y_C . The BNT canonical-form hypotheses supply the required common block-injectivity length. Transitivity gives the stated equality on every minimal representative; the repeated copies have already been combined through $\sum_q \mu_{j,q}^N$. $\square$

Definition 17.4.28 (Doubled-index forms of the three first-site contractions). [Lean](#) [Lean](#) [Lean](#)
[Stmnt](#) Let $P \in M_d(\mathbb{C})$ and write doubled indices as $p = (a, b)$ and $q = (c, e)$. The first-site matrices corresponding respectively to $PH^{(N)}$, $H^{(N)}P$, and $PH^{(N)}P$ are

$$\begin{aligned}(Y_L)_{p,q} &= P_{a,c} \delta_{b,e}, \\ (Y_R)_{p,q} &= \delta_{a,c} P_{e,b}, \\ (Y_C)_{p,q} &= P_{a,c} P_{e,b}.\end{aligned}$$

The horizontal operator $H^{(N)}(\widetilde{M})$ is the cyclic word $H^{(N)}(\widetilde{M}) = \text{tr}_v(\widetilde{M}_1 \cdots \widetilde{M}_N)$:

$$H^{(N)}(\widetilde{M}) = \begin{array}{c} \sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_N \\ | \quad | \quad \cdots \quad | \\ \bullet \quad \bullet \quad \cdots \quad \bullet \\ | \quad | \quad \cdots \quad | \\ \tau_1 \quad \tau_2 \quad \cdots \quad \tau_N \\ \hline \widetilde{M}_1 \quad \widetilde{M}_2 \quad \cdots \quad \widetilde{M}_N \end{array}$$

Its virtual indices are contracted around the row, while every ket and bra index remains external. Reading each ket–bra pair as one doubled index identifies its entries with the matrix product vector of the doubled-index tensor. This is the contraction in [CPGSV16, Proposition 4.13, lines 1876–1880].

Lemma 17.4.29 (Coefficient form of the three first-site contractions). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Stmnt](#) For ket and bra configurations σ, τ , $V^{(N)}(\widehat{M})_{(\sigma, \tau)} = H^{(N)}(M)_{\sigma, \tau}$. If $p_0 = (a, b)$ is the first doubled index and σ', τ' are the ket and bra configurations on the remaining N sites, contraction with Y_L , Y_R , and Y_C gives, respectively,

$$\begin{aligned} & \sum_c P_{a,c} H_{(c, \sigma'), (b, \tau')}^{(N+1)}, \\ & \sum_e H_{(a, \sigma'), (e, \tau')}^{(N+1)} P_{e,b}, \\ & \sum_{c,e} P_{a,c} H_{(c, \sigma'), (e, \tau')}^{(N+1)} P_{e,b}. \end{aligned} \tag{17.31}$$

Proof. [Proof](#) Decode the doubled index at every site as a ket–bra pair. The defining trace of the resulting matrix word is exactly the corresponding entry of $H^{(N)}(M)$. Summing the first doubled index against Y_L , Y_R , or Y_C leaves the single or double sums in (17.31). $\square$

Lemma 17.4.30 (Doubled-index insertions are vertical multiplications). [Lean](#) [Lean](#) [Lean](#)
[Stmnt](#) Let $\widehat{M}$ be the doubled-index tensor associated with M . Then insertion of Y_L , Y_R , and Y_C gives, respectively, the doubled-index tensors associated with $P\widehat{M}$, $\widehat{M}P$, and $P\widehat{M}P$.

Proof. [Proof](#) Write the output and summed doubled indices as (a, b) and (c, e) . The Kronecker deltas in Y_L and Y_R remove, respectively, the sums over e and c ; Y_C leaves both multiplications. The resulting coefficients are $\sum_c P_{a,c} M^{cb}$, $\sum_e P_{e,b} M^{ae}$, and $\sum_{c,e} P_{a,c} P_{e,b} M^{ce}$. $\square$

Theorem 17.4.31 (Representative reduction for the MPO contractions). [Lean](#) [Stmnt](#) Let the doubled-index tensor $\widehat{M}$ generate the same positive-length MPV family as a BNT canonical form

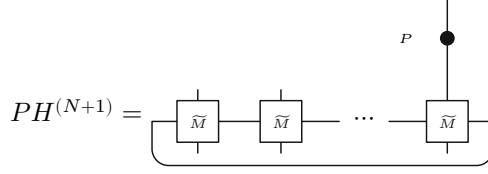


Figure 17.4: The periodic contraction representing $PH^{(N+1)}$. The analogous contractions with Y_R and Y_C place P on the bra leg or on both legs and represent $H^{(N+1)}P$ and $PH^{(N+1)}P$. These are the three first-site contractions in [CPGSV16, Proposition 4.13, lines 1881–1887].

$\bigoplus_{j,q} \mu_{j,q} A_j$. If, entrywise at every positive length,

$$\begin{aligned} PH^{(N)} &= PH^{(N)}P, \\ PH^{(N)} &= H^{(N)}P, \end{aligned}$$

then for every minimal representative A_j the insertions associated with $H^{(N)}P$ and $PH^{(N)}P$ agree. Repeated copies over A_j are grouped through the coefficient $\sum_q \mu_{j,q}^N$.

Proof. [✓ Proof](#) Lemma 17.4.29 turns the two operator identities into equality of the corresponding first-site contractions. Equality of matrix product vectors gives these equalities for the chosen canonical-form decomposition. Apply Theorem 17.4.27 with (Y_L, Y_C, Y_R) . $\square$

Definition 17.4.32 (One-site actions of a matrix on the vertically viewed tensor). [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) View the MPO tensor M vertically as the family of physical-space operators M_{ab} of Definition 17.4.4, indexed by the virtual indices, with matrix elements $(M_{ab})_{ij} = M_{ab}^{ij}$. A matrix $P \in M_d(\mathbb{C})$ then acts on the tensor by

$$\begin{aligned} (PM)^{ij} &= \sum_k P_{ik} M^{kj}, \\ (MP)^{ij} &= \sum_k P_{kj} M^{ik}, \end{aligned}$$

and on an $(N+1)$ -site chain by the first-spin operator $P \otimes \mathbb{1}^{\otimes N}$. On the letters of the vertically viewed tensor these actions are the left and right matrix products, $(PM)_{ab} = PM_{ab}$ and $(MP)_{ab} = M_{ab}P$. The first-spin operators compose site by site, $P_1Q_1 = (PQ)_1$.

Theorem 17.4.33 (One-sided invariance and density-operator commutation). [Lean](#) [Lean](#)

[✓ Stmt](#) Let M generate an MPDO and let $P = P^\dagger$ satisfy $PM = PMP$ on the vertically viewed tensor. For every total length $L \geq 1$, write $P_1 = P \otimes \mathbb{1}^{\otimes (L-1)}$. Then $P_1H^{(L)} = P_1H^{(L)}P_1 = H^{(L)}P_1$. This is the equality chain in the proof of [CPGSV16, Proposition 4.13]; the proposition assumes P is an orthogonal projector, but only self-adjointness of P is used.

Proof. [✓ Proof](#) Put $L = N+1$. Splitting off the first site of a density-operator entry gives

$$H_{(i,\sigma),(j,\tau)}^{(N+1)} = \text{tr}(M^{ij} M^{\sigma_0 \tau_0} \dots M^{\sigma_{N-1} \tau_{N-1}}).$$

Summing the first ket index against P therefore pairs the operator $(PM)^{ij}$ against the remaining word; the hypothesis $PM = PMP$ replaces it by $(PMP)^{ij}$, which unfolds to the entry of $P_1H^{(N+1)}P_1$. This proves $P_1H^{(N+1)} = P_1H^{(N+1)}P_1$. Since $H^{(N+1)}$ is positive semidefinite and P is self-adjoint, Theorem 17.4.26 gives $(\mathbb{1} - P_1)H^{(N+1)}P_1 = 0$, hence $H^{(N+1)}P_1 = P_1H^{(N+1)}P_1 = P_1H^{(N+1)}$. $\square$

Theorem 17.4.34 (One-sided invariance on BNT representatives). [Lean](#) [✓ Stmt](#) *Let M generate an MPDO whose doubled-index tensor $\widehat{M}$ generates the same positive-length MPV family as a BNT canonical form $\bigoplus_{j,q} \mu_{j,q} A_j$, and let $P = P^\dagger$ satisfy $PM = PMP$ on the vertically viewed tensor. Then on every minimal representative A_j the insertions of MP and PMP agree; equivalently, $(\mathbb{1} - P)MP = 0$ on every representative.*

Proof. [✓ Proof](#) For every tail length N , Theorem 17.4.33 turns $PM = PMP$ into the entrywise identities $P_1 H^{(N+1)} = P_1 H^{(N+1)} P_1$ and $P_1 H^{(N+1)} = H^{(N+1)} P_1$. These are exactly the hypotheses of Theorem 17.4.31, whose conclusion is the equality of the insertions of MP and PMP on every minimal representative. $\square$

Theorem 17.4.35 (Commutation with a power of a positive semidefinite matrix). [Lean](#) [✓ Stmt](#) *Let $H \geq 0$, let $p \geq 1$, and let $QH^p = H^p Q$. Then $QH = HQ$.*

Proof. [✓ Proof](#) Diagonalize H . The function $x \mapsto x^p$ is injective on the non-negative reals, so a matrix commutes with the diagonal matrix of the p -th powers exactly when it commutes with the diagonal matrix itself. $\square$

Theorem 17.4.36 (Commutation with a power of a positive semidefinite matrix). [Lean](#) [✓ Stmt](#) *A matrix commuting with a nonzero power of a nonempty-chain density operator generated by an MPDO commutes with the density operator itself. This is the final operator implication in the periodic-sector contradiction in the proof of [CPGSV16, Proposition 4.13].*

Proof. [✓ Proof](#) Diagonalize $H = U\Lambda U^\dagger$ with U unitary and Λ the diagonal matrix of eigenvalues $\lambda_i \geq 0$. Conjugating the hypothesis by U gives, entrywise for $Q' = U^\dagger Q U$,

$$Q'_{ij} \lambda_j^p = \lambda_i^p Q'_{ij}.$$

Since $x \mapsto x^p$ is injective on the non-negative reals, every entry Q'_{ij} with $\lambda_i \neq \lambda_j$ vanishes, so $Q'\Lambda = \Lambda Q'$. Conjugating back gives $QH = HQ$. $\square$

Definition 17.4.37 (Periodic-sector projector). [Lean](#) [✓ Stmt](#) *Let M be an MPO tensor. A periodic-sector projector for M with exponent $p \geq 1$ is an orthogonal projector Q on the physical space such that for every $N \geq 0$, with $Q_1 = Q \otimes \mathbb{1}^{\otimes N}$ acting on the first site,*

$$Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1, \tag{17.32}$$

$$Q_1 [H^{(N+1)}]^p = [H^{(N+1)}]^p Q_1. \tag{17.33}$$

These are the commutation identities that the proof of [CPGSV16, Proposition 4.13], lines 1889–1891, attaches to a nontrivial vertical period; the second family is the displayed identity at lines 1890–1891.

Theorem 17.4.38 (No periodic-sector projector for a matrix product density operator). [Lean](#) [✓ Stmt](#) *Let M generate matrix product density operators. Then M admits no periodic-sector projector.*

Proof. [✓ Proof](#) Suppose Q were such a projector. Every $H^{(N+1)}$ is positive semidefinite, so commutation of Q_1 with $[H^{(N+1)}]^p$ implies commutation with $H^{(N+1)}$ (Theorem 17.4.36), contradicting $Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1$. This is the contradiction concluding the periodic-sector step in the proof of [CPGSV16, Proposition 4.13], lines 1892–1893. $\square$

Definition 17.4.39 (Periodic vectors supply projectors). [Lean](#) [✓ Stmt](#) Say that *periodic vectors of M supply projectors* if for every nontrivial p -periodic vector of the vertically viewed tensor $\widetilde{M}$ — an irreducible restriction B of $\widetilde{M}$ to a minimal invariant subspace whose transfer map has a peripheral eigenvalue different from its spectral radius, as in Definition 8.7.3 — there is an exponent $q \geq 1$ for which M admits a periodic-sector projector. The proof of [CPGSV16, Proposition 4.13], lines 1889–1891, asserts this implication from the general theory of the peripheral spectrum of completely positive maps, with q equal to the period p of the vector; only the existence of some exponent enters the exclusion argument. The projector, its word invariance, and its single-letter displacement are constructed from the cyclic decomposition of the peripheral spectrum (Theorem 17.4.47); the passage from the displacement to the all-length non-commutation family is not needed for the periodic-sector contradiction below.

Theorem 17.4.40 (No periodic vectors in the vertical direction). [Lean](#) [✓ Stmt](#) *Let M generate matrix product density operators and suppose that periodic vectors of M supply projectors. Then the vertically viewed tensor $\widetilde{M}$ has no nontrivial p -periodic vectors. This is the periodic-sector step in the proof of [CPGSV16, Proposition 4.13], lines 1888–1893, granted the projector that the source draws from the general theory of the peripheral spectrum.*

Proof. [✓ Proof](#) A nontrivial p -periodic vector of $\widetilde{M}$ supplies a periodic-sector projector for M , and Theorem 17.4.38 shows that no such projector exists. $\square$

Definition 17.4.41 (Vertically stacked layers). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Placing one MPO layer on top of another and contracting the shared physical index gives the *layer product* of two MPO tensors M and M' with the same physical dimension, $(MM')^{ij} = \sum_k M^{ik} \otimes M'^{kj}$, an MPO tensor on the tensor product of the virtual spaces. The p -fold vertical stack $M^{[p]}$ of M is the iterated layer product of p copies of M ; the empty stack $M^{[0]}$ is the trivial-bond tensor with letters δ_{ij} , which generates the identity operator at every system size.

Theorem 17.4.42 (Stacked layers generate powers of the density operator). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *For every system size N , the operator generated by a layer product is the product of the density operators generated by the layers, $H^{(N)}(MM') = H^{(N)}(M)H^{(N)}(M')$. Consequently the p -fold vertical stack generates $H^{(N)}(M^{[p]}) = [H^{(N)}(M)]^p$ for all N and p ; this is the reading of the powers in the displayed identity of [CPGSV16, Proposition 4.13, lines 1890–1891] as density operators of stacked layers. In particular, the vertical stack of a tensor generating matrix product density operators again generates matrix product density operators on every nonempty chain.*

Proof. [✓ Proof](#) The word product of layer-product letters along a ket–bra configuration expands, by induction on the length and the mixed-product property of the Kronecker product, as a sum over configurations κ of the contracted physical index of Kronecker products of the layers' word products. Taking traces, each summand factors into the product of the two traces, so the entry of $H^{(N)}(MM')$ at (σ, τ) equals $\sum_{\kappa} H^{(N)}(M)_{\sigma\kappa} H^{(N)}(M')_{\kappa\tau}$, the entry of the matrix product. The power identity follows by induction on p , and positive semidefiniteness of $[H^{(N)}]^p$ follows from that of $H^{(N)}$. $\square$

Theorem 17.4.43 (Vertical letters of the stacked tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The vertically viewed letters of a layer product are the products of the layers' vertically viewed letters, $(MM')_{(a,a'),(b,b')} = M_{ab}M'_{a'b'}$. Hence every vertically viewed letter of the p -fold stack $M^{[p]}$ is the evaluation of a length- p word of $\widetilde{M}$. In particular, if a matrix Q satisfies the one-sided invariance $QA_w = QA_wQ$ for every product A_w of p letters of $\widetilde{M}$, then $QM^{[p]} = QM^{[p]}Q$ in the sense of Definition 17.4.32.*

Proof. ✓ Proof The letter identity is a direct computation from the layer product: contracting the shared physical index composes the physical-space operators. Induction on p writes each vertical letter of $M^{[p]}$ as a length- p word. Thus, for every bond index v , there is a word w of length p such that $\widetilde{M}^{[p]}_v = A_w$, and $QA_w = QA_wQ$ implies $Q\widetilde{M}^{[p]}_v = Q\widetilde{M}^{[p]}_vQ$. Applying this identity to the two one-site actions gives $QM^{[p]} = QM^{[p]}Q$. $\square$

Definition 17.4.44 (Periodic-sector projector from cyclic data). Lean ✓ Stmt Let M generate matrix product density operators, let $p \geq 1$, and let Q be an orthogonal projector on the physical space satisfying $QA_w = QA_wQ$ for every product A_w of p letters of $\widetilde{M}$ and $Q_1H^{(N)} \neq H^{(N)}Q_1$ for every length. Then Q is a periodic-sector projector for M with exponent p .

Proof. ✓ Proof By Theorem 17.4.43 the word invariance gives $QM^{[p]} = QM^{[p]}Q$ on the stacked tensor, which generates matrix product density operators by Theorem 17.4.42. Theorem 17.4.33 applied to $M^{[p]}$ yields $Q_1H^{(N)}(M^{[p]}) = H^{(N)}(M^{[p]})Q_1$ for every length, and $H^{(N)}(M^{[p]}) = [H^{(N)}]^p$ by the stacked-layers identity. Together with the assumed non-commutation family this is exactly the periodic-sector projector of Definition 17.4.37. $\square$

Definition 17.4.45 (Periodic vectors supply cyclic projectors). Lean ✓ Stmt Say that *periodic vectors of M supply cyclic projectors* if for every nontrivial p -periodic vector of the vertically viewed tensor $\widetilde{M}$ — presented as in Definition 8.7.3 — there are an exponent $q \geq 1$ and an orthogonal projector Q on the physical space satisfying the one-sided invariance $QA_w = QA_wQ$ for every product A_w of q letters of $\widetilde{M}$, together with $Q_1H^{(N)} \neq H^{(N)}Q_1$ for every length. This is the shape of the cyclic decomposition of the peripheral spectrum [Wol12, Theorem 6.6] applied to the vertical transfer map: each cyclic projection of a period- p sector returns to itself after p applications of the map, while its nontrivial cyclic displacement under one application prevents commutation with $H^{(N)}$. Compared with Definition 17.4.39, the commutation of Q with $[H^{(N)}]^p$ is no longer assumed; it is derived through the stacked-layers identity.

Theorem 17.4.46 (Reduction of the periodic-sector step to the cyclic decomposition). Lean ✓ Stmt Let M generate matrix product density operators. If periodic vectors of M supply cyclic projectors, then they supply periodic-sector projectors, and the vertically viewed tensor $\widetilde{M}$ has no nontrivial p -periodic vectors. This reduces the hypothesis of the periodic-sector step in the proof of [CPGSV16, Proposition 4.13, lines 1888–1893] to the cyclic decomposition of the peripheral spectrum of the vertical transfer map.

Proof. ✓ Proof A nontrivial p -periodic vector supplies an invariant non-commuting projector, which is a periodic-sector projector by Definition 17.4.44. The exclusion of periodic vectors is then Theorem 17.4.40. $\square$

Theorem 17.4.47 (The displaced invariant projector of a nontrivial periodic vector). Lean ✓ Stmt Let M be an MPO tensor and let $\widetilde{M}$ carry a nontrivial p -periodic vector, presented as in Definition 8.7.3: an irreducible restriction B of $\widetilde{M}$ to a minimal invariant subspace, reached through an isometry V , with a positive definite transfer eigenvector ρ of eigenvalue $r > 0$ and a transfer eigenvalue $\mu \neq r$ of modulus r . Then there are an exponent $q \geq 1$ and an orthogonal projector Q on the physical space such that $QA_w = QA_wQ$ for every product A_w of q letters of $\widetilde{M}$, while $Q\widetilde{M} \neq Q\widetilde{M}Q$ in the sense of Definition 17.4.32. This realizes the projector that the proof of [CPGSV16, Proposition 4.13, lines 1889–1891] draws from the general theory of the peripheral spectrum.

Proof. ✓ Proof The rescaled unital gauge K of the corner satisfies the hypotheses of the cyclic decomposition of the peripheral spectrum (Theorem D.3.5), and the eigenvalue $\mu/r \neq 1$ of

modulus one (Lemma D.3.6) forces the peripheral cyclic group to have order $m \geq 2$. Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”] produces orthogonal projections $P_0, \dots, P_{m-1}$ summing to the identity with $\mathcal{E}_K(P_{k+1}) = P_k$, and the letter-level shift $K_v P_{k+1} = P_k K_v$ holds (Lemma 6.8.1.1). Set $q = m$ and let Q be the complement of the support projection π of $YY^\dagger$ for $Y = V\rho^{1/2}P_0$. A word A_w of m letters of $\widetilde{M}$ satisfies $A_w Y = Y G_w$, where G_w is a scalar multiple of the corresponding word in the letters of K , which commutes with P_0 by the full-period shift; so the range of Y is invariant and Lemma D.3.3 gives $Q A_w = Q A_w Q$. If a single letter also satisfied the invariance, then π would fix $A_v Y$, whose columns lie in the sector $V\rho^{1/2}P_{m-1}$ by the cyclic shift; cancelling the injective factor $V\rho^{1/2}$ and using orthogonality of P_{m-1} and P_0 would force $P_{m-1} K_v = 0$ for every letter, hence $P_{m-1} = \mathcal{E}_K(P_0) = 0$, contradicting the nonvanishing of the cyclic projections. $\square$

Theorem 17.4.48 (A displaced idempotent fails to commute at some length). [Lean](#) [Stmt](#) *Let M be an MPO tensor whose doubled-index MPS tensor is in literal CPSV canonical form. If $Q^2 = Q$ and $Q\widetilde{M} \neq Q\widetilde{M}Q$, then there is an integer $N \geq 0$ such that, for $Q_1 = Q \otimes \mathbb{1}^{\otimes N}$,*

$$Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1. \quad (17.34)$$

Thus a displaced idempotent has a noncommuting chain length. The proof of [CPGSV16, Proposition 4.13, lines 1888–1893] only requires this existential conclusion, not noncommutation at every length.

Proof. [Proof](#) Suppose instead that $Q_1 H^{(N+1)} = H^{(N+1)} Q_1$ for every $N \geq 0$. Idempotence gives

$$Q_1 H^{(N+1)} = Q_1^2 H^{(N+1)} = Q_1 H^{(N+1)} Q_1.$$

Taking matrix coefficients yields equality of the one-sided first-site contractions. The literal form of Lemma L in Theorem 17.4.23 identifies the two inserted tensors on the original bond space. By Lemma 17.4.30, this equality is $Q\widetilde{M} = Q\widetilde{M}Q$, contradicting the displacement. $\square$

Theorem 17.4.49 (Literal canonical form excludes vertical periodic sectors). [Lean](#) [Stmt](#) *Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. Then the vertically viewed tensor $\widetilde{M}$ has no nontrivial periodic vectors.*

Proof. [Proof](#) A nontrivial periodic vector supplies an integer $p \geq 1$ and an orthogonal projector Q such that every word A_w of length p in the letters of $\widetilde{M}$ satisfies $Q A_w = Q A_w Q$, while $Q\widetilde{M} \neq Q\widetilde{M}Q$ (Theorem 17.4.47). By Theorem 17.4.48, there is a length $N + 1$ for which

$$Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1.$$

The word invariance and the vertical-stack identity give

$$Q_1 [H^{(N+1)}]^p = [H^{(N+1)}]^p Q_1.$$

Since $H^{(N+1)}$ is positive semidefinite, Theorem 17.4.36 implies $Q_1 H^{(N+1)} = H^{(N+1)} Q_1$, a contradiction. This is the periodic exclusion in [CPGSV16, Proposition 4.13, lines 1888–1893], with the single noncommuting length supplied by Lemma L. $\square$

Theorem 17.4.50 (BNT-refined horizontal form excludes vertical periodic sectors). [Lean](#) [Stmt](#) *Let M be in normalized BNT-refined horizontal form. If an idempotent Q satisfies $Q\widetilde{M} \neq Q\widetilde{M}Q$, then for some $N \geq 0$, $Q_1 H^{(N+1)} \neq H^{(N+1)} Q_1$. Consequently, if M generates matrix product density operators, then $\widetilde{M}$ has no nontrivial periodic vectors.*

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Proof. ✓ **Proof** Suppose first that Q_1 commutes with $H^{(N+1)}$ for every N . Since $Q^2 = Q$, this gives $Q_1 H^{(N+1)} = Q_1 H^{(N+1)} Q_1$ at every positive length. Theorem 17.4.23 and Lemma 17.4.30 then imply $Q\tilde{M} = Q\tilde{M}Q$, a contradiction. Thus a displaced idempotent has a noncommuting length $N + 1$.

A nontrivial periodic vector supplies the orthogonal projector Q of Theorem 17.4.47, invariant under every word of some full period $p \geq 1$. At the noncommuting length just obtained, the stacked-layer identity and the invariant-projection calculation give $Q_1 [H^{(N+1)}]^p = [H^{(N+1)}]^p Q_1$. Positive semidefiniteness and Theorem 17.4.36 imply that Q_1 commutes with $H^{(N+1)}$, which is the required contradiction. This uses the pointwise contradiction in [CPGSV16, Proposition 4.13, lines 1888–1893]; the stronger all-length inequality printed there is unnecessary. $\square$

Definition 17.4.51 (Displaced projectors never commute). Lean ✓ **Stmnt** Say that *displaced projectors of M never commute* if every orthogonal projector Q on the physical space with $Q\tilde{M} \neq Q\tilde{M}Q$ satisfies $Q_1 H^{(N)} \neq H^{(N)} Q_1$ at every length. This is the canonical-form step of the periodic-sector argument: the proof of [CPGSV16, Proposition 4.13] assumes the tensor is in canonical form in the horizontal direction, and the argument of lines 1874–1887 turns all-length commutation of Q_1 with $H^{(N)}$ into the letter-level invariance $Q\tilde{M} = Q\tilde{M}Q$. This argument uses the normalized BNT-refined horizontal hypothesis. The all-length condition is stronger than is needed for either Theorem 17.4.49 or Theorem 17.4.50; each uses a single noncommuting length supplied by the contrapositive of Lemma L.

Theorem 17.4.52 (Reduction of the periodic-sector step to the non-commutation family). Lean

Lean ✓ **Stmnt** *Let M generate matrix product density operators. If displaced projectors of M never commute, then periodic vectors of M supply cyclic projectors, and the vertically viewed tensor $\tilde{M}$ has no nontrivial p -periodic vectors. This narrows the hypothesis of the periodic-sector step in the proof of [CPGSV16, Proposition 4.13, lines 1888–1893] to the canonical-form implication of lines 1874–1887: every other ingredient of the step is proved.*

Proof. ✓ **Proof** A nontrivial p -periodic vector supplies the invariant, displaced projector of Theorem 17.4.47; the non-commutation hypothesis upgrades the displacement to the all-length non-commutation family, so periodic vectors supply cyclic projectors and Theorem 17.4.46 applies. $\square$

Theorem 17.4.53 (Commutation at chain length two forces letter invariance, for an injective tensor). Lean ✓ **Stmnt** *Let M be an MPO tensor whose doubled-index tensor is injective, and let Q be an idempotent on the physical space (Hermiticity is not needed). If $Q_1 H^{(2)} = H^{(2)} Q_1$, then $Q\tilde{M} = Q\tilde{M}Q$.*

Proof. ✓ **Proof** Idempotence of Q upgrades the commutation to the one-sided reduction $Q_1 H^{(2)} = Q_1 H^{(2)} Q_1$. Unfolding this identity against every trailing letter of M gives, for every pair of physical indices a, b , a trace identity pairing $(Q\tilde{M})_{ab}$ against $(Q\tilde{M}Q)_{ab}$ over every letter of the doubled-index tensor. Injectivity's spanning property and nondegeneracy of the trace pairing force the two matrices to agree. $\square$

Theorem 17.4.54 (No periodic vectors for an injective matrix product density operator). Lean

✓ **Stmnt** *Let M generate matrix product density operators and let M 's doubled-index tensor be injective. Then the vertically viewed tensor $\tilde{M}$ has no nontrivial periodic vectors, unconditionally.*

Proof. ✓ **Proof** A nontrivial periodic vector supplies the invariant, displaced projector Q of Theorem 17.4.47. Its word invariance transfers to the stacked tensor, giving commutation with $[H^{(2)}]^p$

The marked site closes to the loop with P inserted, and the remaining N sites close to plain loops. If P is idempotent, then also $\text{tr}(P_1 H^{(N+1)} P_1) = \text{tr}(E_P E^N)$.

Proof. ✓ **Proof** For a length- N configuration ρ , write $E_\rho^{(N)} = M^{\rho_0 \rho_0} \dots M^{\rho_{N-1} \rho_{N-1}}$. Expanding at the first site gives

$$(P_1 H^{(N+1)})_{(a,\rho)(a,\rho)} = \sum_i P_{ai} \text{tr}(M^{ia} E_\rho^{(N)}).$$

Summing over a and ρ and collecting closes the remaining loops by Lemma 17.4.58. For the two-sided compression, first-site actions compose site by site, $P_1 Q_1 = (PQ)_1$, so cyclicity of the trace and idempotency reduce $\text{tr}(P_1 H^{(N+1)} P_1)$ to $\text{tr}(P_1 H^{(N+1)})$. $\square$

Definition 17.4.60 (Sector projectors of the vertical decomposition). Lean Lean ✓ **Stmnt** A matrix $P \in M_d(\mathbb{C})$ is a sector projector for the weight $\mu \in \mathbb{C}$ and the matrix $E_\alpha \in M_D(\mathbb{C})$ if P is an orthogonal projector and inserting P into the single-site vertical loop selects the sector contribution, $E_P = \mu E_\alpha$. In the vertical decomposition

$$\widetilde{M} = \bigoplus_{\alpha,k} \mu_{\alpha,k} X_{\alpha,k} M_\alpha X_{\alpha,k}^{-1},$$

of the vertically viewed tensor, carried by the physical space in the proof of [CPGSV16, Proposition 4.13], the projector $P_{\alpha,k}$ selecting the sector (α, k) has this property with $\mu = \mu_{\alpha,k}$ and $(E_\alpha)_{ab} = \text{tr}(M_\alpha^{(a,b)})$, independently of k . Indeed, cyclicity of trace cancels the internal similarity $X_{\alpha,k} M_\alpha^{(a,b)} X_{\alpha,k}^{-1}$. Applied to the physical isometry of a grouped normal sector, the gauge-corner identity constructs the projector and proves its loop identity.

Theorem 17.4.61 (Literal sector-compression separation). Lean ✓ **Stmnt** Let M be an MPO tensor whose doubled-index MPS tensor is in literal CPSV canonical form, and let $P \in M_d(\mathbb{C})$. If one vertical letter has a nonzero two-sided corner, then some finite-chain first-site compression is nonzero:

$$(\exists v, P \widetilde{M}_v P \neq 0) \implies (\exists N \geq 0, P_1 H^{(N+1)} P_1 \neq 0), \quad (17.36)$$

where $P_1 = P \otimes \mathbb{1}^{\otimes N}$. This is the separation assertion at [CPGSV16, Proposition 4.13, line 1898], obtained from Lemma L in [CPGSV16, Appendix C.3, lines 1835–1858]. It uses neither positivity of $H^{(N+1)}$ nor a grouping of the vertical sectors.

Proof. ✓ **Proof** Suppose that $P_1 H^{(N+1)} P_1 = 0$ for every $N \geq 0$. By the coefficient formula for the doubled-index first-site contraction, the two-sided action $Y_C(P)$ and the zero action agree on every positive-length matrix product vector of the doubled-index tensor. Literal Lemma L therefore identifies their inserted tensors. For every vertical letter v , the insertion identity then gives

$$P \widetilde{M}_v P = 0. \quad (17.37)$$

This contradicts the hypothesis in (17.36). $\square$

Theorem 17.4.62 (A gauge corner gives a nonzero range-projector corner). Lean ✓ **Stmnt** Let M be an MPO tensor of physical dimension d and bond dimension D . Let A be a CPSV normal tensor of positive bond dimension n and physical dimension D^2 , let $V \in \mathbb{C}^{d \times n}$ be an isometry, let $X \in \mathbb{C}^{n \times n}$ be invertible, and let ω be nonzero. Suppose that, for every vertical letter v ,

$$\omega X A_v X^{-1} = V^\dagger \widetilde{M}_v V. \quad (17.38)$$

Then the range projector $VV^\dagger$ has a nonzero vertical corner:

$$(VV^\dagger)\widetilde{M}_v(VV^\dagger) \neq 0 \quad (17.39)$$

for some v . This is the algebraic nonvanishing step used in [CPGSV16, Proposition 4.13, line 1898].

Proof. ✓ Proof Normality supplies a letter v for which $A_v \neq 0$. Since ω is nonzero and X is invertible, the left-hand side of (17.38) is nonzero. If (17.39) vanished, multiplying it by $V^\dagger$ and V and using $V^\dagger V = 1$ would contradict (17.38). $\square$

Theorem 17.4.63 (A nonzero vertical corner has a nonzero sector compression). Lean ✓ Stmt

Let M be in normalized BNT-refined horizontal form and let P be a matrix for which $\widetilde{P}M_v P$ is nonzero for some vertical letter v . Then the first-site compression $P_1 H^{(N+1)} P_1$ is nonzero for some N . This is the separation step used in [CPGSV16, Proposition 4.13, line 1898].

The normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis. The literal separation statement is Theorem 17.4.61; it does not by itself construct the grouped vertical sectors used here.

Proof. ✓ Proof If every compression vanished, its matrix entries would say that the first-site insertion by P has the same matrix product vectors as zero. Lemma L on the normalized BNT-refined horizontal form would then force the inserted tensor itself to vanish, contrary to the assumed nonzero corner. $\square$

Theorem 17.4.64 (The sector trace identity). Lean ✓ Stmt Let P be a sector projector for the weight μ and the matrix E_α . Then for every chain length $N + 1$,

$$\text{tr}(P_1 H^{(N+1)} P_1) = \mu d_\alpha^{(N+1)}. \quad (17.40)$$

Once spectral normalization and grouping identify the reducing projectors with sector projectors $P_{\alpha,k}$ satisfying Definition 17.4.60, this specializes to the trace identification $\text{tr}(P_{\alpha,k} H^{(N)} P_{\alpha,k}) = \mu_{\alpha,k} d_\alpha^{(N)}$ used in the proof of [CPGSV16, Proposition 4.13]; the source states it at the particular length where the compression does not vanish.

Proof. ✓ Proof By Theorem 17.4.59 the trace of the compression is $\text{tr}(E_P E^N)$, and the sector-projector property replaces E_P by μE_α . $\square$

Theorem 17.4.65 (Positivity of the vertical weights). Lean Lean Lean ✓ Stmt Let M generate an MPDO, let Q and P be sector projectors for the weights $\mu_{\alpha,0}$ and $\mu_{\alpha,k}$ and a common matrix E_α , and suppose $\mu_{\alpha,0} > 0$. If the compression of $H^{(N+1)}$ by P at the first site is nonzero, then

$$\begin{aligned} \mu_{\alpha,k} d_\alpha^{(N+1)} &> 0, \\ d_\alpha^{(N+1)} &> 0, \\ \mu_{\alpha,k} &> 0. \end{aligned}$$

Granted such sector projectors, this yields the conclusion of the trace argument in the proof of [CPGSV16, Proposition 4.13]: since w.l.o.g. $\mu_{\alpha,0} > 0$, the positive trace forces $d_\alpha > 0$ and hence $\mu_{\alpha,k} > 0$.

Proof. ✓ Proof By Theorem 17.4.56 the nonzero compression has positive trace, which Theorem 17.4.64 identifies with $\mu_{\alpha,k} d_{\alpha}^{(N+1)}$. In particular $d_{\alpha}^{(N+1)} \neq 0$, so the trace of the compression by the sector projector of $(\alpha, 0)$ at the same length equals $\mu_{\alpha,0} d_{\alpha}^{(N+1)} \neq 0$; that compression is therefore also nonzero, and its trace is a positive real. Dividing by $\mu_{\alpha,0} > 0$ gives $d_{\alpha}^{(N+1)} > 0$, and dividing the first inequality by $d_{\alpha}^{(N+1)}$ gives $\mu_{\alpha,k} > 0$. $\square$

Theorem 17.4.66 (Unitary normalization of a scalar Gram matrix). Lean ✓ Stmt If $\omega > 0$ and $X^{\dagger}X = \omega \mathbb{1}$, then $\omega^{-1/2}X$ is unitary.

Proof. ✓ Proof Divide the Gram identity by ω . $\square$

Theorem 17.4.67 (Gram rigidity of normal tensors). Lean Lean Lean ✓ Stmt Let M_{α} be a normal tensor. Any matrix commuting with every matrix of M_{α} is a scalar multiple of the identity. In particular, if $X \neq 0$ and $X^{\dagger}X$ commutes with every matrix of M_{α} , then $X^{\dagger}X = \omega \mathbb{1}$ for a necessarily positive constant ω , and $\omega^{-1/2}X$ is unitary. This is the normalized form $X_{\alpha,k}^{\dagger}X_{\alpha,k} = \omega_{\alpha,k} \mathbb{1}$ of the Gram-matrix proportionality in the proof of [CPGSV16, Proposition 4.13], with $X_{\alpha,0} = \mathbb{1}$ and $U_{\alpha,k} = \omega_{\alpha,k}^{-1/2}X_{\alpha,k}$. Deriving the commutation from self-adjointness of the sector compressions is a separate step.

Proof. ✓ Proof Commutation with the matrices of M_{α} extends to all word evaluations, and after blocking these span the full matrix algebra, so the commuting matrix is central and hence scalar. The Gram matrix $X^{\dagger}X$ is positive semidefinite and nonzero, so its scalar value is a positive real ω ; dividing X by $\sqrt{\omega}$ makes the Gram matrix the identity. $\square$

Theorem 17.4.68 (Reflected marked-chain identity). Lean Lean Lean Lean ✓ Stmt Let M generate positive semidefinite operators on every nonempty chain, let A be a vertical tensor, and suppose that a matrix V selects the vertical corner $V^{\dagger}\widetilde{M}_vV = cA^v$, where $c > 0$. For vertical indices r, s , write $(\mathcal{R}(A)^{rs})_{ab} = (A^{(a,b)})_{rs}$, and set $(A^{\sharp})^v = (A^{v^{\text{op}}})^{\dagger}$. Then, for every pair of equal-length tail words σ, τ ,

$$\text{tr}(\mathcal{R}(A)^{rs} M^{\sigma_1 \tau_1} \dots M^{\sigma_L \tau_L}) = \text{tr}(\mathcal{R}(A^{\sharp})^{rs} (M^{\dagger})^{\sigma_L \tau_L} \dots (M^{\dagger})^{\sigma_1 \tau_1}). \quad (17.41)$$

Thus the adjoint exchanges the oriented horizontal bond pair and reverses the unmarked tail. In particular, this is not a same-tail identity and does not identify A with $A^{\sharp}$ letterwise.

Proof. ✓ Proof Compress the first site of the positive operator by V . The resulting matrix is Hermitian. Expanding one matrix entry gives the marked chain on the left and the complex conjugate of the entry with r, s and the ket–bra words exchanged. Taking the conjugate transpose reverses the tail product, cyclicity of trace returns the marked factor to the first position, and positivity of c permits cancellation without a phase. $\square$

Theorem 17.4.69 (Gram-dressed reflected marked-chain identity). Lean Lean ✓ Stmt Let M generate positive semidefinite operators on every nonempty chain, and suppose that one vertical corner is a positive scalar multiple of a similarity of a tensor A ,

$$V^{\dagger}\widetilde{M}^vV = cXA^vX^{-1}, \quad c > 0.$$

Then the marked-chain coefficient of the Gram-dressed tensor $(X^{\dagger}X)A^v(X^{\dagger}X)^{-1}$ equals the reflected-adjoint marked-chain coefficient of A in the adjoint tensor, with the two tail words reversed. This is the single-corner, open-index conjugation step from Figures 7–8 of [CPGSV16, Proposition 4.13, lines 1909–1919].

Proof. ✓ Proof Apply the reflected marked-chain identity to the similar tensor XA^vX^{-1} . Conjugate its two open virtual indices by $X^\dagger$ and $(X^{-1})^\dagger$. On the original side this gives the Gram dressing of A ; on the reflected side the gauge factors cancel, leaving the reflected adjoint of A . $\square$

Theorem 17.4.70 (Pairwise Gram conjugation of grouped vertical corners). Lean Lean Lean
✓ Stmt Let M be in normalized BNT-refined horizontal form and generate positive semidefinite operators on every nonempty chain. Suppose two vertical corners are positive scalar multiples of similarities of the same tensor A :

$$\begin{aligned} V_X^\dagger \widetilde{M}^v V_X &= c_X X A^v X^{-1}, & c_X > 0, \\ V_Y^\dagger \widetilde{M}^v V_Y &= c_Y Y A^v Y^{-1}, & c_Y > 0. \end{aligned}$$

Then the two Gram conjugations agree for every vertical letter,

$$(X^\dagger X) A^v (X^\dagger X)^{-1} = (Y^\dagger Y) A^v (Y^\dagger Y)^{-1}. \quad (17.42)$$

Normality of A is not required for this equality; it enters only in the subsequent proportionality of the Gram matrices. This is the pairwise conclusion of Figures 7–8 in [CPGSV16, Proposition 4.13, lines 1909–1919].

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Proof. ✓ Proof Conjugate the open indices in the reflected marked-chain identity for the first corner by $X^\dagger$. Its left side becomes the marked chain of $(X^\dagger X) A^v (X^\dagger X)^{-1}$, while the gauge factors on the reflected side cancel and leave the marked chain of $A^\dagger$. The same calculation with Y gives the identical reflected chain. Thus the two Gram-dressed marks have equal closed chains for every tail word. The reflected identity also holds for the empty tail, but marked separation uses only nonempty tails, as in the proof of Lemma L in [CPGSV16, Appendix C.3, lines 1848–1858].

These marks lie in the physical-letter span. Indeed, with $L_X = V_X X$ and $R_X = V_X (X^{-1})^\dagger$, their coefficients are $f_{(r,s),(i,j)}^X = c_X^{-1} \overline{(L_X)_{ir}} (R_X)_{js}$, and similarly for Y . The corner identities show that these linear combinations are precisely the horizontal slices of the two Gram-dressed tensors. Physical-letter marked separation therefore identifies every slice, hence every vertical letter. $\square$

Definition 17.4.71 (Pairwise Gram dressing for grouped corners). Lean ✓ Stmt An MPO tensor M has the *grouped-corner Gram-dressing property* if the following holds. Let A be a tensor, let V_X, V_Y be two physical maps, let X, Y be invertible matrices, and let $c_X, c_Y > 0$. Suppose that, for every vertical letter v , the following two corner identities hold:

$$\begin{aligned} V_X^\dagger \widetilde{M}^v V_X &= c_X X A^v X^{-1}, \\ V_Y^\dagger \widetilde{M}^v V_Y &= c_Y Y A^v Y^{-1} \end{aligned}$$

Then, for every vertical letter v ,

$$(X^\dagger X) A^v (X^\dagger X)^{-1} = (Y^\dagger Y) A^v (Y^\dagger Y)^{-1}. \quad (17.43)$$

Theorem 17.4.72 (Distinguished reference corner of a grouped phase class). Lean ✓ Stmt
 Suppose a phase-class grouping has positive effective coefficients $c_{\alpha,q}$, corner identities

$$V_{\alpha,q}^\dagger \widetilde{M}^v V_{\alpha,q} = c_{\alpha,q} X_{\alpha,q} A_{\alpha,q}^v X_{\alpha,q}^{-1},$$

and distinguished gauges $X_{\alpha,0} = \mathbb{1}$. Fix a class α and a copy q . After identifying the representative bond space with that of copy q , there are a physical map $W_{\alpha,q}$ and a scalar $c_{\alpha,0} > 0$ such that, for every vertical letter v ,

$$W_{\alpha,q}^\dagger \widetilde{M}^v W_{\alpha,q} = c_{\alpha,0} A_{\alpha,q}^v. \quad (17.44)$$

Proof. ✓ Proof Identify the distinguished bond space with that of copy q and define the transported physical map by

$$W_{\alpha,q} = V_{\alpha,0}.$$

The distinguished gauge is the identity, so its corner equation becomes

$$W_{\alpha,q}^\dagger \widetilde{M}^v W_{\alpha,q} = c_{\alpha,0} A_{\alpha,q}^v.$$

Positivity of the distinguished effective coefficient gives $c_{\alpha,0} > 0$, and the second displayed identity is (17.44). $\square$

Theorem 17.4.73 (Gram conjugation from grouped-corner dressing). Lean ✓ Stmt Suppose a phase-class grouping has positive effective coefficients $c_{\alpha,q}$ and corner identities

$$V_{\alpha,q}^\dagger \widetilde{M}^v V_{\alpha,q} = c_{\alpha,q} X_{\alpha,q} A_{\alpha,q}^v X_{\alpha,q}^{-1}.$$

Assume that the distinguished copy in each class has identity gauge and that M has the grouped-corner Gram-dressing property. Then for every class α , copy q , and vertical letter v ,

$$(X_{\alpha,q}^\dagger X_{\alpha,q}) A_{\alpha,q}^v (X_{\alpha,q}^\dagger X_{\alpha,q})^{-1} = A_{\alpha,q}^v. \quad (17.45)$$

Proof. ✓ Proof Theorem 17.4.72 transports the distinguished identity-gauge corner to the bond space of copy q . The grouped-corner Gram-dressing property gives

$$(X_{\alpha,0}^\dagger X_{\alpha,0}) A_{\alpha,q}^v (X_{\alpha,0}^\dagger X_{\alpha,0})^{-1} = (X_{\alpha,q}^\dagger X_{\alpha,q}) A_{\alpha,q}^v (X_{\alpha,q}^\dagger X_{\alpha,q})^{-1}. \quad (17.46)$$

The left side is $A_{\alpha,q}^v$ because $X_{\alpha,0} = \mathbb{1}$. This gives (17.45). $\square$

Theorem 17.4.74 (Gram conjugation for an actual grouped sector). Lean ✓ Stmt Consider the grouped vertical decomposition furnished by Theorem 17.4.100. For a phase class α and a copy q , transport its representative tensor to the bond space of that copy. If $X_{\alpha,q}$ is the corresponding gauge, then for every vertical letter v ,

$$(X_{\alpha,q}^\dagger X_{\alpha,q}) A_{\alpha,q}^v (X_{\alpha,q}^\dagger X_{\alpha,q})^{-1} = A_{\alpha,q}^v. \quad (17.47)$$

This is Figure 8 for the actual witnesses of the grouped decomposition.

Proof. ✓ Proof Theorem 17.4.70 supplies the grouped-corner Gram-dressing property. Apply Theorem 17.4.73 to the grouped vertical decomposition. $\square$

Theorem 17.4.75 (Literal Gram conjugation for an actual grouped sector). Lean ✓ Stmt Consider the actual phase-class grouping furnished under literal CPSV canonical form. For a class α and copy q , identify its normal representative with the bond space of that copy using the equality of dimensions. Then for every vertical letter v ,

$$(X_{\alpha,q}^\dagger X_{\alpha,q}) A_{\alpha,q}^v (X_{\alpha,q}^\dagger X_{\alpha,q})^{-1} = A_{\alpha,q}^v. \quad (17.48)$$

Proof. [✓ Proof](#) The literal pairwise Figure 8 theorem supplies the grouped-corner Gram-dressing property. Apply Theorem 17.4.73 to the literal phase-class grouping. This gives

$$(X_{\alpha,q}^\dagger X_{\alpha,q}) A_{\alpha,q}^v (X_{\alpha,q}^\dagger X_{\alpha,q})^{-1} = A_{\alpha,q}^v. \quad (17.49)$$

□

Theorem 17.4.76 (Gram conjugation from the adjoint dressing). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let X be invertible and suppose that B and a target C satisfy

$$XBX^{-1} = (X^{-1})^\dagger CX^\dagger.$$

Then conjugation by the Gram matrix carries B to C ,

$$(X^\dagger X)B(X^\dagger X)^{-1} = C. \quad (17.50)$$

Hence any two gauges satisfying this conditional identity with the same target have equal Gram conjugations, and the ratio of their Gram matrices commutes with B . This gives the relative identity from an abstract common target in [CPGSV16, Proposition 4.13, lines 1909–1919]. The reflected marked-chain argument in the source does not identify the target separately for either sector. The same-letter choice $C = B^\dagger$ and the further specialization $B^\dagger = B$ are useful corollaries, but the latter is not assumed by the source.

Proof. [✓ Proof](#) Multiply the dressed-target identity by $X^\dagger$ on the left and by $(X^{-1})^\dagger$ on the right. The left side becomes $(X^\dagger X)B(X^\dagger X)^{-1}$ and the right side collapses to C . Eliminating the common target for two gauges and multiplying by the inverse Gram matrix gives the relative commutation relation. □

$$X_\alpha^\dagger X_\alpha = Y_\alpha^\dagger Y_\alpha$$

Figure 17.5: Equality of the Gram conjugations for two copies of the same vertical normal tensor: $(X_{\alpha,k}^\dagger X_{\alpha,k})M_\alpha(X_{\alpha,k}^\dagger X_{\alpha,k})^{-1} = (X_{\alpha,0}^\dagger X_{\alpha,0})M_\alpha(X_{\alpha,0}^\dagger X_{\alpha,0})^{-1}$. Theorem 17.4.70 gives this equality from self-adjointness of the corresponding sector compressions and horizontal marked separation. Normality is used only in the subsequent proportionality argument. This is [CPGSV16, Proposition 4.13, lines 1915–1921].

Theorem 17.4.77 (Gram-matrix proportionality from relative Gram conjugations). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M_α be a normal tensor, and let X and Y be invertible gauges whose Gram matrices induce the same conjugation on each letter M_α^v . Then there is a positive real number ω such that

$$X^\dagger X = \omega Y^\dagger Y, \quad (17.51)$$

and $\omega^{-1/2}XY^{-1}$ is unitary. Theorem 17.4.70 gives the Figure 8 comparison in [CPGSV16, Proposition 4.13, lines 1909–1921]. This comparison supplies precisely the relative hypothesis for a sector and the distinguished sector. In the normalization $Y = \mathbb{1}$, the resulting unitary is $\omega^{-1/2}X$. The common-target declarations attached to this entry are conditional algebraic corollaries, not the reflected marked-chain statement itself.

Proof. ✓ Proof By Theorem 17.4.76, the ratio $(Y^\dagger Y)^{-1}(X^\dagger X)$ commutes with every letter of M_α . Theorem 17.4.67 makes this ratio scalar. When the matrix dimension is positive, positive definiteness of the two Gram matrices makes the scalar a positive real number ω . In dimension zero the equality is vacuous and one may choose $\omega = 1$. Conjugating the relative Gram identity by Y^{-1} shows that XY^{-1} has Gram matrix $\omega \mathbb{1}$; division by $\sqrt{\omega}$ makes it unitary. $\square$

Theorem 17.4.78 (Positive scalar Gram matrix from grouped-corner dressing). Lean ✓ Stmt *Under the hypotheses of Theorem 17.4.73, suppose in addition that every representative $A_{\alpha,q}$ is normal and has positive bond dimension. Then for every class α and copy q there is a real number $\omega_{\alpha,q} > 0$ such that*

$$X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}. \quad (17.52)$$

Proof. ✓ Proof Theorem 17.4.73 shows that conjugation by $X_{\alpha,q}^\dagger X_{\alpha,q}$ fixes every letter of the normal representative. Apply the identity-gauge specialization of Theorem 17.4.77; it gives a real number $\omega_{\alpha,q} > 0$ with $X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}$. $\square$

Theorem 17.4.79 (Normalization of the grouped vertical gauges). Lean Lean ✓ Stmt *In the grouped vertical decomposition of a matrix-product density operator in normalized BNT-refined horizontal form, let $X_{\alpha,k}$ be the gauge of a copy of the normal representative M_α . Fix the distinguished gauge to $X_{\alpha,0} = \mathbb{1}$. Then there is a positive real number $\omega_{\alpha,k}$ such that*

$$\begin{aligned} X_{\alpha,k}^\dagger X_{\alpha,k} &= \omega_{\alpha,k} \mathbb{1}, \\ U_{\alpha,k} &= \omega_{\alpha,k}^{-1/2} X_{\alpha,k}, \end{aligned}$$

and $U_{\alpha,k}$ is unitary. This is the normalization in [CPGSV16, Proposition 4.13, lines 1903–1921].

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Proof. ✓ Proof The pairwise Figure 8 theorem supplies the grouped-corner Gram-dressing property. Apply Theorem 17.4.78 to the grouped vertical decomposition, then divide each gauge by the positive square root of its Gram scalar. $\square$

Theorem 17.4.80 (Literal normalization of the actual grouped vertical gauges). Lean Lean ✓ Stmt *In the actual phase-class grouping under literal CPSV canonical form, every gauge $X_{\alpha,q}$ has a positive real Gram scalar $\omega_{\alpha,q}$ such that*

$$X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}. \quad (17.53)$$

Consequently $U_{\alpha,q} = \omega_{\alpha,q}^{-1/2} X_{\alpha,q}$ is unitary. This proves the gauge-normalization step of [CPGSV16, Proposition 4.13, lines 1903–1921] and supplies the positive Gram scalars used to normalize the physical sector maps.

Proof. ✓ Proof The literal pairwise Figure 8 theorem supplies the grouped-corner Gram-dressing property. Apply Theorem 17.4.78 to the literal phase-class grouping to obtain

$$X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}. \quad (17.54)$$

Dividing each gauge by the positive square root of its Gram scalar gives the unitary normalization. $\square$

Definition 17.4.81 (Normalized grouped-sector map). [Lean](#) [✓ Stmt](#) After identifying the bond space of a copy with that of its representative, define its normalized physical map by

$$W_{\alpha,q} = V_{\alpha,q} \omega_{\alpha,q}^{-1/2} X_{\alpha,q}.$$

Theorem 17.4.82 (Normalized grouped-sector maps from Gram scalars). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let a phase-class grouping have gauges $X_{\alpha,q}$, physical maps $V_{\alpha,q}$, and effective coefficients $c_{\alpha,q}$. Suppose that, for every class α and copy q , $\omega_{\alpha,q} > 0$ and

$$X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}.$$

Suppose also that the maps $V_{\alpha,q}$ are isometries with pairwise orthogonal ranges, they intertwine $\widetilde{M}$ with the corresponding gauged representatives, and their grouped corner sum reconstructs every letter of $\widetilde{M}$. Then there are maps $W_{\alpha,q}$ such that, for every class α and copy q ,

$$W_{\alpha,q}^\dagger W_{\alpha,q} = \mathbb{1}. \quad (17.55)$$

For distinct pairs $(\alpha, q) \neq (\beta, p)$,

$$W_{\alpha,q}^\dagger W_{\beta,p} = 0. \quad (17.56)$$

For every vertical letter v , these maps also satisfy

$$\begin{aligned} \widetilde{M}_v W_{\alpha,q} &= W_{\alpha,q} (c_{\alpha,q} M_\alpha^v), \\ \widetilde{M}_v &= \sum_\alpha \sum_q W_{\alpha,q} (c_{\alpha,q} M_\alpha^v) W_{\alpha,q}^\dagger. \end{aligned} \quad (17.57)$$

Proof. [✓ Proof](#) After identifying the copy bond space with the representative bond space, define

$$\begin{aligned} Q_{\alpha,q} &= \omega_{\alpha,q}^{-1/2} X_{\alpha,q}, \\ W_{\alpha,q} &= V_{\alpha,q} Q_{\alpha,q}. \end{aligned}$$

Since $\omega_{\alpha,q} > 0$, the Gram identity implies

$$\begin{aligned} X_{\alpha,q}^\dagger &= \omega_{\alpha,q} X_{\alpha,q}^{-1}, \\ Q_{\alpha,q}^\dagger Q_{\alpha,q} &= \mathbb{1}. \end{aligned}$$

Hence $W_{\alpha,q}^\dagger W_{\alpha,q} = \mathbb{1}$, and inserting the two normalized gauges on either side of the original orthogonality equation preserves pairwise orthogonality. The original intertwining identity gives

$$\widetilde{M}_v W_{\alpha,q} = W_{\alpha,q} (c_{\alpha,q} M_\alpha^v).$$

The same Gram identity and the positive square-root normalization give the corner equality

$$Q_{\alpha,q} (c_{\alpha,q} M_\alpha^v) Q_{\alpha,q}^\dagger = c_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}. \quad (17.58)$$

Multiplying (17.58) on the left by $V_{\alpha,q}$ and on the right by $V_{\alpha,q}^\dagger$ gives

$$W_{\alpha,q} (c_{\alpha,q} M_\alpha^v) W_{\alpha,q}^\dagger = V_{\alpha,q} (c_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}) V_{\alpha,q}^\dagger. \quad (17.59)$$

Summing these physical-space identities over all copies and classes, then applying the original grouped reconstruction, proves (17.57). $\square$

Theorem 17.4.83 (Normalized physical maps of the grouped vertical sectors). [Lean](#) [✓ Stmt](#) In the grouped vertical decomposition, put

$$\begin{aligned} Q_{\alpha,q} &= \omega_{\alpha,q}^{-1/2} X_{\alpha,q}, \\ W_{\alpha,q} &= V_{\alpha,q} Q_{\alpha,q}, \end{aligned}$$

and identify the bond space of each copy with that of its chosen representative. Then

$$\begin{aligned} W_{\alpha,q}^\dagger W_{\alpha,q} &= \mathbb{1}, \\ W_{\alpha,q}^\dagger W_{\beta,p} &= 0 \quad ((\alpha, q) \neq (\beta, p)). \end{aligned}$$

Writing $c_{\alpha,q} = \nu_{\alpha,q} \zeta_{\alpha,q} > 0$, the normalized maps satisfy

$$\widetilde{M}_v W_{\alpha,q} = W_{\alpha,q} (c_{\alpha,q} M_\alpha^v),$$

and retain the literal reconstruction

$$\widetilde{M}_v = \sum_\alpha \sum_q W_{\alpha,q} (c_{\alpha,q} M_\alpha^v) W_{\alpha,q}^\dagger. \quad (17.60)$$

These are the normalized sector identities in [CPGSV16, Proposition 4.13, lines 1895–1921].

Proof. [✓ Proof](#) Theorem 17.4.79 supplies the positive Gram scalars. Apply Theorem 17.4.82 to the grouped vertical decomposition. $\square$

Theorem 17.4.84 (Literal normalized physical maps of the grouped vertical sectors). [Lean](#) [✓ Stmt](#) In the phase-class grouping obtained from literal CPSV canonical form, let $Q_{\alpha,q} = \omega_{\alpha,q}^{-1/2} X_{\alpha,q}$ and $W_{\alpha,q} = V_{\alpha,q} Q_{\alpha,q}$, after identifying the bond space of each copy with that of its representative. Then

$$\begin{aligned} W_{\alpha,q}^\dagger W_{\alpha,q} &= \mathbb{1}, \\ W_{\alpha,q}^\dagger W_{\beta,p} &= 0 \quad ((\alpha, q) \neq (\beta, p)), \\ \widetilde{M}_v W_{\alpha,q} &= W_{\alpha,q} (c_{\alpha,q} M_\alpha^v), \end{aligned}$$

where $c_{\alpha,q} = \nu_{\alpha,q} \zeta_{\alpha,q} > 0$. Moreover, every vertical letter is reconstructed exactly:

$$\widetilde{M}_v = \sum_\alpha \sum_q W_{\alpha,q} (c_{\alpha,q} M_\alpha^v) W_{\alpha,q}^\dagger. \quad (17.61)$$

Proof. [✓ Proof](#) Theorem 17.4.80 supplies the positive Gram scalars. Apply Theorem 17.4.82 to the literal phase-class grouping. Its corner identity preserves the original grouped reconstruction and gives (17.61). $\square$

Lemma 17.4.85 (Matrix product vector of the diagonal tensor). [Lean](#) [✓ Stmt](#) For an MPO tensor M and a configuration σ , the matrix product vector of the diagonal tensor $i \mapsto M^{ii}$ equals the matrix product vector of the doubled-index tensor $\widehat{M}$ evaluated on the diagonal-paired configuration $k \mapsto (\sigma_k, \sigma_k)$: $V^{(N)}(i \mapsto M^{ii})(\sigma) = V^{(N)}(\widehat{M})(k \mapsto (\sigma_k, \sigma_k))$.

Proof. [✓ Proof](#) At each site value i , the diagonal tensor agrees with the doubled-index tensor on the diagonal pair, $M^{ii} = \widehat{M}^{(i,i)}$. The matrix product vector is multiplicative over sites, so replacing each factor $M^{\sigma_k \sigma_k}$ by $\widehat{M}^{(\sigma_k, \sigma_k)}$ reindexes the configuration from σ to $k \mapsto (\sigma_k, \sigma_k)$, which is the stated equality. $\square$

Lemma 17.4.86 (Diagonal tensor as a block decomposition under horizontal form). [Lean](#) [✓ Stmt](#)

Suppose the doubled-index tensor $\widehat{M}$ shares its positive-length matrix product vector family with a block-diagonal direct sum $\bigoplus_k \mu_k A_k$ of blocks A_k on the doubled index, as in a horizontal canonical form. Then the diagonal tensor of M generates the same positive-length matrix product vector family as the block-diagonal direct sum $\bigoplus_k \mu_k A_k^{\text{diag}}$ on the single-spin index, where $A_k^{\text{diag}}(i) = A_k(i, i)$ is the diagonal restriction of each block. For every $N \geq 1$,

$$V^{(N)}(i \mapsto M^{ii})(\sigma) = V^{(N)}\left(\bigoplus_k \mu_k A_k^{\text{diag}}\right)(\sigma).$$

Proof. [✓ Proof](#) By Lemma 17.4.85, the diagonal matrix product vector equals that of $\widehat{M}$ on the diagonal-paired configuration, which by hypothesis equals that of the block-diagonal direct sum there. Expanding the direct sum as the weighted sum $\sum_k \mu_k^N V^{(N)}(A_k)$ and restricting each block to the diagonal pair gives $\sum_k \mu_k^N V^{(N)}(A_k^{\text{diag}})$, the matrix product vector of the single-spin block-diagonal direct sum. $\square$

Theorem 17.4.87 (Per-copy horizontal hypotheses do not control diagonal normality). [Lean](#)

[✓ Stmt](#) There is a left-canonical, injective doubled-index tensor with a nonzero block weight satisfying the older per-copy finite-length separation hypothesis whose restriction to the diagonal physical pairs is not normal. On bond space $\mathbb{C}^2$, take

$$A^{(a,b)} = c_a E_{ab}, \quad c_0 = \frac{3}{5}, \quad c_1 = \frac{4}{5}.$$

Then the four matrices $A^{(a,b)}$ span $M_2(\mathbb{C})$ and $\sum_{a,b} (A^{(a,b)})^* A^{(a,b)} = \mathbb{1}$. For a single block, injectivity also gives finite-length block separation. The diagonal restriction, however, consists only of

$$A^{\text{diag}}(0) = \frac{3}{5} E_{00}, \quad A^{\text{diag}}(1) = \frac{4}{5} E_{11}.$$

Every product of these two matrices is diagonal, so no blocking length spans $M_2(\mathbb{C})$.

Proof. [✓ Proof](#) The four nonzero scalar multiples of the matrix units span the full matrix algebra. The identity $(3/5)^2 + (4/5)^2 = 1$ gives left-canonicity. After diagonal restriction, $E_{ii} E_{jj} = \delta_{ij} E_{ii}$. Hence every word is either zero or a scalar multiple of some E_{ii} , and therefore has vanishing $(0, 1)$ entry. Since E_{01} does not, it lies outside the span at every blocking length. $\square$

Lemma 17.4.88 (Diagonal tensor word evaluation). [Lean](#) [✓ Stmt](#) For any word w , the matrix product of the diagonal tensor along w equals the operator word evaluation with equal ket and bra words, since both multiply the matrices $M^{w_k w_k}$ site by site.

Proof. [✓ Proof](#) By induction on the word. $\square$

Lemma 17.4.89 (Diagonal tensor evaluates the density-operator diagonal). [Lean](#) [✓ Stmt](#) For an MPO tensor M and a configuration σ , the matrix product vector of the diagonal tensor equals the diagonal entry of the generated operator:

$$V^{(N)}(i \mapsto M^{ii})(\sigma) = \langle \sigma | \rho^{(N)}(M) | \sigma \rangle. \quad (17.62)$$

Both sides of (17.62) equal $\text{tr}(M^{\sigma_1 \sigma_1} \dots M^{\sigma_N \sigma_N})$.

Proof. [✓ Proof](#) Word evaluation of the diagonal tensor multiplies the matrices $M^{\sigma_k \sigma_k}$, which is exactly the operator word evaluation with equal ket and bra indices; taking the trace gives the diagonal entry. $\square$

Lemma 17.4.90 (Nonnegativity of the diagonal matrix product vector). [Lean](#) [✓ Stmt](#) If $\rho^{(N)}(M)$ is positive semidefinite, then $V^{(N)}(i \mapsto M^{ii})(\sigma) \geq 0$ for every configuration σ .

Proof. [✓ Proof](#) By Lemma 17.4.89, the value is a diagonal entry of $\rho^{(N)}(M)$, and the diagonal entries of a positive semidefinite matrix are non-negative. $\square$

Lemma 17.4.91 (Matrix product vector of the vertical direct-sum tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let g be a block count, $\text{dim}, \text{mult} : \{0, \dots, g-1\} \rightarrow \mathbb{N}$, weights $\omega_{\alpha j} \in \mathbb{C}$ for $0 \leq j < \text{mult}_\alpha$, and blocks A_α of bond dimension dim_α . The vertical direct-sum tensor is the block-diagonal direct sum over the flattened index (α, j) that repeats A_α with weight $\omega_{\alpha j}$, written $\bigoplus_\alpha \bigoplus_j \omega_{\alpha j} A_\alpha$. Its matrix product vector decomposes as

$$V^{(N)}\left(\bigoplus_\alpha \bigoplus_j \omega_{\alpha j} A_\alpha\right)(\sigma) = \sum_\alpha \sum_j (\omega_{\alpha j})^N V^{(N)}(A_\alpha)(\sigma). \quad (17.63)$$

Proof. [✓ Proof](#) The vertical direct-sum tensor is the block-diagonal direct sum over the single flattened index q , with flattened weights $\tilde{\omega}_q$ and blocks $\tilde{A}_q$. By the block matrix-product-vector decomposition (Theorem 2.5.9),

$$V^{(N)}\left(\bigoplus_q \tilde{\omega}_q \tilde{A}_q\right)(\sigma) = \sum_q (\tilde{\omega}_q)^N V^{(N)}(\tilde{A}_q)(\sigma).$$

The flattening identifies q with the pair (α, j) via the bijection

$$\{(\alpha, j) : 0 \leq \alpha < g, 0 \leq j < \text{mult}_\alpha\} \simeq \{0, \dots, \sum_\alpha \text{mult}_\alpha - 1\},$$

under which $\tilde{\omega}_q = \omega_{\alpha j}$ and $\tilde{A}_q = A_\alpha$. Reindexing the sum along this bijection yields the double sum over (α, j) . $\square$

Lemma 17.4.92 (Regrouping a block family by repeated representatives). [Lean](#) [✓ Stmt](#) Let $(B_k)_{k \in I}$ be a finite family of blocks with weights ν_k , and let $(A_\alpha)_{\alpha \in J}$ be a finite family of representative blocks. Suppose there is a bijection

$$e : I \simeq \{(\alpha, j) : \alpha \in J, 0 \leq j < m_\alpha\}$$

such that, writing $e(k) = (\alpha, j)$, one has $\nu_k = \omega_{\alpha j}$ and, for every positive length N , $V^{(N)}(B_k) = V^{(N)}(A_\alpha)$. Then

$$V^{(N)}\left(\bigoplus_{k \in I} \nu_k B_k\right) = V^{(N)}\left(\bigoplus_{\alpha \in J} \bigoplus_{j=0}^{m_\alpha-1} \omega_{\alpha j} A_\alpha\right)$$

for every positive length N . This is the finite-sum reindexing after the vertical sectors have been grouped in [CPGSV16, Proposition 4.13].

Proof. [✓ Proof](#) For $N > 0$ and every configuration σ , the two sides expand as

$$\begin{aligned} & \sum_{k \in I} \nu_k^N V^{(N)}(B_k)(\sigma), \\ & \sum_{\alpha \in J} \sum_{j=0}^{m_\alpha-1} \omega_{\alpha j}^N V^{(N)}(A_\alpha)(\sigma). \end{aligned}$$

The summand indexed by k in the first sum equals the summand indexed by $e(k)$ in the second, so reindexing by e proves the equality. $\square$

Theorem 17.4.93 (Invariant vertical projections in literal CPSV canonical form). [Lean](#) [Lean](#) [✓ Stmt](#) Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. If an orthogonal projection P satisfies $P\widetilde{M}_v = P\widetilde{M}_vP$ for every vertical letter v , then

$$\widetilde{M}_vP = P\widetilde{M}_vP \quad (17.64)$$

for every v . Thus the vertically viewed tensor has invariant-projector closure.

Proof. [✓ Proof](#) The left-invariance hypothesis gives $P_1H^{(N)} = P_1H^{(N)}P_1$ at every positive length. Since $H^{(N)}$ is positive and $P = P^\dagger$,

$$P_1H^{(N)} = P_1H^{(N)}P_1 = H^{(N)}P_1. \quad (17.65)$$

The matrix coefficients of (17.65) give agreement of the corresponding first-site actions. In particular,

$$P_1H^{(N)} = H^{(N)}P_1. \quad (17.66)$$

The original-coordinate marked Lemma L gives equality of their inserted tensors on the original bond space. Reading this equality as a vertical one-site action yields

$$\widetilde{M}_vP = P\widetilde{M}_vP. \quad (17.67)$$

□

Theorem 17.4.94 (Invariant vertical projections are reducing). [Lean](#) [✓ Stmt](#) Let M be in normalized BNT-refined horizontal form and generate matrix product density operators. If an orthogonal projection P satisfies $P\widetilde{M}_v = P\widetilde{M}_vP$ for every vertical letter v , then it also satisfies $\widetilde{M}_vP = P\widetilde{M}_vP$ for every vertical letter v . Thus the vertically viewed tensor has invariant-projector closure.

Proof. [✓ Proof](#) Translate the left-invariance identity into the one-site ket and bra actions of the original tensor. The normalized BNT-refined version of Lemma L gives the opposite one-sided identity. Translating back to the vertically viewed tensor proves the claim. □

Theorem 17.4.95 (Irreducible reducing sectors of the vertically viewed tensor). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 17.4.94, there are positive integers d_k , tensors B_k of bond dimension d_k , and isometries $V_k : \mathbb{C}^{d_k} \rightarrow \mathbb{C}^d$ such that

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ \sum_k V_k V_k^\dagger &= \mathbb{1}, \\ V_k^\dagger V_l &= 0 \quad (k \neq l). \end{aligned}$$

Every B_k is irreducible, and for every vertical letter v ,

$$\begin{aligned} \widetilde{M}_v V_k &= V_k B_k^v, \\ V_k^\dagger \widetilde{M}_v &= B_k^v V_k^\dagger, \\ B_k^v &= V_k^\dagger \widetilde{M}_v V_k. \end{aligned}$$

Moreover, $\widetilde{M}$ and the unit-weight direct sum $\bigoplus_k B_k$ have the same matrix product vectors at every positive length.

Proof. ✓ Proof Apply the irreducible-corner decomposition under invariant-projector closure to $\widetilde{M}$. □

Theorem 17.4.96 (Complete vertical reducing projectors). Lean ✓ Stmt *Under the same hypotheses, there are orthogonal projections P_k that are complete and pairwise orthogonal and, for every vertical letter v , reduce the corresponding letter:*

$$\begin{aligned}\sum_k P_k &= \mathbb{1}, \\ P_k P_l &= 0 \quad (k \neq l), \\ \widetilde{M}_v P_k &= P_k \widetilde{M}_v.\end{aligned}$$

Proof. ✓ Proof For the isometries of Theorem 17.4.95, set $P_k = V_k V_k^\dagger$. The isometry, orthogonality, completeness, and intertwining identities give the four assertions directly. □

$$U \mathcal{K} U^\dagger = \bigoplus_\alpha M_\alpha.$$

Figure 17.6: The vertically viewed tensor decomposes through complete orthogonal physical sectors. Each V_k embeds the irreducible tensor B_k , while $P_k = V_k V_k^\dagger$ is its reducing range projection. The displayed identity is equivalent, for every vertical letter v , to $\widetilde{M}_v = \sum_k V_k B_{k,v} V_k^\dagger$. This is the iterated invariant-subspace construction in [CPGSV16, lines 200–219], applied to the reduction in [CPGSV16, Proposition 4.13, lines 1873–1887].

Theorem 17.4.97 (Spectrally normalized vertical sectors with physical isometries). Lean ✓ Stmt *Under the same hypotheses, there are positive integers d_k , positive scalars ν_k , normal tensors B_k of bond dimension d_k , and isometries $V_k : \mathbb{C}^{d_k} \rightarrow \mathbb{C}^d$ with pairwise orthogonal ranges. For every sector k and every vertical letter v ,*

$$\begin{aligned}\widetilde{M}_v V_k &= V_k (\nu_k B_k^v), \\ V_k^\dagger \widetilde{M}_v &= (\nu_k B_k^v) V_k^\dagger, \\ \nu_k B_k^v &= V_k^\dagger \widetilde{M}_v V_k.\end{aligned}$$

Moreover, every vertical letter is reconstructed literally by

$$\widetilde{M}_v = \sum_k V_k (\nu_k B_k^v) V_k^\dagger. \tag{17.68}$$

Proof. ✓ Proof Retain exactly the irreducible corners containing a nonzero letter. The Perron–Frobenius eigenvalue of each retained corner is positive, and the absence of nontrivial periodic vectors makes the inverse-square-root rescaling normal. Put the removed square root into ν_k and reindex the retained corners. The original complete reducing decomposition expresses each vertical letter as the sum of all compressed corners; every omitted summand vanishes, which gives (17.68). Thus the retained range projections need not sum to the identity, but their orthogonal complement is a zero sector. □

Theorem 17.4.98 (Normal vertical corners in literal CPSV canonical form). Lean ✓ Stmt *Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. Then there are positive integers d_k , positive scalars ν_k , normal*

tensors B_k of bond dimension d_k , and isometries $V_k: \mathbb{C}^{d_k} \rightarrow \mathbb{C}^d$ with pairwise orthogonal ranges. For every sector k and every vertical letter v ,

$$\begin{aligned}\widetilde{M}_v V_k &= V_k(\nu_k B_k^v), \\ V_k^\dagger \widetilde{M}_v &= (\nu_k B_k^v) V_k^\dagger, \\ \nu_k B_k^v &= V_k^\dagger \widetilde{M}_v V_k.\end{aligned}$$

Every vertical letter is reconstructed by

$$\widetilde{M}_v = \sum_k V_k(\nu_k B_k^v) V_k^\dagger. \quad (17.69)$$

The omitted orthogonal complement is an all-zero sector. This is the normal-corner intermediate between the periodic exclusion at lines 1889–1893 and the grouped vertical representation used at line 1898 of [CPGSV16, Proposition 4.13].

Proof. ✓ **Proof** Theorem 17.4.93 gives invariant-projector closure for $\widetilde{M}$, and Theorem 17.4.49 excludes its nontrivial periodic vectors. Applying Theorem 8.7.4 then gives normal nonzero corners with positive spectral weights and orthogonal isometries, together with the stated intertwining and compression formulas. Their reconstruction is

$$\widetilde{M}_v = \sum_k V_k(\nu_k B_k^v) V_k^\dagger. \quad (17.70)$$

□

Definition 17.4.99 (Grouped vertical BNT decomposition with physical isometries). Lean

✓ **Stmnt** An MPO tensor M has a *grouped vertical BNT decomposition with physical isometries* if there are positive bond dimensions d_k , positive weights μ_k , normal tensors B_k , and isometries $V_k: \mathbb{C}^{d_k} \rightarrow \mathbb{C}^d$ with pairwise orthogonal ranges such that

$$\begin{aligned}\widetilde{M}_v V_k &= V_k(\mu_k B_k^v), \\ V_k^\dagger \widetilde{M}_v &= (\mu_k B_k^v) V_k^\dagger, \\ \mu_k B_k^v &= V_k^\dagger \widetilde{M}_v V_k, \\ \widetilde{M}_v &= \sum_k V_k(\mu_k B_k^v) V_k^\dagger.\end{aligned}$$

Let $k = (\alpha, q)$ be the phase-class enumeration. There are equalities of the representative and copy bond dimensions, invertible gauges $X_{\alpha, q}$, and nonzero phases $\zeta_{\alpha, q}$ of unit modulus, with $X_{\alpha, 0} = \mathbb{1}$ and $\zeta_{\alpha, 0} = 1$, such that

$$B_{\alpha, q}^v = \zeta_{\alpha, q} X_{\alpha, q} M_\alpha^v X_{\alpha, q}^{-1}.$$

The representatives M_α form a basis of normal tensors for the weighted block tensor and are pairwise inequivalent up to gauge and phase. Every effective coefficient $c_{\alpha, q} = \mu_{\alpha, q} \zeta_{\alpha, q}$ is nonzero and positive.

For $P_{\alpha, q} = V_{\alpha, q} V_{\alpha, q}^\dagger$, the triple consisting of $P_{\alpha, q}$, $c_{\alpha, q}$, and the representative loop of M_α satisfies the sector-projector identity of Definition 17.4.60. Moreover, for every (α, q) there is a finite chain length for which the first-site compression by $P_{\alpha, q}$ is nonzero. Finally, the grouped physical maps remain isometries with pairwise orthogonal ranges, satisfy both grouped intertwining identities and the grouped corner identity, and reconstruct each vertical letter exactly:

$$\widetilde{M}_v = \sum_\alpha \sum_q V_{\alpha, q}(c_{\alpha, q} X_{\alpha, q} M_\alpha^v X_{\alpha, q}^{-1}) V_{\alpha, q}^\dagger. \quad (17.71)$$

Theorem 17.4.100 (Vertical normal sectors grouped into BNT representatives). [Lean](#) [Stmnt](#)
Under the same hypotheses, partition the normalized vertical sectors by equality of their matrix product vector families up to a fixed nonzero scalar per site. There are normal representatives M_α , an enumeration $k = (\alpha, q)$ of the original sectors, invertible matrices $X_{\alpha,q}$, and complex numbers $\zeta_{\alpha,q}$ of unit modulus, with $X_{\alpha,0} = \mathbb{1}$ and $\zeta_{\alpha,0} = 1$, such that

$$B_{\alpha,q}^v = \zeta_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}. \quad (17.72)$$

The representatives are pairwise gauge-phase distinct and form a basis of normal tensors for the weighted block tensor $\bigoplus_k \nu_k B_k$. Every grouped coefficient $\nu_{\alpha,q} \zeta_{\alpha,q}$ is positive. With the physical isometries from Theorem 17.4.97 retained unchanged, one has both reducing identities, exact compression, and the literal reconstruction

$$\widetilde{M}_v = \sum_\alpha \sum_q V_{\alpha,q} (\nu_{\alpha,q} \zeta_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}) V_{\alpha,q}^\dagger. \quad (17.73)$$

For each pair (α, q) , the range projector of $V_{\alpha,q}$ obeys the representative-loop identity and gives a nonzero compression at some finite length. This is the decomposition and positivity argument of [CPGSV16, Proposition 4.13, lines 1898–1902]. The relative Gram identity and proportional-unitary normalization of lines 1903–1908 are not asserted here.

Proof. [Proof](#) Take one representative from each phase class. Theorem 9.2.8 gives equality of bond dimensions and (17.72) for every copy. Distinct representatives are gauge-phase separated by construction, so Theorem 9.2.13 gives eventual linear independence. The phase-class regrouping of the finite sum gives the BNT expansion. The representative copy is assigned the identity gauge and phase one, while the self-overlap normalization of normal tensors shows that every phase has unit modulus. Substitution into the two intertwining identities, the compression identity, and the reconstruction (17.68) proves all remaining algebraic assertions without altering any physical isometry. Cyclicity of trace gives the sector-loop identities. The nonzero-corner separation theorem supplies a nonzero compression for every sector, and the sector trace identity and MPDO positivity then make every grouped coefficient positive. $\square$

Theorem 17.4.101 (Literal vertical normal sectors grouped into BNT representatives). [Lean](#) [Stmnt](#)
Under the same hypotheses, partition the normalized vertical sectors by equality of their matrix product vector families up to a fixed nonzero scalar per site. There are normal representatives M_α , an enumeration $k = (\alpha, q)$ of the original sectors, invertible matrices $X_{\alpha,q}$, and complex numbers $\zeta_{\alpha,q}$ of unit modulus, with $X_{\alpha,0} = \mathbb{1}$ and $\zeta_{\alpha,0} = 1$, such that

$$B_{\alpha,q}^v = \zeta_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}. \quad (17.74)$$

The representatives are pairwise gauge-phase distinct and form a basis of normal tensors for the weighted block tensor $\bigoplus_k \nu_k B_k$. Every grouped coefficient $\nu_{\alpha,q} \zeta_{\alpha,q}$ is positive. With the physical isometries from Theorem 17.4.98 retained unchanged, one has both reducing identities, exact compression, and the literal reconstruction

$$\widetilde{M}_v = \sum_\alpha \sum_q V_{\alpha,q} (\nu_{\alpha,q} \zeta_{\alpha,q} X_{\alpha,q} M_\alpha^v X_{\alpha,q}^{-1}) V_{\alpha,q}^\dagger. \quad (17.75)$$

For each pair (α, q) , the range projector of $V_{\alpha,q}$ obeys the representative-loop identity and gives a nonzero compression at some finite length. This is the decomposition and positivity argument of [CPGSV16, Proposition 4.13, lines 1898–1902]. The relative Gram identity and proportional-unitary normalization of lines 1903–1908 are not asserted here.

Proof. ✓ Proof Take one representative from each phase class. Theorem 9.2.8 gives equality of bond dimensions and (17.74) for every copy. Distinct representatives are gauge-phase separated by construction, so Theorem 9.2.13 gives eventual linear independence. The phase-class regrouping of the finite sum gives the BNT expansion. The representative copy is assigned the identity gauge and phase one, while the self-overlap normalization of normal tensors shows that every phase has unit modulus. Substitution into the two intertwining identities, the compression identity, and the reconstruction (17.69) proves all remaining algebraic assertions without altering any physical isometry. Cyclicity of trace gives the sector-loop identities. The nonzero-corner separation theorem supplies a nonzero compression for every sector, and the sector trace identity and MPDO positivity then make every grouped coefficient positive. $\square$

Definition 17.4.102 (Block-row map of sector isometries). Lean ✓ Stmt Let $W_k : \mathbb{C}^{D_k} \rightarrow \mathbb{C}^d$ be a finite family of rectangular matrices. Their block-row map $U : \mathbb{C}^d \rightarrow \bigoplus_k \mathbb{C}^{D_k}$ is defined by $Ux = (W_k^\dagger x)_k$. This realizes the coisometry in the conclusion of [CPGSV16, Proposition 4.13, lines 1863–1870] once the sector isometries have been normalized.

Theorem 17.4.103 (Conjugation and reconstruction from orthogonal sectors). Lean Lean ✓ Stmt Suppose that

$$\begin{aligned} W_k^\dagger W_k &= \mathbb{1}, \\ W_k^\dagger W_l &= 0 \quad (k \neq l), \end{aligned}$$

and, for every letter v and every sector k , $T_v W_k = W_k B_k^v$. Then the block-row map U is a coisometry and conjugates T to the direct sum of the sector tensors:

$$\begin{aligned} UU^\dagger &= \mathbb{1}, \\ UT_v U^\dagger &= \bigoplus_k B_k^v. \end{aligned}$$

If the letters also satisfy the literal sector decomposition

$$T_v = \sum_k W_k B_k^v W_k^\dagger, \tag{17.76}$$

then

$$T_v = U^\dagger \left(\bigoplus_k B_k^v \right) U.$$

This last equality is a conditional consequence of (17.76); no additional assertion from the source is used.

Proof. ✓ Proof The (k, l) block of $UU^\dagger$ is $W_k^\dagger W_l$, hence the first identity follows from orthonormality. The corresponding block of $UT_v U^\dagger$ is $W_k^\dagger T_v W_l = W_k^\dagger W_l B_l^v$, which is B_k^v for $k = l$ and zero otherwise. Finally, multiplication of the block diagonal matrix by the block row gives the sum $\sum_k W_k B_k^v W_k^\dagger$, which is T_v by hypothesis. The row indices of U form the dependent disjoint union $\coprod_k \{0, \dots, D_k - 1\}$. Its canonical order-preserving bijection with $\{0, \dots, \sum_k D_k - 1\}$ gives the coordinate reindexing required by the vertical canonical-form definition. $\square$

Theorem 17.4.104 (Vertical canonical form from orthogonal grouped sectors). [Lean](#) [✓ Stmt](#)
 Let $\{A_\alpha\}_{\alpha=0}^{g-1}$ be a BNT for the vertically viewed tensor $\widetilde{M}$. For each α , let $r_\alpha > 0$, let $\omega_{\alpha,q} > 0$ for $0 \leq q < r_\alpha$, and let $W_{\alpha,q} : \mathbb{C}^{D_\alpha} \rightarrow \mathbb{C}^d$ satisfy

$$\begin{aligned} W_{\alpha,q}^\dagger W_{\alpha,q} &= \mathbb{1}, \\ W_{\alpha,q}^\dagger W_{\beta,p} &= 0 \quad ((\alpha,q) \neq (\beta,p)). \end{aligned}$$

Suppose that, for every vertical letter v and every pair (α,q) , $\widetilde{M}_v W_{\alpha,q} = W_{\alpha,q}(\omega_{\alpha,q} A_\alpha^v)$, and that the letters have the exact sector decomposition

$$\widetilde{M}_v = \sum_{\alpha=0}^{g-1} \sum_{q=0}^{r_\alpha-1} W_{\alpha,q}(\omega_{\alpha,q} A_\alpha^v) W_{\alpha,q}^\dagger.$$

Then M is in vertical canonical form. This is the final finite-dimensional implication in [CPGSV16, Proposition 4.13, lines 1863–1870].

Proof. [✓ Proof](#) Form the block-row map U from the adjoints $W_{\alpha,q}^\dagger$. Theorem 17.4.103 gives

$$\begin{aligned} UU^\dagger &= \mathbb{1}, \\ U \widetilde{M}_v U^\dagger &= \bigoplus_{\alpha,q} \omega_{\alpha,q} A_\alpha^v, \\ \widetilde{M}_v &= U^\dagger \left(\bigoplus_{\alpha,q} \omega_{\alpha,q} A_\alpha^v \right) U. \end{aligned}$$

Identifying the dependent pairs (α,q) and their bond indices with the standard coordinates of the two finite direct sums gives the required matrix dimensions. Positivity of the multiplicities and weights, together with the BNT hypothesis, supplies the remaining conditions of vertical canonical form. $\square$

Lemma 17.4.105 (A retained vertical phase class exists). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 17.4.100, the retained vertical decomposition has at least one sector and hence at least one phase class.

Proof. [✓ Proof](#) The unit-modulus weight witness in the normalized BNT-refined horizontal form, together with left canonicity of its representative, makes the doubled-index tensor nonzero. Hence its vertical reshaping $\widetilde{M}$ is nonzero. If the retained decomposition had no sector, (17.73) would make every letter of $\widetilde{M}$ zero, a contradiction. Passing from the nonempty finite set of retained sectors to its phase partition gives a retained representative. $\square$

Theorem 17.4.106 (Nonvanishing of the vertical reshaping). [Lean](#) [✓ Stmt](#) If an MPO tensor M is in normalized BNT-refined horizontal form, then its vertical reshaping is nonzero: $\widetilde{M} \neq 0$. This is the nonvanishing observation used in [CPGSV16, Proposition 4.13, lines 1895–1902].

Proof. [✓ Proof](#) Write the doubled-index tensor in normalized BNT-refined horizontal form as $\widehat{M} = GSG^{-1}$, where S is the weighted direct sum of its basis tensors. Choose a copy whose weight has unit modulus. Its basis tensor cannot have every letter equal to zero, since left canonicity gives $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. The corresponding nonzero weight and nonzero letter give $S \neq 0$, hence $\widehat{M} \neq 0$. If $\widetilde{M} = 0$, the bijection between its vertical index and the two horizontal bond indices would give $\widehat{M} = 0$, a contradiction. $\square$

Theorem 17.4.107 (Algebraic BNT from a grouped vertical decomposition). [Lean](#) [Stmt](#) Let M be an MPO tensor and put $T = \widetilde{M}$. Let B_k have bond dimension d_k , let $\mu_k \in \mathbb{C}$, and let $V_k : \mathbb{C}^{d_k} \rightarrow \mathbb{C}^d$ be an isometry for $0 \leq k < r$. Assume, for every vertical letter v , that

$$\begin{aligned} V_k^\dagger T_v &= (\mu_k B_k^v) V_k^\dagger, \\ T_v &= \sum_{k=0}^{r-1} V_k (\mu_k B_k^v) V_k^\dagger. \end{aligned}$$

Partition the tensors B_k into matrix-product-vector phase classes and choose one representative A_α from each class. If $\{A_\alpha\}_\alpha$ is a basis of normal tensors in the spectral sense for the weighted direct sum $A_{\text{ret}} = \bigoplus_{k=0}^{r-1} \mu_k B_k$, then $\{A_\alpha\}_\alpha$ is an algebraic basis of normal tensors for T .

This is the basis-of-normal-tensors conclusion in [CPGSV16, Proposition 4.13, lines 1863–1921], stated for the grouped decomposition before the final coisometry is formed.

Proof. [Proof](#) Put $C_k^v = \mu_k B_k^v$. Induction on a word w gives $V_k^\dagger T^w = C_k^w V_k^\dagger$. For a nonempty word $w = (v, w')$, reconstruction of its first letter, cyclicity of trace, and $V_k^\dagger V_k = \mathbb{1}$ therefore give

$$\begin{aligned} \text{tr}(T^w) &= \sum_k \text{tr}(V_k C_k^v V_k^\dagger T^{w'}) \\ &= \sum_k \text{tr}(C_k^v C_k^{w'}) = \sum_k \mu_k^{|w|} \text{tr}(B_k^w) = V^{(|w|)}(A_{\text{ret}})_w. \end{aligned}$$

Thus T and A_{ret} have the same matrix product vectors at every positive length. The spectral BNT hypothesis gives an algebraic BNT for A_{ret} by Theorem 9.2.5; in particular, it supplies the required positive bond dimensions. Lemma E.1.5 then gives the same algebraic BNT for T . $\square$

Theorem 17.4.108 (Pairwise Figure 8 identity under literal canonical form). [Lean](#) [Stmt](#) Let M be an MPO tensor whose doubled-index MPS tensor is in the literal CPSV canonical form, and suppose that M generates positive semidefinite operators on every nonempty chain. Let A be a tensor, let X and Y be invertible matrices, and suppose that two vertical corners satisfy

$$\begin{aligned} V_X^\dagger \widetilde{M}^v V_X &= c_X X A^v X^{-1}, & c_X &> 0, \\ V_Y^\dagger \widetilde{M}^v V_Y &= c_Y Y A^v Y^{-1}, & c_Y &> 0. \end{aligned}$$

Then the Gram-dressed tensors agree:

$$(X^\dagger X) A^v (X^\dagger X)^{-1} = (Y^\dagger Y) A^v (Y^\dagger Y)^{-1} \tag{17.77}$$

for every vertical letter v .

Proof. [Proof](#) For $Z = X, Y$, set

$$f_{(r,s),(i,j)}^Z = c_Z^{-1} \overline{(V_Z Z)_{ir}} (V_Z (Z^{-1})^\dagger)_{js}.$$

The corner identity identifies the mark with coefficients f^Z with the horizontal slice of $(Z^\dagger Z) A^v (Z^\dagger Z)^{-1}$. Self-adjointness of the two corner compressions identifies both closed marked chains with the same reflected chain. The original-coordinate marked Lemma L for literal canonical form therefore gives equality of the two marks, hence (17.77). $\square$

Theorem 17.4.109 (Positive relative Gram scalar under literal canonical form). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor whose doubled-index MPS tensor is in the literal CPSV canonical form, and suppose that M generates positive semidefinite operators on every nonempty chain. Let A be normal, let X and Y be invertible matrices, and suppose that

$$\begin{aligned} V_X^\dagger \widetilde{M}^v V_X &= c_X X A^v X^{-1}, & c_X &> 0, \\ V_Y^\dagger \widetilde{M}^v V_Y &= c_Y Y A^v Y^{-1}, & c_Y &> 0. \end{aligned}$$

Then there is a real number $\omega > 0$ such that

$$X^\dagger X = \omega Y^\dagger Y. \quad (17.78)$$

Proof. [✓ Proof](#) Equation (17.77) makes the ratio of the two Gram matrices commute with every letter of A . Normality makes this ratio scalar, and positive definiteness makes the scalar a positive real number. $\square$

Theorem 17.4.110 (Vertical canonical form from grouped sectors and Gram dressing). [Lean](#) [✓ Stmt](#) If M has a grouped vertical BNT decomposition with physical isometries in the sense of Definition 17.4.99, and M has the grouped-corner Gram-dressing property of Definition 17.4.71, then M is in vertical canonical form.

Proof. [✓ Proof](#) Theorem 17.4.78 gives $X_{\alpha,q}^\dagger X_{\alpha,q} = \omega_{\alpha,q} \mathbb{1}$ with $\omega_{\alpha,q} > 0$. Absorb $\omega_{\alpha,q}^{-1/2} X_{\alpha,q}$ into the physical corner maps. The normalized maps are isometries with orthogonal ranges, intertwine $\widetilde{M}$ with the undressed representatives, and preserve the exact reconstruction. The grouped representatives form a BNT, so Theorem 17.4.104 gives the vertical canonical form. $\square$

Theorem 17.4.111 (Vertical canonical form of a canonical MPDO). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor whose doubled-index MPS tensor is in the literal CPSV canonical form. If M generates matrix product density operators, then M is in canonical form in the vertical direction. More precisely, there is a basis of normal tensors $\{M_\alpha\}_\alpha$ for the vertically viewed tensor $\widetilde{M}$, together with positive diagonal matrices μ_α and a coisometry U from the physical space onto the retained nonzero sectors. Writing $B = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha$, one has

$$UU^\dagger = \mathbb{1}. \quad (17.79)$$

$$U \widetilde{M} U^\dagger = B. \quad (17.80)$$

$$\widetilde{M} = U^\dagger B U. \quad (17.81)$$

The source explicitly displays (17.80) and calls U an isometry [CPGSV16, Proposition 4.13]. In this rectangular orientation, however, U maps the physical space onto the retained sector space and satisfies (17.79); thus U is a coisometry and $U^\dagger$ is the isometric inclusion. The reverse identity (17.81) is not displayed separately in the proposition. It records the exact reconstruction implicit in the preceding assertion that $\widetilde{M}$ is itself in canonical form, including the permitted zero orthogonal complement.

Proof. [✓ Proof](#) Theorem 17.4.101 supplies the grouped vertical BNT decomposition with physical isometries. The literal pairwise Figure 8 theorem supplies the grouped-corner Gram-dressing property. Apply Theorem 17.4.110. $\square$

Theorem 17.4.112 (BNT-refined horizontal form implies vertical canonical form). [Lean](#) [Lean](#)
 $\checkmark$ *Stmt* A tensor in normalized BNT-refined horizontal form that generates matrix product density operators is also in canonical form in the vertical direction: there exist a basis of normal tensors $\{M_\alpha\}_\alpha$ for the vertically viewed tensor $\widetilde{M}$, positive diagonal matrices μ_α , and a coisometry U from the physical space onto the retained nonzero sectors. Writing $B = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha$, one has

$$\begin{aligned} UU^\dagger &= \mathbb{1}, \\ U\widetilde{M}U^\dagger &= B, \\ \widetilde{M} &= U^\dagger BU. \end{aligned}$$

The conclusion includes the basis-of-normal-tensors condition in [CPGSV16, Proposition 4.13], together with the positive weights and the coisometric realization. Its hypothesis is the normalized BNT-refined form of Definition 17.4.1, rather than the literal CPSV canonical form assumed in Theorem 17.4.111.

Proof. $\checkmark$ *Proof* Theorem 17.4.100 supplies the grouped vertical decomposition. The pairwise Figure 8 theorem supplies its grouped-corner Gram-dressing property. Apply Theorem 17.4.110. $\square$

Chapter 18

Pure States: Renormalization of Matrix Product States

This chapter develops renormalization fixed points for matrix product vectors. It compares physical two-site blocking, idempotence of the transfer map, and zero correlation length, and then gives the square-root fixed-point form of the fixed points. The organization follows the pure-state analysis of [CPGSV16, Section 3].

18.1 Physical blocking and transfer idempotence

Definition 18.1.1 (Physical blocking isometry). [Lean](#) [✓ Stmt](#) An MPS tensor A is a *renormalization fixed point* if there is an isometry $V: \mathbb{C}^d \rightarrow \mathbb{C}^{d^2}$, with coefficients $V_{(i_1, i_2), j}$ and $V^\dagger V = \mathbb{1}$, such that, for all physical indices i_1, i_2 ,

$$A^{i_1} A^{i_2} = \sum_{j=0}^{d-1} V_{(i_1, i_2), j} A^j. \quad (18.1)$$

This is the relation $AA = A$ in [CPGSV16, Definition 3.2].

Definition 18.1.2 (Idempotent transfer-map criterion). [Lean](#) [✓ Stmt](#) The transfer map of A satisfies the idempotence criterion when $\mathcal{E}_A \circ \mathcal{E}_A = \mathcal{E}_A$.

Theorem 18.1.3 (A physical blocking isometry gives transfer idempotence). [Lean](#) [✓ Stmt](#) Suppose there exists an isometry $V: \mathbb{C}^d \rightarrow \mathbb{C}^{d^2}$, written as coefficients $V_{(i_1, i_2), j}$, such that

$$A^{i_1} A^{i_2} = \sum_{j=0}^{d-1} V_{(i_1, i_2), j} A^j \quad (18.2)$$

for all $i_1, i_2 \in \{0, \dots, d-1\}$. Then $\mathcal{E}_A^2 = \mathcal{E}_A$.

Proof. [✓ Proof](#) Substituting the decomposition (18.2) into the double Kraus sum gives, for every

bond matrix X ,

$$\begin{aligned}\mathcal{E}_A^2(X) &= \sum_{i_1, i_2} (A^{i_1} A^{i_2}) X (A^{i_1} A^{i_2})^\dagger \\ &= \sum_{j, k} \left(\sum_{i_1, i_2} V_{(i_1, i_2), j} \overline{V_{(i_1, i_2), k}} \right) A^j X (A^k)^\dagger \\ &= \sum_{j, k} (V^\dagger V)_{j, k} A^j X (A^k)^\dagger = \sum_{j, k} \delta_{j, k} A^j X (A^k)^\dagger = \sum_j A^j X (A^j)^\dagger = \mathcal{E}_A(X).\end{aligned}$$

Hence $\mathcal{E}_A^2 = \mathcal{E}_A$. $\square$

Theorem 18.1.4 (Kraus-isometry criterion for transfer idempotence). [Lean](#) [✓ Stmt](#) *The transfer map of A is idempotent if and only if there exists an isometry $V: \mathbb{C}^d \rightarrow \mathbb{C}^{d^2}$, written as coefficients $V_{(i_1, i_2), j}$ with $V^\dagger V = \mathbb{1}$, such that*

$$A^{i_1} A^{i_2} = \sum_{j=0}^{d-1} V_{(i_1, i_2), j} A^j \quad (18.3)$$

for all $i_1, i_2 \in \{0, \dots, d-1\}$.

Proof. [✓ Proof](#) The backward implication is Theorem 18.1.3. For the forward implication, expanding $\mathcal{E}_A^2 = \mathcal{E}_A$ as a Kraus sum gives, for every X ,

$$\sum_{i_1, i_2} (A^{i_1} A^{i_2}) X (A^{i_1} A^{i_2})^\dagger = \sum_j A^j X (A^j)^\dagger.$$

the corresponding theorem in the companion quantum-channel volume [LTC26, “Rectangular Kraus freedom with general finite index types”] then supplies the isometry V relating the two Kraus families. $\square$

Theorem 18.1.5 (Physical and transfer-map fixed-point criteria). [Lean](#) [✓ Stmt](#) *The transfer map of an MPS tensor is idempotent if and only if the tensor has a physical blocking isometry.*

Proof. [✓ Proof](#) This is Theorem 18.1.4, with the physical blocking relation written as Definition 18.1.1. $\square$

Definition 18.1.6 (Limit of the renormalization flow). [Lean](#) [✓ Stmt](#) Let $A = (A^i)_{i=0}^{d-1}$ have bond dimension D . Its physical-isometry class *appears as a limit of the renormalization flow* if there are a physical dimension e and a tensor $B = (B^a)_{a=0}^{e-1}$ of the same bond dimension D such that

$$\hat{\mathcal{E}}_B^{2^n} \longrightarrow \hat{\mathcal{E}}_A.$$

Here $\hat{\mathcal{E}}_A$ and $\hat{\mathcal{E}}_B$ are the transfer matrices, and convergence is taken in the standard topology on the finite-dimensional matrix space $M_{D^2}(\mathbb{C})$. The physical dimension is quantified because blocking changes it, whereas the bond dimension remains fixed. Transfer matrices represent the quotient by physical isometries described in [CPGSV16, lines 382–394 and 1205–1209].

Theorem 18.1.7 (A dyadic power limit is idempotent). [Lean](#) [✓ Stmt](#) *Let R be a Hausdorff topological monoid with continuous multiplication. If $E^{2^n} \rightarrow P$, then $P^2 = P$.*

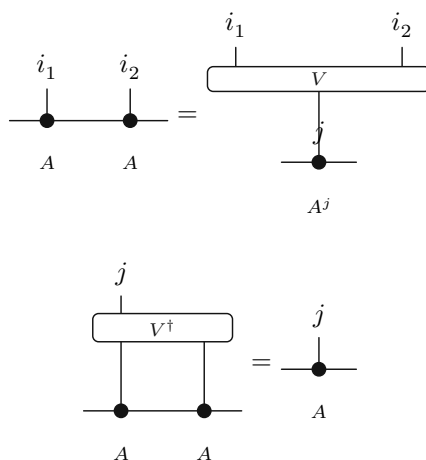
Proof. [✓ Proof](#) Continuity of multiplication gives $E^{2^n} E^{2^n} \rightarrow P^2$. On the other hand, $E^{2^n} E^{2^n} = E^{2^{n+1}} \rightarrow P$. Uniqueness of limits gives $P^2 = P$. $\square$

Theorem 18.1.8 (Limits of the renormalization flow). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) *A tensor class appears as a limit of the two-site renormalization flow if and only if it has a physical blocking isometry. Equivalently, there is an isometry $V: \mathbb{C}^d \rightarrow \mathbb{C}^{d^2}$ such that*

$$A^{i_1} A^{i_2} = \sum_{j=0}^{d-1} V_{(i_1, i_2), j} A^j$$

for every i_1, i_2 . This is [CPGSV16, Theorem 3.1 and lines 1205–1209].

Proof. [Stmnt](#) [Proof](#) If $\hat{\mathcal{E}}_B^{2^n} \rightarrow \hat{\mathcal{E}}_A$, then Theorem 18.1.7 gives $\hat{\mathcal{E}}_A^2 = \hat{\mathcal{E}}_A$. Since the transfer-matrix representation is injective, $\mathcal{E}_A^2 = \mathcal{E}_A$. Theorem 18.1.5 supplies the physical isometry. Conversely, if A has such an isometry, its transfer map is idempotent. Taking $B = A$ makes the dyadic orbit constant: $\hat{\mathcal{E}}_A^{2^n} = \hat{\mathcal{E}}_A$ for every n . $\square$



$$\sum_{i_1, i_2} (V^\dagger)_{i_1, i_2}^j A^{i_1} A^{i_2} = A^j. \quad (18.4)$$

Figure 18.1: The fusion identity of the renormalization fixed-point characterization: two contracted copies of A equal a single copy of A with the isometry V merging the summed index j into the physical pair (i_1, i_2) , as stated in (18.3); conversely, applying $V^\dagger$ to the two physical legs of the contracted pair recovers the single tensor, as stated in (18.4). The source draws the isometry as U [CPGSV16, Theorem 3.1, lines 396–412].

18.2 Zero-correlation-length conditions

Definition 18.2.1 (Virtual one-block idempotence convention). [Lean](#) [Stmnt](#) This auxiliary one-block convention means $\mathcal{E}_A^2 = \mathcal{E}_A$. It is not local orthogonality in the sense of the source, which is the family of mixed-sector equations $\mathcal{E}_{j, j'} = 0$ for distinct BNT components.

Definition 18.2.2 (BNT local orthogonality). Lean ✓ Stmt Let $(A_j)_j$ be a basis of normal tensors. The family is *locally orthogonal* when, for every pair of distinct components,

$$\mathcal{E}_{j,j'} = \sum_i A_j^i \otimes \overline{A_{j'}^i} = 0.$$

This is the mixed-sector condition of [CPGSV16, Definition 3.5].

Definition 18.2.3 (Virtual-insertion auxiliary BNT condition). Lean ✓ Stmt Let $(A_j)_j$ be a basis of normal tensors for A . This auxiliary condition requires virtual-insertion distance independence and $\mathcal{E}_{j,j'} = 0$ for every pair of distinct BNT components. It is not [CPGSV16, Definition 3.6], which quantifies over physical observables on all disjoint regions.

Definition 18.2.4 (Positive-gap restriction of BNT zero correlation length). Lean ✓ Stmt A family $(A_j)_j$ of MPS tensors has positive-gap BNT zero correlation length for A when $(A_j)_j$ is a basis of normal tensors for A in the sense of Definition 9.1.2, its components are locally orthogonal, and the physical correlations of A are independent of separation whenever both complementary gaps are positive. Adjacent regions are not included.

Lemma 18.2.5 (Positive-gap restriction, unfolded). Lean ✓ Stmt *The positive-gap restriction is equivalent to the conjunction of the BNT relation, positive-gap physical distance independence, and BNT local orthogonality.*

Proof. ✓ Proof This is the defining conjunction. □

Definition 18.2.6 (Physical BNT zero correlation length). Lean ✓ Stmt A BNT family has zero correlation length when the associated matrix-product vectors have correlations independent of distance for all disjoint physical regions and the BNT components are locally orthogonal. This is [CPGSV16, Definition 3.6].

Theorem 18.2.7 (Two source gaps in the zero-correlation-length characterization). Not ready CPSV asserts that a tensor A in canonical form has physical BNT zero correlation length if and only if

$$\mathcal{E}_A^2 = \mathcal{E}_A,$$

and, more generally, that for such A the following are equivalent: A is a renormalization fixed point; A has physical zero correlation length; and, for every $N > 2$, the vector $|V^{(N)}(A)\rangle$ belongs to the ground space of a nearest-neighbor commuting parent Hamiltonian [CPGSV16, Theorems 3.8 and 3.10]. Both directions of the biconditional fail as printed, and with them the three-way equivalence.

For the reverse implication, let $C^0 = (1)$ and form the direct sum of two scalar copies with weights 1 and $1/2$. The resulting tensor is

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The scalar tensor C is normal, and the weights obey the source convention $|\mu_k| \leq 1$, with at least one weight of modulus one. The singleton BNT family (C) is locally orthogonal and gives A a physical BNT zero correlation length, whereas

$$\mathcal{E}_A^2 \neq \mathcal{E}_A.$$

Thus zero correlation length does not imply transfer idempotence under the source's raw weight normalization.

Independently, the Bell-pair chain of Theorem 18.2.9 is a single normal block. Taking its weight to be 1 and the ambient coisometry to be the identity gives a literal CPSV canonical-form tensor. Its transfer map is idempotent, so it is a renormalization fixed point, but its adjacent two-region expectation is 1, while the expectation after an allowed one-site shift is 0. Hence transfer idempotence does not imply physical zero correlation length for the unrestricted quantification over disjoint regions, regardless of the commuting-parent condition.

Source gap (adjacent regions). The printed forward argument uses $\mathcal{E}_A^n = \mathcal{E}_A$, which follows from idempotence only for $n \geq 1$. Adjacent regions insert $\mathcal{E}_A^0 = \mathbb{1}$ and are included in the source definition. This gap and its scope are recorded in [con26e]. Theorem 18.5.18 proves the repaired forward implication when both complementary gaps are positive. It does not prove the unrestricted biconditional of [CPGSV16, Theorem 3.8] or the unrestricted three-way equivalence of [CPGSV16, Theorem 3.10]. The converse and equivalence results below require their stated unit-weight, multiplicity-one, or spectral hypotheses. The printed statements are recorded here as refuted, not as theorems awaiting proof.

Proof. Theorem 18.2.8 refutes the reverse implication. Theorem 18.2.9 refutes the forward implication, and with it the implication from fixed points to zero correlation length. $\square$

Theorem 18.2.8 (Auxiliary weighted-sector example). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Let P be the auxiliary BNT sector decomposition with one scalar basis tensor $C^0 = (1)$, two one-dimensional copies, and weights 1 and $1/2$. Let $A = \mathcal{A}(P)$, so

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Then P satisfies the auxiliary BNT sector hypotheses, $A = \mathcal{A}(P)$, and A and $\mathcal{A}(P)$ generate the same matrix-product vectors. Moreover, the singleton family (C) is a physical BNT zero-correlation-length family for A , while

$$\mathcal{E}_A^2 \neq \mathcal{E}_A.$$

Its length- N matrix-product-vector coefficient is $V^{(N)}(A) = 1 + 2^{-N}$.

Proof. [✓ Proof](#) The unique basis block has bond dimension one. It is irreducible and left-canonical, its self-overlap is identically 1, and its singleton MPV family is linearly independent. Pairwise distinctness is vacuous. The two weights have moduli at most 1, and the first has modulus 1; hence P satisfies the auxiliary BNT sector hypotheses.

The two one-dimensional weighted copies assemble exactly to $A^0 = \text{diag}(1, 1/2)$. This equality gives equality of the generated matrix-product vectors at every length, and therefore at every positive length. The sector power sum is

$$1^N + \left(\frac{1}{2}\right)^N = 1 + 2^{-N}.$$

The scalar block is a normal tensor, the displayed coefficient spans the MPV of A , and the singleton family is eventually linearly independent. Since $d = 1$, every physical observable is scalar and each two-point expectation depends only on the total gap length. Thus physical correlations are independent of distance, while local orthogonality is vacuous for the singleton BNT family.

Finally, if $\mathcal{E}_A$ were idempotent, the one-letter blocking equation would give $(A^0)^2 = vA^0$ for some scalar v . Its two diagonal entries yield $1 = v$ and $1/4 = v/2$, respectively, which is impossible. $\square$

Theorem 18.2.12 (Basis direct sum is in literal CPSV canonical form). [Lean](#) [✓ Stmt](#) Let P be an auxiliary BNT sector decomposition with distinct basis tensors $(A_j)_j$. The multiplicity-one unit-weight tensor $A = \bigoplus_j A_j$ is in literal CPSV canonical form, with retained weights all equal to 1, retained blocks $(A_j)_j$, and the identity ambient coisometry. This statement does not reconstruct the raw weighted repeated-copy tensor represented by P .

Proof. [✓ Proof](#) Every retained dimension is positive. Each A_j is irreducible and left-canonical, hence normal by Corollary 9.6.1.18. The unit-weight retained direct sum equals A , so the identity coisometry gives the required exact reconstruction. $\square$

Theorem 18.2.13 (Auxiliary BNT basis gives a basis for its direct sum). [Lean](#) [Lean](#) [✓ Stmt](#) Let P be an auxiliary BNT sector decomposition with distinct basis tensors $(A_j)_j$, and let $A = \bigoplus_j A_j$ be their direct sum with one unit-weight copy of each sector. Then $(A_j)_j$ is a basis of normal tensors for A : every A_j is a normal tensor, at every positive system length N

$$V^{(N)}(A) = \sum_j V^{(N)}(A_j),$$

and the states $\{|V^{(N)}(A_j)\rangle_j$ are linearly independent for all sufficiently large N .

Proof. [✓ Proof](#) Every basis block is irreducible and left-canonical with normalized self-overlap converging to 1, hence a normal tensor by Corollary 9.6.1.18. The direct sum is the unit-weight block-diagonal tensor, so its length- N matrix-product vector splits as the sum of the block vectors by Theorem 2.5.9. Eventual linear independence is part of the auxiliary BNT sector hypotheses. $\square$

Theorem 18.2.14 (Multiplicity-one fixed point implies positive-gap ZCL). [Lean](#) [✓ Stmt](#) Let P be an auxiliary BNT sector decomposition with distinct basis tensors $(A_j)_j$, and let $A = \bigoplus_j A_j$ be their direct sum with one unit-weight copy of each sector. If A is a renormalization fixed point,

$$\mathcal{E}_A^2 = \mathcal{E}_A,$$

then the following restricted conclusion holds: its physical correlations are independent of the separation whenever both complementary gaps are positive, and for all distinct BNT components $j \neq j'$

$$\mathcal{E}_{j,j'} = 0.$$

This is a multiplicity-one, unit-weight, positive-gap statement. It does not assert the physical zero-correlation-length conclusion of [CPGSV16, Theorem 3.8].

Proof. [✓ Proof](#) By Theorem 18.2.13, the distinct basis tensors of P are a basis of normal tensors for A . The basis blocks are irreducible, left-canonical, and pairwise gauge-phase distinct, so Theorem 18.2.11 applies to the direct sum. $\square$

Theorem 18.2.15 (Simultaneous inverse for BNT word evaluations). [Lean](#) [✓ Stmt](#) Suppose the simultaneous length- L word evaluations of $(A_j)_j$ span $\bigoplus_j M_{D_j}(\mathbb{C})$. There are coefficients $C_{(j,a,b),w}$ such that, for every sector k ,

$$\sum_{|w|=L} C_{(j,a,b),w} (A_k^w)_{x,y} = \delta_{j,k} \delta_{a,x} \delta_{b,y}.$$

Proof. [✓ Proof](#) The tuple whose j -th component is the matrix unit $|a\rangle\langle b|$ and whose other components vanish belongs to the full simultaneous word span. Reading its (k, x, y) entry gives the displayed identity. $\square$

Theorem 18.2.16 (Sector-supported physical observable realization). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $A = \bigoplus_k A_k$, and suppose the simultaneous length- L word evaluations span $\bigoplus_k M_{D_k}(\mathbb{C})$. For every sector j and matrices $R, l \in M_{D_j}(\mathbb{C})$, there is an observable O on L sites whose inserted transfer map vanishes on every sector pair except (j, j) and satisfies

$$\mathcal{E}_O(X)_{j,j} = R \operatorname{tr}(lX_{j,j}). \quad (18.5)$$

In particular, arbitrary virtual matrix-unit maps supported on one sector are realized by physical observables.

Proof. [✓ Proof](#) Apply Theorem 18.2.15 independently to the left and right word evaluations. The observable with coefficients

$$O_{\tau,\sigma} = C_{(j,a,b),\sigma} \overline{C_{(j,c,e),\tau}}$$

sends the (b, e) entry of $X_{j,j}$ to the (a, c) entry and annihilates every other entry and sector pair. Summing these matrix-unit observables with coefficients $R_{a,c} l_{e,b}$ gives (18.5). $\square$

Theorem 18.2.17 (Multiplicity-one BNT rank-one observables at the sharp blocking length). [Lean](#) [✓ Stmt](#) Let P be an auxiliary BNT sector decomposition of total bond dimension D , and form the direct sum $A = \bigoplus_j A_j$ with one unit-weight copy of each BNT basis tensor. There is a positive length $L \leq 3D^5$ such that, for every basis sector j and every $R, l \in M_{D_j}(\mathbb{C})$, a physical observable on L sites realizes the sector-supported insertion (18.5). This is the multiplicity-one specialization of the block-injective physical-observable assertion used in [CPGSV16, proof of Theorem 3.8], equations (1252) and (1256) in the local source. It does not realize the weighted copy-pair insertions for the full weighted tensor represented by P .

Proof. [✓ Proof](#) Theorem 9.6.1.22 gives a positive $L \leq 3D^5$ for which the simultaneous word evaluations span the product matrix algebra. Apply Theorem 18.2.16 at this length. $\square$

Lemma 18.2.18 (Sector compression calculus for the direct sum). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $A = \bigoplus_k A_k$. Denote compression to the (j, j) bond block by C_j and its trace-pairing adjoint by I_j . Then

$$C_j I_j = \operatorname{id}.$$

If $\mathcal{E}_A$ and $\mathcal{E}_j$ are the transfer maps of A and A_j , respectively, then, for every $n \geq 0$,

$$C_j \mathcal{E}_A^n = \mathcal{E}_j^n C_j.$$

The sector-supported rank-one map $\mathcal{R}_{j;R,l}(X) = I_j(R) \operatorname{tr}(lC_j(X))$ has operator trace

$$\operatorname{Tr}_{\operatorname{End}}(\mathcal{R}_{j;R,l}) = \operatorname{tr}(lR).$$

Finally, if $\mathcal{E}_j^*(l) = \lambda l$, then

$$\operatorname{tr}(l \mathcal{E}_j^n(X)) = \lambda^n \operatorname{tr}(lX).$$

Proof. ✓ Proof The inclusion is the trace-pairing adjoint of compression; expanding it in matrix units gives $C_j I_j = \text{id}$ and the rank-one trace formula. The block-diagonal form of A gives $C_j \mathcal{E}_A = \mathcal{E}_j C_j$, and induction gives the identity for every power. Moving one factor of $\mathcal{E}_j$ across the trace pairing and iterating proves the eigenvector formula. $\square$

Theorem 18.2.19 (Subleading spectral pair excludes positive-gap CID). Lean ✓ Stmt *Let P be an auxiliary BNT sector decomposition, let $A = \bigoplus_k A_k$, and fix a sector j . Suppose there are matrices $r, l \in M_{D_j}(\mathbb{C})$ and a scalar $\lambda \neq 0$ with $|\lambda| < 1$ such that*

$$(\mathcal{E}_j(r), \mathcal{E}_j^*(l), \text{tr}(lr)) = (\lambda r, \lambda l, 1).$$

Then A does not have positive-gap physical CID.

Proof. ✓ Proof Let ρ_j be the trace-one positive fixed point of $\mathcal{E}_j$. At the common blocking length, choose physical observables with inserted transfer maps

$$(\mathcal{E}_{O_1}(X), \mathcal{E}_{O_2}(X)) = (\rho_j \text{tr}(lX_{j,j}), r \text{tr}(X_{j,j})).$$

The left eigenvector equation, together with $\mathcal{E}_j(\rho_j) = \rho_j$, $\text{tr}(\rho_j) = 1$, and $\text{tr}(lr) = 1$, gives

$$\langle O_1 O_2 \rangle_{n_1, n_2} = \lambda^{n_1}.$$

Positive-gap CID compares $(n_1, n_2) = (1, 2)$ with $(2, 1)$, so $\lambda = \lambda^2$. Since $\lambda \neq 0$, this forces $\lambda = 1$, contrary to $|\lambda| < 1$. $\square$

Theorem 18.2.20 (Conditional positive-gap equivalence at the multiplicity-one representative). Lean Lean ✓ Stmt *Let P be an auxiliary BNT sector decomposition and $A = \bigoplus_j A_j$. Assume that every non-idempotent block $\mathcal{E}_j^2 \neq \mathcal{E}_j$ has matrices r_j, l_j and a scalar λ_j satisfying the five spectral conditions in Theorem 18.2.19. Then*

$$A \text{ has positive-gap BNT ZCL} \iff \mathcal{E}_A^2 = \mathcal{E}_A.$$

The spectral-pair hypothesis is assumed explicitly.

Proof. ✓ Proof If some $\mathcal{E}_j$ were not idempotent, its assumed spectral pair would contradict physical CID by Theorem 18.2.19. Thus every diagonal mixed transfer map is idempotent. Local orthogonality makes every off-diagonal mixed transfer map zero, so Theorem 18.4.6 gives $\mathcal{E}_A^2 = \mathcal{E}_A$. The reverse implication is Theorem 18.2.11. $\square$

Theorem 18.2.21 (Conditional positive-gap ZCL implication for commuting parent ground spaces). Lean ✓ Stmt *Let P and $A = \bigoplus_j A_j$ satisfy the multiplicity-one unit-weight hypotheses of Theorem 18.2.20, including its normalized nonzero subleading spectral pair for every non-idempotent block. If A has positive-gap BNT zero correlation length, then for every $N > 2$ its translated two-site parent interactions satisfy all three ground-space conditions:*

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(A) &= 0, \\ \ker H_2^{(N)}(A) &= \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{aligned}$$

This is only the conditional corrected implication from (ii) to (iii) of [CPGSV16, Theorem 3.10], not a three-way equivalence.

Proof. ✓ **Proof** Theorem 18.2.20 gives $\mathcal{E}_A^2 = \mathcal{E}_A$. The multiplicity-one fixed-point theorem 13.1.33 then gives the commuting parent interactions, zero-energy direct-sum state, and stated common kernel for every $N > 2$. $\square$

Theorem 18.2.22 (Physical ZCL implies idempotence under the spectral assumption). Lean

✓ **Stmt** Under the spectral-pair hypotheses of Theorem 18.2.20, physical BNT zero correlation length for $A = \bigoplus_j A_j$ implies $\mathcal{E}_A^2 = \mathcal{E}_A$.

Proof. ✓ **Proof** Source physical CID includes all positive-gap configurations. Apply the forward implication in Theorem 18.2.20. $\square$

Lemma 18.2.23 (Sector-supported probes without spectral assumptions). Lean ✓ **Stmt** Let P be an auxiliary BNT sector decomposition in BNT canonical form, let $A = \bigoplus_j A_j$, and fix a sector j . For every $u, v \in M_{D_j}(\mathbb{C})$, there are a positive integer L and observables O_1, O_2 on L sites such that, for all $n_1, n_2 \in \mathbb{N}$,

$$\mathrm{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \mathcal{E}_A^{n_2} \mathcal{E}_{O_1} \mathcal{E}_A^{n_1}) = \mathrm{tr}(u \mathcal{E}_{A_j}^{n_1}(v)).$$

Proof. ✓ **Proof** Choose the trace-one stationary state ρ_j of $\mathcal{E}_{A_j}$. Simultaneous block injectivity supplies observables with inserted maps

$$\mathcal{E}_{O_1} = |\rho_j\rangle\langle u|, \quad \mathcal{E}_{O_2} = |v\rangle\langle \mathbb{1}|.$$

Compression to sector j intertwines powers of $\mathcal{E}_A$ with powers of $\mathcal{E}_{A_j}$. Hence $\mathcal{E}_A^{n_2}$ fixes the middle state ρ_j , and the composed inserted map is

$$|v\rangle\langle (\mathcal{E}_{A_j}^{n_1})^*(u)|.$$

The operator trace of this rank-one map is $\mathrm{tr}((\mathcal{E}_{A_j}^{n_1})^*(u)v) = \mathrm{tr}(u \mathcal{E}_{A_j}^{n_1}(v))$. $\square$

Theorem 18.2.24 (Unconditional reverse implication at the multiplicity-one representative).

Lean ✓ **Stmt** Let P be an auxiliary BNT sector decomposition in BNT canonical form and $A = \bigoplus_j A_j$. If A has positive-gap BNT zero correlation length, then $\mathcal{E}_A^2 = \mathcal{E}_A$.

Proof. ✓ **Proof** Fix j and arbitrary $u, v \in M_{D_j}(\mathbb{C})$. Apply positive-gap CID to the observables of Lemma 18.2.23, comparing complementary gaps $(1, 2)$ and $(2, 1)$. This gives

$$\mathrm{tr}(u \mathcal{E}_{A_j}(v)) = \mathrm{tr}(u \mathcal{E}_{A_j}^2(v)).$$

Since u and v are arbitrary and the trace pairing is nondegenerate, $\mathcal{E}_{A_j}^2 = \mathcal{E}_{A_j}$. Local orthogonality makes every mixed map $\mathcal{E}_{A_j, A_k}$ with $j \neq k$ equal to zero. Therefore every diagonal and off-diagonal mixed map is idempotent, and the direct-sum criterion gives $\mathcal{E}_A^2 = \mathcal{E}_A$. $\square$

Theorem 18.2.25 (Positive-gap BNT zero correlation length at the multiplicity-one representative). Lean ✓ **Stmt** Let P be an auxiliary BNT sector decomposition in BNT canonical form and $A = \bigoplus_j A_j$. Then

$$A \text{ has positive-gap BNT zero correlation length} \iff \mathcal{E}_A^2 = \mathcal{E}_A.$$

Proof. ✓ **Proof** The forward implication is Theorem 18.2.24. Conversely, transfer idempotence implies positive-gap physical correlation independence, while the BNT hypotheses give local orthogonality for the distinct sectors. $\square$

Theorem 18.2.26 (Physical BNT zero correlation length at the multiplicity-one representative). Lean ✓ Stmt Let P be an auxiliary BNT sector decomposition in BNT canonical form and $A = \bigoplus_j A_j$. If A has physical BNT zero correlation length, then $\mathcal{E}_A^2 = \mathcal{E}_A$.

Proof. ✓ Proof Physical correlation independence includes all positive-gap configurations. Restrict to those configurations and apply Theorem 18.2.24. $\square$

Theorem 18.2.27 (Positive-gap BNT zero correlation length gives commuting parent ground spaces). Lean ✓ Stmt Let P be an auxiliary BNT sector decomposition in BNT canonical form and $A = \bigoplus_j A_j$. If A has positive-gap BNT zero correlation length, then for every $N > 2$ its translated two-site parent interactions satisfy

$$\begin{aligned} h_i h_j &= h_j h_i, \\ h_i V^{(N)}(A) &= 0, \\ \ker H_2^{(N)}(A) &= \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \end{aligned}$$

Proof. ✓ Proof Theorem 18.2.24 gives $\mathcal{E}_A^2 = \mathcal{E}_A$. The multiplicity-one fixed-point theorem 13.1.33 gives the stated interactions and ground spaces. $\square$

Theorem 18.2.28 (Positive-gap zero correlation length and commuting parent ground spaces for the basis direct sum). Lean ✓ Stmt Let $A_1, \dots, A_g$ be the distinct basis tensors of a BNT canonical form, and set $A = \bigoplus_{j=1}^g A_j$. Then

$$A \text{ has positive-gap BNT zero correlation length} \iff A \text{ has all-chain commuting parent ground spaces spanned by} \quad (18.6)$$

This statement concerns one unit-weight copy of each distinct sector and positive complementary gaps. It does not include repeated copies, arbitrary raw weights, or adjacent complementary regions.

Proof. ✓ Proof By Theorem 18.2.25, positive-gap BNT zero correlation length is equivalent to transfer idempotence. The latter is equivalent to the all-chain commuting parent ground-space condition by Theorem 13.1.99. $\square$

Remark 18.2.29 (Remaining obstructions in the printed converse). For the multiplicity-one unit-weight representative, the sector-supported observables require no additional hypothesis. The full literal CPSV canonical-form equations (1252) and (1256) also contain all repeated-copy pairs and their raw weight powers; the theorem above does not realize those weighted insertions. Moreover, the preceding claim that $\mathcal{E}_j^2 \neq \mathcal{E}_j$ yields a nonzero eigenvalue λ with $|\lambda| < 1$ is not valid for a non-idempotent operator with a nilpotent Jordan part at eigenvalue zero. Hence the printed argument does not prove the unconditional converse in Theorem 3.8. At multiplicity one, Theorem 18.2.24 repairs the argument by arbitrary trace probes, without deriving the nonzero subleading left and right eigenvectors used at lines 1260–1268. The conditional theorems above remain the formal record of the printed spectral argument.

18.3 Normal tensors and fixed-point isometries

18.3.1 Auxiliary block-family hypotheses

The next predicate is an auxiliary hypothesis on an indexed block family. It is not the literal CPSV canonical form of Definition 8.6.7 or canonical form II of Definition 8.8.3. In particular,

it starts from injective left-canonical blocks, orders nonzero weights, and imposes a self-overlap limit; it does not reconstruct a tensor from normal blocks in an ambient bond space.

Definition 18.3.1.1 (Auxiliary injective block-family hypotheses). [Lean](#) [✓ Stmt](#) A collection of scaling factors $(\mu_k)_{k=1}^r$ and block tensors $(A_k)_{k=1}^r$ satisfies the *auxiliary block-family hypotheses* if:

1. each A_k is injective,
2. each A_k satisfies the TP normalization $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$,
3. the moduli are non-increasing: $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_r|$,
4. all $\mu_k \neq 0$,
5. every block bond dimension is positive: $D_k \geq 1$ for every k ,
6. the self-overlap converges: $O_{A_k A_k}(N) \rightarrow 1$ for each k .

Only this one-sided normalization is assumed; no unital condition is assumed here. Condition (6) is a primitivity hypothesis: for a primitive channel ([Wol12, Theorem 6.7(3)]), $T^n(\rho) \rightarrow \rho_\infty$ for every initial state, which forces the self-overlap to converge to 1. See also [Wol12, Theorem 6.8] for the completely-positive characterisation and Chapter 6 for the transfer-operator gap proof.

Theorem 18.3.1.2 (Auxiliary block-family hypotheses from peripheral primitivity). [Lean](#) [✓ Stmt](#) Let $(\mu_k, A_k)_{k=1}^r$ be nonzero weights and injective left-canonical blocks with positive bond dimensions, non-increasing moduli $|\mu_1| \geq \dots \geq |\mu_r| > 0$, and $\sigma_\partial(\mathcal{E}_{A_k}) = \{1\}$. Then $(\mu_k, A_k)_{k=1}^r$ satisfies the auxiliary hypotheses of Definition 18.3.1.1, and $O_{A_k A_k}(N) \rightarrow 1$ for every k .

Proof. [✓ Proof](#) The injectivity, left-canonical identity, weight ordering, and nonzero-weight clauses are exactly the hypotheses. For each block, peripheral primitivity gives the complementary spectral gap $\rho_{\text{spec}}(\mathcal{E}_{A_k} - P) < 1$ by the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]. Theorem 6.10.2 then yields $O_{A_k A_k}(N) \rightarrow 1$. $\square$

Theorem 18.3.1.3 (RFP normal tensor is injective). [Lean](#) [✓ Stmt](#) If A is a normal RFP tensor, then A is injective.

Proof. [✓ Proof](#) The RFP condition $\mathcal{E}_A^2 = \mathcal{E}_A$ gives coefficients $V_{i_1 i_2, j}$ such that

$$A^{i_1} A^{i_2} = \sum_j V_{i_1 i_2, j} A^j.$$

Induction on the word length therefore gives $A^{i_1} \dots A^{i_n} \in \text{span}\{A^j : 0 \leq j < d\}$ for $n \geq 1$. By normality, the words of some positive length N span $M_D(\mathbb{C})$. The preceding inclusion therefore shows that the one-site span is already $M_D(\mathbb{C})$. $\square$

Theorem 18.3.1.4 (Left-canonical normal RFP tensor is injective). [Lean](#) [✓ Stmt](#) Assume $D > 0$. Let A be a normal renormalization fixed point such that $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then A is injective. This is the left-canonical injectivity step in [CPGSV16, Appendix B].

Proof. [✓ Proof](#) The conclusion follows immediately from Theorem 18.3.1.3. Normality already supplies a positive block-injectivity length, so this implication does not require the additional normalization. $\square$

Theorem 18.3.1.5 (Unitary diagonal fixed point for a left-canonical normal RFP). [Lean](#) [✓ Stmt](#) Assume $D > 0$. Let A be a normal renormalization fixed point such that $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then there exist a unitary U and a diagonal positive definite matrix Λ such that, for $B^i := U^\dagger A^i U$, $\sum_i (B^i)^\dagger B^i = \mathbb{1}$ and $\mathcal{E}_B(\Lambda) = \Lambda$. This is the diagonal fixed-point reduction in [CPGSV16, Appendix B].

Proof. [✓ Proof](#) By Theorem 18.3.1.4, the tensor A is injective. By Theorem 4.1.2, its transfer map $\mathcal{E}_A$ is irreducible, and hence A is irreducible as a tensor. Applying Theorem 8.8.6 gives a unitary conjugate B for which the trace-preserving normalization remains valid and a diagonal positive definite matrix Λ with $\mathcal{E}_B(\Lambda) = \Lambda$. $\square$

Theorem 18.3.1.6 (Rank-one classification for RFP transfer maps). [Lean](#) [✓ Stmt](#) Let A be an injective left-canonical RFP tensor with positive-definite fixed point ρ of the transfer map $\mathcal{E}_A$. Then $\mathcal{E}_A = P_\rho$, where $P_\rho(X) = \frac{\text{tr}(X)}{\text{tr}(\rho)} \rho$ is the rank-one fixed-point projection.

Proof. [✓ Proof](#) The exponential convergence bound gives $\|\mathcal{E}_A^n(X) - P_\rho(X)\| \leq C(1 - \delta)^n \|X\|$. Idempotence of $\mathcal{E}_A$ implies $\mathcal{E}_A^{n+1} = \mathcal{E}_A$ for all $n \geq 0$, so $\|\mathcal{E}_A(X) - P_\rho(X)\| \leq C(1 - \delta)^{n+1} \|X\|$ for all n . Taking $n \rightarrow \infty$ gives $\mathcal{E}_A = P_\rho$. $\square$

Theorem 18.3.1.7 (Diagonal Kraus form for a normal fixed point). [Lean](#) [✓ Stmt](#) A normal left-canonical renormalization fixed-point tensor A admits a decomposition $A^i = X \Lambda U^i X^{-1}$, where Λ is diagonal positive, $\sum_i (U^i)^\dagger U^i = I$, and

$$\sum_i \overline{(U^i)_{\alpha,\beta}} (U^i)_{\alpha',\beta'} = D^{-1} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

Proof. [✓ Proof](#) After the diagonal fixed-point reduction, write the conjugated tensor as B and its positive diagonal fixed point as ρ . The rank-one classification gives $\mathcal{E}_B(Y) = P_\rho(Y) = \frac{\text{tr}(Y)}{\text{tr}(\rho)} \rho$. Set

$$\begin{aligned} \Lambda_\alpha &= \sqrt{\frac{D \rho_{\alpha,\alpha}}{\text{tr}(\rho)}}, \\ E_{\alpha,\beta} &= D^{-1/2} e_{\alpha,\beta}, \\ K_{\alpha,\beta} &= \Lambda E_{\alpha,\beta}. \end{aligned}$$

Direct multiplication of matrix units yields

$$\sum_{\alpha,\beta} K_{\alpha,\beta} Y K_{\alpha,\beta}^\dagger = P_\rho(Y).$$

Kraus freedom therefore gives an isometry V such that $B^i = \sum_{\alpha,\beta} V_{i,(\alpha,\beta)} K_{\alpha,\beta}$. With $U^i = \sum_{\alpha,\beta} V_{i,(\alpha,\beta)} E_{\alpha,\beta}$, one has $B^i = \Lambda U^i$ and

$$\begin{aligned} U_{\alpha,\beta}^i &= D^{-1/2} V_{i,(\alpha,\beta)}, \\ \sum_i \overline{U_{\alpha,\beta}^i} U_{\alpha',\beta'}^i &= D^{-1} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \end{aligned}$$

Conjugating back gives $A^i = X \Lambda U^i X^{-1}$. $\square$

Theorem 18.3.1.8 (Diagonal Kraus form with square-sum normalization). [Lean](#) [✓ Stmt](#) The decomposition in Theorem 18.3.1.7 additionally records that the diagonal weights satisfy $\sum_{\alpha} \Lambda_{\alpha}^2 = D$. This square-sum identity is the trace-normalization seed: the explicit weights are $\Lambda_{\alpha} = \sqrt{D \rho_{\alpha,\alpha} / \text{tr } \rho}$, and $\sum_{\alpha} \rho_{\alpha,\alpha} = \text{tr } \rho$ for the diagonal fixed point ρ .

Proof. [✓ Proof](#) The diagonal fixed-point reduction and the rank-one classification of injective left-canonical renormalization fixed points produce the diagonal weights $\Lambda_{\alpha} = \sqrt{D \rho_{\alpha,\alpha} / \text{tr } \rho}$, so $\Lambda_{\alpha}^2 = D \rho_{\alpha,\alpha} / \text{tr } \rho$. Summing over α and using $\sum_{\alpha} \rho_{\alpha,\alpha} = \text{tr } \rho$ gives $\sum_{\alpha} \Lambda_{\alpha}^2 = D$. $\square$

Theorem 18.3.1.9 (Unit pair-index form for a normal fixed point). [Lean](#) [✓ Stmt](#) A normal left-canonical renormalization fixed-point tensor A admits a decomposition $A^i = X \Lambda U^i X^{-1}$, where Λ is diagonal positive and

$$\sum_i \overline{(U^i)_{\alpha,\beta}} (U^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

This is the unit pair-index convention of [CPGSV16, Section 3]. The statement still does not impose the trace-normalization $\text{tr}(\Lambda) = 1$.

Proof. [✓ Proof](#) Apply Theorem 18.3.1.14, which gives $A^i = X \sqrt{\Lambda} U^i X^{-1}$ with U already a unit pair-index isometry. Set $\tilde{\Lambda}_{\alpha} = \sqrt{\Lambda_{\alpha}}$, so that $A^i = X \tilde{\Lambda} U^i X^{-1}$ with $\tilde{\Lambda}$ diagonal positive; the decomposition and the pair-index condition are immediate. $\square$

Theorem 18.3.1.10 (Per-block diagonal Kraus form). [Lean](#) [✓ Stmt](#) When each block A_k of a multi-block tensor is a normal, left-canonical renormalization fixed point, that block admits the isometry decomposition $A_k^i = X_k \Lambda_k U_k^i X_k^{-1}$, with X_k invertible, Λ_k diagonal positive, and $U_k = (U_k^i)$ satisfies $\sum_i (U_k^i)^{\dagger} U_k^i = I$ and the pair-index orthonormality condition

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}} (U_k^i)_{\alpha',\beta'} = D_k^{-1} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

This is the blockwise form of the diagonal Kraus decomposition (Theorem 18.3.1.7); see [CPGSV16, Section 3]. The source additionally imposes the normalization $\text{tr}(\Lambda_k) = 1$; the statement here gives positive Λ_k without it. That normalization is genuine rather than a conjugation gauge, since rescaling $\Lambda_k \mapsto \Lambda_k / \text{tr}(\Lambda_k)$ factors out as an overall scalar on A_k^i (conjugation by X_k preserves the scale); the statement is thus the unnormalized diagonal Kraus form. Scope restriction (source isometry). Corollary 3.12 also invokes the joint isometry condition

$$\sum_i (U_k^i)_{\alpha,\beta} \overline{(U_{\ell}^i)_{\alpha',\beta'}} = \delta_{k,\ell} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

The theorem above records the contracted per-block condition $\sum_i (U_k^i)^{\dagger} U_k^i = I$ and the diagonal pair-index equation with right-hand side $D_k^{-1} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}$. It does not impose the trace-normalization $\text{tr}(\Lambda_k) = 1$, and it does not include the cross-block equations for $k \neq \ell$.

Proof. [✓ Proof](#) Apply Theorem 18.3.1.7 to each block. $\square$

Theorem 18.3.1.11 (Per-block unit pair-index decomposition). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 18.3.1.10, each block admits a decomposition $A_k^i = X_k \Lambda_k U_k^i X_k^{-1}$ with Λ_k diagonal positive and

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}} (U_k^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

This is a blockwise unit pair-index decomposition. It still does not impose the source trace-normalization of Λ_k or the cross-block orthogonality equations for distinct blocks.

Proof. ✓ **Proof** Apply Theorem 18.3.1.9 to each block. □

Definition 18.3.1.12 (Square-root fixed-point form). Lean ✓ **Stmt** Let $\lambda_\alpha > 0$ satisfy $\sum_\alpha \lambda_\alpha = 1$, and set $\rho = \text{diag}(\lambda_\alpha)$. A normal tensor A has the *square-root fixed-point form* when there are an invertible matrix X and a tensor U satisfying the unit pair-index isometry

$$\sum_i \overline{(U^i)_{\alpha, \beta}} (U^i)_{\alpha', \beta'} = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}.$$

such that

$$A^i = X \sqrt{\rho} U^i X^{-1}.$$

Local fix (square-root diagonal). The display at line 1278 uses the bare diagonal Λ , while lines 1281–1283 impose a unit pair-index isometry and line 1300 constructs the reference tensor with coefficients $\sqrt{\Lambda_\alpha}$. These equations force the repaired form $A^i = X \sqrt{\rho} U^i X^{-1}$ used in this definition. The correction is documented in [con26p].

Explicitly, the reference tensor is

$$\widehat{A}_{\alpha, \beta}^{(\alpha', \beta')} = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} \sqrt{\lambda_\alpha}.$$

Thus $\rho = \text{diag}(\lambda_\alpha)$ is the trace-one diagonal fixed point. This records the $j = j'$ part of the joint isometry condition [CPGSV16, Appendix B, lines 1278, 1281–1283, 1300].

Lemma 18.3.1.13 (Gauge invariance of the square-root fixed-point form). Lean ✓ **Stmt** *Gauge-equivalent tensors have the square-root fixed-point form simultaneously.*

Proof. ✓ **Proof** Write $A^i = X \sqrt{\rho} U^i X^{-1}$ and $B^i = Y A^i Y^{-1}$. Then

$$B^i = (Y X) \sqrt{\rho} U^i (Y X)^{-1}.$$

Thus the diagonal weight and pair-index isometry are unchanged. Applying the same argument to the inverse gauge proves the converse. □

Theorem 18.3.1.14 (Trace-normalized square-root form for a normal fixed point). Lean ✓ **Stmt** *A normal left-canonical renormalization fixed-point tensor has the square-root fixed-point form. With $\lambda_\alpha = \rho_{\alpha, \alpha} / \text{tr } \rho$ and $\rho_0 = \text{diag}(\lambda_\alpha)$, it admits $A^i = X \sqrt{\rho_0} U^i X^{-1}$, where ρ_0 is positive and trace-normalized and U is a unit pair-index isometry. Trace normalization is the identity $\sum_\alpha \rho_{\alpha, \alpha} = \text{tr } \rho$. This is the normal-tensor statement corresponding to lines 1278 and 1281–1283, with the square-root correction dictated by the reference tensor at line 1300 [CPGSV16, Appendix B].*

Proof. ✓ **Proof** The structural form with the square-sum identity (Theorem 18.3.1.8) supplies a positive diagonal weight $\tilde{\Lambda}$ with $\sum_\alpha \tilde{\Lambda}_\alpha^2 = D$. Rescaling to the unit pair-index convention by $U^i \mapsto \sqrt{D} U^i$, set

$$\rho_0 = \text{diag} \left(\left(\frac{\tilde{\Lambda}_\alpha}{\sqrt{D}} \right)^2 \right).$$

Then $\text{tr}(\rho_0) = D^{-1} \sum_\alpha \tilde{\Lambda}_\alpha^2 = D^{-1} \cdot D = 1$, while $\sqrt{(\rho_0)_{\alpha, \alpha}} = \tilde{\Lambda}_\alpha / \sqrt{D}$ recovers the decomposition $A^i = X \sqrt{\rho_0} U^i X^{-1}$. □

$$A = X\sqrt{\rho}U X^{-1}$$

Figure 18.2: The square-root form of a normal renormalization fixed point: $\rho = \text{diag}(\lambda_\alpha)$ and $A^i = X\sqrt{\rho}U^iX^{-1}$. The pair-index isometry is the one in lines 1281–1283, and the square root follows from the reference tensor at line 1300 [CPGSV16, Appendix B].

Lemma 18.3.1.15 (Reference-tensor transfer identity for a unit pair-index isometry). [Lean](#)

[✓ Stmt](#) Let U be a family of bond matrices satisfying the unit pair-index isometry condition

$$\sum_i \overline{(U^i)_{\alpha,\beta}} (U^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

Then for every bond matrix Z ,

$$\sum_i U^i Z (U^i)^\dagger = \text{tr}(Z) I.$$

Proof. [✓ Proof](#) Reading off the (x, y) entry, the pair-index condition gives $\sum_i (U^i)_{x,\alpha} \overline{(U^i)_{y,\beta}} = \delta_{x,y} \delta_{\alpha,\beta}$, so the entry collapses to $\delta_{x,y} \sum_\alpha Z_{\alpha,\alpha} = \delta_{x,y} \text{tr}(Z)$, which is the (x, y) entry of $\text{tr}(Z) I$. $\square$

Theorem 18.3.1.16 (The square-root form implies the fixed-point property). [Lean](#) [✓ Stmt](#) A tensor in the square-root fixed-point form is a renormalization fixed point. This is the backward direction of the structural characterization of pure-state renormalization fixed points in [CPGSV16, Section 3], which the source states as immediate.

Proof. [✓ Proof](#) Write $A^i = X\sqrt{\rho}U^iX^{-1}$. Substituting into the transfer map and pulling the index-independent factors $X\sqrt{\rho}$ and $\sqrt{\rho}X^\dagger$ outside the sum, the unit pair-index isometry identity (Lemma 18.3.1.15) gives the rank-one form

$$\begin{aligned} \mathcal{E}_A(Y) &= \text{tr}(X^{-1}Y(X^{-1})^\dagger) R, \\ R &= X\rho X^\dagger. \end{aligned}$$

Idempotence then reduces to $\text{tr}(X^{-1}R(X^{-1})^\dagger) = \text{tr}(\rho) = 1$, the trace normalization, so $\mathcal{E}_A \circ \mathcal{E}_A = \mathcal{E}_A$. $\square$

Theorem 18.3.1.17 (Normal-tensor fixed-point characterization). [Lean](#) [✓ Stmt](#) A normal tensor is a renormalization fixed point if and only if it has the square-root fixed-point form of Definition 18.3.1.12. Its bond dimension is necessarily positive, rather than subject to an additional hypothesis, and no left-canonical hypothesis is imposed on the original tensor. This is the corrected form of Lemma B.1 in [CPGSV16, Appendix B, lines 1274–1301].

Proof. [✓ Proof](#) Normality first gives positive bond dimension by Theorem 9.2.1. For the forward implication, choose the pure trace-preserving Perron gauge from Theorem 9.2.3. Idempotence and algebraic normality are invariant under this gauge, so Theorem 18.3.1.14 gives the square-root form for the gauged tensor. Lemma 18.3.1.13 gives the form for the original tensor. The reverse implication is Theorem 18.3.1.16. $\square$

Theorem 18.3.1.18 (Per-block trace-normalized square-root form). [Lean](#) [✓ Stmt](#) *Each block of a multi-block tensor whose blocks are normal, left-canonical renormalization fixed points has the square-root fixed-point form, with a trace-normalized diagonal weight and a unit pair-index isometry. This is the single-block trace-normalized form applied to each block; it omits the cross-block orthogonality between distinct normal-tensor blocks.*

Proof. [✓ Proof](#) Apply Theorem 18.3.1.14 to each block. □

Across all blocks, the repaired trace-normalized decomposition has the following representation. Here j labels the normal-tensor block and q its repeated representation; the isometries

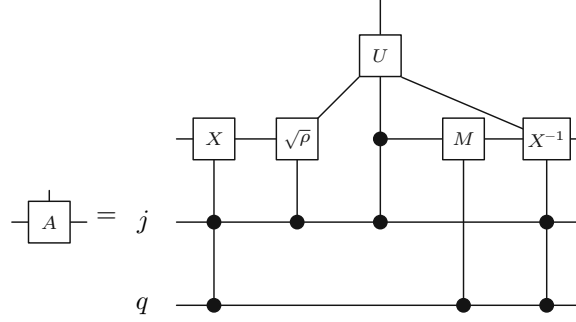


Figure 18.3: Block and multiplicity indices in the fixed-point decomposition. The independent rails j and q pass through the coefficient network without being fused: X and X^{-1} depend on both, $\sqrt{\rho}$ and U carry the block label j , and M carries the multiplicity label q . The upper leg of U is the physical index, as in the source figure [CPGSV16, Section 3.4, lines 543–563].

in [CPGSV16, Section 3.4, lines 543–561] also satisfy cross-block physical orthogonality. Those additional equations are not part of the per-block theorem above.

18.4 Direct sums and the joint isometry condition

Lemma 18.4.1 (Block decomposition of the direct-sum transfer map). [Lean](#) [✓ Stmt](#) *Let $(B_k)_k$ be a family of tensors and let $\bigoplus_k B_k$ be their direct sum on the total bond space. For every bond matrix X and every pair of blocks, the (j, j') bond block of the transfer sum*

$$\sum_i \left(\bigoplus_k B_k^i \right) X \left(\bigoplus_k B_k^i \right)^\dagger$$

equals $\mathcal{E}_{j,j'}$ applied to the (j, j') bond block of X .

Proof. [✓ Proof](#) A block-diagonal factor connects only matching blocks, so the (j, j') block of the product is

$$\left(\sum_i \left(\bigoplus_k B_k^i \right) X \left(\bigoplus_k B_k^i \right)^\dagger \right)_{j,j'} = \sum_i B_j^i X_{j,j'} (B_{j'}^i)^\dagger = \mathcal{E}_{j,j'}(X_{j,j'}).$$

□

Theorem 18.4.2 (Reindexing the direct-sum transfer map). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of tensors, let Φ be the canonical identification of the block-labelled bond space with the total bond space, and let $\mathcal{S}_B$ be the block transfer sum. Then, for every block matrix Y ,

$$\mathcal{E}_{\oplus_k B_k}(\Phi(Y)) = \Phi(\mathcal{S}_B(Y)).$$

Proof. [✓ Proof](#) Since Φ preserves products and adjoints, expanding the transfer map gives

$$\begin{aligned} \mathcal{E}_{\oplus_k B_k}(\Phi(Y)) &= \sum_i \Phi \left(\bigoplus_k B_k^i \right) \Phi(Y) \Phi \left(\bigoplus_k B_k^i \right)^\dagger \\ &= \Phi \left(\sum_i \left(\bigoplus_k B_k^i \right) Y \left(\bigoplus_k B_k^i \right)^\dagger \right) \\ &= \Phi(\mathcal{S}_B(Y)). \end{aligned}$$

□

Theorem 18.4.3 (Direct-sum quasi-idempotence on the block transfer sum). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of tensors and let $c \in \mathbb{C}$. If

$$\mathcal{E}_{\oplus_k B_k}^2 = c \mathcal{E}_{\oplus_k B_k},$$

then the corresponding block transfer sum $\mathcal{S}_B$ satisfies, for every bond matrix Y ,

$$\mathcal{S}_B(\mathcal{S}_B(Y)) = c \mathcal{S}_B(Y).$$

Proof. [✓ Proof](#) Reindex the total bond space by its block labels. By Theorem 18.4.2, the direct-sum transfer map becomes $\mathcal{S}_B$. Transporting the assumed equation through this identification gives the displayed identity. □

Theorem 18.4.4 (Direct-sum quasi-idempotence on every mixed corner). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of tensors and let $c \in \mathbb{C}$. If

$$\mathcal{E}_{\oplus_k B_k}^2 = c \mathcal{E}_{\oplus_k B_k},$$

then, for every ordered pair of blocks (j, j') whose two bond dimensions are positive,

$$\mathcal{E}_{j, j'}^2 = c \mathcal{E}_{j, j'}.$$

Proof. [✓ Proof](#) Let Z be an arbitrary rectangular matrix and realize it as the (j, j') block of a matrix Y on the total bond space. By Lemma 18.4.1, $[\mathcal{S}_B(Y)]_{j, j'} = \mathcal{E}_{j, j'}(Z)$. Theorem 18.4.3 gives $\mathcal{S}_B^2 = c \mathcal{S}_B$, and hence

$$\mathcal{E}_{j, j'}(\mathcal{E}_{j, j'}(Z)) = [\mathcal{S}_B(\mathcal{S}_B(Y))]_{j, j'} = [c \mathcal{S}_B(Y)]_{j, j'} = c \mathcal{E}_{j, j'}(Z).$$

□

Lemma 18.4.5 (Pairwise idempotence of the block transfer sum). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of tensors. If every mixed transfer operator $\mathcal{E}_{j, j'}$ is idempotent, then the block transfer sum $\mathcal{S}_B$ is idempotent: $\mathcal{S}_B(\mathcal{S}_B(X)) = \mathcal{S}_B(X)$.

Proof. ✓ Proof By Lemma 18.4.1, the (j, j') block satisfies

$$(\mathcal{S}_B^2(X))_{j,j'} = \mathcal{E}_{j,j'}^2(X_{j,j'}) = \mathcal{E}_{j,j'}(X_{j,j'}) = (\mathcal{S}_B(X))_{j,j'}.$$

Equality on every block gives the asserted equality on the total bond space. $\square$

Theorem 18.4.6 (Pairwise mixed-transfer criterion for a direct sum). Lean ✓ Stmt *Suppose every bond dimension is positive. The transfer map of $\bigoplus_k B_k$ is idempotent if and only if, for every pair j, j' , the mixed transfer operator is idempotent:*

$$\mathcal{E}_{\bigoplus_k B_k}^2 = \mathcal{E}_{\bigoplus_k B_k} \iff \mathcal{E}_{j,j'}^2 = \mathcal{E}_{j,j'}.$$

Proof. ✓ Proof By Lemma 18.4.1, the (j, j') block of the direct-sum transfer map depends only on the (j, j') block of its argument and is obtained by applying $\mathcal{E}_{j,j'}$. Thus pairwise idempotence gives

$$(\mathcal{E}_{\bigoplus_k B_k}^2(X))_{j,j'} = \mathcal{E}_{j,j'}^2(X_{j,j'}) = \mathcal{E}_{j,j'}(X_{j,j'}) = (\mathcal{E}_{\bigoplus_k B_k}(X))_{j,j'}$$

for every pair of blocks, and hence gives idempotence on the total bond space, as in Lemma 18.4.5. Conversely, every matrix M of the appropriate rectangular size occurs as the (j, j') block of some Y on the total bond space. Applying Lemma 18.4.1 and whole-tensor idempotence gives

$$\mathcal{E}_{j,j'}^2(M) = (\mathcal{E}_{\bigoplus_k B_k}^2(Y))_{j,j'} = (\mathcal{E}_{\bigoplus_k B_k}(Y))_{j,j'} = \mathcal{E}_{j,j'}(M).$$

$\square$

Lemma 18.4.7 (Scaling of a rectangular mixed transfer operator). Lean ✓ Stmt *For complex scalars c, e and tensors A, B of possibly different bond dimensions, $\mathcal{E}_{cA, eB} = c\bar{e} \mathcal{E}_{A, B}$.*

Proof. ✓ Proof Conjugate transposition changes the second scalar to its complex conjugate: $\sum_i (cA^i)X(eB^i)^\dagger = c\bar{e} \sum_i A^i X(B^i)^\dagger$. $\square$

Theorem 18.4.8 (Phase coherence for literal repeated blocks). Lean ✓ Stmt *Let A have positive bond dimension and a nonzero idempotent transfer map. Let μ_q have unit modulus. If the literal bond direct sum $\bigoplus_q \mu_q A$ has an idempotent transfer map, then $\mu_q = \mu_{q'}$ for every pair q, q' .*

Proof. ✓ Proof The pairwise criterion (Theorem 18.4.6) gives that $\mathcal{E}_{\mu_q A, \mu_{q'} A}$ is idempotent. Lemma 18.4.7 and $\mathcal{E}_{A, A} = \mathcal{E}_A$ give $\mathcal{E}_{\mu_q A, \mu_{q'} A} = \mu_q \overline{\mu_{q'}} \mathcal{E}_{A, A} = \mu_q \overline{\mu_{q'}} \mathcal{E}_A$. Hence $(\mu_q \overline{\mu_{q'}}) \mathcal{E}_A$ is idempotent. Since $\mathcal{E}_A$ is nonzero and $|\mu_q \overline{\mu_{q'}}| = |\mu_q| |\mu_{q'}| = 1$,

$$(\mu_q \overline{\mu_{q'}})^2 \mathcal{E}_A = \mu_q \overline{\mu_{q'}} \mathcal{E}_A \implies \mu_q \overline{\mu_{q'}} (\mu_q \overline{\mu_{q'}} - 1) = 0 \implies \mu_q \overline{\mu_{q'}} = 1.$$

Since $|\mu_{q'}| = 1$ gives $\overline{\mu_{q'}} \mu_{q'} = 1$, one has $\mu_q = \mu_q (\overline{\mu_{q'}} \mu_{q'}) = (\mu_q \overline{\mu_{q'}}) \mu_{q'} = \mu_{q'}$. $\square$

Theorem 18.4.9 (Cross-block transfer vanishing for a direct-sum fixed point). Lean ✓ Stmt *Let $(A_j)_j$ be a family of distinct irreducible left-canonical blocks, no two of which are gauge-phase equivalent, and suppose the direct sum $\bigoplus_j A_j$ is a renormalization fixed point. Then for every pair of distinct components the mixed transfer operator vanishes, $\mathcal{E}_{j,j'} = \sum_i A_j^i \otimes \overline{A_{j'}^i} = 0$. This is the cross-block $(\delta_{j,j'})$ content of the isometry condition of [CPGSV16, Theorem 3.11 and Corollary 3.12] at the level of the normal-tensor blocks. The passage to the residual tensors U_j is given by Theorem 18.4.16.*

Proof. ✓ Proof By Lemma 18.4.1 the (j, j') bond block of the direct-sum transfer map acts as $\mathcal{E}_{j, j'}$. Whole-tensor idempotence $\mathcal{E}^2 = \mathcal{E}$ therefore restricts to each block as $\mathcal{E}_{j, j'}^2 = \mathcal{E}_{j, j'}$. For distinct irreducible left-canonical blocks, Theorem 6.6.2 gives $\rho_{\text{spec}}(\mathcal{E}_{j, j'}) < 1$ when the bond dimensions agree, because the blocks are not gauge-phase equivalent; Theorem 6.6.4 gives the same inequality when the bond dimensions differ. Consequently, Lemma 6.5.3 gives

$$\mathcal{E}_{j, j'}^2 = \mathcal{E}_{j, j'}, \quad \rho_{\text{spec}}(\mathcal{E}_{j, j'}) < 1 \quad \implies \quad \mathcal{E}_{j, j'} = 0.$$

□

Definition 18.4.10 (Joint isometry condition). Lean ✓ Stmt A family $(U_j)_j$ satisfies the *joint isometry condition* when

$$\sum_i (U_j^i)_{\alpha, \beta} \overline{(U_{j'}^i)_{\alpha', \beta'}} = \delta_{j, j'} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}.$$

Each block satisfies the within-block pair-index orthonormality, and the cross-block sums between distinct blocks vanish. This is the isometry condition of [CPGSV16, Theorem 3.11], split into its $j = j'$ and $j \neq j'$ cases.

Theorem 18.4.11 (One-letter Gram matrix under the joint isometry condition). Lean ✓ Stmt Let $\mathcal{V} = \bigsqcup_j \{j\} \times [D_j] \times [D_j]$ be the disjoint union of the within-sector virtual pairs. For $x = (j, \alpha, \beta)$ in $\mathcal{V}$, set $u_x(i) = (U_j^i)_{\alpha, \beta}$. Then $\sum_i u_x(i) \overline{u_y(i)} = \delta_{x, y}$.

Proof. ✓ Proof If x and y belong to the same sector, the assertion is the within-sector part of the joint isometry condition. If they belong to distinct sectors, it is the cross-sector vanishing condition. □

Lemma 18.4.12 (Covariance of the mixed transfer operator). Lean ✓ Stmt For decompositions $A^i = X_A D_A U^i X_A^{-1}$ and $B^i = X_B D_B V^i X_B^{-1}$, the mixed transfer operator of A, B is the mixed transfer operator of the tensors U, V conjugated by the outer factors:

$$\mathcal{E}_{A, B}(Y) = X_A D_A \mathcal{E}_{U, V}(X_A^{-1} Y (X_B^{-1})^\dagger) (X_B D_B)^\dagger.$$

Proof. ✓ Proof Substitute the two decompositions into $\mathcal{E}_{A, B}(Y) = \sum_i A^i Y (B^i)^\dagger$ and collect the outer factors, which are independent of the summation index, outside the sum. □

Lemma 18.4.13 (Vanishing after removing the diagonal factors). Lean ✓ Stmt With decompositions $A^i = X_A D_A U^i X_A^{-1}$ and $B^i = X_B D_B V^i X_B^{-1}$ whose factors X_A, D_A, X_B, D_B are invertible, if $\mathcal{E}_{A, B} = 0$ then $\mathcal{E}_{U, V} = 0$.

Proof. ✓ Proof The covariance identity Lemma 18.4.12 expresses $\mathcal{E}_{A, B}$ as $\mathcal{E}_{U, V}$ conjugated by the invertible outer factors $X_A D_A$ and $(X_B D_B)^\dagger$, so $\mathcal{E}_{A, B} = 0$ forces $\mathcal{E}_{U, V} = 0$. □

Lemma 18.4.14 (Entrywise cross-block isometry sum). Lean ✓ Stmt If $\mathcal{E}_{U, V} = 0$, then, for all virtual indices,

$$\sum_i (U^i)_{\alpha, \beta} \overline{(V^i)_{\alpha', \beta'}} = 0.$$

Proof. ✓ Proof Apply $\mathcal{E}_{U, V} = 0$ to the matrix unit supported at (β, β') and read the (α, α') entry. □

Lemma 18.4.15 (Each block of a direct-sum fixed point is a fixed point). [Lean](#) [✓ Stmt](#) If the direct sum $\bigoplus_k B_k$ is a renormalization fixed point, with $\dim_k \geq 1$ for all k , then each block B_j is a renormalization fixed point.

Proof. [✓ Proof](#) The diagonal mixed transfer operator $\mathcal{E}_{j,j}$ is the transfer map of B_j , and whole-tensor idempotence makes it idempotent, which is exactly the renormalization fixed-point condition for B_j . $\square$

Theorem 18.4.16 (Joint isometry condition for a direct-sum fixed point). [Lean](#) [✓ Stmt](#) Let $(A_j)_j$ be a family of normal, irreducible, left-canonical blocks, no two of which are gauge-phase equivalent, with $\dim_j \geq 1$ for all j , and suppose the direct sum $\bigoplus_j A_j$ is a renormalization fixed point. Then there are invertible matrices X_j , trace-one positive diagonal matrices $\rho_j = \text{diag}(\lambda_{j,\alpha})$, and tensors U_j with

$$A_j^i = X_j \sqrt{\rho_j} U_j^i X_j^{-1},$$

such that $(U_j)_j$ satisfies the joint isometry condition. These are the isometry equations of [CPGSV16, Corollary 3.12] for the direct sum of distinct normal-tensor blocks.

Proof. [✓ Proof](#) Whole-tensor idempotence makes the diagonal mixed transfer operator of each block its own transfer map, hence each block is a renormalization fixed point; the square-root fixed-point form (Theorem 18.3.1.14) supplies the decomposition $A_j^i = X_j \sqrt{\rho_j} U_j^i X_j^{-1}$ with the within-block orthonormality, which is the $j = j'$ case. For $j \neq j'$, Theorem 18.4.9 gives $\mathcal{E}_{A_j, A_{j'}} = 0$; by the covariance Lemma 18.4.12 and invertibility of the outer factors $\mathcal{E}_{U_j, U_{j'}} = 0$, and reading the entry of this operator at a matrix unit yields the cross-block sum $\sum_i (U_j^i)_{\alpha, \beta} \overline{(U_{j'}^i)_{\alpha', \beta'}} = 0$. $\square$

Theorem 18.4.17 (Direct sum of square-root fixed-point blocks). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of tensors, each in the square-root fixed-point form, whose cross-block mixed transfer operators vanish, $\mathcal{E}_{j,j'} = 0$ for $j \neq j'$. Then the direct sum $\bigoplus_k B_k$ is a renormalization fixed point. This is the backward direction of the structural characterization of pure-state renormalization fixed points [CPGSV16, Section 3], in the distinct-blocks case where each normal tensor appears once; the cross-block vanishing is the $j \neq j'$ part of the isometry condition of that characterization, so it is part of the source joint isometry condition.

Proof. [✓ Proof](#) The direct-sum transfer map decouples block by block: by Lemma 18.4.1 its (j, j') bond block acts as $\mathcal{E}_{j,j'}$ on the (j, j') bond block of the argument. Each diagonal block $\mathcal{E}_{j,j}$ is the transfer map of B_j , which is idempotent because B_j is a renormalization fixed point (Theorem 18.3.1.16); each off-diagonal block $\mathcal{E}_{j,j'}$ with $j \neq j'$ vanishes by hypothesis. Therefore, for every bond block,

$$[\mathcal{E}_{\bigoplus B}^2(Y)]_{j,j'} = \mathcal{E}_{j,j'}(\mathcal{E}_{j,j'}(Y_{j,j'})) = \mathcal{E}_{j,j'}(Y_{j,j'}) = [\mathcal{E}_{\bigoplus B}(Y)]_{j,j'},$$

so $\mathcal{E}_{\bigoplus B}^2 = \mathcal{E}_{\bigoplus B}$ and the direct sum is a renormalization fixed point. $\square$

Theorem 18.4.18 (Distinct-block fixed-point characterization). [Lean](#) [✓ Stmt](#) Let $(B_k)_k$ be a family of normal, irreducible, left-canonical blocks with $\dim_k \geq 1$ for all k , no two of which are gauge-phase equivalent. Then the direct sum $\bigoplus_k B_k$ is a renormalization fixed point if and only if each block is in the square-root fixed-point form and the mixed transfer operators between distinct blocks vanish:

$$\left(\bigoplus_k B_k \text{ is a RFP} \right) \iff (\forall k, B_k \text{ in square-root form}) \wedge (\forall j \neq j', \mathcal{E}_{j,j'} = 0).$$

This is the distinct-blocks (multiplicity-one, phase-one) case of the structural characterization of pure-state renormalization fixed points [CPGSV16, Theorem 3.11], combining its forward and backward directions.

Proof. ✓ Proof For the forward direction, whole-tensor idempotence makes each diagonal mixed transfer operator the transfer map of its block (Lemma 18.4.15), so each block is a renormalization fixed point and hence in the square-root fixed-point form (Theorem 18.3.1.14), while the off-diagonal operators vanish, $\mathcal{E}_{j,j'} = 0$ for $j \neq j'$, by Theorem 18.4.9. Conversely, a direct sum of square-root fixed-point blocks whose cross-block operators vanish is a renormalization fixed point (Theorem 18.4.17). $\square$

Corollary 18.4.19 (BNT basis direct-sum characterization). Lean Lean ✓ Stmt Let P satisfy the auxiliary BNT sector hypotheses, with distinct basis tensors $(B_j)_j$. Then the direct sum containing one copy of each basis tensor is a renormalization fixed point if and only if every B_j has the square-root fixed-point form and $\mathcal{E}_{j,j'} = 0$ for $j \neq j'$. This is the multiplicity-one, phase-one specialization of [CPGSV16, Theorem 3.11 and Corollary 3.12]. It concerns the direct sum of the basis representatives, not the repeated-copy tensor carrying the weights $\mu_{j,q}$.

If the basis direct sum is a renormalization fixed point, there are also invertible matrices X_j , trace-one positive diagonal matrices $\rho_j = \text{diag}(\lambda_{j,\alpha})$, and tensors U_j such that $B_j^i = X_j \sqrt{\rho_j} U_j^i X_j^{-1}$, and the family $(U_j)_j$ satisfies the joint isometry condition, including the cross-block equations for $j \neq j'$.

Proof. ✓ Proof Positive basis dimensions supply the nonzero bond spaces. For each basis tensor, irreducibility and left-canonical normalization are part of the auxiliary BNT sector hypotheses. If $m_j > 0$ is the peripheral period of B_j , the normalized self-overlap hypothesis and the periodic self-overlap formula used in Theorem 17.4.20 give, along the same subsequence,

$$\begin{aligned} \lim_{k \rightarrow \infty} O_{B_j B_j}(m_j k) &= 1, \\ \lim_{k \rightarrow \infty} O_{B_j B_j}(m_j k) &= m_j. \end{aligned}$$

Hence $m_j = 1$. The transfer map is primitive by the corresponding theorem in the companion quantum-channel volume [LTC26, “Primitivity equals period one”], and normality follows from Theorem 8.11.1.1. Gauge-phase distinctness is also part of the auxiliary BNT sector hypotheses. The equivalence now follows from Theorem 18.4.18; the same derived hypotheses in Theorem 18.4.16 gives the joint isometry condition. $\square$

Theorem 18.4.20 (Transfer idempotence under coisometric reconstruction). Lean ✓ Stmt Let A and B be tensors with bond dimensions D and R . Suppose that $U \in \mathbb{C}^{R \times D}$ satisfies $UU^\dagger = I_R$ and, for every physical index i ,

$$A^i = U^\dagger B^i U.$$

Then A is a renormalization fixed point if and only if B is a renormalization fixed point.

Proof. ✓ Proof For every bond matrix X ,

$$\mathcal{E}_A(X) = U^\dagger \mathcal{E}_B(UXU^\dagger)U.$$

The identity $UU^\dagger = I_R$ gives $U(U^\dagger Y U)U^\dagger = Y$. Applying these two identities twice proves each implication. $\square$

Theorem 18.4.21 (A fixed nonzero scaling of a normal tensor). [Lean](#) [✓ Stmt](#) Let B be a normal tensor and let $c \in \mathbb{C} \setminus \{0\}$. If cB is a renormalization fixed point, then

$$|c| = 1, \quad \mathcal{E}_B^2 = \mathcal{E}_B.$$

Proof. [✓ Proof](#) Normality gives $r(\mathcal{E}_B) = 1$, while $\mathcal{E}_{cB} = |c|^2 \mathcal{E}_B$. Since $c \neq 0$, this operator is nonzero. Theorem 6.5.2 gives $r(\mathcal{E}_{cB}) = 1$, and Theorem 6.5.1 gives

$$1 = r(\mathcal{E}_{cB}) = |c|^2 r(\mathcal{E}_B) = |c|^2.$$

Thus $|c| = 1$, so $\mathcal{E}_{cB} = \mathcal{E}_B$ and the idempotence of $\mathcal{E}_B$ follows. $\square$

Theorem 18.4.22 (Blocks of a CPSV fixed point). [Lean](#) [✓ Stmt](#) Suppose A has CPSV canonical-form data

$$A^i = U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U, \quad UU^\dagger = I,$$

where every A_k is normal. If A is a renormalization fixed point, then every block k satisfies

$$|\mu_k| = 1, \quad \mathcal{E}_{A_k}^2 = \mathcal{E}_{A_k}.$$

Proof. [✓ Proof](#) Theorem 18.4.20 makes the retained direct sum $\bigoplus_k \mu_k A_k$ a renormalization fixed point. Lemma 18.4.15 then makes each scaled block $\mu_k A_k$ a renormalization fixed point. Since $\mu_k \neq 0$, Theorem 18.4.21 gives both conclusions. $\square$

Theorem 18.4.23 (Phase-class residual isometries). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be a CPSV canonical-form renormalization fixed point. Choose one representative C_j from each matrix-product-vector phase class. Then there are invertible matrices X_j , positive numbers $\lambda_{j,\alpha}$ with $\sum_\alpha \lambda_{j,\alpha} = 1$, and tensors U_j such that

$$C_j^i = X_j \text{diag} \left(\sqrt{\lambda_{j,\alpha}} \right) U_j^i X_j^{-1},$$

$$\sum_i (U_j^i)_{\alpha,\beta} \overline{(U_{j'}^i)_{\alpha',\beta'}} = \delta_{j,j'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

The conclusion also covers the case in which there are no classes.

Local fix (square-root diagonal). CPSV prints a bare Λ_j in Corollary 3.12, whereas the reference tensor at line 1300 has coefficients $\sqrt{(\rho_j)_{\alpha,\alpha}}$, which forces the factor $\sqrt{\lambda_{j,\alpha}}$ used above. This repair is documented in [\[con26p\]](#).

Proof. [✓ Proof](#) The retained weighted direct sum is a renormalization fixed point. For each representative, Theorem 18.4.22 gives

$$|\mu_j| = 1, \quad \mathcal{E}_{C_j}^2 = \mathcal{E}_{C_j}.$$

The normal-tensor characterization supplies the displayed square-root form. Put the representatives independently in left-canonical gauge and retain the unit-modulus factors μ_j . Distinct phase-class representatives are not gauge-phase equivalent, while idempotence of the retained direct sum restricts to every mixed pair. Hence Theorem 18.4.9 gives

$$\mathcal{E}_{\mu_j C_j, \mu_{j'} C_{j'}} = 0 \quad (j \neq j').$$

Cancelling the nonzero scalar factors and the invertible gauges yields $\mathcal{E}_{C_j, C_{j'}} = 0$. Removing the positive diagonal factors and evaluating at matrix units gives the off-diagonal residual equation. Together with the within-block orthonormality, this is the joint isometry condition. $\square$

Theorem 18.4.24 (Square-root characterization of fixed points). Not ready CPSV asserts that a tensor A in canonical form is a renormalization fixed point if and only if it can be written as

$$A^i = \bigoplus_{j=1}^g \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} \sqrt{\rho_j} U_j^i X_{j,q}^{-1},$$

where $|\mu_{j,q}| = 1$, each ρ_j is positive diagonal with $\text{tr}(\rho_j) = 1$, and

$$\sum_i (U_j^i)_{\alpha,\beta} \overline{(U_{j'}^i)_{\alpha',\beta'}} = \delta_{j,j'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}.$$

The source specifies that $\bigoplus_{j,q}$ is a simultaneous direct sum in the physical and virtual spaces, not an ordinary virtual block diagonal at each fixed physical letter i [CPGSV16, Theorem 3.11, lines 561–562]. It does not provide a coordinate map for the physical routing of q , a corresponding dimension hypothesis, or a q -indexed isometry equation. Consequently the display does not yet determine a source-faithful coordinate-level predicate. The routed characterization therefore remains not ready pending a source clarification or erratum.

Local fix (square-root diagonal). CPSV prints a bare Λ_j in Theorem 3.11, whereas the reference tensor at line 1300 has coefficients $\sqrt{(\rho_j)_{\alpha,\alpha}}$, which forces the factor $\sqrt{\rho_j}$ used above. This repair is documented in [con26p].

18.5 Renormalization flow and physical correlations

Theorem 18.5.1 (Injectivity in the auxiliary block family). Lean ✓ Stmt Every block satisfying the auxiliary hypotheses of Definition 18.3.1.1 is injective. This is an immediate projection of that auxiliary predicate, not a theorem about literal CPSV canonical form.

Proof. ✓ Proof Injectivity is one of the defining auxiliary hypotheses. □

Theorem 18.5.2 (Primitive-transfer convergence for each auxiliary block). Lean ✓ Stmt Fix one block A_k in a family satisfying Definition 18.3.1.1. Then the primitive transfer map of that block satisfies: $\mathcal{E}_{A_k}^{2^n}$ converges pointwise to an idempotent linear map $\mathcal{E}_{k,\infty}$ as $n \rightarrow \infty$. Thus there exists $\mathcal{E}_{k,\infty}$ with $\mathcal{E}_{k,\infty}^2 = \mathcal{E}_{k,\infty}$ such that $\mathcal{E}_{A_k}^{2^n}(\rho) \rightarrow \mathcal{E}_{k,\infty}(\rho)$ for every bond matrix ρ . This is the per-block primitive convergence used in the Appendix B discussion, not convergence of the full weighted canonical-family transfer matrix [CPGSV16, Appendix B, lines 1211–1244].

Proof. ✓ Proof Each block is injective and left-canonical, so its transfer map is a primitive channel with a unique positive-definite fixed point ρ_0 . The exponential convergence bound $\|\mathcal{E}^n(X) - P(X)\| \leq C(1 - \delta)^n \|X\|$ (Theorem 15.3.1) then gives pointwise convergence $\mathcal{E}^n(X) \rightarrow P(X)$, and composing with the subsequence $2^n \rightarrow \infty$ yields the result. The witness $\mathcal{E}_{k,\infty} = P_k$ is the rank-one fixed-point projection, which is idempotent. □

Theorem 18.5.3 (Phase obstruction to canonical-form flow convergence). Lean ✓ Stmt Let $\omega = e^{2\pi i/3}$ and consider the tensor with one physical component

$$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}.$$

This tensor is in canonical form, with two one-dimensional normal components of weights 1 and ω . For the matrix unit e_{21} ,

$$\mathcal{E}_A^{2^n}(e_{21}) = \omega^{2^n} e_{21}.$$

The right-hand side alternates between ωe_{21} and $\omega^2 e_{21}$. Hence the dyadic renormalization flow does not converge. This refutes the unrestricted canonical-form convergence assertion in [CPGSV16, lines 417 and 1211–1244]; it does not affect the primitive single-component convergence above.

Proof. [✓ Proof](#) The two one-dimensional components are normal and both weights have modulus one. Direct multiplication gives $\mathcal{E}_A(e_{21}) = \omega e_{21}$. Since 2^n is congruent alternately to 1 and 2 modulo 3, the displayed sequence has the two distinct values claimed. $\square$

Definition 18.5.4 (One-letter doubled-MPO virtual-gauge condition). [Lean](#) [Lean](#) [✓ Stmt](#) For an MPO tensor M with one physical component, the Appendix D blocking diagram holds up to virtual gauge if $M^{[2]}$ and M are related by simultaneous similarity on the virtual indices. For a pure tensor A , this condition is imposed on the doubled tensor

$$M = A \otimes \bar{A}.$$

Both physical matrix algebras are one-dimensional, so every physical unitary-conjugation channel in the diagram is the identity. Similarity is symmetric and hence gives both directions of the diagram in [CPGSV16, Appendix D, equation RFP-gauge].

Definition 18.5.5 (One-letter MPS blocking witness). [Lean](#) [✓ Stmt](#) A tensor A with one physical component has a blocking witness if there are an invertible matrix X and a scalar ζ with $|\zeta| = 1$ such that

$$A = \zeta X A^{[2]} X^{-1}.$$

Definition 18.5.6 (Doubled virtual gauge). [Lean](#) [✓ Stmt](#) Given an invertible virtual matrix X , the induced gauge on the doubled bond space is $X \otimes \bar{X}$.

Definition 18.5.7 (Cube-phase swap gauge). [Lean](#) [✓ Stmt](#) The permutation matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines an invertible virtual gauge that exchanges the 1 and ω coordinates of the cube-phase tensor.

Theorem 18.5.8 (An MPS blocking witness gives the doubled-MPO condition). [Lean](#) [✓ Stmt](#) Every one-letter MPS blocking witness induces the exact doubled-MPO virtual-gauge condition.

Proof. [✓ Proof](#) Tensor the identity $A = \zeta X A^{[2]} X^{-1}$ with its complex conjugate. The virtual matrices combine to $X \otimes \bar{X}$, while the amplitude factor is $\zeta \bar{\zeta} = |\zeta|^2 = 1$. $\square$

Theorem 18.5.9 (The printed pure blocking-up-to-gauge equivalence is false). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\omega = e^{2\pi i/3}$, and set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\omega^3 = 1$, $\omega \neq 1$, and

$$A = \omega S A^2 S^{-1}.$$

The doubled MPO tensor generates positive semidefinite operators on every nonempty chain, is in CPSV canonical form, and has four scalar-block weights satisfying the source normalization convention. It also satisfies the literal one-letter virtual-gauge condition in the source diagram, but its transfer map is not idempotent. Hence the canonical-restricted pure-state equivalence claimed after [CPGSV16, Appendix D, equation RFP-gauge] is false.

Proof. ✓ Proof On every nonempty chain, $M = A \otimes \bar{A}$ generates the rank-one operator $|V^{(N)}(A)\rangle\langle V^{(N)}(A)|$, which is positive semidefinite. Moreover, M is the direct sum of four bond-one scalar blocks with weights $\mu_p \bar{\mu}_q$, where $\mu_0 = 1$ and $\mu_1 = \omega$. Each block is normal and every weight has modulus one, proving both the CPSV canonical-form condition and its normalization.

Swapping the two virtual coordinates changes $\text{diag}(1, \omega^2)$ into $\text{diag}(\omega^2, 1)$; multiplication by ω gives A . Writing $M = A \otimes \bar{A}$, doubling cancels the phase through

$$(\omega S A^{[2]} S^{-1}) \otimes (\bar{\omega} \bar{S} \bar{A}^{[2]} \bar{S}^{-1}) = (S \otimes \bar{S}) M^{[2]} (S \otimes \bar{S})^{-1},$$

since $\omega \bar{\omega} = 1$. For every $n \geq 0$, idempotence would imply

$$\mathcal{E}_A^2 = \mathcal{E}_A \implies \mathcal{E}_A^{2^n} = \mathcal{E}_A.$$

The dyadic powers would therefore form a constant, hence convergent, sequence. This contradicts the preceding phase-obstruction theorem, which shows that the dyadic orbit of e_{21} alternates between the distinct factors ω and ω^2 . □

Remark 18.5.10. The counterexample refutes the printed pure equivalence. It does not decide the mixed-state continuation in the following source paragraph.

Remark 18.5.11. When the transfer map is idempotent ($\mathcal{E}^2 = \mathcal{E}$), connected correlations become separation-independent for all separations $n \geq 1$. Informally, this corresponds to zero correlation length ($\xi = 0$). The full spectral equivalence (idempotence iff vanishing subleading spectrum) is deferred here.

Definition 18.5.12 (Virtual-insertion auxiliary condition). Lean ✓ Stmt For every positive-definite right fixed point $\rho^R > 0$, every pair of virtual bond matrices X, Y , and all separations $n, m \geq 1$, the virtual-insertion expression is independent of distance: $C(X, Y; n) = C(X, Y; m)$.

Definition 18.5.13 (Physical observable transfer). Lean ✓ Stmt For an observable O on a block of L physical spins, define the linear map on the virtual matrix space by

$$\mathcal{E}_O(X) = \sum_{i,j} \langle j|O|i \rangle A^i X (A^j)^\dagger.$$

This is the observable transfer map of [CPGSV16, lines 490–496].

Definition 18.5.14 (Two-observable periodic expectation). Lean ✓ Stmt Let O_1, O_2 act on blocks of L_1, L_2 physical spins, separated around the periodic chain by complementary gaps of lengths n_1, n_2 . Their two-observable expectation is

$$\text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{n_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{n_1}).$$

The trace is the operator trace on the virtual matrix space, as in [CPGSV16, lines 490–496].

Definition 18.5.15 (Physical correlations independent of distance). Lean ✓ Stmt Let O_1, O_2 act on two disjoint contiguous regions of positive lengths $L_1, L_2 \geq 1$ in a periodic chain, with complementary gap lengths $n_1, n_2 \geq 0$. Physical correlations are independent of distance when, for every $m_1, m_2 \geq 0$ satisfying $n_1 + n_2 = m_1 + m_2$, one has

$$\text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{n_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{n_1}) = \text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{m_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{m_1}).$$

Thus either region may be translated without crossing the other. Zero gaps, corresponding to adjacent regions, are included. This is [CPGSV16, Definition 3.3 and lines 490–496].

Definition 18.5.16 (Positive-gap restriction of physical distance independence). [Lean](#) [Stmnt](#) For an observable O on a block of L physical spins, write

$$\mathcal{E}_O(X) = \sum_{i,j} \langle j|O|i\rangle A^i X(A^j)^\dagger.$$

Place two observables O_1, O_2 on nonempty finite blocks of a periodic chain, with positive complementary gap lengths n_1, n_2 . The tensor has *positive-gap physical correlations independent of distance* when the expectation is unchanged upon replacing these gaps by positive m_1, m_2 satisfying $n_1 + n_2 = m_1 + m_2$:

$$\text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{n_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{n_1}) = \text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{m_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{m_1}).$$

This positive-gap restriction is the transfer-matrix form of [CPGSV16, Definition 3.3 and the correlation formula preceding Theorem 3.8], restricted to positive complementary gaps. It excludes adjacent regions.

Lemma 18.5.17 (Physical CID implies positive-gap physical CID). [Lean](#) [Stmnt](#) *Correlations independent of distance for all disjoint physical regions are independent of distance when both complementary gaps are positive.*

Proof. [Proof](#) Restrict the four gap lengths to positive integers. □

Theorem 18.5.18 (Idempotent transfer implies positive-gap physical CID). [Lean](#) [Stmnt](#) *If $\mathcal{E}_A^2 = \mathcal{E}_A$, then physical correlations are independent of distance whenever both complementary gaps are positive.*

Proof. [Proof](#) Idempotence gives $\mathcal{E}_A^n = \mathcal{E}_A$ for every $n \geq 1$. Substituting $\mathcal{E}_A^{n_1} = \mathcal{E}_A^{n_2} = \mathcal{E}_A^{m_1} = \mathcal{E}_A^{m_2} = \mathcal{E}_A$ in the two-observable transfer formula gives

$$\begin{aligned} \text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{n_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{n_1}) &= \text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A) \\ &= \text{Tr}_{M_D(\mathbb{C})}(\mathcal{E}_{O_2} \circ \mathcal{E}_A^{m_2} \circ \mathcal{E}_{O_1} \circ \mathcal{E}_A^{m_1}). \end{aligned}$$
□

Definition 18.5.19 (Virtual-insertion one-block auxiliary condition). [Lean](#) [Stmnt](#) This auxiliary one-block predicate is the conjunction of $\mathcal{E}_A^2 = \mathcal{E}_A$ and virtual-insertion distance independence. It is not the physical zero-correlation-length condition of Definition 18.2.6.

Theorem 18.5.20 (Virtual-insertion distance independence implies RFP). [Lean](#) [Stmnt](#) *Suppose $\rho^R > 0$ is a right fixed point of the transfer map. If the virtual-insertion condition holds for every positive-definite right fixed point, then the transfer map is idempotent.*

Proof. [Proof](#) Let $\rho > 0$ be the fixed point. For every matrix Z , invertibility gives $Z = (Z\rho^{-1})\rho$. Apply distance independence with $X = Z\rho^{-1}$, the second insertion N , and separations 2 and 1. The identical disconnected terms cancel. Thus, for every N , $\text{tr}(N\mathcal{E}_A^2(Z)) = \text{tr}(N\mathcal{E}_A(Z))$. Nondegeneracy of the trace pairing yields $\mathcal{E}_A^2(Z) = \mathcal{E}_A(Z)$ for every Z , hence $\mathcal{E}_A^2 = \mathcal{E}_A$. □

Theorem 18.5.21 (Redundancy of the one-block auxiliary condition). [Lean](#) [Stmnt](#) *The auxiliary conjunction ($\mathcal{E}_A^2 = \mathcal{E}_A$ and virtual-insertion distance independence) is equivalent to $\mathcal{E}_A^2 = \mathcal{E}_A$. This is a logical simplification of the auxiliary convention, not the physical ZCL theorem of [CPGSV16, Theorem 3.8].*

Proof. ✓ Proof The forward direction extracts the idempotence hypothesis. Conversely, idempotence gives $\mathcal{E}_A^n = \mathcal{E}_A$ for every $n \geq 1$. Hence, for every positive-definite right fixed point ρ^R , every pair of virtual matrices X, Y , and every $n \geq 1$,

$$\begin{aligned} C(X, Y; n) &= \text{tr}(Y \mathcal{E}_A^n(X \rho^R)) - \text{tr}(X \rho^R) \text{tr}(Y \rho^R) \\ &= \text{tr}(Y \mathcal{E}_A(X \rho^R)) - \text{tr}(X \rho^R) \text{tr}(Y \rho^R) \\ &= C(X, Y; 1). \end{aligned}$$

Thus the connected correlator is independent of every positive separation. □

Theorem 18.5.22 (Per-block auxiliary equivalence). Lean ✓ Stmt *Let $(A_k)_k$ satisfy the auxiliary block-family hypotheses of Definition 18.3.1.1. For each k , the transfer map of A_k is idempotent if and only if A_k satisfies the one-block auxiliary conjunction of Definition 18.5.19. The block-family hypothesis is unused; this is Theorem 18.5.21 applied to one block. It should not be read as the BNT-family ZCL assertion in [CPGSV16, Theorem 3.10].*

Proof. ✓ Proof Apply Theorem 18.5.21 to the chosen block. □

Chapter 19

Matrix Product Unitaries

This chapter develops the algebraic prerequisites for matrix product unitaries. The first result supplies the corrected nil-matrix bound used in the blocking argument of [CPGSV17].

Theorem 19.1 (Dimension bound for a nilpotent Lie action). [Lean](#) [✓ Stmt](#) *Let a Lie algebra L act nilpotently on a finite-dimensional vector space V over a field K . Write $C_r(L, V)$ for the r th term of the lower central series of this action. Then*

$$C_{\dim_K(V)}(L, V) = 0. \quad (19.1)$$

Proof. [✓ Proof](#) As long as $C_r(L, V)$ is nonzero, nilpotency prevents two consecutive terms from being equal: equality would make every later term equal, in contradiction with eventual vanishing. Thus each nonzero step strictly lowers dimension, and at most $\dim_K(V)$ such steps are possible. $\square$

Theorem 19.2 (Finite-dimensional nil-matrix theorem). [Lean](#) [✓ Stmt](#) *Let V be a finite-dimensional vector space over a field K , and let $A \subseteq \text{End}_K(V)$ be a not necessarily unital associative subalgebra. If every element of A is nilpotent, then every ordered product of $\dim_K(V)$ elements of A is zero. The empty product in dimension zero is also zero because the identity endomorphism of the zero space is zero.*

Proof. [✓ Proof](#) Regard A as a Lie algebra under the commutator bracket. Engel's theorem makes V a nilpotent Lie module, so the dimension bound gives

$$C_{\dim_K(V)}(A, V) = 0. \quad (19.2)$$

For every nonnegative integer r , every ordered family $a_1, \dots, a_r$ in A , and every vector v in V , induction on r gives

$$(a_1 \cdots a_r)v \in C_r(A, V). \quad (19.3)$$

Taking $r = \dim_K(V)$ proves that the product annihilates every vector and is therefore zero. $\square$

Theorem 19.3 (Nil matrix subalgebras). [Lean](#) [✓ Stmt](#) *Let A be a not necessarily unital associative subalgebra of $n \times n$ matrices over a field. If every matrix in A is nilpotent, then every ordered product of n matrices from A vanishes.*

Proof. [✓ Proof](#) Transport the matrix algebra to the endomorphisms of the coordinate space through the standard algebra equivalence. Nilpotency, membership, and the ordered product are preserved, so the finite-dimensional nil-matrix theorem applies; injectivity of the equivalence returns the zero matrix. $\square$

The length n is a weak bound, not a strict bound below n ; the latter is false for strictly upper-triangular matrices. This corrects the intermediate strict inequality in [CPGSV17] without changing its final strict D^4 estimate.

19.1 The matrix product unitary condition

Let $\mathcal{U} = \{U^{ij}\}_{i,j=0}^{d-1}$ be a matrix product operator tensor of bond dimension D , and write $U^{(N)}$ for the periodic operator obtained by closing a chain of N copies of $\mathcal{U}$.

Theorem 19.1.1 (Unitarity under simultaneous reindexing). [Lean](#) [✓ Stmt](#) *Let I and J be finite sets, let $e: I \rightarrow J$ be a bijection, and let A be a unitary matrix indexed by I . The matrix indexed by J whose (j, k) entry is*

$$A_{e^{-1}(j), e^{-1}(k)} \tag{19.4}$$

is unitary.

Proof. [✓ Proof](#) Simultaneous reindexing preserves the identity and matrix multiplication, and it commutes with the conjugate transpose. Reindex the equation $AA^\dagger = \mathbb{1}$. $\square$

Definition 19.1.2 (Matrix product unitary tensor). [Lean](#) [Lean](#) [✓ Stmt](#) The tensor $\mathcal{U}$ generates matrix product unitaries if $U^{(N)}$ is unitary for every integer $N > 1$. This is the matrix product unitary condition in [CPGSV17, Section III, equation (10)].

Lemma 19.1.3 (First matrix product unitarity equation). [Lean](#) [✓ Stmt](#) *If $\mathcal{U}$ generates matrix product unitaries and $N > 1$, then*

$$U^{(N)}(U^{(N)})^\dagger = \mathbb{1}. \tag{19.5}$$

Proof. [✓ Proof](#) Expand the unitary-group condition for $U^{(N)}$. $\square$

Lemma 19.1.4 (Second matrix product unitarity equation). [Lean](#) [✓ Stmt](#) *If $\mathcal{U}$ generates matrix product unitaries and $N > 1$, then*

$$(U^{(N)})^\dagger U^{(N)} = \mathbb{1}. \tag{19.6}$$

This is the unitarity equation displayed in [CPGSV17, Section III, equation (12)].

Proof. [✓ Proof](#) Use the equivalent left-multiplied characterization of a unitary matrix. $\square$

Theorem 19.1.5 (Composition preserves matrix product unitarity). [Lean](#) [✓ Stmt](#) *Let $\mathcal{U}$ and $\mathcal{V}$ be matrix product unitary tensors with the same physical dimension. Contracting the output index of $\mathcal{U}$ with the input index of $\mathcal{V}$ gives a matrix product unitary tensor whose periodic operator is $U^{(N)}V^{(N)}$, in that order. This is the concatenation tensor $\mathcal{W}$ in the proof of [CPGSV17, Theorem IV.6(ii)].*

Proof. [✓ Proof](#) For every $N > 1$, both $U^{(N)}$ and $V^{(N)}$ are unitary. The defining contraction of the product tensor gives the periodic operator $U^{(N)}V^{(N)}$, which is unitary because the unitary matrices form a group under multiplication. $\square$

Lemma 19.1.6 (Physical blocking preserves matrix product unitarity). Lean ✓ Stmt Suppose that $\mathcal{U}$ generates matrix product unitaries. For every positive integer L , the tensor $\mathcal{U}^{[L]}$ obtained by blocking L adjacent sites also generates matrix product unitaries. This is the blocking operation of [CPGSV17, Definition II.5 and equation (9)]; preservation of the MPU property under blocking is used in the proof of [CPGSV17, Proposition III.3].

Proof. ✓ Proof At system size N , closing the blocked tensor gives $U^{(NL)}$, up to the canonical bijection between N blocked configurations and NL unblocked configurations. If $N > 1$ and $L > 0$, then $NL > 1$, so $U^{(NL)}$ is unitary. Simultaneous reindexing preserves unitarity. $\square$

19.2 Simple matrix product unitary tensors

Definition 19.2.1 (Double layer). Lean Lean ✓ Stmt For an MPO tensor $\mathcal{U}$, let $\mathcal{W}$ be the double-layer tensor obtained by placing the physical adjoint above $\mathcal{U}$ and contracting the adjacent physical legs:

$$\mathcal{W}^{ik} = \sum_j (\mathcal{U}^\sharp)^{ij} \otimes \mathcal{U}^{jk}. \quad (19.7)$$

The bond space of $\mathcal{W}$ is the product of the two original bond spaces. The physical adjoint exchanges and conjugates the physical indices without reversing the order of the virtual chain, exactly as in the barred upper layer of Figures II_Simple1.png and II_Simple2.png of [CPGSV17].

Theorem 19.2.2 (Closed double layer). Lean ✓ Stmt For every chain length N , closing the double layer gives

$$W^{(N)} = (U^{(N)})^\dagger U^{(N)}. \quad (19.8)$$

Proof. ✓ Proof The operator family is multiplicative under the local product tensor, and the physical adjoint generates the conjugate transpose without reflecting the chain. $\square$

Theorem 19.2.3 (Physical-adjoint identities for matrix product unitaries). Lean Lean Lean ✓ Stmt The physical adjoint is involutive. If $\mathcal{U}$ generates matrix product unitaries, then so does $\mathcal{U}^\sharp$, and for every chain length N ,

$$(\mathcal{U}^\sharp)^\sharp = \mathcal{U}, \quad (\mathcal{U}^\sharp)^{(N)} = (\mathcal{U}^{(N)})^\dagger. \quad (19.9)$$

Proof. ✓ Proof Applying the physical adjoint twice conjugates every entry twice and exchanges the physical indices twice. Closing a physical-adjoint tensor exchanges the two physical configurations and conjugates the resulting scalar. Hence its periodic operator is the conjugate transpose. The first MPU unitarity equation for $U^{(N)}$ is therefore the second unitarity equation for $(U^\sharp)^{(N)}$. $\square$

Theorem 19.2.4 (Output-first double-layer entry). Lean ✓ Stmt Order the left doubled bond as (α, γ) and the right doubled bond as (β, δ) . Then the double layer whose lower tensor is $\mathcal{U}^\sharp$ has entries

$$\mathcal{W}(\mathcal{U}^\sharp)_{(\alpha, \gamma), (\beta, \delta)}^{ik} = \sum_j \mathcal{U}_{\alpha, \beta}^{ij} \overline{\mathcal{U}_{\gamma, \delta}^{kj}}. \quad (19.10)$$

Proof. ✓ Proof Expand the double layer and then apply the physical adjoint to its upper factor. The two exchanges of physical indices leave the contracted index j in the second physical position of both factors. The product-index convention keeps (α, γ) and (β, δ) in the displayed order. $\square$

Definition 19.2.5 (Simple matrix product unitary tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The tensor $\mathcal{U}$ is *simple* if there are vectors a, b in the double-layer bond space such that, for all physical indices i, j, k, ℓ ,

$$(a|W^{ij}|b) = \delta_{ij}, \quad (\text{simple1})$$

$$W^{ij}W^{k\ell} = W^{ij}|b)(a|W^{k\ell}. \quad (\text{simple2})$$

Thus (simple1) closes one double-layer letter to the physical identity, while (simple2) inserts the rank-one operator $|b)(a|$ between adjacent letters. These are precisely the two diagrams in [CPGSV17, Definition III.2].

Theorem 19.2.6 (Sequential contraction of a simple double layer). [Lean](#) [✓ Stmt](#) If $\mathcal{U}$ is simple and $N > 1$, then

$$W^{(N)} = \mathbb{1}. \quad (19.11)$$

Proof. [✓ Proof](#) Apply (simple2) sequentially between neighboring letters. The inserted rank-one operators split the periodic contraction into the product of the one-letter contractions $(a|W^{\sigma_x \tau_x}|b)$. Identity (simple1) turns this product into $\prod_x \delta_{\sigma_x, \tau_x}$, which is the matrix element of the physical identity. $\square$

Theorem 19.2.7 (A simple tensor generates matrix product unitaries). [Lean](#) [✓ Stmt](#) Every simple tensor generates matrix product unitaries. This is [CPGSV17, Proposition III.3(i)].

Proof. [✓ Proof](#) For every $N > 1$, the closed-double-layer identity and the sequential contraction give

$$(U^{(N)})^\dagger U^{(N)} = \mathbb{1}. \quad (19.12)$$

Hence $U^{(N)}$ is unitary. $\square$

19.3 Double-layer blocking and mixed physical traces

Theorem 19.3.1 (Physical adjoints and double layers commute with blocking). [Lean](#) [Lean](#) [✓ Stmt](#) For every blocking length L , physical adjunction commutes with blocking, and consequently

$$\mathcal{W}(\mathcal{U}^{[L]}) = \mathcal{W}(\mathcal{U})^{[L]}. \quad (19.13)$$

The virtual order is unchanged on both sides.

Proof. [✓ Proof](#) Physical adjunction conjugates each matrix entry without reversing the virtual word. The first identity follows entrywise. The second follows because physical blocking commutes with the local operator product. $\square$

Theorem 19.3.2 (Blocked double-layer diagonal is a transfer power). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Assume $d > 0$, and let E be the normalized transfer matrix of $\mathcal{U}$. After blocking L sites, the normalized physical diagonal of the double layer is exactly E^L . More precisely, if

$$q : \{0, \dots, D-1\} \times \{0, \dots, D-1\} \longrightarrow \{0, \dots, D^2-1\} \quad (19.14)$$

is the standard product encoding, then its entries are

$$(E^L)_{q^{-1}(a), q^{-1}(b)}. \quad (19.15)$$

Thus the row and column reindexings have the same explicit orientation. If $\mathcal{U}$ is an MPU and $D > 1$, then at the stabilization exponent J this diagonal is the same reindexing of the rank-one projector $|\rho)(\Phi|$.

Proof. ✓ Proof On one site, expanding the diagonal double-layer contraction gives the transfer matrix after the stated product reindexing. The factor d^{-1} is the square of the local factor $d^{-1/2}$. Summing blocked diagonal letters over words and distributing the ordered product shows that normalized diagonals turn blocking into powers. Simultaneous reindexing preserves multiplication, giving the displayed formula. Transfer stabilization gives the final rank-one specialization. $\square$

Theorem 19.3.3 (Normalized diagonal as a diagonal-word sum). Lean ✓ Stmt Let $\mathcal{W}$ be an MPO tensor of physical dimension d , and let

$$R = \frac{1}{d} \sum_{i=0}^{d-1} W^{ii} \quad (19.16)$$

be its normalized physical diagonal. Here and in the two diagonal-tail identities below, scalar inverses are the totalized complex-field inverses, so $0^{-1} = 0$. For every $K \geq 0$,

$$R^K = d^{-K} \sum_{\tau \in \{0, \dots, d-1\}^K} W^{\tau_0 \tau_0} \dots W^{\tau_{K-1} \tau_{K-1}}. \quad (19.17)$$

At $K = 0$, the word product is the identity matrix. This is the algebraic diagonal-tail expansion used in the source-factor networks of [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Proof. ✓ Proof Expand the K th power of the finite sum defining R . Distributivity indexes the resulting terms by physical words τ , and each term is the ordered diagonal word displayed above. $\square$

Theorem 19.3.4 (A matrix basis adapted to the trace). Lean ✓ Stmt If $d > 0$, the identity matrix extends to a basis of $M_d(\mathbb{C})$ whose other elements are traceless.

Proof. ✓ Proof The trace functional is nonzero on the identity. Extend the identity by a basis of the kernel of the trace. $\square$

Theorem 19.3.5 (Identity plus traceless physical decomposition). Lean Lean ✓ Stmt Assume $d > 0$. For every double-layer tensor $\mathcal{W}$, there is a basis $\{\sigma_0, \sigma_\alpha\}$ of the physical matrix algebra and virtual matrices E, S_α such that $\sigma_0 = \mathbb{1}$ and, for every nonzero index α , $\text{tr}(\sigma_\alpha) = 0$, while

$$\mathcal{W} = E \otimes \mathbb{1} + \sum_{\alpha \neq 0} S_\alpha \otimes \sigma_\alpha. \quad (19.18)$$

Here E is the normalized physical diagonal. Equivalently, contraction against a physical matrix X separates as

$$\text{contr}_{\text{phys}}(\mathcal{W}, X) = \text{tr}(X)E + R_{\mathcal{W}}(X), \quad (19.19)$$

$$R_{\mathcal{W}}(X) = \text{contr}_{\text{phys}}\left(\mathcal{W}, X - \frac{\text{tr}(X)}{d} \mathbb{1}\right), \quad (19.20)$$

where the second physical argument is traceless. This is equation *WIsom* in [CPGSV17, Proposition III.3(ii)].

Proof. ✓ **Proof** The trace functional is nonzero on the identity. Extend the identity to a basis adapted to the kernel of the trace; all remaining basis vectors are therefore traceless. The bilinear pairing $(X, Y) \mapsto \text{tr}(XY)$ is nondegenerate. Its dual basis gives the virtual coefficients by physical contraction. The dual vector paired with the identity is $d^{-1}\mathbb{1}$, so its virtual coefficient is precisely E . Expanding matrix units in the dual basis yields the decomposition. Subtracting the trace component gives the equivalent residual formula. $\square$

Theorem 19.3.6 (Multilinear physical contraction). Lean Lean ✓ Stmt Suppose a length- N closed tensor equals the physical identity. For arbitrary physical matrices $X_0, \dots, X_{N-1}$,

$$\text{tr} \left(\prod_{k=0}^{N-1} \text{contr}_{\text{phys}}(\mathcal{W}, X_k) \right) = \prod_{k=0}^{N-1} \text{tr}(X_k). \quad (19.21)$$

In particular this applies to the double layer of an MPU whenever $N > 1$.

Proof. ✓ **Proof** Expand each contraction and distribute the ordered product. The closed tensor coefficient is a Kronecker delta between the two physical words, so only equal words survive. The remaining independent diagonal sums factor into the product of the physical traces. $\square$

Theorem 19.3.7 (Positive-length mixed residual traces). Lean Lean ✓ Stmt Let $\mathcal{U}$ be an MPU with physical dimension $d > 0$. Every closed product of positive length that contains at least one residual coefficient has zero trace. In particular, this includes a single residual coefficient. This is the mixed-trace assertion preceding equation *ESE=0* in [CPGSV17, Proposition III.3(ii)].

Proof. ✓ **Proof** At lengths greater than one, represent identity factors by $d^{-1}\mathbb{1}$ and residual factors by traceless physical arguments. The multilinear contraction identity then vanishes. For a single residual coefficient A , repeat the same coefficient at every length $q > 1$ to obtain $\text{tr}(A^q) = 0$. Thus the one-letter case is recovered after the longer words: the shifted zero-trace-power theorem makes A nilpotent, and therefore $\text{tr}(A) = 0$. $\square$

Theorem 19.3.8 (Controlled mixed residual traces and rank-one sandwich). Lean Lean Lean ✓ Stmt Let $\mathcal{U}$ be an MPU with physical dimension $d > 0$, and let $N > 1$. Every closed product of N identity and residual coefficients that contains at least one residual coefficient has zero trace. If the normalized diagonal is rank one, then every nonempty residual word P satisfies

$$EPE = 0. \quad (19.22)$$

The same conclusion holds after blocking $J > 0$ sites whenever the J th normalized transfer power is $|\rho)(\Phi|$, with both virtual indices reindexed by $q^{-1} : \{0, \dots, D^2-1\} \rightarrow \{0, \dots, D-1\} \times \{0, \dots, D-1\}$.

Proof. ✓ **Proof** Represent identity factors by $d^{-1}\mathbb{1}$ and residual factors by their traceless physical arguments. Since $N > 1$, the multilinear contraction identity applies, and one trace factor vanishes. For a nonempty residual word, adjoin E as one extra factor; its closed trace has length at least two and therefore vanishes. Cyclicity of trace and

$$|\rho)(\Phi|P|\rho)(\Phi| = \text{tr}(P|\rho)(\Phi|) |\rho)(\Phi| \quad (19.23)$$

give $EPE = 0$. The blocked statement follows from the transfer-power formula with the same product-index reindexing in both matrix indices. $\square$

Definition 19.3.9 (Residual nonunital algebra). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ have bond dimension D . Contract the double layer against arbitrary physical matrices, subtract the normalized diagonal coefficient, and let $\mathcal{S}$ be the resulting set of residual matrices in $M_{D^2}(\mathbb{C})$. The residual algebra $\mathfrak{S}$ is the not necessarily unital algebra generated by $\mathcal{S}$. This is the algebra generated by the matrices S'_α in [CPGSV17, Proposition III.3(ii)].

Theorem 19.3.10 (Trace and nilpotence of the residual algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $\mathcal{U}$ is an MPU with positive physical dimension, then every nonempty product of residual generators has zero trace. Consequently, for every $A \in \mathfrak{S}$ and every integer $N > 1$,

$$\mathrm{tr}(A) = 0, \quad \mathrm{tr}(A^N) = 0, \quad (19.24)$$

and every element of $\mathfrak{S}$ is nilpotent.

Proof. [✓ Proof](#) A nonempty product in the multiplicative semigroup generated by $\mathcal{S}$ is a nonempty residual word, so its trace vanishes by the positive-length mixed residual identity. The generated nonunital algebra is the linear span of this semigroup. Linearity of the trace therefore gives $\mathrm{tr}(A) = 0$ for every $A \in \mathfrak{S}$. Since every positive power of A remains in $\mathfrak{S}$, the same argument gives $\mathrm{tr}(A^N) = 0$ for $N > 1$. The shifted zero-trace-power theorem now implies that A is nilpotent. $\square$

Theorem 19.3.11 (Length- D^2 residual products vanish). [Lean](#) [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ be an MPU of bond dimension D and positive physical dimension. Every ordered product of exactly D^2 elements of $\mathfrak{S}$ vanishes. In particular, for arbitrary residual generators,

$$S_{\alpha_1} \cdots S_{\alpha_{D^2}} = 0. \quad (19.25)$$

This is equation `Sprime=0` in [CPGSV17, Proposition III.3(ii)], with the corrected exact length D^2 in place of the unsupported strict intermediate bound.

Proof. [✓ Proof](#) The algebra $\mathfrak{S}$ acts on a space of dimension D^2 , and every element is nilpotent. The finite-dimensional nil-matrix theorem therefore annihilates every ordered product of D^2 elements. Each residual generator belongs to $\mathfrak{S}$, giving the stated specialization. $\square$

Definition 19.3.12 (Three-form residual subspace). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let E be the normalized diagonal after the first blocking and let $\mathfrak{S}$ be its residual nonunital algebra. Define

$$\mathcal{T} = \mathrm{span}\{EA : A \in \mathfrak{S}\} + \mathrm{span}\{AE : A \in \mathfrak{S}\} \quad (19.26)$$

$$+ \mathrm{span}\{AEB : A, B \in \mathfrak{S}\}. \quad (19.27)$$

This is the basis-independent sum of the three forms in equation `sprimeforms` of [CPGSV17, Proposition III.3(ii)].

Theorem 19.3.13 (Ordered second-block expansion). [Lean](#) [✓ Stmt](#) Suppose $E^2 = E$, $EAE = 0$ for every $A \in \mathfrak{S}$, and every ordered product of $N > 0$ elements of $\mathfrak{S}$ vanishes. For scalars c_k and elements $S_k \in \mathfrak{S}$,

$$\prod_{k=0}^{N-1} (c_k E + S_k) - \left(\prod_{k=0}^{N-1} c_k \right) E \in \mathcal{T}. \quad (19.28)$$

Products are taken in increasing site order and need not commute.

Proof. ✓ **Proof** Expand the product as a sum over binary choices of E or S_k , retaining the original order. The all- E choice is the subtracted term because $E^q = E$ for $q > 0$. The all-residual choice vanishes by the length- N hypothesis. In every remaining word, contiguous copies of E collapse. If two resulting copies of E are separated by a nonempty residual word, the factor EAE makes the word zero. Thus each surviving word has one contiguous projector block and belongs to one of the three summands of $\mathcal{T}$. $\square$

Theorem 19.3.14 (Second-block residuals lie in the three-form subspace). Lean Lean Lean ✓ **Stmt** Let the first-block tensor be an MPU of bond dimension $D > 0$. Assume its normalized double-layer diagonal E is a rank-one idempotent. After a second blocking of exactly D^2 sites, every residual coefficient belongs to $\mathcal{T}$.

The conclusion is membership in the sum of the three subspaces. It does not assign one exclusive type to a coefficient in an arbitrary physical basis.

Proof. ✓ **Proof** First extend $EPE = 0$ from nonempty residual words P to every element of $\mathfrak{S}$ by writing the generated algebra as the linear span of the multiplicative semigroup of residual generators. For a blocked matrix unit, each local double-layer entry is

$$\delta_{ij}E + R_{ij}. \quad (19.29)$$

The ordered expansion theorem applies with $N = D^2$. Its all- E coefficient is the blocked Kronecker delta and is exactly removed by the residual slice. Hence every blocked matrix-unit residual lies in $\mathcal{T}$. Expanding an arbitrary physical matrix in matrix units and using linearity proves the result. $\square$

Theorem 19.3.15 (Three-form contraction identities). Lean Lean ✓ **Stmt** Suppose $E^2 = E$ and $EAE = 0$ for every $A \in \mathfrak{S}$. For all $X, Y \in \mathcal{T}$,

$$XY = XEY, \quad \text{tr}(XE) = 0. \quad (19.30)$$

Proof. ✓ **Proof** It is enough by bilinearity and linearity of trace to consider the generators EA , AE , and AEB of the three summands. Representative products reduce as

$$(EA)(BE) = E(AB)E = 0, \quad (19.31)$$

$$(AE)(EB) = AEB = (AE)E(EB), \quad (19.32)$$

$$\text{tr}((AEB)E) = \text{tr}(A(EBE)) = 0. \quad (19.33)$$

The remaining generator pairs follow by the same use of associativity, $E^2 = E$, and $EAE = 0$ for $A \in \mathfrak{S}$. $\square$

Theorem 19.3.16 (Exact simplicity after the second blocking). Lean Lean ✓ **Stmt** Let $\mathcal{V}$ be an MPU of positive physical and bond dimensions whose normalized double-layer diagonal is

$$E = |\rho\rangle\langle\Phi|, \quad \langle\Phi|\rho\rangle = 1. \quad (19.34)$$

Then blocking $\mathcal{V}$ by exactly D^2 sites produces a simple tensor with witnesses $a = \Phi$ and $b = \rho$.

Proof. ✓ **Proof** The pairing normalization makes E idempotent. Write every blocked double-layer coefficient as $\delta_{ij}E + R_{ij}$, where $R_{ij} \in \mathcal{T}$. The trace identity for $R_{ij}E$ gives the diagonal contraction, and $R_{ij}R_{k\ell} = R_{ij}ER_{k\ell}$, together with $E^2 = E$, gives the rank-one insertion identity after expanding both coefficients. $\square$

Definition 19.3.17 (Physical-basis reindexing). [Lean](#) [Lean](#) [✓ Stmt](#) For a bijection e between two physical index sets, the reindexed tensor and its action on length- N configurations are

$$(e^*\mathcal{U})^{ij} = \mathcal{U}^{e(i)e(j)}, \quad (e_*\sigma)(n) = e(\sigma(n)). \quad (19.35)$$

Theorem 19.3.18 (Words under physical reindexing). [Lean](#) [✓ Stmt](#) For ket and bra words i, j , physical reindexing gives

$$(e^*\mathcal{U})[i, j] = \mathcal{U}[e_*i, e_*j]. \quad (19.36)$$

Proof. [✓ Proof](#) Induct simultaneously along the two words. The empty pair gives the identity matrix, mismatched lengths give zero, and the nonempty case applies the defining equality to the first letters and the induction hypothesis to the tails. $\square$

Theorem 19.3.19 (Finite-size operators under physical reindexing). [Lean](#) [✓ Stmt](#) At every length N , the sitewise configuration bijection identifies

$$(e^*\mathcal{U})^{(N)} \cong \mathcal{U}^{(N)}. \quad (19.37)$$

Proof. [✓ Proof](#) Apply the word identity to every pair of configurations and take the virtual trace. This is simultaneous row and column reindexing by e_* . $\square$

Theorem 19.3.20 (Double layer under physical reindexing). [Lean](#) [✓ Stmt](#) For every physical-index bijection e , taking the double layer commutes with physical reindexing:

$$W(e^*\mathcal{U})^{ij} = W(\mathcal{U})^{e(i)e(j)}. \quad (19.38)$$

Proof. [✓ Proof](#) This follows entrywise from the definitions of physical reindexing and the double layer. $\square$

Lemma 19.3.21 (Iterated physical blocking). [Lean](#) [✓ Stmt](#) Let e be the canonical bijection which groups a word of length mn into n consecutive words of length m . Iterating blocking at lengths m and n , then applying this physical reindexing, agrees with direct blocking at length mn :

$$e^*((\mathcal{U}_m)_n) = \mathcal{U}_{mn}. \quad (19.39)$$

Proof. [✓ Proof](#) Expand the two blocked words. The canonical regrouping concatenates their n words of length m into the direct word of length mn . $\square$

Theorem 19.3.22 (Bounded blocking to simplicity). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Every MPU of positive physical and bond dimensions has a positive blocking length $k \leq D^4$ for which the blocked tensor is simple. If $D > 1$, one may choose

$$k = JD^2 < D^4, \quad 0 < J \leq D^2 - 1. \quad (19.40)$$

If $D = 1$, the length-one block is simple. Simplicity is preserved under every further iterated blocking by a positive factor and under a bijective relabeling of the physical basis.

Proof. [✓ Proof](#) At the stabilization exponent J , the normalized double-layer diagonal is the rank-one fixed projector $|\rho\rangle\langle\Phi|$. The preceding theorem makes the subsequent length- D^2 block simple. If e is the canonical grouping bijection, the two blocking stages satisfy

$$e^*((\mathcal{U}_J)_{D^2}) = \mathcal{U}_{JD^2}. \quad (19.41)$$

For $D > 1$,

$$JD^2 < (D^2)D^2 = D^4. \quad (19.42)$$

The one-dimensional transfer matrix is already the identity. For a further block with letters $W_1, \dots, W_L$ and $R = |\rho\rangle\langle\Phi|$, repeated insertion gives

$$W_1 \cdots W_L = W_1 R W_2 R \cdots R W_L, \quad (19.43)$$

$$(\Phi|W_1 \cdots W_L|\rho) = \prod_{q=1}^L (\Phi|W_q|\rho). \quad (19.44)$$

The second identity multiplies the sitewise Kronecker deltas. $\square$

Theorem 19.3.23 (Matching contractions at the direct stabilized block). [Lean](#) [Stmt](#) Let $\mathcal{U}$ be an MPU of positive physical and bond dimensions. Suppose that ρ has trace one and that, for some positive integer J , the normalized transfer matrix satisfies

$$E^J = |\rho\rangle\langle\mathbb{1}|. \quad (19.45)$$

Put $K = JD^2$. After the standard product reindexing of the physical coordinates, the direct length- K block satisfies the contractions [\(simple1\)](#) and [\(simple2\)](#) with the specific witnesses ρ and $\mathbb{1}$ from this transfer-power identity.

Proof. [Proof](#) First block J sites. The normalized diagonal of this block is the reindexed rank-one matrix $|\rho\rangle\langle\mathbb{1}|$, and the trace-one condition is exactly $(\mathbb{1}|\rho) = 1$. The exact second-block contraction theorem then supplies both identities after a further D^2 sites. The canonical equivalence between iterated blocking and direct length- JD^2 blocking transports each double-layer coefficient without changing either virtual witness. $\square$

Theorem 19.3.24 (Normalized output-tail coisometry). [Lean](#) [Stmt](#) Let $d > 0$, let $\mathcal{U}$ be an MPU, and let $K \geq 0$. For $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$, separate the first two output sites from a tail $\tau \in \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$. Then

$$d^{-K} \sum_{\tau} \sum_{\eta} U_{(q,\tau),\eta}^{(K+2)} \overline{U_{(p,\tau),\eta}^{(K+2)}} = \delta_{p,q}. \quad (19.46)$$

The retained matrix entry has the reversed order (q, p) , and the global contraction is the output-first equation $U^{(K+2)}(U^{(K+2)})^\dagger = \mathbb{1}$.

Proof. [Proof](#) Apply output-first unitarity before performing any source-factor rearrangement. For each fixed tail, the inner sum is the corresponding identity-matrix entry, hence it is $\delta_{p,q}$. There are exactly

$$|\{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}| = d^K \quad (19.47)$$

tails, and $d^{-K}d^K = 1$ because $d > 0$. $\square$

Theorem 19.3.25 (Normalized input-tail isometry). [Lean](#) [Stmt](#) Let $d > 0$, let $\mathcal{U}$ be an MPU, and let $K \geq 0$. For $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$, separate the first two input sites from a common input tail $\tau \in \{0, \dots, K-1\} \rightarrow \{0, \dots, d-1\}$. Then

$$d^{-K} \sum_{\tau} \sum_{\eta} \overline{U_{\eta,(p,\tau)}^{(K+2)}} U_{\eta,(q,\tau)}^{(K+2)} = \delta_{p,q}. \quad (19.48)$$

The global contraction is the input-first equation $(U^{(K+2)})^\dagger U^{(K+2)} = \mathbb{1}$. This is the input-oriented tail contraction supporting the network calculation in [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Proof. ✓ Proof Apply input-first unitarity for each fixed tail. The inner output sum is $\delta_{p,q}$. Summing over the d^K tails and multiplying by d^{-K} leaves the same Kronecker delta. $\square$

Theorem 19.3.26 (Simplicity at every later direct blocking). Lean Lean Lean ✓ Stmt Let $\mathcal{U}$ generate an MPU, and suppose its direct length- k block is simple for some $k > 0$. Then its direct length- k' block is simple for every $k' \geq k$, without any divisibility condition on k' by k . This is the corollary following [CPGSV17, Proposition III.3].

Proof. ✓ Proof First suppose vectors a, b satisfy the rank-one insertion identity

$$W^{ij}W^{k\ell} = W^{ij}|b)(a|W^{k\ell}. \quad (19.49)$$

Set $x_{ij} = (a|W^{ij}|b)$. Applying this identity around the constant two-site and three-site words, and then using MPU unitarity, gives

$$x_{ij}^2 = \delta_{ij}, \quad (19.50)$$

$$x_{ij}^3 = \delta_{ij}. \quad (19.51)$$

Hence $x_{ii} = 1$, while $x_{ij} = 0$ for $i \neq j$, so the same vectors obey the one-letter contraction $(a|W^{ij}|b) = \delta_{ij}$.

For the successor step, take four words I, J, K, L of length $k+1$ and write $W^{I,J} = W^{i_0 j_0} \dots W^{i_k j_k}$, with the analogous notation for $W^{K,L}$. Apply the length- k rank-one insertion identity to the suffixes $I_+ = (i_1, \dots, i_k)$, $J_+ = (j_1, \dots, j_k)$ and the prefixes $K_- = (k_0, \dots, k_{k-1})$, $L_- = (\ell_0, \dots, \ell_{k-1})$. Then

$$W^{I,J}W^{K,L} = W^{i_0 j_0} (W^{I_+, J_+} W^{K_-, L_-}) W^{k_k \ell_k} \quad (19.52)$$

$$= W^{I,J}|b)(a|W^{K,L}. \quad (19.53)$$

Thus the length- $(k+1)$ block has the same insertion witnesses, and the two- and three-site argument gives their one-letter contraction. Induction on $k' - k$ proves the claim. $\square$

Theorem 19.3.27 (Common simple blocking along a continuous MPU path). Lean Lean Lean ✓ Stmt Let $d, D > 0$. Every bond- D MPU tensor becomes simple after blocking exactly D^4 sites. For each fixed L , the map

$$\mathcal{U} \mapsto \mathcal{U}_L \quad (19.54)$$

is continuous. Consequently, if $x \mapsto \mathcal{W}(x)$ is a continuous family of bond- D MPU tensors, then $x \mapsto \mathcal{W}(x)_{D^4}$ is a continuous family of simple tensors. This is only the common-blocking step in the proof of Proposition 19.9.59; it does not choose reduced representatives or assert constancy of their source ranks.

Proof. ✓ Proof Choose for each tensor a positive simple blocking length $k \leq D^4$. Simplicity persists at every later direct blocking, hence at D^4 . For fixed blocked ket and bra words, every matrix entry of the blocked tensor is a finite sum of products of entries of the original tensor. These entries are continuous, and the finite product and sum operations preserve continuity. Applying this fixed blocking map to the original path proves the path statement. $\square$

Theorem 19.3.28 (Supplied-projector simple2 contraction). [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ be simple with positive physical dimension, let W be its double layer, and let E be the normalized diagonal of W . Suppose that for some $J > 0$,

$$E^J = |\rho)(\Phi|. \quad (19.55)$$

Then the simple insertion identity holds with these same supplied witnesses:

$$W^{ij}W^{kl} = W^{ij}|\rho)(\Phi|W^{kl}. \quad (19.56)$$

Proof. [✓ Proof](#) Choose the witnesses a, b from simplicity and put $R = |b)(a|$. Expanding the normalized diagonal as the average of the diagonal letters, the two defining simple contractions give

$$W^{ij}EW^{kl} = W^{ij}W^{kl}. \quad (19.57)$$

Indeed, insert R on both sides of each diagonal letter; its middle sandwich is $RW^{aa}R = R$ by the first contraction, and the normalized sum contains d equal terms. The same identity makes $W^{ij}E^N = W^{ij}E$ for every $N > 0$. Hence

$$W^{ij}E^JW^{kl} = W^{ij}W^{kl}. \quad (19.58)$$

Substituting the supplied equality $E^J = |\rho)(\Phi|$ proves the claim. No uniqueness of rank-one factorizations is used. $\square$

Theorem 19.3.29 (Fixed boundaries of a normalized rank-one power). [Lean](#) [✓ Stmt](#) Let R be the normalized physical diagonal of the double layer of an MPO tensor. Suppose

$$(\Phi|\rho) = 1, \quad R^J = |\rho)(\Phi|. \quad (19.59)$$

Then the two boundaries of the stabilized diagonal tail are fixed:

$$R^J|\rho) = |\rho), \quad (\Phi|R^J = (\Phi|. \quad (19.60)$$

This is the fixed-pair normalization from the canonical-form-II fixed-point equations in [CPGSV17, equations (6a)–(6b)] and underlies the tail contractions in [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Proof. [✓ Proof](#) Substitute $R^J = |\rho)(\Phi|$ in each expression and use $(\Phi|\rho) = 1$. $\square$

Theorem 19.3.30 (Rank-one insertion as a normalized diagonal tail). [Lean](#) [✓ Stmt](#) Let W be the double layer of an MPO tensor, and let R be its normalized physical diagonal. If

$$R^J = |\rho)(\Phi|, \quad (19.61)$$

then, for all physical indices i, j, k, ℓ ,

$$W^{ij}|\rho)(\Phi|W^{kl} = d^{-J} \sum_{\tau \in \{0, \dots, d-1\}^J} W^{ij}W^{\tau_0\tau_0} \dots W^{\tau_{J-1}\tau_{J-1}}W^{kl}. \quad (19.62)$$

This is the exact diagonal-word tail between the two retained letters in [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Proof. [✓ Proof](#) Replace the rank-one insertion by R^J , apply the diagonal-word expansion, and distribute the two retained letters across the normalized finite sum. $\square$

Theorem 19.3.31 (Simple two-letter product as a normalized diagonal tail). [Lean](#) [✓ Stmt](#) In the setting of the preceding theorem, suppose in addition that the same fixed pair satisfies the supplied two-letter contraction

$$W^{ij}W^{k\ell} = W^{ij}|\rho\rangle(\Phi|W^{k\ell} \quad (19.63)$$

for all physical indices. Then

$$W^{ij}W^{k\ell} = d^{-J} \sum_{\tau \in \{0, \dots, d-1\}^J} W^{ij}W^{\tau_0\tau_0} \dots W^{\tau_{J-1}\tau_{J-1}}W^{k\ell}. \quad (19.64)$$

Thus the supplied insertion and its stabilized transfer power occur in one aligned network, as required in the calculation of [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Proof. [✓ Proof](#) Apply the supplied two-letter contraction and then replace its rank-one insertion by the normalized diagonal-word tail. $\square$

Theorem 19.3.32 (The fixed-point simple1 contraction). [Lean](#) [Lean](#) [✓ Stmt](#) For every MPU of positive physical and bond dimensions there are normalized left and right fixed vectors Φ and ρ of the transfer matrix such that the unblocked double layer satisfies

$$(\Phi|W^{ij}|\rho) = \delta_{ij}. \quad (19.65)$$

The unblocked tensor need not satisfy the rank-one insertion identity.

Proof. [✓ Proof](#) Let $E^J = |\rho\rangle(\Phi|$ be a stabilized transfer power. Decompose $W^{ij} = \delta_{ij}E + R_{ij}$. Fixedness and normalization give

$$\text{tr}(E|\rho)(\Phi|) = 1. \quad (19.66)$$

The mixed residual trace identity applied to one residual factor followed by J diagonal factors gives

$$\text{tr}(R_{ij}E^J) = 0. \quad (19.67)$$

Since $E^J = |\rho\rangle(\Phi|$, these two trace identities are exactly the displayed contraction. $\square$

19.4 Trace-power lemma

For the transfer-matrix spectrum argument of [CPGSV17, Proposition 1] we first record the algebraic fact that a matrix whose positive powers all have trace 1 must have characteristic polynomial $X^{n-1}(X - 1)$. The source assumes only the powers with exponent strictly greater than one; this weaker hypothesis determines the nonzero spectrum as a set.

Definition 19.4.1 (Rank-one diagonal idempotent). [Lean](#) [✓ Stmt](#) For an index i_0 in a finite set I of cardinality n , let M_{i_0} be the $n \times n$ diagonal matrix with a 1 at position i_0 and zeros elsewhere.

Lemma 19.4.2 (Properties of the rank-one diagonal idempotent). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The matrix M_{i_0} from definition 19.4.1 satisfies

- $M_{i_0}^k = M_{i_0}$ for every $k \geq 1$,

- $\text{tr}(M_{i_0}^k) = 1$ for every $k \geq 1$,
- $\chi_{M_{i_0}}(X) = X^{n-1}(X-1)$.

Proof. [✓ Proof](#) The first two items follow from $1^k = 1$ and $0^k = 0$ for $k \geq 1$. The characteristic polynomial is computed from $\chi_{\text{diag}(d)}(X) = \prod_i (X - d_i)$. $\square$

Theorem 19.4.3 (Finite-range trace-power lemma). [Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix with $n \geq 1$. If $\text{tr}(A^k) = 1$ for every k with $1 \leq k \leq n$, then

$$\chi_A(X) = X^{n-1}(X-1). \quad (19.68)$$

Proof. [✓ Proof](#) Choose any index i_0 (the hypothesis $n \geq 1$ makes this possible) and let B be the rank-one diagonal idempotent from Lemma 19.4.2. For $1 \leq k \leq n$ we have $\text{tr}(A^k) = 1 = \text{tr}(B^k)$. The finite-range half of the Newton–Girard the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”] then yields $\chi_A = \chi_B = X^{n-1}(X-1)$. $\square$

Theorem 19.4.4 (All-positive-powers trace-power lemma). [Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix. If $\text{tr}(A^k) = 1$ for every $k \geq 1$, then

$$\chi_A(X) = X^{n-1}(X-1). \quad (19.69)$$

(The hypothesis already forces $n \geq 1$ because the trace of a 0×0 matrix is 0, contradicting $\text{tr}(A) = 1$.)

Proof. [✓ Proof](#) If $n = 0$ the hypothesis $\text{tr}(A) = 1$ contradicts the fact that the trace of an empty matrix is 0; the conclusion follows vacuously. When $n \geq 1$, the hypothesis on all positive powers implies the finite-range hypothesis of Theorem 19.4.3. $\square$

Theorem 19.4.5 (Shifted moments under matrix powers). [Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix such that $\text{tr}(A^k) = 1$ for every $k > 1$, and let $m > 1$. Then $\text{tr}((A^m)^r) = 1$ for every $r \geq 1$.

Proof. [✓ Proof](#) For $r \geq 1$, one has $(A^m)^r = A^{mr}$ and $mr > 1$, so the hypothesis at exponent mr gives the result. $\square$

Theorem 19.4.6 (Trace powers above one determine the nonzero spectrum). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix. If $\text{tr}(A^k) = 1$ for every integer $k > 1$, then

$$\sigma(A) \setminus \{0\} = \{1\}. \quad (19.70)$$

Thus 1 occurs and every nonzero spectral value equals 1. This is the set-spectrum consequence of [CPGSV17, Proposition III.1]; set equality alone does not assert algebraic multiplicity.

Proof. [✓ Proof](#) For every positive integer r , the shifted moments give

$$\text{tr}((A^2)^r) = \text{tr}(A^{2r}) = 1, \quad \text{tr}((A^3)^r) = \text{tr}(A^{3r}) = 1. \quad (19.71)$$

Hence

$$\chi_{A^2}(X) = \chi_{A^3}(X) = X^{n-1}(X-1). \quad (19.72)$$

If $\mu \in \sigma(A)$ is nonzero, spectral mapping places μ^2 in $\sigma(A^2)$ and μ^3 in $\sigma(A^3)$, so

$$\mu^2 = \mu^3 = 1, \quad \mu = 1. \quad (19.73)$$

Conversely, spectral mapping from $1 \in \sigma(A^2)$ supplies a nonzero spectral value of A , which must be 1. $\square$

Theorem 19.4.7 (Generalized eigenspace inclusion under squaring). [Lean](#) [✓ Stmt](#) For a complex square matrix A , write $G_1(A)$ for its maximal generalized eigenspace at the eigenvalue 1. Then

$$G_1(A) \subseteq G_1(A^2). \quad (19.74)$$

Proof. [✓ Proof](#) The commuting factorization

$$A^2 - \mathbb{1} = (A + \mathbb{1})(A - \mathbb{1}) \quad (19.75)$$

gives, for every nonnegative integer k ,

$$(A^2 - \mathbb{1})^k = (A + \mathbb{1})^k (A - \mathbb{1})^k. \quad (19.76)$$

Therefore every vector annihilated by a power of $A - \mathbb{1}$ is annihilated by the corresponding power of $A^2 - \mathbb{1}$. $\square$

Theorem 19.4.8 (Multiplicity at one does not decrease under squaring). [Lean](#) [✓ Stmt](#) For every complex square matrix A ,

$$\text{mult}_1(\chi_A) \leq \text{mult}_1(\chi_{A^2}). \quad (19.77)$$

Proof. [✓ Proof](#) With G_1 as above, algebraic multiplicity equals the dimension of the maximal generalized eigenspace. Hence

$$\text{mult}_1(\chi_A) = \dim G_1(A) \quad (19.78)$$

$$\leq \dim G_1(A^2) = \text{mult}_1(\chi_{A^2}). \quad (19.79)$$

$\square$

Theorem 19.4.9 (Trace powers above one determine the characteristic polynomial). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix. If $\text{tr}(A^k) = 1$ for every integer $k > 1$, then

$$\text{mult}_1(\chi_A) = 1, \quad \chi_A(X) = X^{n-1}(X - 1). \quad (19.80)$$

Thus 1 is the sole nonzero eigenvalue and has algebraic multiplicity one. This is the exact shifted-moment conclusion of [CPGSV17, Proposition III.1].

Proof. [✓ Proof](#) Applying the all-positive trace-power theorem to A^2 gives

$$\chi_{A^2}(X) = X^{n-1}(X - 1), \quad \text{mult}_1(\chi_{A^2}) = 1. \quad (19.81)$$

The multiplicity inequality therefore bounds $\text{mult}_1(\chi_A)$ by one. The set-spectrum theorem shows that 1 occurs, so this multiplicity is positive and hence equals one. The same set-spectrum theorem excludes every other nonzero root. Splitting the monic characteristic polynomial now yields

$$\text{mult}_1(\chi_A) = 1, \quad \chi_A(X) = X^{n-1}(X - 1). \quad (19.82)$$

$\square$

Theorem 19.4.10 (Vanishing positive trace powers determine the characteristic polynomial).

[Lean](#) [✓ Stmt](#) Let A be an $n \times n$ complex matrix. If $\text{tr}(A^k) = 0$ for every integer $k \geq 1$, then

$$\chi_A(X) = X^n. \quad (19.83)$$

Proof. ✓ Proof The zero matrix has the same positive trace powers as A and characteristic polynomial X^n . Newton–Girard therefore identifies the two characteristic polynomials. This is the characteristic-polynomial consequence of the all-positive trace condition obtained in [CPGSV17, proof of Proposition III.3]. $\square$

Theorem 19.4.11 (Shifted vanishing moments under matrix powers). Lean ✓ Stmt *Let A be an $n \times n$ complex matrix such that $\text{tr}(A^k) = 0$ for every $k > 1$, and let $m > 1$. Then $\text{tr}((A^m)^r) = 0$ for every $r \geq 1$. This auxiliary result supports the generalization below to a weaker hypothesis; the source itself supplies vanishing for every positive exponent.*

Proof. ✓ Proof For $r \geq 1$, one has $(A^m)^r = A^{mr}$ and $mr > 1$, so the hypothesis at exponent mr gives the result. $\square$

Theorem 19.4.12 (Shifted zero trace powers force nilpotence). Lean Lean Lean ✓ Stmt *Let A be an $n \times n$ complex matrix. If $\text{tr}(A^N) = 0$ for every integer $N > 1$, then A is nilpotent and*

$$\chi_A(X) = X^n. \quad (19.84)$$

The source obtains the all-positive trace hypothesis in [CPGSV17, proof of Proposition III.3]; its specialization is the elementwise nilpotence assertion used before the nil-matrix theorem. The result above is a stronger generalization because it assumes vanishing only for $N > 1$. It is useful independently and requires no positivity, normality, or diagonalizability.

Proof. ✓ Proof Every positive power of A^2 has vanishing trace. Hence

$$\chi_{A^2}(X) = X^n. \quad (19.85)$$

If μ is a spectral value of A , spectral mapping shows that μ^2 is a root of this characteristic polynomial. Therefore $\mu^2 = 0$, and hence $\mu = 0$. Thus every root of the split monic polynomial χ_A is zero, so $\chi_A(X) = X^n$. Cayley–Hamilton now proves that A is nilpotent. $\square$

The standard-form construction uses the compact singular-value decomposition in which the intermediate dimension is the rank rather than the smaller ambient dimension.

Theorem 19.4.13 (Compact rectangular singular-value decomposition). Lean ✓ Stmt *Let $M \in M_{m,n}(\mathbb{C})$ be a complex matrix of rank r . There are matrices $V \in M_{r,m}(\mathbb{C})$ and $U \in M_{r,n}(\mathbb{C})$ and strictly positive real numbers $s_0, \dots, s_{r-1}$ such that, for $D = \text{diag}(s_0, \dots, s_{r-1})$,*

$$M = V^\dagger D U, \quad V V^\dagger = \mathbb{1}_r, \quad U U^\dagger = \mathbb{1}_r. \quad (19.86)$$

Thus no zero singular values occur in the intermediate space. This is the compact decomposition used in the standard-form construction of [CPGSV17, Section III].

Proof. ✓ Proof Apply the spectral theorem to $MM^\dagger$. Its nonzero eigenvalues are positive, and their number is $\text{rank}(MM^\dagger) = \text{rank}(M) = r$. For an orthonormal eigenvector e_j with eigenvalue $\lambda_j > 0$, put $w_j = M^\dagger e_j$. Then $\langle w_j, w_k \rangle = \lambda_j \delta_{jk}$. Hence the vectors $\lambda_j^{-1/2} w_j$ are orthonormal. Expanding each column of M in the eigenbasis and discarding the zero-eigenvalue terms gives

$$M = \sum_{\lambda_j > 0} \sqrt{\lambda_j} e_j (\lambda_j^{-1/2} w_j)^\dagger, \quad (19.87)$$

which has the form in (19.86). $\square$

We now introduce the two matrices used for the singular-value decompositions of a Matrix Product Unitary tensor in Section III of [CPGSV17]. Write $\mathcal{U}_{j,\alpha,\beta}^i$ for the entries of the tensor, where i, j are physical indices and α, β are auxiliary indices.

Definition 19.4.14 (First matrix associated to a Matrix Product Unitary tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The matrix M_1 is obtained by combining the left auxiliary index with the lower physical index, and the upper physical index with the right auxiliary index. Thus

$$(M_1)_{(\alpha,j),(i,\beta)} = \mathcal{U}_{j,\alpha,\beta}^i. \quad (19.88)$$

Transporting the row and column indices along the standard product enumerations gives a matrix $\widetilde{M}_1$ indexed by $\{0, \dots, Dd-1\} \times \{0, \dots, dD-1\}$. Its entries are obtained by applying M_1 to the inverse images of the enumerated indices, and

$$\text{rank}(\widetilde{M}_1) = \text{rank}(M_1). \quad (19.89)$$

Definition 19.4.15 (Second matrix associated to a Matrix Product Unitary tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The matrix M_2 is obtained by combining the left auxiliary index with the upper physical index, and the lower physical index with the right auxiliary index. Thus

$$(M_2)_{(\alpha,i),(j,\beta)} = \mathcal{U}_{j,\alpha,\beta}^i. \quad (19.90)$$

Transporting the row and column indices along the standard product enumerations gives a matrix $\widetilde{M}_2$ indexed by $\{0, \dots, Dd-1\} \times \{0, \dots, dD-1\}$. Its entries are obtained by applying M_2 to the inverse images of the enumerated indices, and

$$\text{rank}(\widetilde{M}_2) = \text{rank}(M_2). \quad (19.91)$$

Definition 19.4.16 (Right rank). [Lean](#) [Lean](#) [✓ Stmt](#) The right rank is

$$r = \text{rank}(M_1). \quad (19.92)$$

Definition 19.4.17 (Left rank). [Lean](#) [Lean](#) [✓ Stmt](#) The left rank is

$$\ell = \text{rank}(M_2). \quad (19.93)$$

Theorem 19.4.18 (Source cuts under physical adjunction). [Lean](#) [Lean](#) [✓ Stmt](#) *Physical adjunction exchanges the two source cuts and conjugates every entry:*

$$M_1(\mathcal{U}^\sharp) = \overline{M_2(\mathcal{U})}, \quad (19.94)$$

$$M_2(\mathcal{U}^\sharp) = \overline{M_1(\mathcal{U})}. \quad (19.95)$$

Proof. [✓ Proof](#) The physical adjoint exchanges the upper and lower physical indices while leaving both auxiliary indices fixed. Thus the first regrouping becomes the entrywise conjugate of the second, and conversely. $\square$

Definition 19.4.19 (Physical-pair swap). [Lean](#) [Lean](#) [✓ Stmt](#) On the flattened physical alphabet, define the involution

$$s(i, j) = (j, i). \quad (19.96)$$

This acts on physical coordinates and is distinct from the doubled-bond exchange used in reflected transfer calculations.

Theorem 19.4.20 (Normalized flattening under physical adjunction). [Lean](#) [✓ Stmt](#) *The normalized flattening of the physical adjoint satisfies*

$$\widetilde{\mathcal{U}}^\sharp{}^{(i,j)} = \overline{\widetilde{\mathcal{U}}^{(j,i)}}. \quad (19.97)$$

Proof. [✓ Proof](#) For every pair of physical indices (i, j) ,

$$\widetilde{\mathcal{U}}^\sharp{}^{(i,j)} = \frac{1}{\sqrt{d}} (\mathcal{U}^\sharp)^{(i,j)} = \frac{1}{\sqrt{d}} \overline{\mathcal{U}^{(j,i)}} = \overline{\widetilde{\mathcal{U}}^{(j,i)}}. \quad (19.98)$$

The first equality is the normalized flattening, the second is physical adjunction, and the third uses the reality of $d^{-1/2}$. $\square$

Definition 19.4.21 (Canonical-form-II data under physical adjunction). [Lean](#) [✓ Stmt](#) Given canonical-form-II data for $\widetilde{\mathcal{U}}$, define the transformed tuple by

$$r' = r, \quad (19.99)$$

$$D'_k = D_k, \quad (19.100)$$

$$\mu'_k = \overline{\mu_k}, \quad (19.101)$$

$$(A'_k)^{(i,j)} = \overline{A_k^{(j,i)}}, \quad (19.102)$$

$$V' = \overline{V}. \quad (19.103)$$

These data reconstruct $\widetilde{\mathcal{U}}^\sharp$.

Theorem 19.4.22 (Canonical form II under physical adjunction). [Lean](#) [✓ Stmt](#) *If $\widetilde{\mathcal{U}}$ is in canonical form II, then $\widetilde{\mathcal{U}}^\sharp$ is in canonical form II.*

Proof. [✓ Proof](#) Theorem 8.6.4 independently preserves normality and left-canonical normalization under entrywise conjugation, and Theorem 8.6.8 preserves these properties under the physical-pair swap. Conjugating the coisometry equation gives

$$V' V'^\dagger = \overline{V V^\dagger} = \mathbb{1}. \quad (19.104)$$

Conjugating the reconstruction at the swapped physical index gives

$$\widetilde{\mathcal{U}}^\sharp{}^{(i,j)} = (V')^\dagger \left(\bigoplus_k \mu'_k (A'_k)^{(i,j)} \right) V'. \quad (19.105)$$

Finally, each diagonal positive matrix Λ_k is real, and therefore

$$\mathcal{E}_{\overline{A_k}}(\Lambda_k) = \overline{\mathcal{E}_{A_k}(\Lambda_k)} = \overline{\Lambda_k} = \Lambda_k. \quad (19.106)$$

Physical relabeling leaves this transfer map unchanged, which proves the transformed fixed-point equation. $\square$

Theorem 19.4.23 (Full support under physical adjunction). [Lean](#) [✓ Stmt](#) *If canonical-form-II data for $\widetilde{\mathcal{U}}$ have full support, then the data from Definition 19.4.21 also have full support.*

Proof. [✓ Proof](#) Complex conjugation of the weights leaves the block indices and their bond dimensions unchanged, so the full-support sum is unchanged. $\square$

Theorem 19.4.24 (Source ranks under physical adjunction). [Lean](#) [Lean](#) [✓ Stmt](#) *Physical adjunction exchanges the two source ranks:*

$$r(\mathcal{U}^\sharp) = \ell(\mathcal{U}), \quad (19.107)$$

$$\ell(\mathcal{U}^\sharp) = r(\mathcal{U}). \quad (19.108)$$

Proof. [✓ Proof](#) Apply Theorem 19.4.18 and use invariance of matrix rank under entrywise conjugation. $\square$

Theorem 19.4.25 (Source cuts under physical reindexing). [Lean](#) [Lean](#) [✓ Stmt](#) *A bijective re-labeling of the physical basis simultaneously relabels the physical component of every source-cut row and column. Hence*

$$M_1(e^*\mathcal{U}) \cong M_1(\mathcal{U}), \quad M_2(e^*\mathcal{U}) \cong M_2(\mathcal{U}), \quad (19.109)$$

with the right and left orientations unchanged.

Proof. [✓ Proof](#) Evaluate each cut after relabeling. In M_1 , the lower physical index lies in the row and the upper physical index lies in the column; in M_2 these roles are exchanged. Applying e to both entries gives the two displayed row-and-column equivalences. $\square$

Theorem 19.4.26 (Source ranks under physical reindexing). [Lean](#) [Lean](#) [✓ Stmt](#) *Bijective physical relabeling preserves both source ranks:*

$$r(e^*\mathcal{U}) = r(\mathcal{U}), \quad \ell(e^*\mathcal{U}) = \ell(\mathcal{U}). \quad (19.110)$$

Proof. [✓ Proof](#) Matrix rank is unchanged by bijective row and column relabelings. Apply this fact to the two source-cut identities. $\square$

Definition 19.4.27 (Independent tensor product of matrix product tensors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $A = \{A^i\}_{i \in \{0, \dots, d-1\}}$ and $B = \{B^k\}_{k \in \{0, \dots, e-1\}}$ have auxiliary dimensions D and E , respectively. Write

$$\pi_{m,n} : \{0, \dots, m-1\} \times \{0, \dots, n-1\} \longrightarrow \{0, \dots, mn-1\} \quad (19.111)$$

for the standard finite-product coordinate bijection. Their independent tensor product $A \boxtimes B$ has physical dimension de , auxiliary dimension DE , and local matrices determined by

$$(A \boxtimes B)_{\pi_{D,E}(\alpha,\gamma), \pi_{D,E}(\beta,\delta)}^{\pi_{d,e}(i,k)} = A_{\alpha,\beta}^i B_{\gamma,\delta}^k. \quad (19.112)$$

For every N , the same coordinates give the pointwise splitting

$$(\{0, \dots, N-1\} \rightarrow \{0, \dots, de-1\}) \simeq (\{0, \dots, N-1\} \rightarrow \{0, \dots, d-1\}) \times (\{0, \dots, N-1\} \rightarrow \{0, \dots, e-1\}), \quad (19.113)$$

$$\sigma \mapsto (\sigma_d, \sigma_e), \quad (19.114)$$

$$\sigma(n) = \pi_{d,e}(\sigma_d(n), \sigma_e(n))$$

This is infrastructure for the tensoring clause in the proof of the MPU Index Theorem IV.6(ii) [CPGSV17, lines 824–845], not a separately stated theorem of that paper. The notation $A = \{A^i\}_i$ agrees with the canonical-form block notation in [CPGSV16, Section 2.3, lines 214–245].

Theorem 19.4.28 (Word evaluation of an independent tensor product). [Lean](#) [✓ Stmt](#) If w has length n and

$$w = (\pi_{d,e}(i_1, k_1), \dots, \pi_{d,e}(i_n, k_n)), \quad (19.115)$$

$$w_d = (i_1, \dots, i_n), \quad w_e = (k_1, \dots, k_n), \quad (19.116)$$

then

$$(A \boxtimes B)^w = \text{reind}_{\pi_{D,E}}(A^{w_d} \otimes B^{w_e}). \quad (19.117)$$

Proof. [✓ Proof](#) Induct on the length of w . For the empty word, use $\mathbb{1}_D \otimes \mathbb{1}_E = \mathbb{1}_{DE}$ and invariance of the identity under the coordinate bijection. For a leading letter $\pi_{d,e}(i, k)$, equation (19.112) and the induction hypothesis reduce the claim to

$$(A^i \otimes B^k)(X \otimes Y) = (A^i X) \otimes (B^k Y), \quad (19.118)$$

after transporting multiplication through the same coordinate bijection. $\square$

Theorem 19.4.29 (Homogeneous word spans of an independent tensor product). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $S_N(A) = \text{span}_{\mathbb{C}}\{A^w : |w| = N\}$. The pointwise splitting of length- N words gives

$$S_N(A \boxtimes B) = \text{span}_{\mathbb{C}}\{\text{reind}_{\pi_{D,E}}(A^u \otimes B^v) : |u| = |v| = N\}. \quad (19.119)$$

In particular, if $S_N(A) = M_D(\mathbb{C})$ and $S_N(B) = M_E(\mathbb{C})$, then $S_N(A \boxtimes B) = M_{DE}(\mathbb{C})$. If A and B are algebraically normal, then $A \boxtimes B$ is algebraically normal.

This is the homogeneous-word infrastructure for the tensoring sentence in the proof of the MPU Index Theorem IV.6(ii) [CPGSV17, lines 824–845], not a separate theorem stated there. It applies to the retained blocks in the canonical-form decomposition [CPGSV16, Section 2.3, lines 214–245].

Proof. [✓ Proof](#) For every product word w , let (u, v) be its pointwise component words. The bijection between w and (u, v) and Theorem 19.4.28 give

$$(A \boxtimes B)^w = \text{reind}_{\pi_{D,E}}(A^u \otimes B^v), \quad (19.120)$$

which proves (19.119) after taking linear spans. Pure tensors span $M_D(\mathbb{C}) \otimes M_E(\mathbb{C})$. The Kronecker linear equivalence, followed by simultaneous reindexing along $\pi_{D,E}$, is an isomorphism onto $M_{DE}(\mathbb{C})$; this proves the fixed-length consequence. If $S_{N_A}(A) = M_D(\mathbb{C})$ and $S_{N_B}(B) = M_E(\mathbb{C})$ for positive N_A, N_B , then persistence at positive multiples gives

$$S_{N_A N_B}(A) = M_D(\mathbb{C}), \quad S_{N_A N_B}(B) = M_E(\mathbb{C}). \quad (19.121)$$

Applying the fixed-length consequence at $N_A N_B$ proves algebraic normality of $A \boxtimes B$. $\square$

Theorem 19.4.30 (Left-canonical normal blocks under independent tensor products). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A and B be CPSV normal tensors, and suppose that

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}_D, \quad \sum_k (B^k)^\dagger B^k = \mathbb{1}_E. \quad (19.122)$$

Then $A \boxtimes B$ is a CPSV normal tensor and is left-canonical:

$$\sum_{i,k} ((A \boxtimes B)^{\pi_{d,e}(i,k)})^\dagger (A \boxtimes B)^{\pi_{d,e}(i,k)} = \mathbb{1}_{DE}. \quad (19.123)$$

This is the retained-normal-block infrastructure for the tensoring sentence in the proof of the MPU Index Theorem IV.6(ii) [CPGSV17, lines 824–845], not a separate theorem stated there. The retained-normal-block terminology is that of the canonical-form decomposition in [CPGSV16, Section 2.3, lines 214–245]; the two left-canonical identities are explicit infrastructure hypotheses, not an additional assertion attributed to that passage.

Proof. ✓ Proof The adjoint and multiplication laws for Kronecker products, together with the fact that simultaneous reindexing preserves adjoints, products, and finite sums, give

$$\sum_{i,k} ((A \boxtimes B)^{\pi_{d,e}(i,k)})^\dagger (A \boxtimes B)^{\pi_{d,e}(i,k)} \quad (19.124)$$

$$= \text{reind}_{\pi_{D,E}} \left(\sum_{i,k} (A^i \otimes B^k)^\dagger (A^i \otimes B^k) \right) \quad (19.125)$$

$$= \text{reind}_{\pi_{D,E}} \left(\left(\sum_i (A^i)^\dagger A^i \right) \otimes \left(\sum_k (B^k)^\dagger B^k \right) \right) \quad (19.126)$$

$$= \text{reind}_{\pi_{D,E}} (\mathbb{1}_D \otimes \mathbb{1}_E) = \mathbb{1}_{DE}, \quad (19.127)$$

proving (19.123). Spectral normality gives positive lengths N_A, N_B at which the two homogeneous word spans are full. Theorem 19.4.29 therefore makes $A \boxtimes B$ algebraically normal at the common length $N_A N_B$. The algebraically normal, left-canonical tensor is CPSV normal by Theorem 8.6.3. □

Definition 19.4.31 (Independent tensor product of matrix product operator tensors). Lean

Lean Lean ✓ Stmt Let $\mathcal{U}$ and $\mathcal{V}$ have physical dimensions d, e and auxiliary dimensions D, E , respectively. Their independent tensor product $\mathcal{U} \boxtimes \mathcal{V}$ has dimensions de and DE and entries

$$(\mathcal{U} \boxtimes \mathcal{V})_{(\alpha,\gamma),(\beta,\delta)}^{(i,k),(j,l)} = \mathcal{U}_{\alpha,\beta}^{i,j} \mathcal{V}_{\gamma,\delta}^{k,l}. \quad (19.128)$$

A product-valued physical configuration splits sitewise into one configuration for $\mathcal{U}$ and one for $\mathcal{V}$.

Definition 19.4.32 (Blocked product-alphabet equivalence). Lean ✓ Stmt A blocked word of L product letters separates canonically into two blocked words:

$$\{0, \dots, (de)^L - 1\} \simeq \{0, \dots, d^L e^L - 1\}. \quad (19.129)$$

The map decodes the length- L word, splits every letter into its d and e components, and re-encodes the two resulting words.

Theorem 19.4.33 (Decoding the blocked product alphabet). Lean Lean ✓ Stmt If q denotes the blocked product-alphabet equivalence, then each site s satisfies

$$\text{dec}_d(q(I)_1, s) = \text{dec}_{de}(I, s)_1, \quad (19.130)$$

$$\text{dec}_e(q(I)_2, s) = \text{dec}_{de}(I, s)_2. \quad (19.131)$$

Proof. ✓ Proof Each equality follows by composing the inverse product encoding at every site with the corresponding projection. □

Theorem 19.4.34 (Finite-size operator of an independent tensor product). Lean ✓ Stmt For every system size N , the sitewise configuration bijection identifies the finite-size operator with

$$(\mathcal{U} \boxtimes \mathcal{V})^{(N)} \cong \mathcal{U}^{(N)} \otimes \mathcal{V}^{(N)}. \quad (19.132)$$

Proof. ✓ Proof Grouping the two auxiliary indices makes every closed-word matrix a Kronecker product. The identity

$$\text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B) \quad (19.133)$$

then gives the stated finite-size operator after the sitewise configuration bijection. $\square$

Theorem 19.4.35 (Independent tensor products preserve matrix product unitarity). Lean ✓ Stmt *The independent tensor product of two matrix product unitary tensors is a matrix product unitary tensor. This is the tensoring operation in the proof of [CPGSV17, Theorem IV.6(ii)].*

Proof. ✓ Proof The multiplication rule $(A \otimes B)(C \otimes D) = AC \otimes BD$ gives

$$(\mathcal{U}^{(N)} \otimes \mathcal{V}^{(N)})(\mathcal{U}^{(N)} \otimes \mathcal{V}^{(N)})^\dagger \quad (19.134)$$

$$= \mathcal{U}^{(N)}(\mathcal{U}^{(N)})^\dagger \otimes \mathcal{V}^{(N)}(\mathcal{V}^{(N)})^\dagger = \mathbb{1} \otimes \mathbb{1} = \mathbb{1}. \quad (19.135)$$

Simultaneous reindexing preserves this unitarity equation. $\square$

Theorem 19.4.36 (Blocking an independent tensor product). Lean ✓ Stmt *For every blocking length L , separating each blocked product letter by the canonical equivalence q gives the exact identity*

$$(\mathcal{U} \boxtimes \mathcal{V})^{[L]} = q^*(\mathcal{U}^{[L]} \boxtimes \mathcal{V}^{[L]}). \quad (19.136)$$

The ordering of the two component words agrees with the independent tensor-product convention.

Proof. ✓ Proof Evaluate both sides on blocked ket and bra letters. Word multiplication of local Kronecker products gives

$$(\mathcal{U} \boxtimes \mathcal{V})[I, J] = \mathcal{U}[I_1, J_1] \otimes \mathcal{V}[I_2, J_2]. \quad (19.137)$$

The two decoding identities identify I_1, I_2 and J_1, J_2 with the corresponding letters of $q(I)$ and $q(J)$. $\square$

Definition 19.4.37 (Source-cut shuffle for an independent tensor product). Lean ✓ Stmt The source-cut shuffle sends

$$((\alpha, j), (\alpha', j')) \mapsto ((\alpha, \alpha'), (j, j')). \quad (19.138)$$

The analogous map applies to column indices.

Theorem 19.4.38 (Source cuts of an independent tensor product). Lean Lean ✓ Stmt *The source cuts satisfy*

$$M_1(\mathcal{U} \boxtimes \mathcal{V}) \cong M_1(\mathcal{U}) \otimes M_1(\mathcal{V}), \quad (19.139)$$

$$M_2(\mathcal{U} \boxtimes \mathcal{V}) \cong M_2(\mathcal{U}) \otimes M_2(\mathcal{V}). \quad (19.140)$$

Thus the paper's right and left orientations are unchanged.

Proof. ✓ Proof Evaluate both source cuts entry by entry after the displayed row and column shuffles. Each entry is the product of the corresponding entries of the two source cuts. $\square$

Theorem 19.4.39 (Source-cut ranks of an independent tensor product). [Lean](#) [Lean](#) [✓ Stmt](#) *The right and left source-cut ranks are multiplicative:*

$$r(\mathcal{U} \boxtimes \mathcal{V}) = r(\mathcal{U})r(\mathcal{V}), \quad (19.141)$$

$$\ell(\mathcal{U} \boxtimes \mathcal{V}) = \ell(\mathcal{U})\ell(\mathcal{V}). \quad (19.142)$$

This is the rank-multiplication ingredient for the tensoring case in the proof of [CPGSV17, Theorem IV.6(ii)].

Proof. [✓ Proof](#) Matrix rank is unchanged by row and column bijections and satisfies $\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B)$. $\square$

Theorem 19.4.40 (One-site blocking bounds for the source-cut ranks). [Lean](#) [Lean](#) [✓ Stmt](#) *Let $\mathcal{U}^{[L]}$ denote the direct blocking of L sites of a tensor $\mathcal{U}$ with physical dimension d , and let r_L and ℓ_L be its right and left source-cut ranks. Then*

$$r_{L+1} \leq d r_L, \quad (19.143)$$

$$\ell_{L+1} \leq d \ell_L. \quad (19.144)$$

These are the one-site forms of the two upper bounds in the proof of [CPGSV17, Proposition IV.2].

Proof. [✓ Proof](#) For the first cut, choose a compact factorization of the source matrix of $\mathcal{U}^{[L]}$ through a space of dimension r_L . Split each physical word of length $L+1$ into its first letter and its remaining word of length L . For suitable rectangular matrices $A_{1,L}$ and $B_{1,L}$, the successor source matrix factors as

$$M_{1,L+1} = A_{1,L} B_{1,L}. \quad (19.145)$$

The intermediate space is $\mathbb{C}^d \otimes \mathbb{C}^{r_L}$, and hence

$$\text{rank}(M_{1,L+1}) \leq \dim(\mathbb{C}^d \otimes \mathbb{C}^{r_L}) = d r_L. \quad (19.146)$$

Interchanging the upper and lower physical indices gives matrices $A_{2,L}$ and $B_{2,L}$ with

$$M_{2,L+1} = A_{2,L} B_{2,L}. \quad (19.147)$$

This factorization passes through $\mathbb{C}^d \otimes \mathbb{C}^{\ell_L}$, so

$$\text{rank}(M_{2,L+1}) \leq \dim(\mathbb{C}^d \otimes \mathbb{C}^{\ell_L}) = d \ell_L. \quad (19.148)$$

$\square$

Theorem 19.4.41 (Source-cut rank bounds between two blocking lengths). [Lean](#) [Lean](#) [✓ Stmt](#) *If $L_0 \leq L$, then*

$$r_L \leq d^{L-L_0} r_{L_0}, \quad (19.149)$$

$$\ell_L \leq d^{L-L_0} \ell_{L_0}. \quad (19.150)$$

These are the two blocking upper bounds used in the proof of [CPGSV17, Proposition IV.2].

Proof. ✓ Proof Induct on $L - L_0$. The case $L = L_0$ is equality. For the right rank, the successor bound and the induction hypothesis give

$$r_{L+1} \leq dr_L \leq d(d^{L-L_0}r_{L_0}) = d^{L+1-L_0}r_{L_0}. \quad (19.151)$$

The same multiplication by d gives

$$\ell_{L+1} \leq d\ell_L \leq d(d^{L-L_0}\ell_{L_0}) = d^{L+1-L_0}\ell_{L_0}. \quad (19.152)$$

□

Theorem 19.4.42 (Exact blocking growth from endpoint rank products). Lean Lean Lean ✓ Stmt Let $d > 0$ and $L_0 \leq L$. Suppose the endpoint source-cut ranks satisfy the supplied product identities

$$r_{L_0}\ell_{L_0} = d^{2L_0}, \quad r_L\ell_L = d^{2L}. \quad (19.153)$$

Then the blocking bounds are exact:

$$r_L = d^{L-L_0}r_{L_0}, \quad \ell_L = d^{L-L_0}\ell_{L_0}. \quad (19.154)$$

This is the saturation step in the proof of [CPGSV17, Proposition IV.2].

Proof. ✓ Proof The blocking bounds give

$$r_L \leq d^{L-L_0}r_{L_0}, \quad \ell_L \leq d^{L-L_0}\ell_{L_0}. \quad (19.155)$$

Their upper bounds have product

$$(d^{L-L_0}r_{L_0})(d^{L-L_0}\ell_{L_0}) = d^{2(L-L_0)}d^{2L_0} = d^{2L} = r_L\ell_L. \quad (19.156)$$

Since $d > 0$, the latter product is positive, so both r_L and ℓ_L are positive. Hence neither upper bound can be strict, and both equalities follow. □

Definition 19.4.43 (Product-index source weight). Lean ✓ Stmt For $\rho \in M_D(\mathbb{C})$, let

$$\chi: \mathbb{C}^D \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^D, \quad \chi(e_\alpha \otimes e_j) = e_j \otimes e_\alpha. \quad (19.157)$$

In the virtual–physical row order (α, j) of M_1 , define

$$W_\rho = \rho \otimes \mathbb{1}_d = \chi^\dagger(\mathbb{1}_d \otimes \rho)\chi. \quad (19.158)$$

Thus the source expression $\mathbb{1}_d \otimes \rho$ is first written in the physical–virtual order (j, α) and then reindexed by χ to the virtual–physical order (α, j) . The two Kronecker products are not identified without this reindexing. This is the source weight in [CPGSV17, equation labelled Y1Y1X1X1].

Theorem 19.4.44 (Positivity of the product-index source weight). Lean ✓ Stmt If ρ is positive definite, then W_ρ is positive definite.

Proof. ✓ Proof The Kronecker product of the positive-definite matrices ρ and $\mathbb{1}_d$ is positive definite. □

Proof. ✓ Proof The source-factor construction supplies $M_1 = X_1 Y_1$. The second compact singular-value decomposition gives $M_2 = X_2 Y_2$ directly. $\square$

Theorem 19.4.49 (Entry forms of the source-cut factorizations). Lean Lean Lean Lean ✓ Stmt *The entries of the two products satisfy*

$$(X_1 Y_1)_{(\alpha, j), (i, \beta)} = \mathcal{U}_{j, \alpha, \beta}^i, \quad (19.171)$$

$$(X_2 Y_2)_{(\alpha, i), (j, \beta)} = \mathcal{U}_{j, \alpha, \beta}^i. \quad (19.172)$$

These are the two graphical entry identifications in [CPGSV17, Section III].

Proof. ✓ Proof Evaluate the two matrix factorizations at $((\alpha, j), (i, \beta))$ and $((\alpha, i), (j, \beta))$, respectively. $\square$

Theorem 19.4.50 (Source-factor normalizations). Lean Lean Lean Lean ✓ Stmt *The left factors satisfy*

$$X_1^\dagger W_\rho X_1 = \mathbb{1}_r, \quad X_2^\dagger X_2 = \mathbb{1}_\ell, \quad (19.173)$$

$$\sum_{x, y} \overline{(X_1)_{x, r}} (W_\rho)_{x, y} (X_1)_{y, r'} = \delta_{r, r'}, \quad (19.174)$$

$$\sum_{\beta, i} \overline{(X_2)_{(\beta, i), l}} (X_2)_{(\beta, i), l'} = \delta_{l, l'}. \quad (19.175)$$

These are [CPGSV17, Section III].

Proof. ✓ Proof The source-factor construction supplies $X_1^\dagger W_\rho X_1 = \mathbb{1}_r$; evaluating at (r, r') gives the displayed weighted sum. Since $X_2 = V_2^\dagger$, the coisometry identity $V_2 V_2^\dagger = \mathbb{1}_\ell$ gives $X_2^\dagger X_2 = \mathbb{1}_\ell$; evaluating at (l, l') gives the second displayed entry sum. $\square$

Theorem 19.4.51 (Rotated second source-cut Gram). Lean ✓ Stmt *For every tensor $\mathcal{U}$, physical indices p, q , and auxiliary indices a, b ,*

$$\sum_{\beta, z} \overline{\mathcal{U}_{p, \beta, a}^z} \mathcal{U}_{q, \beta, b}^z = \sum_l \overline{(Y_2)_{l, (p, a)}} (Y_2)_{l, (q, b)}. \quad (19.176)$$

Thus the factor indexed by (p, a) is starred and the factor indexed by (q, b) is unstarred. This is the second-cut rotation used in [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)]. The identity requires no Matrix Product Unitary, dimension, positivity, or simple-tensor hypothesis.

Proof. ✓ Proof Substitute the entry factorization $M_2 = X_2 Y_2$ into the left-hand side and reorder the finite sums. The contraction of the two X_2 factors is $X_2^\dagger X_2 = \mathbb{1}_\ell$, so the two source indices coincide and the remaining sum is the displayed Gram matrix of Y_2 . $\square$

Theorem 19.4.52 (Recovery of the right source factors). Lean Lean ✓ Stmt *The right source factors are recovered from the two source matrices by*

$$Y_1 = X_1^\dagger W_\rho M_1, \quad Y_2 = X_2^\dagger M_2. \quad (19.177)$$

Here $W_\rho = \rho \otimes I_d$ in the product-index order (left virtual, physical).

Proof. ✓ Proof Multiply $M_1 = X_1 Y_1$ on the left by $X_1^\dagger W_\rho$ and use $X_1^\dagger W_\rho X_1 = I_r$. Similarly, multiply $M_2 = X_2 Y_2$ on the left by $X_2^\dagger$ and use $X_2^\dagger X_2 = I_\ell$. $\square$

Theorem 19.4.56 (Open two-site source factorization). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#)
For $q = (q_1, q_2)$ and $j = (j_1, j_2)$, the untraced two-site product is

$$(\mathcal{U}^{q_1 j_1} \mathcal{U}^{q_2 j_2})_{\alpha, \gamma} = \sum_{r, l} (X_1)_{(\alpha, j_1), r} u_{(l, r), q} (Y_2)_{l, (j_2, \gamma)}. \quad (19.187)$$

The intermediate partial contraction is

$$\sum_r (X_1)_{(\alpha, j_1), r} u_{(l, r), (q_1, q_2)} = \sum_{\beta} \mathcal{U}_{j_1, \alpha, \beta}^{q_1} (X_2)_{(\beta, q_2), l}. \quad (19.188)$$

Proof. [Stmnt](#) Use the three identities

$$u_{(l, r), q} = \sum_{\beta} (Y_1)_{r, (q_1, \beta)} (X_2)_{(\beta, q_2), l}, \quad (19.189)$$

$$(X_1 Y_1)_{(\alpha, j_1), (q_1, \beta)} = \mathcal{U}_{j_1, \alpha, \beta}^{q_1}, \quad (19.190)$$

$$(X_2 Y_2)_{(\beta, q_2), (j_2, \gamma)} = \mathcal{U}_{j_2, \beta, \gamma}^{q_2}. \quad (19.191)$$

Substitution and reordering of the scalar finite sums give the displayed partial contraction and the untraced two-site identity, with the virtual indices α and γ left open. Encoding both physical pairs by the standard equivalence $\{0, \dots, d-1\} \times \{0, \dots, d-1\} \simeq \{0, \dots, (-1)d^2\}$ gives the corresponding formula for the concrete two-site block. $\square$

Theorem 19.4.57 (Reflected open two-site source factorization). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#)
The reflected factorization of the same untraced two-site product is

$$(\mathcal{U}^{i_1 j_1} \mathcal{U}^{i_2 j_2})_{\alpha, \gamma} = \sum_{l, r} (X_2)_{(\alpha, i_1), l} v_{(j_2, j_1), (r, l)} (Y_1)_{r, (i_2, \gamma)}. \quad (19.192)$$

Thus the physical row of v is (j_2, j_1) , while its source column is (r, l) . Both orders remain explicit in the formula for the concrete two-site block under $\{0, \dots, d-1\} \times \{0, \dots, d-1\} \simeq \{0, \dots, (-1)d^2\}$.

Proof. [Stmnt](#) Expand the first tensor letter through $M_2 = X_2 Y_2$ and the second through $M_1 = X_1 Y_1$. Then

$$\sum_{\beta} (X_2)_{(\alpha, i_1), l} (Y_2)_{l, (j_1, \beta)} (X_1)_{(\beta, j_2), r} (Y_1)_{r, (i_2, \gamma)} = (X_2)_{(\alpha, i_1), l} v_{(j_2, j_1), (r, l)} (Y_1)_{r, (i_2, \gamma)}. \quad (19.193)$$

Summing over l and r gives the result. The two-site block statement follows by decoding its ket and bra indices into ordered pairs. $\square$

Theorem 19.4.58 (Open periodic-tail source expansion). [Lean](#) [Lean](#) [Stmnt](#) Separate the first two physical sites of a length- $(K+2)$ configuration. For an output pair q , an input pair j , and tails τ, ζ , the periodic entry first expands as

$$U_{(q, \tau), (j, \zeta)}^{(K+2)} = \sum_{\alpha, \gamma, r, l} (X_1)_{(\alpha, j_1), r} u_{(l, r), q} (Y_2)_{l, (j_2, \gamma)} (T_{\tau, \zeta})_{\gamma, \alpha}, \quad (19.194)$$

where $T_{\tau, \zeta}$ is the open K -site virtual word. Reconstructing Y_2 gives

$$U_{(q, \tau), (j, \zeta)}^{(K+2)} = \sum_{\alpha, r, l, \beta, i} (X_1)_{(\alpha, j_1), r} u_{(l, r), q} \overline{(X_2)_{(\beta, i), l}} (\mathcal{U}^{i j_2} T_{\tau, \zeta})_{\beta, \alpha}. \quad (19.195)$$

No power of d occurs in either algebraic expansion.

Proof. [✓ Proof](#) Reindexing each configuration into its first two coordinates and its tail gives

$$U_{(q,\tau),(j,\zeta)}^{(K+2)} = \text{tr}[(\mathcal{U}^{q_1 j_1} \mathcal{U}^{q_2 j_2}) T_{\tau,\zeta}]. \quad (19.196)$$

Insert the open two-site factorization from the preceding theorem. The recovery identity and its entrywise evaluation are

$$Y_2 = X_2^\dagger M_2, \quad (19.197)$$

$$(Y_2)_{l,(j_2,\gamma)} = \sum_{\beta,i} \overline{(X_2)_{(\beta,i),l}} \mathcal{U}_{j_2,\beta,\gamma}^i. \quad (19.198)$$

Substitution combines the last raw tensor letter with $T_{\tau,\zeta}$ by matrix multiplication and gives the second displayed expansion. $\square$

Definition 19.4.59 (Normalized open-tail coefficient). [Lean](#) [✓ Stmt](#) For

$$A_{i,j_2}^{\tau,\zeta} = \mathcal{U}^{ij_2} T_{\tau,\zeta}, \quad (19.199)$$

define the coefficient indexed by l, r, l', r' to be

$$C_{lr,l'r'}^{(K)} = d^{-K} \sum_{\substack{\tau,\zeta,j_1,j_2,\alpha,\alpha', \\ \beta,\beta',i,i'}} (X_1)_{(\alpha,j_1),r} \overline{(X_1)_{(\alpha',j_1),r'}} (X_2)_{(\beta,i),l} \overline{(X_2)_{(\beta',i'),l'}} \quad (19.200)$$

$$\cdot (A_{i,j_2}^{\tau,\zeta})_{\beta,\alpha} \overline{(A_{i',j_2}^{\tau,\zeta})_{\beta',\alpha'}}. \quad (19.201)$$

This is the coefficient obtained from the two open-tail expansions in the normalized Gram contraction in the proof of the source-tensor isometry lemma in [CPGSV17, Section III]. The external factor $u_{(l,r),q}$ is unstarred, while $u_{(l',r'),p}$ is starred. Thus the displayed coefficient contains the unique normalization factor d^{-K} .

Definition 19.4.60 (Forward source- u kernel). [Lean](#) [✓ Stmt](#) For a tail of length K , define

$$F_{ii';\beta\beta',\alpha\alpha'}^{(K)} = d^{-K} \sum_{\tau,\zeta,j_2} (\mathcal{U}^{ij_2} T_{\tau,\zeta})_{\beta,\alpha} \overline{(\mathcal{U}^{i'j_2} T_{\tau,\zeta})_{\beta',\alpha'}}. \quad (19.202)$$

The expression contains exactly one normalization factor d^{-K} .

Theorem 19.4.61 (Normalized output-layer tail entry). [Lean](#) [✓ Stmt](#) Let W_{out} be the output-first double layer. Then

$$(E_{\text{out}}^K)_{(\gamma,\gamma'),(\alpha,\alpha')} = d^{-K} \sum_{\tau,\zeta} (T_{\tau,\zeta})_{\gamma,\alpha} \overline{(T_{\tau,\zeta})_{\gamma',\alpha'}}. \quad (19.203)$$

Proof. [✓ Proof](#) Expand the normalized diagonal of the blocked output layer, reindex a blocked physical index by its decoded word, and evaluate the product double layer. The physical adjoint contributes the conjugated second word, while the blocked physical dimension supplies d^{-K} . $\square$

Theorem 19.4.62 (Forward-kernel double-layer packaging). [Lean](#) [✓ Stmt](#) The forward kernel is

$$F_{ii'}^{(K)} = W_{\text{out}}^{ii'} E_{\text{out}}^K. \quad (19.204)$$

Both doubled bond indices put the unstarred component before the starred component.

Proof. [✓ Proof](#) Expand the matrix product over its doubled bond index. The first output-layer letter supplies the sum over j_2 , and the normalized diagonal power supplies the common output word and contracted input word. Reordering finite sums gives the defining forward kernel without reversing the virtual chain. $\square$

Theorem 19.4.63 (First source-cut range projection). [Lean](#) [✓ Stmt](#) *On the range of the first source cut,*

$$X_1 X_1^\dagger W_\rho M_1 = M_1, \quad W_\rho = \rho \otimes 1_d. \quad (19.205)$$

Proof. [✓ Proof](#) Substitute $X_1^\dagger W_\rho M_1 = Y_1$ and then use $M_1 = X_1 Y_1$. $\square$

Theorem 19.4.64 (Second source-cut range projection). [Lean](#) [✓ Stmt](#) *On the range of the second source cut,*

$$X_2 X_2^\dagger M_2 = M_2. \quad (19.206)$$

This does not assert $X_2 X_2^\dagger = 1$ on the ambient product-index space.

Proof. [✓ Proof](#) Substitute $X_2^\dagger M_2 = Y_2$ and then use $M_2 = X_2 Y_2$. $\square$

Theorem 19.4.65 (Expanded normalized source- u metric). [Lean](#) [✓ Stmt](#) *Assume $d > 0$, let $\mathcal{U}$ be an MPU, and let ρ be positive definite. Expanding both $(K+2)$ -site periodic entries through X_1, u, X_2 and the open tail gives*

$$d^{-K} \sum_{\tau, j, \zeta} \mathcal{A}_{q; \tau, j, \zeta} \overline{\mathcal{A}_{p; \tau, j, \zeta}} = \delta_{p, q}, \quad (19.207)$$

where $\mathcal{A}_{q; \tau, j, \zeta}$ is the full source- u expansion with the q entry unstarred. This theorem evaluates the displayed expansion directly. It neither regroups the sum through $C_{lr, l'r'}^{(K)}$ nor identifies it with the ordinary Gram matrix $u^\dagger u$.

Proof. [✓ Proof](#) Replace each source-factor sum by its periodic MPO entry. Reindex the input configuration into its first two coordinates and its length- K tail, then apply normalized output-first MPU coisometry. $\square$

Theorem 19.4.66 (Closed output-letter formula for source- u Gram entries). [Lean](#) [✓ Stmt](#) *Let $\mathcal{U}$ be an MPO tensor, let ρ be positive definite, and let $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$. Write W_{out} for the output-first double layer of $\mathcal{U}$. Then*

$$\sum_{l, r} u_{(l, r), q} \overline{u_{(l, r), p}} = \sum_{\beta, \delta, \gamma, \alpha} \rho_{\alpha, \gamma} (W_{\text{out}}^{q_1 p_1})_{(\gamma, \alpha), (\beta, \delta)} \quad (19.208)$$

$$\cdot (X_2 X_2^\dagger)_{(\beta, q_2), (\delta, p_2)}. \quad (19.209)$$

Thus q is the unstarred ket index, p is the starred bra index, and each doubled bond is ordered as (ket, bra) . This is the first closed network in [CPGSV17, Eq. **uUnitary** and lines 545–556].

Proof. [✓ Proof](#) Expand both source- u entries and contract their common source indices. Weighted recovery of Y_1 , followed by the first source-cut range identity, replaces $Y_1^\dagger Y_1$ by $M_1^\dagger W_\rho M_1$. Expanding M_1 and the output-first double layer gives the displayed ρ -weighted letter, while the remaining X_2 factors form the stated range projector. $\square$

Theorem 19.4.67 (Closed double-layer trace for the normalized output tail). [Lean](#) [✓ Stmt](#) Assume $d > 0$. For any MPO tensor $\mathcal{U}$, tail length K , and $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$, let W_{out} be its output-first double layer and let E_{out} be the normalized physical diagonal of W_{out} . Then

$$d^{-K} \sum_{\tau, \eta} U_{(q, \tau), \eta}^{(K+2)} \overline{U_{(p, \tau), \eta}^{(K+2)}} = \text{tr}[W_{\text{out}}^{q_1 p_1} W_{\text{out}}^{q_2 p_2} E_{\text{out}}^K]. \quad (19.210)$$

The retained letters occur in (q, p) order, so q is unstarred and p is starred; their doubled bonds are in (ket, bra) order. This is the normalized closed output tail in [CPGSV17, Eq. `uUnitary` and lines 550–556].

Proof. [✓ Proof](#) Close the contracted input configuration as the periodic MPO of the output-first double layer. Splitting off the two retained sites leaves a common diagonal word of length K . Expanding the normalized diagonal power as the normalized sum of those words and using linearity of the trace gives the displayed closed trace. $\square$

Theorem 19.4.68 (Second-cut metric factorization). [Lean](#) [✓ Stmt](#) Define the second-cut metric by $H_2 = Z_2 Z_2^\dagger$. Then the range projector of the second source cut factors as

$$X_2 X_2^\dagger = M_2 H_2 M_2^\dagger. \quad (19.211)$$

This is a factorization through the chosen right inverse; it does not assert that H_2 , $X_2 X_2^\dagger$, or $M_2 M_2^\dagger$ is the identity.

Proof. [✓ Proof](#) From $M_2 = X_2 Y_2$ and $Y_2 Z_2 = 1$ one obtains $M_2 Z_2 = X_2$. Multiply this identity by its adjoint and reassociate the three factors. $\square$

Theorem 19.4.69 (Source- u Gram entry with the second-cut metric). [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ be an MPO tensor, let ρ be positive definite, and let $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$ be the retained physical pairs. No MPU or positive-dimension hypothesis is required. With $H_2 = Z_2 Z_2^\dagger$, the ordinary source- u Gram entry is

$$\sum_{l, r} u_{(l, r), q} \overline{u_{(l, r), p}} = \sum_{\beta, \delta, \gamma, \alpha, j, a, k, c} \rho_{\alpha, \gamma} (W_{\text{out}}^{q_1 p_1})_{(\gamma, \alpha), (\beta, \delta)} \quad (19.212)$$

$$\cdot U_{\beta, a}^{q_2 j} (H_2)_{(j, a), (k, c)} \overline{U_{\delta, c}^{p_2 k}}. \quad (19.213)$$

The q factors are unstarred and the p factors are starred.

Proof. [✓ Proof](#) Substitute the second-cut metric factorization into the closed output-letter formula and expand both matrix products. Reordering the finite sums gives the displayed eight-index expression. $\square$

Theorem 19.4.70 (Supplied-witness overlapping reblocking). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose a direct block of length K satisfies the two-letter contraction with specified vectors $|a\rangle$ and $|b\rangle$. For an arbitrary MPO tensor, one suffix/prefix extension preserves the same insertion $|b\rangle|a\rangle$ at direct length $K + 1$. For any MPU tensor, specified vectors satisfying this two-letter identity also satisfy the one-letter identity with the same vectors. Applying that implication after two extensions gives, at direct length $K + 2$,

$$(a|W^{IJ}|b) = \delta_{I, J}, \quad (19.214)$$

$$W^{IJ} W^{LM} = W^{IJ} |b\rangle \langle a| W^{LM}. \quad (19.215)$$

No positivity or nonzero-dimension hypothesis is required. This is the corollary following [CPGSV17, Proposition III.3].

Proof. ✓ Proof Extend the supplied two-letter identity twice by overlapping suffix and prefix windows. For $x = (a|W^{IJ}|b)$, the constant configurations of lengths two and three give $x^2 = \delta_{I,J}$ and $x^3 = \delta_{I,J}$, which force $x = \delta_{I,J}$. Physical blocking preserves MPU unitarity, so this one-letter identity and the twice-extended two-letter identity hold simultaneously at direct length $K + 2$ with the original supplied witnesses. $\square$

Definition 19.4.71 (Reflected blocked-transfer coordinate maps). Lean Lean ✓ Stmt The doubled-bond coordinate reflection exchanges the two virtual factors,

$$(a, b) \mapsto (b, a). \quad (19.216)$$

Independently, spatial reflection reverses a blocked word of length K ,

$$(i_0, \dots, i_{K-1}) \mapsto (i_{K-1}, \dots, i_0). \quad (19.217)$$

These are support coordinates for the proof of [CPGSV17, Lemma III.7 in Section III.B], rather than separate definitions printed there. The blocked-word reversal is not used in the transfer transport below; conjugate transposition there reverses only the order of the two blocked letters.

Theorem 19.4.72 (Reflected blocked-transfer coordinate identities). Lean Lean Lean Lean Lean ✓ Stmt *The doubled-bond equivalence and the blocked-word reversal are involutions. The latter sends the letter in position k to the letter in position $K - 1 - k$ and reverses the encoded word.*

Proof. ✓ Proof Decode the doubled bond, apply the product swap, and encode it again. Regard a blocked index as a word indexed by the K ordered positions, precompose with $k \mapsto K - 1 - k$, and reverse the corresponding finite list. $\square$

Theorem 19.4.73 (Normalized diagonal under output reflection). Lean ✓ Stmt *Let $R(\mathcal{U})$ denote the normalized physical diagonal of the double layer of an MPO tensor $\mathcal{U}$, and let $s(a, b) = (b, a)$ be the doubled-bond swap. Then*

$$R(\mathcal{U}^\sharp)_{x,y} = R(\mathcal{U})_{s(x),s(y)}. \quad (19.218)$$

This identity requires neither the MPU property nor a positivity or positive-dimension hypothesis. It is a coordinate identity supporting the calculation in the proof of [CPGSV17, Lemma III.7 in Section III.B], not a separate theorem printed there.

Proof. ✓ Proof Expand the two normalized diagonals. Physical adjunction exchanges the two physical indices, and exchanging the two finite physical sums leaves precisely the simultaneous doubled-bond swap on the row and column. $\square$

Theorem 19.4.74 (Reflected transfer-power identities). Lean Lean Lean Lean Lean ✓ Stmt *Let $\mathcal{U}$ be an MPO tensor. Write E for the transfer matrix of its normalized flattening, $E^\sharp$ for that of $\mathcal{U}^\sharp$, and E_{out} for the normalized diagonal of the double layer of $\mathcal{U}^\sharp$. If $d > 0$ and*

$$E^J = |\rho\rangle\langle 1|, \quad (19.219)$$

then

$$(E^\sharp)^J = |\rho^\top\rangle\langle 1|. \quad (19.220)$$

No MPU, positivity, trace, positive bond-dimension, or positivity of J is needed for this implication.

Suppose in addition that $D > 0$ and $\text{tr}(\rho) = 1$, and set $K = JD^2$. Then the normalized diagonal R_K of the double layer of the K -site block of $\mathcal{U}$ satisfies

$$R_K = |\rho)(1|. \quad (19.221)$$

More generally, if $d > 0$, $\rho > 0$, and the normalized diagonal R_L of an L -site block satisfies $R_L = |\rho)(1|$, write $R_L^\sharp$ for the normalized diagonal of the corresponding block of $\mathcal{U}^\sharp$. Then

$$R_L^{\sharp\dagger} = |1)(\rho|. \quad (19.222)$$

Consequently, under the combined assumptions $d, D > 0$, $\rho > 0$, $\text{tr}(\rho) = 1$, and $E^J = |\rho)(1|$,

$$R_K^{\sharp\dagger} = |1)(\rho|, \quad E_{\text{out}}^K = |\rho^\top)(1|. \quad (19.223)$$

Here all rank-one operators use the column-stacking identification $M_D(\mathbb{C}) \cong \mathbb{C}^{D^2}$.

Proof. ✓ Proof The blocked normalized diagonal is the corresponding transfer power. Reindex the reflected diagonal by the doubled-bond swap to obtain the transpose weight. When ρ is positive definite, its Hermitian symmetry converts the conjugate-transposed reflected diagonal to $|1)(\rho|$. The trace-one condition makes $|\rho)(1|$ idempotent, so the additional positive D^2 power leaves it unchanged. Conjugating once more gives the stated output-layer orientation. $\square$

Theorem 19.4.75 (Reflected supplied contractions). Lean Lean ✓ Stmt Let $\mathcal{U}$ be an MPU with $d, D > 0$. Let E be the transfer matrix of its normalized flattening, and suppose

$$E^J = |\rho)(1|, \quad (19.224)$$

where $\rho > 0$, $\text{tr}(\rho) = 1$, and $J > 0$. Set $K = JD^2$. For $n \in \{K, K + 2\}$, let $\widehat{W}_n$ be the double layer of the n -site block of $\mathcal{U}^\sharp$. Then, for all blocked physical indices,

$$(\rho|\widehat{W}_n^{IJ\dagger}|1) = \delta_{I,J}, \quad (19.225)$$

$$\widehat{W}_n^{LM\dagger}\widehat{W}_n^{IJ\dagger} = \widehat{W}_n^{LM\dagger}|1)(\rho|\widehat{W}_n^{IJ\dagger}. \quad (19.226)$$

The same supplied vectors occur at both lengths; the $K + 2$ identities do not replace them by existentially chosen witnesses.

Proof. ✓ Proof Apply the direct-block simple contractions to the reflected transfer power $|\rho^\top)(1|$. Positivity identifies the conjugate-transposed vectors with $|\rho|$ and $|1|$, while conjugate transposition reverses the order of the two blocked letters. Overlapping-window reblocking preserves these witnesses through the two additional sites. These contractions are support identities for the graphical proof of [CPGSV17, Lemma III.7 and its proof in Section III.B]; they are not asserted as a separate printed theorem of that paper. $\square$

Remark 19.4.76 (Scope of the closed source- u formulas). The first theorem rewrites the ordinary source- u Gram entry, while the second independently rewrites the normalized closed output tail. They do not identify these two expressions or justify inserting the source-cut range projections through the intervening chain. That range-restricted equality remains separate. Neither theorem asserts an ambient $X_2X_2^\dagger = \mathbb{1}$, $M_2M_2^\dagger = \mathbb{1}$, or $Y_2Y_2^\dagger = \mathbb{1}$ identity.

Theorem 19.4.77 (Terminal source-factor boundary contraction). Lean ✓ Stmt If the normalized internal tail supplies

$$\rho_{\alpha',\alpha} \delta_{\beta,\beta'} \delta_{i,i'}, \quad (19.227)$$

then the remaining boundary sum is

$$\sum_{j,\alpha,\alpha',\beta,i} (X_1)_{(\alpha,j),r} \overline{(X_1)_{(\alpha',j),r'}} (X_2)_{(\beta,i),l} \overline{(X_2)_{(\beta,i),l'}} \rho_{\alpha',\alpha} = \delta_{l,l'} \delta_{r,r'}. \quad (19.228)$$

Proof. ✓ Proof Reorder the scalar factors so that the sum separates into the weighted X_1 contraction and the unweighted X_2 contraction. Evaluating

$$X_1^\dagger W_\rho X_1 = \mathbb{1}_r, \quad X_2^\dagger X_2 = \mathbb{1}_\ell \quad (19.229)$$

at (r', r) and (l, l') , respectively, gives the two Kronecker deltas. This is precisely the terminal boundary simplification in the proof of the source-tensor isometry lemma in [CPGSV17, Section III]. □

Remark 19.4.78 (Scope of the open-tail algebra). The preceding results evaluate the fully expanded normalized metric, but do not prove that regrouping it through $C_{lr,l'r'}^{(K)}$ gives the ordinary Gram matrix $u^\dagger u$. The remaining independent equality is between that Gram matrix and the normalized closed $(K+2)$ -site output tail, with both external source- u tensors retained before moving the cut. No result uses $X_2 X_2^\dagger = \mathbb{1}$, $M_2 M_2^\dagger = \mathbb{1}$, or $Y_2 Y_2^\dagger = \mathbb{1}$.

Definition 19.4.79 (Tensorized source dressing). Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Put

$$Y = Y_1 \otimes Y_2, \quad Z = Z_1 \otimes Z_2, \quad (19.230)$$

with source-bond order $r \times \ell$. Thus Y maps $(\mathbb{C}^d \otimes \mathbb{C}^D)^{\otimes 2}$ to $\mathbb{C}^r \otimes \mathbb{C}^\ell$, and Z maps in the opposite direction. The regrouping of the open indices is

$$R((i, a), (j, b)) = ((i, j), \text{flat}(a, b)), \quad (19.231)$$

where $\text{flat} : \{0, \dots, D-1\} \times \{0, \dots, D-1\} \simeq \{0, \dots, (-1)D^2\}$ is the canonical product enumeration. The dressed source- v Gram equation is

$$Y^\dagger (v^\dagger v) Y = Y^\dagger Y. \quad (19.232)$$

These are the tensorized factors surrounding $v^\dagger v$ in [CPGSV17, Theorem III.8, Section III.B, equations (31)–(32)].

Theorem 19.4.80 (Four-letter source contractions). Lean Lean Lean Lean ✓ Stmt Let $p = (p_1, p_2)$ be the starred physical pair, $q = (q_1, q_2)$ the unstarred pair, and let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be virtual paths. The weighted first-cut contraction is

$$\sum_r \overline{(Y_1)_{r,(p_1,a_1)}} (Y_1)_{r,(q_1,b_1)} = \sum_{\alpha,\alpha',j_1} \overline{U_{\alpha_1}^{p_1 j_1}} \rho_{\alpha\alpha'} U_{\alpha' b_1}^{q_1 j_1}. \quad (19.233)$$

The entry of two ordinary double-layer letters from row (a_1, b_1) to column (a_2, b_2) is

$$\sum_{\alpha,\beta,j_1,j_2} \overline{U_{\alpha_1}^{j_1 p_1}} U_{b_1 \beta}^{j_1 q_1} \overline{U_{\alpha_2}^{j_2 p_2}} U_{\beta b_2}^{j_2 q_2}. \quad (19.234)$$

Set

$$\rho'_x = \text{vec}(\rho)_{\text{flat}^{-1}(x)}, \quad \Phi'_x = \text{vec}(\mathbb{1}_D)_{\text{flat}^{-1}(x)}. \quad (19.235)$$

Inserting $|\rho'\rangle\langle\Phi'|$ between the two letters gives

$$(W^{p_1 q_1} |\rho'\rangle\langle\Phi'| W^{p_2 q_2})_{(a_1, b_1), (a_2, b_2)} \quad (19.236)$$

$$= \sum_{\alpha, \alpha', \beta, j_1, j_2} \overline{U_{a_1 \alpha}^{j_1 p_1}} U_{b_1 \alpha'}^{j_1 q_1} \rho_{\alpha' \alpha} \overline{U_{\beta a_2}^{j_2 p_2}} U_{\beta b_2}^{j_2 q_2}. \quad (19.237)$$

Finally, the Gram entry of $Y_1 \otimes Y_2$ expands as

$$\sum_{\alpha, \alpha', \beta, j_1, j_2} \overline{U_{\alpha a_1}^{p_1 j_1}} \rho_{\alpha \alpha'} U_{\alpha' b_1}^{q_1 j_1} \overline{U_{\beta a_2}^{j_2 p_2}} U_{\beta b_2}^{j_2 q_2}. \quad (19.238)$$

Proof. [✓ Proof](#) Expand the ordinary and rank-one-inserted matrix products entrywise. For the tensorized Gram entry, use $X_1^\dagger(\rho \otimes \mathbb{1}_d)X_1 = \mathbb{1}_r$ on the first cut and $X_2^\dagger X_2 = \mathbb{1}_\ell$ on the second cut, then reorder the finite sums. No ambient identity $X_2 X_2^\dagger = \mathbb{1}$ is used. $\square$

Theorem 19.4.81 (One-sided contraction of the dressed source operator). [Lean](#) [✓ Stmt](#) [Discussion: #6071](#)

Let $\rho > 0$. For output pair $z = (z_1, z_2)$, physical pair $p = (p_1, p_2)$, and virtual pair $a = (a_1, a_2)$, the entry of $v(Y_1 \otimes Y_2)$ is

$$(v(Y_1 \otimes Y_2))_{z, ((p_1, a_1), (p_2, a_2))} = \sum_{\gamma, l} U_{\gamma a_1}^{p_1 z_1}(Y_2)_{l, (z_2, \gamma)}(Y_2)_{l, (p_2, a_2)}. \quad (19.239)$$

The first source cut has contracted to one local tensor entry, while the two factors from the second cut remain in the same summand. No partial second-cut Gram matrix has been formed. This coupling is the source-cut ordering in the calculation of [\[CPGSV17, Theorem III.8, Section III.B, equations \(31\)–\(32\)\]](#).

Proof. [✓ Proof](#) Expand the matrix product and the tensor product, reorder the three finite sums, and contract $X_1 Y_1$ by the first source-cut entry factorization. Keep both Y_2 factors together throughout. $\square$

Theorem 19.4.82 (Tensorized source right inverse). [Lean](#) [✓ Stmt](#) The tensorized source factors satisfy

$$YZ = \mathbb{1}_{r\ell}. \quad (19.240)$$

Proof. [✓ Proof](#) Multiplicativity of the Kronecker product gives

$$YZ = (Y_1 Z_1) \otimes (Y_2 Z_2) = \mathbb{1}_r \otimes \mathbb{1}_\ell = \mathbb{1}_{r\ell}. \quad (19.241)$$

$\square$

Theorem 19.4.83 (Regrouped entries of the dressed Gram equation). [Lean](#) [✓ Stmt](#) For every positive definite source weight ρ , and for $x = R^{-1}(p, a)$ and $y = R^{-1}(q, b)$, the dressed Gram equation is equivalent to

$$\sum_t \left(\sum_s \overline{Y_{s,x}} \sum_z \overline{v_{z,s}} v_{z,t} \right) Y_{t,y} = \sum_t \overline{Y_{t,x}} Y_{t,y}. \quad (19.242)$$

Here $p, q \in \{0, \dots, d-1\} \times \{0, \dots, d-1\}$, $a, b \in \{0, \dots, (-1)^{D^2}\}$, the indices s, t range over $\{0, \dots, r-1\} \times \{0, \dots, \ell-1\}$, and z ranges over $\{0, \dots, d-1\} \times \{0, \dots, d-1\}$.

Proof. ✓ **Proof** Evaluate both sides of $Y^\dagger(v^\dagger v)Y = Y^\dagger Y$ at (x, y) and expand the three matrix products. Conversely, apply the displayed equality after regrouping each pair of open source-cut indices by R . $\square$

Theorem 19.4.84 (Right-inverse cancellation of a dressed Gram equation). Lean ✓ **Stmnt** *If Y has a right inverse Z , so that $YZ = \mathbb{1}$, then for every square matrix G on the row space of Y ,*

$$Y^\dagger G Y = Y^\dagger Y \iff G = \mathbb{1}. \quad (19.243)$$

Proof. ✓ **Proof** Conjugate the first equality by $Z^\dagger$ on the left and Z on the right. Since $Z^\dagger Y^\dagger = (YZ)^\dagger = \mathbb{1}$, both outer factors cancel. The reverse implication follows by substituting $G = \mathbb{1}$. $\square$

Theorem 19.4.85 (Cancellation of the source- v dressing). Lean ✓ **Stmnt** *For every positive definite source weight ρ ,*

$$Y^\dagger(v^\dagger v)Y = Y^\dagger Y \iff v^\dagger v = \mathbb{1}_{r_\ell}. \quad (19.244)$$

Proof. ✓ **Proof** Apply right-inverse cancellation with $G = v^\dagger v$ and $Z = Z_1 \otimes Z_2$. $\square$

Theorem 19.4.86 (Right-rank bound). Lean ✓ **Stmnt** *If the physical and auxiliary dimensions are d and D , respectively, then*

$$r \leq Dd. \quad (19.245)$$

Proof. ✓ **Proof** The matrix M_1 has dD columns. Its rank is at most its number of columns, hence at most $dD = Dd$. $\square$

Theorem 19.4.87 (Left-rank bound). Lean ✓ **Stmnt** *If the physical and auxiliary dimensions are d and D , respectively, then*

$$\ell \leq Dd. \quad (19.246)$$

Proof. ✓ **Proof** The matrix M_2 has dD columns. Its rank is at most its number of columns, hence at most $dD = Dd$. $\square$

19.5 Full-support canonical representatives

Definition 19.5.1 (Full support). Lean ✓ **Stmnt** Let

$$A^i = V^* \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) V \quad (19.247)$$

be CPSV canonical-form data in ambient bond dimension D , where every A_k has positive bond dimension D_k . The data have *full support* when

$$\sum_{k=1}^r D_k = D. \quad (19.248)$$

Theorem 19.5.2 (Reduced canonical representative of the normalized tensor). Lean ✓ Stmt
 Let $d, D > 0$, let U be an MPU tensor of physical dimension d and bond dimension D , and let $\tilde{U} = d^{-1/2}U$ be its normalized doubled-index tensor. There are an integer $D_{\text{red}} > 0$, a normal tensor A_{red} of bond dimension D_{red} , and canonical-form-II data for A_{red} such that, for every integer $N > 0$ and every word σ of length N ,

$$D_{\text{red}} \leq D, \quad (19.249)$$

$$V^{(N)}(A_{\text{red}})_{\sigma} = V^{(N)}(\tilde{U})_{\sigma}, \quad (19.250)$$

$$\sum_{k=1}^r D_k = D_{\text{red}}. \quad (19.251)$$

This organizes the dimension-bounded canonical replacement used in [CPGSV17, lines 257–294 and 319–326]. Its irreducible reduction and canonical-form-II gauge are those of [CPGSV16, lines 195–255 and 1058–1077]. The shifted transfer argument is the one used in [CPGSV17, lines 344–356].

Proof. ✓ Proof Remove the zero irreducible blocks of $\tilde{U}$ and write

$$A_0^i = \bigoplus_{k=1}^r B_k^i, \quad \sum_{k=1}^r D_k \leq D, \quad (19.252)$$

$$V^{(N)}(A_0)_{\sigma} = V^{(N)}(\tilde{U})_{\sigma}, \quad (19.253)$$

for every $N > 0$ and every word σ of length N . At $N = 2$ the common self-overlap is one, so the retained sum is nonempty and $D_0 = \sum_k D_k > 0$. For every integer $N > 1$, overlap invariance gives

$$\text{tr}(E_{A_0}^N) = \sum_{\sigma} |V^{(N)}(A_0)_{\sigma}|^2 = \sum_{\sigma} |V^{(N)}(\tilde{U})_{\sigma}|^2 = 1. \quad (19.254)$$

Let $J_k : \mathbb{C}^{D_k} \rightarrow \mathbb{C}^{D_0}$ be the inclusion of the k th block. Then

$$J_k^{\dagger} J_k = \mathbb{1}, \quad A_0^i J_k = J_k B_k^i, \quad (19.255)$$

$$\mathcal{E}_{A_0}(J_k X J_k^{\dagger}) = J_k \mathcal{E}_{B_k}(X) J_k^{\dagger}. \quad (19.256)$$

Thus every nonzero transfer eigenvalue of B_k is a nonzero eigenvalue of E_{A_0} and hence equals one by (19.254). Perron–Frobenius theory supplies $\rho_k > 0$ and $s_k > 0$ with $\mathcal{E}_{B_k}(\rho_k) = s_k \rho_k$. Therefore $s_k = 1$, and every peripheral eigenvalue of B_k is also one. Each B_k is consequently normal.

The direct sum now has literal canonical-form data with full support. Apply the nonsingular blockwise canonical-form-II gauge. It preserves the block dimensions and all periodic vectors, so the resulting tensor A_{red} still satisfies

$$\sum_{k=1}^r D_k = D_{\text{red}}, \quad \text{tr}(E_{A_{\text{red}}}^N) = 1, \quad (19.257)$$

for every integer $N > 1$. The shifted traces force $r = 1$, spectral radius one, and primitivity. Full support identifies the unique block with the ambient bond space, so A_{red} is normal. $\square$

Theorem 19.5.3 (Reduced canonical representative of an MPU). [Lean](#) [✓ Stmt](#) Let $d, D > 0$ and let U be an MPU tensor. There are $D_{\text{red}} > 0$ and an MPU tensor U_{red} of bond dimension D_{red} such that, for every integer $N > 0$,

$$D_{\text{red}} \leq D, \quad (19.258)$$

$$U_{\text{red}}^{(N)} = U^{(N)}. \quad (19.259)$$

The normalized doubled-index tensor of U_{red} has canonical-form-II data whose retained block dimensions sum to D_{red} . No equality is asserted at length zero, and no virtual gauge is asserted between the two potentially unequal-dimensional bond spaces.

Proof. [✓ Proof](#) Apply Theorem 19.5.2 and write its representative as A_{red} . Define

$$U_{\text{red}}^{ij} = \sqrt{d} A_{\text{red}}^{(i,j)}. \quad (19.260)$$

Since $d > 0$, normalization recovers A_{red} exactly. For every $N > 0$ and every ket–bra pair (σ, τ) , the doubled-index coefficient identity gives

$$(U_{\text{red}}^{(N)})_{\sigma, \tau} = d^{N/2} V^{(N)}(A_{\text{red}})_{(\sigma, \tau)} \quad (19.261)$$

$$= d^{N/2} V^{(N)}(\tilde{U})_{(\sigma, \tau)} \quad (19.262)$$

$$= (U^{(N)})_{\sigma, \tau}. \quad (19.263)$$

Hence the periodic operators agree at every positive length. For $N > 1$ the right side is unitary, so U_{red} is an MPU. $\square$

Theorem 19.5.4 (Full support under physical blocking). [Lean](#) [✓ Stmt](#) Suppose that canonical-form data have full support:

$$\sum_{k=1}^r D_k = D. \quad (19.264)$$

Then every positive physical blocking of these data also has full support.

Proof. [✓ Proof](#) Positive blocking supplies canonical-form data with the same block dimensions, so the full-support sum is unchanged. $\square$

Theorem 19.5.5 (Full support under physical relabeling). [Lean](#) [✓ Stmt](#) A bijective relabeling of the physical alphabet preserves full support of canonical-form data.

Proof. [✓ Proof](#) Physical relabeling leaves the blocks and their bond dimensions unchanged, so the full-support sum is unchanged. $\square$

Theorem 19.5.6 (An ambient-dimensional block is irreducible). [Lean](#) [✓ Stmt](#) If a block has bond dimension D , then the ambient tensor A is irreducible.

Proof. [✓ Proof](#) If k is the block with $D_k = D$, the total-dimension bound gives

$$D = D_k \leq \sum_{j=1}^r D_j \leq D. \quad (19.265)$$

Since every retained block dimension is positive, these equalities force k to be the only retained block. After reindexing its bond coordinates, the ambient coisometry U is square, hence unitary, and reconstruction reads

$$A^i = U^*(\mu_k B^i)U, \quad (19.266)$$

where B is the reindexed normal block and $\mu_k \neq 0$. Thus Theorem D.3.4 transports irreducibility of B through the nonzero rescaling and unitary conjugation. $\square$

Theorem 19.5.7 (Unique block with full support). [Lean](#) [✓ Stmt](#) *Canonical-form data with one block and full support reconstruct an irreducible ambient tensor.*

Proof. [✓ Proof](#) Let k be the unique block. The singleton sum and full-support identity give

$$\sum_{j=1}^r D_j = D_k = D. \quad (19.267)$$

Thus k has ambient bond dimension, so Theorem 19.5.6 applies. $\square$

Theorem 19.5.8 (Normality of a unique full-support block). [Lean](#) [✓ Stmt](#) *Suppose that canonical-form data have one block and full support. If the transfer map has spectral radius one and is primitive, then the ambient tensor is a normal tensor.*

Proof. [✓ Proof](#) Combine the preceding irreducibility conclusion with the supplied spectral-radius and primitivity properties. $\square$

19.6 Transfer-matrix identities

Let D be a positive integer and let $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ be a linear map. The transfer matrix $\widehat{T}$ of T under the column-stacking identification $M_D(\mathbb{C}) \cong \mathbb{C}^{D^2}$ is defined in the companion quantum-channel volume [LTC26, “Transfer matrix of a matrix endomorphism”], the matrix-units special case of the one-family transfer matrix of the corresponding definition in the companion quantum-channel volume [LTC26, “Transfer matrix in one family”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Functoriality of transfer matrices”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Trace preservation fixes the vectorized identity on the left”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Powers and traces of transfer matrices”].

Theorem 19.6.1 (Eigenvalues of a transfer matrix). [Lean](#) [✓ Stmt](#) *A complex number μ is an eigenvalue of T if and only if it is an eigenvalue of $\widehat{T}$.*

Proof. [✓ Proof](#)

Column stacking is a linear bijection and intertwines T with multiplication by $\widehat{T}$. It therefore carries nonzero eigenmatrices to nonzero eigenvectors and conversely. $\square$

For a matrix product tensor A , write $V^{(N)}(A)_\sigma$ for the coefficient of its length- N periodic vector at the configuration σ .

Theorem 19.6.2 (Transfer-matrix powers as periodic overlaps). [Lean](#) [Lean](#) [✓ Stmt](#) Let A and B be matrix product tensors with the same positive bond dimension, and let $F_{A,B}(X) = \sum_i A^i X (B^i)^\dagger$ be their mixed transfer map. Then

$$\mathrm{tr} \left(\widehat{F_{A,B}}^N \right) = \sum_{\sigma} V^{(N)}(A)_{\sigma} \overline{V^{(N)}(B)_{\sigma}}. \quad (19.268)$$

In particular, taking $B = A$ identifies the trace of the N th power of the ordinary transfer matrix with the length- N self-overlap.

Proof. [✓ Proof](#) Replace the matrix power by the transfer matrix of the corresponding operator power, identify matrix trace with operator trace, and expand the latter over periodic words. For the ordinary self-transfer statement, specialize to $B = A$ and use the identity $F_{A,A} = \mathcal{E}_A$ from Theorem [B.1.1](#). $\square$

Definition 19.6.3 (Normalized flattening of an MPO tensor). [Lean](#) [✓ Stmt](#) For an MPO tensor U of physical dimension $d > 0$, its normalized flattening is the doubled-physical-index MPS tensor

$$\tilde{U}^{ij} = \frac{1}{\sqrt{d}} U^{ij}. \quad (19.269)$$

This is the normalization used in the normal-tensor argument of [[CPGSV17](#), Section III, equation (13)].

Lemma 19.6.4 (Normalized flattening of a composition tensor). [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ and $\mathcal{V}$ be MPO tensors with the same positive physical dimension d , and let

$$W^{ik} = \sum_{j=0}^{d-1} U^{ij} \otimes V^{jk} \quad (19.270)$$

be their composition tensor. Under the canonical identification of its auxiliary space with $\mathbb{C}^{D_U} \otimes \mathbb{C}^{D_V}$, the normalized flattenings satisfy

$$\tilde{W}^{ik} = \sqrt{d} \sum_{j=0}^{d-1} \tilde{U}^{ij} \otimes \tilde{V}^{jk}. \quad (19.271)$$

This identity combines the raw composition tensor used in the proof of [[CPGSV17](#), Theorem IV.6(ii)] with the normalization in [[CPGSV17](#), Section III, equation (13)]; it is not a separately stated result of that proof.

Proof. [✓ Proof](#) Substitute $U^{ij} = \sqrt{d} \tilde{U}^{ij}$ and $V^{jk} = \sqrt{d} \tilde{V}^{jk}$ into the defining contraction for W^{ik} , and then multiply the result by $d^{-1/2}$. $\square$

Theorem 19.6.5 (Normalized flattening under blocking). [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ be an MPO tensor of physical dimension d , and let $L \geq 0$. For the normalized-tensor interpretation assume $d > 0$, as in Definition [19.6.3](#); with the convention $0^{-1} = 0$, the algebraic identity below also holds for $d = 0$. Pairing the ket and bra letters at each site gives a canonical bijection

$$\vartheta_L : \mathrm{Fin}(d^L)^2 \longrightarrow \mathrm{Fin}(d^2)^L. \quad (19.272)$$

If $\widetilde{\mathcal{U}} = d^{-1/2}\mathcal{U}$ denotes normalized flattening, then

$$\widetilde{\mathcal{U}}^{[L]} = \vartheta_L^*(\widetilde{\mathcal{U}}^{[L]}). \quad (19.273)$$

Thus normalized flattening commutes with blocking, up to the canonical doubled-index relabeling. This is the normalized form of the blocking operation in [CPGSV17, Definition II.5 and equation (9)].

Proof. ✓ Proof A blocked letter is a product of L original letters. The identity

$$(\sqrt{d^L})^{-1} = ((\sqrt{d})^{-1})^L \quad (19.274)$$

identifies its normalization with the product of the L one-site normalizations. Theorem 20.14.2 identifies the two physical indexings for every $L \geq 0$; at $L = 0$ both blocked words are empty and both local matrix products are the identity. $\square$

Theorem 19.6.6 (Left-canonical normalization under blocking and relabeling). Lean ✓ Stmt
Suppose that an MPS tensor A is left-canonical:

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}. \quad (19.275)$$

Then for every $L \geq 0$ its blocked tensor is left-canonical:

$$\sum_I ((A^{[L]})^I)^\dagger (A^{[L]})^I = \mathbb{1}. \quad (19.276)$$

If e is a bijection of physical alphabets, then e^*A is left-canonical if and only if A is:

$$\sum_j ((e^*A)^j)^\dagger (e^*A)^j = \mathbb{1} \iff \sum_i (A^i)^\dagger A^i = \mathbb{1}. \quad (19.277)$$

Proof. ✓ Proof Expand (19.276) as a sum over length- L words and sum one letter at a time, repeatedly applying the one-site left-canonical identity. For $L = 0$, the unique empty word evaluates to the identity. The equivalence (19.277) follows by reindexing the finite sum along e . $\square$

Definition 19.6.7 (Canonical-form-II data under blocking and relabeling). Lean Lean Lean
✓ Stmt Let A have canonical-form-II data with retained weights μ_k , blocks A_k , and positive definite diagonal fixed matrices Λ_k . For every positive integer L , the blocked tensor $A^{[L]}$ has canonical-form-II data with weights μ_k^L , blocks $A_k^{[L]}$, and the same matrices Λ_k . The retained blocks remain left-canonical:

$$\sum_I ((A_k^{[L]})^I)^\dagger (A_k^{[L]})^I = \mathbb{1}. \quad (19.278)$$

A bijective relabeling of the physical alphabet likewise preserves canonical-form-II data, including both the matrices Λ_k and the left-canonical identities.

Consequently, if the normalized flattening $\widetilde{\mathcal{U}}$ of an MPO tensor has canonical-form-II data, then for every $L > 0$ the normalized flattening of $\mathcal{U}^{[L]}$ has the data obtained by first blocking and then relabeling by ϑ_L from Theorem 19.6.5. Theorem 19.6.6 preserves the left-canonical identity under both operations. For each retained block, Lemma D.4.8 gives

$$\mathcal{E}_{A_k^{[L]}} = \mathcal{E}_{A_k}^L, \quad \mathcal{E}_{A_k}^L(\Lambda_k) = \Lambda_k. \quad (19.279)$$

The second identity follows by iterating $\mathcal{E}_{A_k}(\Lambda_k) = \Lambda_k$. Thus the same positive definite diagonal matrix is fixed after blocking. A physical bijection leaves the transfer map unchanged. The final construction applies these two operations to (19.273).

Theorem 19.6.8 (Self-overlap scaling). Lean Lean ✓ Stmt For a matrix product tensor A , a complex number c , and a nonnegative integer N , scaling every local matrix by c gives

$$\sum_{\sigma} V^{(N)}(cA)_{\sigma} \overline{V^{(N)}(cA)_{\sigma}} = (c\bar{c})^N \sum_{\sigma} V^{(N)}(A)_{\sigma} \overline{V^{(N)}(A)_{\sigma}}. \quad (19.280)$$

Consequently, if $d > 0$ and U is an MPO tensor of physical dimension d , then

$$\sum_{\sigma} V^{(N)}(\tilde{U})_{\sigma} \overline{V^{(N)}(\tilde{U})_{\sigma}} = d^{-N} \sum_{\sigma} V^{(N)}(U^{\text{MPS}})_{\sigma} \overline{V^{(N)}(U^{\text{MPS}})_{\sigma}}. \quad (19.281)$$

Proof. ✓ Proof Each length- N coefficient is multiplied by c^N . Multiplication by its complex conjugate gives the first identity. Taking $c = 1/\sqrt{d}$ gives the second. $\square$

Theorem 19.6.9 (Normalized MPU transfer trace). Lean ✓ Stmt Let U be an MPU tensor with positive physical and bond dimensions, and let E be the transfer matrix of its normalized flattening. For every integer $N > 1$,

$$\text{tr}(E^N) = 1. \quad (19.282)$$

This is the trace identity in the proof of the normal-tensor result [CPGSV17, Proposition III.1].

Proof. ✓ Proof Scaling both tensor copies by $1/\sqrt{d}$ multiplies the length- N self-overlap by d^{-N} . The doubled-index overlap identity and MPU unitarity give the complete chain

$$\text{tr}(E^N) = \sum_{\sigma} V^{(N)}(\tilde{U})_{\sigma} \overline{V^{(N)}(\tilde{U})_{\sigma}} \quad (19.283)$$

$$= d^{-N} \sum_{\sigma} V^{(N)}(U^{\text{MPS}})_{\sigma} \overline{V^{(N)}(U^{\text{MPS}})_{\sigma}} \quad (19.284)$$

$$= d^{-N} \text{tr}(U^{(N)}(U^{(N)})^{\dagger}) \quad (19.285)$$

$$= d^{-N} \text{tr}(\mathbb{1}) \quad (19.286)$$

$$= d^{-N} d^N \quad (19.287)$$

$$= 1. \quad (19.288)$$

Here the fourth equality uses $N > 1$, so that the MPU equation applies. $\square$

19.7 Block transfer multiplicity

Definition 19.7.1 (Flattened dependent-block inclusion). Lean ✓ Stmt Let the blocks be indexed by $k \in \{0, \dots, r-1\}$, let n_k be their nonnegative dimensions, and put $n = \sum_{k \in \{0, \dots, r-1\}} n_k$. The standard block-order bijection is

$$\phi: \bigsqcup_{k \in \{0, \dots, r-1\}} \{0, \dots, n_k-1\} \longrightarrow \{0, \dots, n-1\}, \quad (19.289)$$

$$\phi(k, a) = \sum_{j=0}^{k-1} n_j + a. \quad (19.290)$$

For each $k \in \{0, \dots, r-1\}$, define $J_k : \mathbb{C}^{n_k} \rightarrow \mathbb{C}^n$ by

$$(J_k)_{x,a} = \begin{cases} 1, & x = \phi(k, a), \\ 0, & x \neq \phi(k, a). \end{cases} \quad (19.291)$$

Theorem 19.7.2 (Dependent-block inclusion identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The inclusions satisfy the following identities, with the second holding for distinct k and ℓ :*

$$J_k^\dagger J_k = \mathbb{1}, \quad (19.292)$$

$$J_k^\dagger J_\ell = 0. \quad (19.293)$$

If $B^i = \bigoplus_k \mu_k A_k^i$ is a weighted block-diagonal tensor, then

$$B^i J_k = J_k (\mu_k A_k^i). \quad (19.294)$$

Proof. [✓ Proof](#) Reindex the canonical dependent-summand inclusion along the equivalence between dependent pairs and flattened coordinates. Transporting its isometry, cross-block orthogonality, and block-diagonal intertwining identities gives the three displayed formulas. $\square$

Definition 19.7.3 (Ambient inclusion of a canonical block). [Lean](#) [✓ Stmt](#) If U is the reconstruction coisometry and J_k is the inclusion of the k th retained block, define

$$V_k = U^\dagger J_k. \quad (19.295)$$

Theorem 19.7.4 (Ambient block isometry and intertwining). [Lean](#) [Lean](#) [✓ Stmt](#) *For every retained block and physical index i ,*

$$V_k^\dagger V_k = \mathbb{1}, \quad (19.296)$$

$$A^i V_k = V_k (\mu_k A_k^i). \quad (19.297)$$

Proof. [✓ Proof](#) The coisometry $UU^\dagger = \mathbb{1}$ and $J_k^\dagger J_k = \mathbb{1}$ give

$$V_k^\dagger V_k = J_k^\dagger U U^\dagger J_k = J_k^\dagger J_k = \mathbb{1}. \quad (19.298)$$

The canonical reconstruction and block-diagonal intertwining give

$$A^i V_k = U^\dagger \left(\bigoplus_j \mu_j A_j^i \right) J_k = U^\dagger J_k (\mu_k A_k^i) = V_k (\mu_k A_k^i). \quad (19.299)$$

$\square$

Definition 19.7.5 (Weighted-block transfer eigenvalue). [Lean](#) [✓ Stmt](#) For a block k , define its weighted transfer eigenvalue by

$$q_k = \mu_k \overline{\mu_k} = |\mu_k|^2. \quad (19.300)$$

Theorem 19.7.6 (Shifted traces normalize every weight). [Lean](#) [✓ Stmt](#) *If the ambient transfer matrix satisfies $\text{tr}(E^N) = 1$ for every integer $N > 1$, then every block satisfies*

$$q_k = 1. \quad (19.301)$$

Proof. ✓ Proof Choose a nonzero fixed matrix X_k for the unweighted normal block and set $Y_k = V_k X_k V_k^\dagger$. The isometry and intertwining identities give

$$Y_k \neq 0, \quad E(Y_k) = q_k Y_k. \quad (19.302)$$

Thus q_k lies in the ambient transfer spectrum; it is nonzero because $\mu_k \neq 0$. The shifted trace-power spectrum is $\{0, 1\}$, hence $q_k = 1$. $\square$

Theorem 19.7.7 (Unique full-support block geometry). Lean Lean ✓ Stmt *If there is exactly one block k and the blocks fill the ambient bond space, then*

$$D_k = D, \quad V_k V_k^\dagger = \mathbb{1}. \quad (19.303)$$

Proof. ✓ Proof The full-support sum reduces to the unique summand:

$$D = \sum_{j=1}^r D_j = D_k. \quad (19.304)$$

Therefore V_k is a square matrix. Since $V_k^\dagger V_k = \mathbb{1}$, finite equal-dimensional inverse symmetry gives $V_k V_k^\dagger = \mathbb{1}$. $\square$

Theorem 19.7.8 (Shifted transfer traces determine one block). Lean ✓ Stmt *Let A have literal CPSV canonical-form data with r blocks. If its transfer matrix E satisfies $\text{tr}(E^N) = 1$ for every integer $N > 1$, then*

$$r = 1. \quad (19.305)$$

This is the block multiplicity consequence used in [CPGSV17, Proposition III.1].

Proof. ✓ Proof Each normal block A_k has a nonzero transfer fixed vector X_k . Let U be the reconstruction coisometry and let J_k be the flattened inclusion of the k th block. Define the ambient inclusion and transported matrix by

$$V_k = U^\dagger J_k, \quad Y_k = V_k X_k V_k^\dagger. \quad (19.306)$$

The matrix Y_k is nonzero and is an ambient transfer eigenmatrix with eigenvalue $q_k = \mu_k \overline{\mu_k} \neq 0$. Theorem 19.4.6 gives $q_k = 1$.

For distinct k and ℓ , the inclusion identities give

$$V_k^\dagger Y_k V_k = X_k, \quad (19.307)$$

$$V_k^\dagger Y_\ell V_k = 0. \quad (19.308)$$

Hence the matrices Y_k , for $1 \leq k \leq r$, are linearly independent. Vectorizing gives

$$\text{vec}(Y_k) \in \ker(E^2 - \mathbb{1}). \quad (19.309)$$

For every $r \geq 1$, the shifted traces and Theorem 19.4.4 give

$$\text{tr}((E^2)^r) = 1, \quad (19.310)$$

$$\chi_{E^2}(X) = X^{D^2-1}(X - 1). \quad (19.311)$$

The eigenspace dimension is bounded by the algebraic multiplicity, so

$$r \leq \dim \ker(E^2 - \mathbb{1}) \leq \text{mult}_1(\chi_{E^2}) = 1. \quad (19.312)$$

Finally, if r were zero, the retained direct sum and therefore A would vanish, contradicting $\text{tr}(E^2) = 1$. Consequently $r = 1$. $\square$

19.8 Normalized transfer spectrum and conditional normality

Theorem 19.8.1 (Shifted transfer traces imply radius one and primitivity). [Lean](#) [✓ Stmt](#) Let $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ be a linear map with $D > 0$. If $\text{tr}(\widehat{T}^N) = 1$ for every integer $N > 1$, then

$$r(T) = 1, \quad \text{peripheral}(T) = \{1\}. \quad (19.313)$$

This is the radius-and-primitivity part of [\[CPGSV17, Proposition III.1\]](#).

Proof. [✓ Proof](#) The shifted traces give

$$\sigma(\widehat{T}) \setminus \{0\} = \{1\}. \quad (19.314)$$

Column stacking identifies the eigenvalues of T and $\widehat{T}$. Consequently every spectral value of T is zero or one, and one occurs, so

$$r(T) = \max_{\lambda \in \sigma(T)} |\lambda| = 1. \quad (19.315)$$

Restricting to eigenvalues of modulus one gives

$$\text{peripheral}(T) = \{1\}, \quad (19.316)$$

which is primitivity. □

Theorem 19.8.2 (Exact spectrum of the normalized MPU transfer matrix). [Lean](#) [Lean](#) [✓ Stmt](#) Let U be an MPU tensor with positive physical and bond dimensions, and let E be the transfer matrix of its normalized flattening. Then

$$\sigma(E) \setminus \{0\} = \{1\}, \quad \text{mult}_1(\chi_E) = 1, \quad \chi_E(X) = X^{D^2-1}(X-1). \quad (19.317)$$

This is the transfer-matrix conclusion of [\[CPGSV17, Proposition III.1\]](#). The first set equality alone does not assert algebraic multiplicity.

Proof. [✓ Proof](#) For every integer $N > 1$, the normalized transfer identity gives

$$\text{tr}(E^N) = 1. \quad (19.318)$$

Applying the exact shifted trace-power theorem in dimension D^2 yields

$$\chi_E(X) = X^{D^2-1}(X-1), \quad \text{mult}_1(\chi_E) = 1, \quad \sigma(E) \setminus \{0\} = \{1\}. \quad (19.319)$$

□

Theorem 19.8.3 (Finite blocking eliminates the zero-primary transfer component). [Lean](#) [✓ Stmt](#) Let U be an MPU tensor with physical dimension $d > 0$ and bond dimension $D > 1$, and let E be the transfer matrix of its normalized flattening. There is an integer J satisfying

$$0 < J \leq D^2 - 1 \quad (19.320)$$

and vectors ρ, Φ satisfying

$$(\Phi \mid \rho) = 1, \quad E\rho = \rho, \quad (\Phi \mid E = (\Phi \mid, \quad (19.321)$$

such that

$$E^J = |\rho\rangle\langle\Phi|. \quad (19.322)$$

Moreover, for every integer $k \geq J$,

$$E^k = E^J. \quad (19.323)$$

Thus blocking J sites removes the entire generalized zero-eigenspace. The one-dimensional case is stated separately below.

This is the Jordan-elimination step in [CPGSV17, proof of Proposition III.2(ii)].

Proof. ✓ Proof The exact characteristic polynomial is

$$\chi_E(X) = X^{D^2-1}(X-1). \quad (19.324)$$

Cayley–Hamilton gives $E^{D^2} = E^{D^2-1}$. Hence every power from $J = D^2 - 1$ onward is equal, and $P = E^J$ is idempotent. Since $J > 1$, the normalized transfer trace identity gives $\text{tr}(P) = 1$. The range of an idempotent has dimension equal to its trace, so P has rank one and factors as $P = |\rho\rangle\langle\Phi|$. Taking the factors with $(\Phi | \rho) = 1$ gives the displayed fixed-point identities from $EP = PE = P$. $\square$

Theorem 19.8.4 (Uniqueness of the normalized fixed-pair factorization). Lean ✓ Stmt Let $P = E^J$ be a stabilized rank-one transfer power with normalized left and right factors. If ρ' and Φ' are any other fixed vectors with $(\Phi' | \rho') = 1$, then

$$P = |\rho'\rangle\langle\Phi'|. \quad (19.325)$$

Proof. ✓ Proof Applying P to ρ' and applying Φ' to P shows that the two pairs of rank-one factors differ by reciprocal scalars. Their normalized pairings force the product of those scalars to be one. $\square$

Theorem 19.8.5 (Identification with chosen reduced-CFII fixed points). Lean ✓ Stmt Suppose $d > 0$, $D > 1$, and $\mathcal{U}$ is an MPU whose normalized flattening is presented by chosen canonical-form-II data with blocks filling the bond space. There is a unique block k . Normalize its diagonal positive definite fixed matrix Λ_k to have trace one, and let V_k be its inclusion into the ambient bond space. Then

$$\rho = V_k \Lambda_k V_k^\dagger, \quad \rho > 0, \quad \text{tr}(\rho) = 1, \quad (19.326)$$

$$E|\rho\rangle = |\rho\rangle, \quad (\Phi|E = (\Phi|, \quad (\Phi | \rho) = 1, \quad (19.327)$$

where $(\Phi|$ is the vectorized identity. Consequently,

$$E^{D^2-1} = |\rho\rangle\langle\Phi|. \quad (19.328)$$

The diagonal matrix is the block matrix Λ_k in the canonical-form-II fixed-point equations in [CPGSV17, equations (6a)–(6b)]; its ambient transport ρ need not be diagonal. This is a statement about the chosen reduced representative and the blocking step in [CPGSV17, proof of Proposition III.2(ii)]; it does not assert that every ambient MPU tensor is itself in canonical form II.

Proof. ✓ Proof The shifted trace identities satisfy $\text{tr}(E^N) = 1$ for every integer $N > 1$ and give

$$\text{tr}(E^N) = 1 \implies r = 1. \quad (19.329)$$

Let k be this block. Canonical form II supplies a diagonal $\Lambda_0 > 0$ fixed by the unweighted block transfer map. Normalize it by

$$\Lambda = \text{tr}(\Lambda_0)^{-1} \Lambda_0, \quad \text{tr}(\Lambda) = 1. \quad (19.330)$$

If ω_k is its weight, then

$$|\omega_k|^2 = 1, \quad E_k(\Lambda) = \Lambda, \quad (19.331)$$

where E_k is the transfer map of the weighted block $\widetilde{B}_k = \omega_k B_k$. Full support makes the block inclusion $V = V_k$ both isometric and onto:

$$V^\dagger V = I, \quad VV^\dagger = I. \quad (19.332)$$

Hence $\rho = V\Lambda V^\dagger$ is positive definite with trace one, and the intertwining equations $A_i V = V\widetilde{B}_k(i)$ give the fixedness transport

$$E(V\Lambda V^\dagger) = VE_k(\Lambda)V^\dagger = V\Lambda V^\dagger = \rho. \quad (19.333)$$

The blockwise left-canonical identity transports similarly:

$$\sum_i \widetilde{B}_k(i)^\dagger \widetilde{B}_k(i) = I \implies \sum_i A_i^\dagger A_i = I \implies (\Phi|E = (\Phi|. \quad (19.334)$$

Finally,

$$(\Phi | \rho) = \text{tr}(\rho) = 1, \quad E^{D^2-1} = |\rho)(\Phi|, \quad (19.335)$$

where the last equality is the uniqueness of the normalized fixed-pair factorization for the stabilized power. $\square$

Theorem 19.8.6 (Reduced-CFII data for the forced simple block). Lean ✓ Stmt Let $d > 0$, $D > 1$, and let $\mathcal{U}$ be an MPU whose normalized flattening has chosen canonical-form-II data with full support. Put $J = D^2 - 1$ and $W = \mathcal{U}_{JD^2}$. There is a matrix $\rho > 0$ with $\text{tr}(\rho) = 1$ such that

$$E|\rho) = |\rho), \quad (\Phi|E = (\Phi|, \quad E^J = |\rho)(\Phi|. \quad (19.336)$$

With the corresponding vectorized pair ρ', Φ' , the same blocked tensor W satisfies the two aligned simple contractions

$$\Phi' \cdot W^{ij} \rho' = \delta_{ij}, \quad (19.337)$$

$$W^{ij} W^{k\ell} = W^{ij} |\rho')(\Phi' | W^{k\ell}. \quad (19.338)$$

This theorem supplies only the fixed pair and the two simple contractions. It does not assert a source-cut factorization or source- v isometry.

Proof. ✓ Proof Theorem 19.8.5 gives the positive trace-one fixed matrix and the identity $E^J = |\rho)(\Phi|$. Applying Theorem 19.3.23 with this pair gives both contractions on the direct block $W = \mathcal{U}_{JD^2}$. $\square$

Theorem 19.8.7 (One-dimensional normalized transfer matrix). [Lean](#) [✓ Stmt](#) If the physical dimension satisfies $d > 0$ and the bond dimension is $D = 1$, then the normalized transfer matrix satisfies

$$E = 1. \quad (19.339)$$

Consequently its powers stabilize at the positive exponent one.

Proof. [✓ Proof](#) In dimension one the characteristic polynomial identity reduces to $\chi_E(X) = X - 1$, and Cayley–Hamilton gives $E = 1$. $\square$

Theorem 19.8.8 (Normality of a full-support normalized MPU representative). [Lean](#) [✓ Stmt](#) Let U be an MPU tensor with positive physical and bond dimensions. Suppose that the normalized flattening is equipped with CPSV canonical-form data whose blocks have full support in the ambient bond space. Then the normalized flattening is a normal tensor.

This is the full-support representative form of [CPGSV17, Proposition III.1]. The full-support hypothesis is essential for this ambient representative: adjoining a zero virtual direct summand preserves every periodic operator but introduces a nontrivial invariant projection. Thus the theorem does not assert that an arbitrary, possibly nonminimal MPU representative is normal.

Proof. [✓ Proof](#) Write $\widetilde{U}$ for the normalized flattening, set $T = \mathcal{E}_{\widetilde{U}}$ for its transfer map, and let $E = \widehat{T}$ be the corresponding transfer matrix. Also write r for the number of blocks and D_k for the bond dimension of the block k . For every integer $N > 1$, the normalized transfer identity gives

$$\mathrm{tr}(E^N) = 1. \quad (19.340)$$

The block-multiplicity theorem and the transfer-spectrum theorem then give

$$r = 1, \quad r(T) = 1, \quad \text{peripheral}(T) = \{1\}. \quad (19.341)$$

Full support supplies

$$\sum_{k=1}^r D_k = D. \quad (19.342)$$

Since $r = 1$, the unique block has ambient dimension D and therefore makes $\widetilde{U}$ irreducible. The displayed radius and peripheral-spectrum identities provide its remaining normality fields. Consequently $\widetilde{U}$ is a normal tensor. $\square$

19.9 Standard form, index, and symmetry classification

Definition 19.9.1 (Reduced full-support source datum). [Not ready](#) Let $\widetilde{\mathcal{U}} = d^{-1/2}\mathcal{U}$ be the normalized flattening, and let E be its transfer matrix. A reduced full-support source datum for $\mathcal{U}$, with physical dimension $d > 0$ and positive bond dimension, consists of an integer $J > 0$, a positive definite matrix ρ with $\mathrm{tr}(\rho) = 1$, and $(\Phi| = \mathrm{vec}(\mathbb{1}_D)^\top$ such that

$$E^J = |\rho)(\Phi|. \quad (19.343)$$

These data are supplied in one of two ways.

1. If $D = 1$, take $J = 1$ and $\rho = \mathbb{1}_1$ using Theorem 19.8.7.

2. If $D > 1$, choose canonical-form-II data for the normalized flattening of this same tensor, with its blocks filling the ambient bond space. Theorem 19.8.5 supplies the fixed pair with $J = D^2 - 1$.

For $D > 1$, this states the fixed-pair convention in [CPGSV17, equations labelled `Erightright` and `II_UTransfer`, and proof of Proposition III.2(ii)]. The $D = 1$ branch is the direct one-dimensional specialization. This definition does not assert that an arbitrary representative with $D > 1$ admits the required canonical-form-II data and does not replace $\mathcal{U}$ by another tensor.

Theorem 19.9.2 (Complete source- u network contraction). Not ready Discussion: #5982 Let $\mathcal{U}$ be an MPU tensor with physical dimension $d > 0$ and bond dimension $D > 0$. Let $\rho > 0$ satisfy $\text{tr}(\rho) = 1$, put $(\Phi| = \text{vec}(\mathbb{1}_D)^\top$, and suppose that for some $J > 0$ the normalized transfer matrix satisfies

$$E^J = |\rho)(\Phi|. \quad (19.344)$$

Construct the source factors and u from $\mathcal{U}$ using ρ , and set $K = JD^2$. For retained physical pairs $p, q \in \{0, \dots, d-1\}^2$, the complete source- u contraction satisfies

$$\sum_{l,r} u_{(l,r),q} \overline{u_{(l,r),p}} = d^{-K} \sum_{\tau,\eta} U_{(q,\tau),\eta}^{(K+2)} \overline{U_{(p,\tau),\eta}^{(K+2)}}. \quad (19.345)$$

Here τ ranges over input words of length K , and η ranges over output words of length $K+2$. This is the supplied-fixed-pair form of the range-insertion equality in [CPGSV17, Lemma III.7].

Proof. Put $W = \mathcal{U}^{[K]}$. The reflected transfer-power identity, the direct matching contractions, and supplied-witness reblocking give the aligned direct and reflected contractions at lengths K and $K+2$, with the same fixed pair at both lengths. The block W is used only inside this contraction argument; the external source tensors are the source- u factors of $\mathcal{U}$.

Expand the ordinary Gram entry while retaining both external source- u tensors and the full ρ -weighted boundary. Insert the first- and second-cut range projections, but contract them only inside the closed $(K+2)$ -site network. The reflected contractions move the second-cut metric through the overlapping windows. After every virtual and source index is closed, the terminal boundary contraction gives the normalized output-first trace displayed above. No separate identity for the second-cut metric or either ambient range projection is asserted. $\square$

Theorem 19.9.3 (Complete reduced source- v contraction). Not ready Discussion: #6071 Let $\mathcal{U}$ be an MPU tensor equipped with a reduced full-support source datum. Let J , ρ , and $(\Phi|$ be its stabilization exponent and normalized fixed pair, so that

$$E^J = |\rho)(\Phi|. \quad (19.346)$$

Compute both source cuts, their ranks r and ℓ , the source operator v , and the right source factors Y_1, Y_2 from this same tensor $\mathcal{U}$ using ρ . Let W be the double layer of $\mathcal{U}$. Then the supplied contraction

$$W^{ij} W^{k\ell} = W^{ij} |\rho)(\Phi| W^{k\ell} \quad (19.347)$$

holds if and only if

$$(Y_1 \otimes Y_2)^\dagger v^\dagger v (Y_1 \otimes Y_2) = (Y_1 \otimes Y_2)^\dagger (Y_1 \otimes Y_2). \quad (19.348)$$

When $\mathcal{U}$ is simple, Theorem 19.3.28 supplies the displayed contraction with this same fixed pair. This is the complete equivalence in [CPGSV17, proof of Theorem III.8].

Proof. Insert $E^J = |\rho)(\Phi|$ into the complete network while retaining the source factors of $\mathcal{U}$. Expand both source cuts and rotate the second cut. Contract the resulting four-letter expression as one closed network to obtain the dressed Gram equation, and reverse the same calculation for the converse. $\square$

Lemma 19.9.4 (The source operator u is an isometry). Not ready Discussion: #5708 For any MPU tensor $\mathcal{U}$ under the source's standing canonical-form-II convention, let r and ℓ be the two source-cut ranks and let u be the source operator. Then

$$u^\dagger u = \mathbb{1}. \quad (19.349)$$

Consequently u is an isometry and $r\ell \geq d^2$. This is the unrestricted statement of [CPGSV17, Lemma III.7].

Lemma 19.9.5 (Admissible source- u isometry). Not ready Discussion: #5708 Let $\mathcal{U}$ be an MPU tensor with physical dimension $d > 0$ and bond dimension $D > 0$. Let $\rho > 0$ satisfy $\text{tr}(\rho) = 1$, put $(\Phi| = \text{vec}(\mathbb{1}_D)^\top$, and suppose that for some $J > 0$ the normalized transfer matrix satisfies

$$E^J = |\rho)(\Phi|. \quad (19.350)$$

Let r and ℓ be the two source-cut ranks, and construct the source factors and u from $\mathcal{U}$ using ρ . Then

$$u^\dagger u = \mathbb{1}. \quad (19.351)$$

Consequently u is an isometry and $r\ell \geq d^2$. This is the reduced-source version of [CPGSV17, Lemma III.7].

Proof. Set $K = JD^2$. The complete-network identity and output-first MPU unitarity give, for all retained physical pairs p, q ,

$$(u^\dagger u)_{p,q} = d^{-K} \sum_{\tau, \eta} U_{(q, \tau), \eta}^{(K+2)} \overline{U_{(p, \tau), \eta}^{(K+2)}} \quad (19.352)$$

$$= \delta_{p,q}. \quad (19.353)$$

Hence $u^\dagger u = \mathbb{1}$. $\square$

Theorem 19.9.6 (Simple-tensor equivalence theorem). Not ready Discussion: #5708 Let $\mathcal{U}$ be an MPU tensor under the source's standing canonical-form-II convention. Let r and ℓ be its source-cut ranks, and let u and v be its source operators. Then the following conditions are equivalent:

1. $\mathcal{U}$ is simple;
2. $r\ell = d^2$;
3. u is unitary;
4. v is unitary.

This is the unrestricted statement of [CPGSV17, Theorem III.8].

Theorem 19.9.7 (Admissible simple-tensor equivalence theorem). Not ready Discussion: #5708 Let $\mathcal{U}$ be an MPU tensor equipped with a reduced full-support source datum, and let ρ and $(\Phi|$ be its normalized fixed pair. Compute the source-cut ranks r and ℓ and the source operators u and v from this same tensor $\mathcal{U}$ using ρ . Then the following conditions are equivalent:

1. $\mathcal{U}$ is simple;
2. $r\ell = d^2$;
3. u is unitary;
4. v is unitary.

This is the reduced-source version of [CPGSV17, Theorem III.8].

Proof. Lemma 19.9.5 gives $u^\dagger u = \mathbb{1}$ before any of the four conditions is assumed. Thus u is a standing isometry and $r\ell \geq d^2$ throughout the proof.

Suppose first that $\mathcal{U}$ is simple. Theorem 19.3.28, applied to the stabilized transfer power, gives the second contraction with the supplied pair $|\rho\rangle\langle\Phi|$. By Theorem 19.9.3,

$$(Y_1 \otimes Y_2)^\dagger v^\dagger v (Y_1 \otimes Y_2) = (Y_1 \otimes Y_2)^\dagger (Y_1 \otimes Y_2). \quad (19.354)$$

Cancelling the dressing by the right inverses of Y_1 and Y_2 gives $v^\dagger v = \mathbb{1}$, hence $r\ell \leq d^2$. Together with the standing source- u isometry, this gives $r\ell = d^2$.

If $r\ell = d^2$, the standing isometry u is square and therefore unitary. If u is unitary, the exact two-site factorization of $U^{(2)}$ and unitarity of $U^{(2)}$ imply that v is unitary. Finally, suppose that v is unitary. Unitarity gives the dressed Gram identity, and Theorem 19.9.3 gives the supplied second contraction for $\mathcal{U}$. The same-witness implication in Theorem 19.4.70 supplies the first contraction, so $\mathcal{U}$ is simple. $\square$

Definition 19.9.8 (Standard form). Not ready Discussion: #6019 Let $\mathcal{U}$ be a simple tensor under the source's standing canonical-form-II convention. Its standard form is the depth-two form of the two-site block $\mathcal{U}_2$ determined by the source unitaries u and v . This is the unrestricted definition of [CPGSV17, Definition III.9].

Definition 19.9.9 (Admissible standard form). Not ready Discussion: #6019 Let $\mathcal{U}$ be a simple tensor equipped with a reduced full-support source datum. The admissible standard form of the two-site blocking $\mathcal{U}_2$ is the depth-two form determined by the source unitaries

$$u: \mathbb{C}^d \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^\ell \otimes \mathbb{C}^r, \quad (19.355)$$

$$v: \mathbb{C}^r \otimes \mathbb{C}^\ell \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d. \quad (19.356)$$

On four consecutive blocked legs its unitary is

$$U = v_{23} v_{41} u_{12} u_{34}. \quad (19.357)$$

Theorem 19.9.10 (Fundamental Theorem of MPU). Not ready Discussion: #6020 Given two simple tensors $\mathcal{U}$ and $\mathcal{V}$ with standard forms, they generate the same MPU for every N if and only if there are unitaries x , y , and z relating the two standard forms by the local gauge equations of [CPGSV17, Eqns. (SFuu) and (SFvv)]. This is the unrestricted statement of the source's Fundamental Theorem of MPU.

Theorem 19.9.11 (Admissible Fundamental Theorem of MPU). Not ready Discussion: #6020 Let $\mathcal{U}$ and $\mathcal{V}$ be simple tensors equipped with reduced full-support source data and written in admissible standard form, with source unitaries $(u_{\mathcal{U}}, v_{\mathcal{U}})$ and $(u_{\mathcal{V}}, v_{\mathcal{V}})$. They generate the same MPU for every N ,

$$U^{(N)} = V^{(N)}, \quad (19.358)$$

if and only if their admissible standard forms are related by the local unitary gauges of [CPGSV17, Eqs. (SFuu) and (SFvv)]: there are unitaries x , y , and z on the three corresponding source and canonical-form-II legs. In particular, the source unitaries obey the local relation

$$u_{\mathcal{V}} = (x \otimes y)u_{\mathcal{U}}, \quad (19.359)$$

while the two canonical-form-II tensors are related on their virtual leg by z and on their source legs by x and y .

Theorem 19.9.12 (Rank of a Kronecker product). [Lean](#) [✓ Stmt](#) Let A and B be finite matrices over a field. Then

$$\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B). \quad (19.360)$$

Proof. [✓ Proof](#) Identify the image of the Kronecker product with the tensor product of the two images and multiply their dimensions. $\square$

Definition 19.9.13 (Specified-tensor source-index value). [Lean](#) [✓ Stmt](#) For a specified tensor $\mathcal{U}$ whose right and left source-cut ranks r and ℓ are positive, define

$$\text{ind}_{\text{src}}(\mathcal{U}) = \frac{1}{2}(\log_2 r - \log_2 \ell). \quad (19.361)$$

This definition makes no choice of a simple blocking.

Theorem 19.9.14 (Auxiliary source-index identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\mathcal{U}$ and $\mathcal{V}$ be specified tensors. Suppose that their source ranks obey

$$r_{\mathcal{V}} = cr_{\mathcal{U}}, \quad \ell_{\mathcal{V}} = c\ell_{\mathcal{U}}, \quad (19.362)$$

where c , $r_{\mathcal{U}}$, and $\ell_{\mathcal{U}}$ are positive. Then

$$\text{ind}_{\text{src}}(\mathcal{V}) = \text{ind}_{\text{src}}(\mathcal{U}). \quad (19.363)$$

Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ instead have positive source ranks and suppose that

$$r_{\mathcal{W}} = cr_{\mathcal{U}}r_{\mathcal{V}}, \quad (19.364)$$

$$\ell_{\mathcal{W}} = c\ell_{\mathcal{U}}\ell_{\mathcal{V}}, \quad (19.365)$$

where c is a positive integer. Then

$$\text{ind}_{\text{src}}(\mathcal{W}) = \text{ind}_{\text{src}}(\mathcal{U}) + \text{ind}_{\text{src}}(\mathcal{V}). \quad (19.366)$$

If d , $r_{\mathcal{U}}$, and $\ell_{\mathcal{U}}$ are positive and a specified tensor has $r_{\mathcal{U}}\ell_{\mathcal{U}} = d^2$, then

$$\text{ind}_{\text{src}}(\mathcal{U}) = \log_2 \frac{r_{\mathcal{U}}}{d} = -\log_2 \frac{\ell_{\mathcal{U}}}{d}. \quad (19.367)$$

Proof. [✓ Proof](#) Expand the definition. For common scaling, apply $\log_2(cx) = \log_2 c + \log_2 x$ to both ranks and cancel the two copies of $\log_2 c$. For the three-tensor identity, apply the product formula twice to each rank and cancel the common term. For the equivalent formulas, apply the same product identity to $r_{\mathcal{U}}\ell_{\mathcal{U}} = d^2$ and use $\log_2(x/y) = \log_2 x - \log_2 y$. $\square$

Theorem 19.9.15 (Conditional blocking invariance of the specified source-index value). [Lean](#)
[✓ Stmt](#) Let $d > 0$ and $k_0 \leq k$. Suppose that the endpoint source ranks satisfy

$$r_{k_0} \ell_{k_0} = d^{2k_0}, \quad (19.368)$$

$$r_k \ell_k = d^{2k}. \quad (19.369)$$

Then the specified source-index values agree:

$$\text{ind}_{\text{src}}(\mathcal{U}_k) = \text{ind}_{\text{src}}(\mathcal{U}_{k_0}). \quad (19.370)$$

This statement assumes the endpoint products; it neither selects a simple blocking nor defines the public index.

Proof. [✓ Proof](#) Exact blocking growth gives the common positive factor d^{k-k_0} :

$$r_k = d^{k-k_0} r_{k_0}, \quad (19.371)$$

$$\ell_k = d^{k-k_0} \ell_{k_0}. \quad (19.372)$$

Its logarithm occurs once in each rank term and therefore cancels from their difference. $\square$

Definition 19.9.16 (Index). [Not ready](#) [Discussion: #6014](#) Let k be any blocking length for which $\mathcal{U}_k$ is simple, and let r and ℓ be its right and left source-cut ranks. The source index is

$$\text{ind}(\mathcal{U}) = \frac{1}{2}(\log_2 r - \log_2 \ell). \quad (19.373)$$

This is the unrestricted definition of [\[CPGSV17, Definition IV.1\]](#).

Definition 19.9.17 (Admissible index). [Not ready](#) [Discussion: #6014](#) Let k be a blocking length for which $\mathcal{U}_k$ is simple and is equipped with a reduced full-support source datum, and let r and ℓ be its right and left source-cut ranks. Define

$$\text{ind}(\mathcal{U}) = \frac{1}{2}(\log_2 r - \log_2 \ell). \quad (19.374)$$

Since $r\ell = d^{2k}$, this is equivalently

$$\text{ind}(\mathcal{U}) = \log_2 \frac{r}{d^k} = -\log_2 \frac{\ell}{d^k}. \quad (19.375)$$

Proposition 19.9.18 (Well-definedness of the index). [Not ready](#) [Discussion: #6014](#) The source index is independent of the chosen simple blocking. More precisely, if $k \geq k_0$ and both $\mathcal{U}_{k_0}$ and $\mathcal{U}_k$ are simple, then

$$r_k = d^{k-k_0} r_{k_0}, \quad \ell_k = d^{k-k_0} \ell_{k_0}. \quad (19.376)$$

This is the unrestricted statement of [\[CPGSV17, Proposition IV.2\]](#).

Proposition 19.9.19 (Well-definedness of the admissible index). [Not ready](#) [Discussion: #6014](#) The admissible index is independent of the chosen admissible simple blocking. More precisely, if $k \geq k_0$, both $\mathcal{U}_{k_0}$ and $\mathcal{U}_k$ are simple, and each is equipped with a reduced full-support source datum, then

$$r_k = d^{k-k_0} r_{k_0}, \quad \ell_k = d^{k-k_0} \ell_{k_0}. \quad (19.377)$$

Lemma 19.9.20 (Tensor-product additivity). Not ready Discussion: #6015 For MPU tensors under the source's standing canonical-form convention,

$$\text{ind}(\mathcal{U} \otimes \mathcal{V}) = \text{ind}(\mathcal{U}) + \text{ind}(\mathcal{V}). \quad (19.378)$$

This is the unrestricted tensor-product statement in [CPGSV17, Theorem IV.6(ii)].

Lemma 19.9.21 (Admissible tensor-product additivity). Not ready Discussion: #6015 Suppose there is one positive blocking length k such that $\mathcal{U}_k$, $\mathcal{V}_k$, and $(\mathcal{U} \otimes \mathcal{V})_k$ are simple and each carries its own reduced full-support source datum. Define all three indices from these blocks. Then

$$\text{ind}(\mathcal{U} \otimes \mathcal{V}) = \text{ind}(\mathcal{U}) + \text{ind}(\mathcal{V}). \quad (19.379)$$

This is the common-block reduced-source version of [CPGSV17, Theorem IV.6(ii)].

Lemma 19.9.22 (Four-site composition source-rank bounds). Lean Lean ✓ Stmt Let $\mathcal{U}$ and $\mathcal{V}$ be arbitrary matrix product operator tensors of common physical dimension d . For their product tensor, four-site blocking satisfies

$$r[(\mathcal{U} \cdot \mathcal{V})_4] \leq d^2 r[\mathcal{U}] r[\mathcal{V}], \quad (19.380)$$

$$\ell[(\mathcal{U} \cdot \mathcal{V})_4] \leq d^2 \ell[\mathcal{U}] \ell[\mathcal{V}]. \quad (19.381)$$

No unitarity or simplicity assumption is required.

Proof. ✓ Proof Use the canonical physical reindexing to identify iterated two-site blocking with direct four-site blocking. For the right cut, factor the two-site block of each tensor through its source tensor u . The diagonal cut through the two blocked factors retains one physical leg and one right source leg from each factor. Thus, for suitable rectangular matrices $A_{1,4}$ and $B_{1,4}$,

$$M_1((\mathcal{U} \cdot \mathcal{V})_4) = A_{1,4} B_{1,4}, \quad (19.382)$$

where the columns of $A_{1,4}$ and the rows of $B_{1,4}$ are indexed by

$$\mathcal{K}_1 = (\mathbb{C}^d \otimes \mathbb{C}^{r[\mathcal{U}]}) \otimes (\mathbb{C}^d \otimes \mathbb{C}^{r[\mathcal{V}]}). \quad (19.383)$$

Consequently,

$$r[(\mathcal{U} \cdot \mathcal{V})_4] = \text{rank}(A_{1,4} B_{1,4}) \leq \dim(\mathcal{K}_1) = d^2 r[\mathcal{U}] r[\mathcal{V}]. \quad (19.384)$$

For the left cut, use the reflected v factorization, with its physical order (j_2, j_1) and source order (r, ℓ) . For suitable rectangular matrices $A_{2,4}$ and $B_{2,4}$,

$$M_2((\mathcal{U} \cdot \mathcal{V})_4) = A_{2,4} B_{2,4}, \quad (19.385)$$

where the intermediate space is

$$\mathcal{K}_2 = (\mathbb{C}^d \otimes \mathbb{C}^{\ell[\mathcal{U}]}) \otimes (\mathbb{C}^d \otimes \mathbb{C}^{\ell[\mathcal{V}]}). \quad (19.386)$$

Hence

$$\ell[(\mathcal{U} \cdot \mathcal{V})_4] = \text{rank}(A_{2,4} B_{2,4}) \leq \dim(\mathcal{K}_2) = d^2 \ell[\mathcal{U}] \ell[\mathcal{V}]. \quad (19.387)$$

Both source-cut ranks are unchanged by the canonical physical relabeling. □

Lemma 19.9.23 (Composition additivity). [Not ready](#) [Discussion: #6016](#) Let $\mathcal{W}$ generate the composition of the MPUs generated by $\mathcal{U}$ and $\mathcal{U}'$. Under the source's standing canonical-form convention,

$$\text{ind}(\mathcal{W}) = \text{ind}(\mathcal{U}) + \text{ind}(\mathcal{U}'). \quad (19.388)$$

This is the unrestricted composition statement in [CPGSV17, Theorem IV.6(ii)].

Lemma 19.9.24 (Admissible composition additivity). [Not ready](#) [Discussion: #6016](#) Suppose there is a positive integer k such that $\mathcal{U}_k$, $\mathcal{U}'_k$, and the composition tensor $\mathcal{W}_k$ are simple and each carries its own reduced full-support source datum. Suppose in addition that $\mathcal{U}_{2k}$, $\mathcal{U}'_{2k}$, and $\mathcal{W}_{4k}$ are simple and each carries its own reduced full-support source datum. Define the three indices initially from $\mathcal{U}_k$, $\mathcal{U}'_k$, and $\mathcal{W}_k$; the supplied data at $2k$ and $4k$ make the later comparison admissible. If $\mathcal{W}$ generates the composition of the MPUs generated by $\mathcal{U}$ and $\mathcal{U}'$, then

$$\text{ind}(\mathcal{W}) = \text{ind}(\mathcal{U}) + \text{ind}(\mathcal{U}'). \quad (19.389)$$

This is the six-block reduced-source version of [CPGSV17, Theorem IV.6(ii)].

Definition 19.9.25 (MPU canonical form). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A tensor A of physical dimension d and bond dimension D is in MPU canonical form if there are positive block dimensions D_k , weights $\mu_k \in \mathbb{C}$, tensors A_k of bond dimension D_k , and a coisometry

$$C: \mathbb{C}^D \rightarrow \bigoplus_{k=1}^r \mathbb{C}^{D_k}, \quad CC^\dagger = \mathbb{1}, \quad (19.390)$$

such that

$$A^i = C^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) C. \quad (19.391)$$

Every A_k is irreducible and the spectral radius of its transfer map is one. The blocks may be periodic, so no uniqueness condition is imposed on their peripheral eigenvalues. This is the canonical form of [CPGSV17, lines 259–267]; normal tensors form a strictly stronger class.

Definition 19.9.26 (Strict equivalence). [Lean](#) [✓ Stmt](#) Fix a virtual bond dimension D . Let $\mathcal{U}$ and $\mathcal{V}$ be canonical-form MPU tensors with physical dimensions d_a and d_b . Given an equality $h: d_a = d_b$, reindex $\mathcal{U}$ from d_a to d_b along h^{-1} . The tensors are strictly equivalent with respect to h if there is a continuous path $\mathcal{W}(p)$, $p \in [0, 1]$, in the fixed-bond MPU locus such that

$$\mathcal{W}(0) = \text{reind}_{h^{-1}}(\mathcal{U}), \quad (19.392)$$

$$\mathcal{W}(1) = \mathcal{V}. \quad (19.393)$$

Only the canonical-form requirement is relaxed along the path. This is the fixed-bond interpretation of [CPGSV17, lines 708–714]; it does not compare unequal raw bond dimensions.

Theorem 19.9.27 (Normalized transfer data under identity ancillas). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $x > 0$. After adjoining an identity ancilla of dimension x , the normalized flattening is the physical reindexing of the original flattening with a normalized diagonal ancilla alphabet. Its transfer map is exactly the original transfer map. Consequently, chosen canonical-form-II data with full support determine canonical-form-II data with full support for $\mathcal{U}^{(x)} = \mathcal{U} \otimes \mathbb{1}_x$.

The product-index equivalence has the orientation

$$((i, a), (j, b)) \mapsto ((i, j), (a, b)). \quad (19.394)$$

This result preserves the normalized transfer data. It does not identify the old and new source cuts, ranks, or chosen compact-SVD factors.

Proof. ✓ Proof Only the x diagonal ancilla letters are nonzero, each with coefficient $x^{-1/2}$. Hence their total transfer contribution is

$$x \left(x^{-1/2} \right)^2 = 1. \quad (19.395)$$

Thus the transfer map and each diagonal positive fixed matrix are unchanged. The block dimensions, weights, and ambient coisometry are unchanged as well, so full support follows from the same block-dimension sum. The displayed physical reindexing gives the normalized flattening of the enlarged MPO. $\square$

Definition 19.9.28 (Equivalence after blocking and ancillas). Lean ✓ Stmt Fix a virtual bond dimension D . Let $\mathcal{U}$ and $\mathcal{V}$ be MPU tensors with physical dimensions d_a and d_b . They are equivalent if there are a positive blocking length k and coprime positive integers p_a, p_b with

$$p_a d_a = p_b d_b \quad (19.396)$$

such that $(\mathcal{U}^{(p_a)})_k$ and $(\mathcal{V}^{(p_b)})_k$ are strictly equivalent. Here the positive-dimensional identity ancillas are attached before the common blocking, and

$$\mathcal{U}^{(p)} = \mathcal{U} \otimes \mathbb{1}_p. \quad (19.397)$$

This is the fixed-bond interpretation of [CPGSV17, lines 716–724]; it does not introduce a stabilization relation for unequal raw bond dimensions.

Definition 19.9.29 (Admissible equivalence and path datum). Not ready An admissible strict-equivalence datum for a path $[0, 1] \ni p \mapsto \mathcal{W}(p)$ consists of a common positive blocking length m such that the actual blocked path tensor $\mathcal{W}(p)_m$ is simple and carries a reduced full-support source datum for every p . The blocked tensors and their source data vary continuously with p , after fixed identifications of the finite-dimensional spaces involved. No equivalent or reduced representative may replace $\mathcal{W}(p)_m$. In particular, the actual blocked endpoint tensors carry their own source data.

When the path is used under a specified symmetry $\mathcal{S}$, the datum also includes continuously varying reduced full-support source data for the fixed family of comparison paths

$$\overline{\mathcal{W}(p)_m}, \quad \mathcal{W}(p)_m^\sharp, \quad \mathcal{W}(p)_m^\top, \quad \mathcal{S}[\mathcal{W}(p)_m]. \quad (19.398)$$

Every tensor in this family is simple. This requirement is part of the datum independently of which comparisons a later proof uses; no preservation under conjugation, physical adjoint, transposition, or $\mathcal{S}$ is assumed.

Two tensors are admissibly strictly equivalent if their strict equivalence is witnessed by such a path. They are admissibly equivalent if the blocked, ancilla-enlarged endpoints in Definition 19.9.28 are joined by an admissible strict-equivalence path. The source arguments motivating these additional path hypotheses are Proposition IV.5, Theorem IV.6, Corollary IV.7, and the lemma following the symmetry-equivalence definitions in [CPGSV17]; the datum itself is introduced here and has no source-labelled counterpart.

Theorem 19.9.30 (Generic two-projection angle block). Lean Lean Lean ✓ Stmt

Let P and Q be orthogonal projections on a complex inner-product space. The compression of Q to $\text{ran } P$ is self-adjoint. If a unit vector $x \in \text{ran } P$ satisfies

$$PQx = \mu x, \quad 0 < \mu < 1, \quad (19.399)$$

then there is a unit vector $w \perp x$ such that

$$Px = x, \quad Pw = 0, \quad (19.400)$$

$$Qx = \mu x + \sqrt{\mu(1-\mu)} w, \quad Qw = \sqrt{\mu(1-\mu)} x + (1-\mu)w. \quad (19.401)$$

Thus $\text{span}\{x, w\}$ is the generic two-dimensional angle block. This lemma concerns one eigenvalue in $(0, 1)$; it does not include the endpoint eigenspaces or an orthogonal direct-sum decomposition of the ambient space.

Proof. [✓ Proof](#) Set $d = Qx - \mu x$. Symmetry and idempotence give

$$Pd = 0, \quad \langle x, d \rangle = 0, \quad \|d\|^2 = \mu(1-\mu), \quad (19.402)$$

$$Qd = \mu(1-\mu)x + (1-\mu)d. \quad (19.403)$$

Since $0 < \mu < 1$, the vector $w = d/\sqrt{\mu(1-\mu)}$ is well defined and has the stated properties. $\square$

Definition 19.9.31 (Compression spectral data). [Lean](#) [Lean](#) [✓ Stmt](#) For orthogonal projections P and Q on a finite-dimensional complex inner-product space, let $(x_i)_i$ be an orthonormal eigenbasis of the compression of Q to $\text{ran } P$, with ordered real eigenvalues $(\mu_i)_i$.

Theorem 19.9.32 (Compression-spectrum endpoint and angle-block trichotomy). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let P and Q be orthogonal projections on a finite-dimensional complex inner-product space, with compression spectral data (x_i, μ_i) . Then, for every i , $\mu_i \in [0, 1]$ and exactly one of the following alternatives holds:

1. $\mu_i = 0$ and $Qx_i = 0$;
2. $\mu_i = 1$ and $Qx_i = x_i$;
3. $0 < \mu_i < 1$, and x_i belongs to the two-dimensional angle block described in Theorem 19.9.30.

This theorem classifies the compression eigenvectors.

Proof. [✓ Proof](#) Apply the finite-dimensional spectral theorem to the self-adjoint compression of Q to $\text{ran } P$. For a unit compression eigenvector x with eigenvalue μ , the angle-defect identity gives

$$\|Qx - \mu x\|^2 = \mu(1-\mu), \quad (19.404)$$

hence $0 \leq \mu \leq 1$. At $\mu = 0$ or 1 the defect vanishes, giving the two common one-dimensional blocks. For $0 < \mu < 1$, apply Theorem 19.9.30. $\square$

Definition 19.9.33 (Compression defect block). [Lean](#) [Lean](#) [✓ Stmt](#) For compression spectral data (x_i, μ_i) , define

$$d_i = Qx_i - \mu_i x_i, \quad (19.405)$$

$$B_i = \text{span}\{x_i, d_i\}. \quad (19.406)$$

Theorem 19.9.34 (Orthogonality of compression angle blocks). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For all indices i, j ,

$$\langle x_i, d_j \rangle = 0, \quad (19.407)$$

$$\langle d_i, x_j \rangle = 0, \quad (19.408)$$

$$\langle d_i, d_j \rangle = \mu_j(1-\mu_j)\langle x_i, x_j \rangle. \quad (19.409)$$

In particular, $\langle d_i, d_j \rangle = 0$ when $i \neq j$, including when $\mu_i = \mu_j$. The block B_i has dimension at most two, is invariant under both P and Q , and satisfies $B_i \perp B_j$ for distinct indices.

Proof. ✓ Proof Since $Px_i = x_i$, symmetry of P and the compression eigenvalue equation give

$$\langle x_i, d_j \rangle = \langle x_i, Qx_j \rangle - \mu_j \langle x_i, x_j \rangle \quad (19.410)$$

$$= \langle Px_i, Qx_j \rangle - \mu_j \langle x_i, x_j \rangle \quad (19.411)$$

$$= \langle x_i, PQx_j \rangle - \mu_j \langle x_i, x_j \rangle = 0. \quad (19.412)$$

Conjugate symmetry gives $\langle d_i, x_j \rangle = 0$. For the exact defect pairing, symmetry of Q and the local action formula give

$$Qd_j = \mu_j(1 - \mu_j)x_j + (1 - \mu_j)d_j, \quad (19.413)$$

$$\langle d_i, d_j \rangle = \langle Qx_i, d_j \rangle - \mu_i \langle x_i, d_j \rangle, \quad (19.414)$$

$$\langle Qx_i, d_j \rangle - \mu_i \langle x_i, d_j \rangle = \langle x_i, Qd_j \rangle, \quad (19.415)$$

$$\langle x_i, Qd_j \rangle = \mu_j(1 - \mu_j)\langle x_i, x_j \rangle + (1 - \mu_j)\langle x_i, d_j \rangle, \quad (19.416)$$

$$\langle d_i, d_j \rangle = \mu_j(1 - \mu_j)\langle x_i, x_j \rangle. \quad (19.417)$$

Orthogonality of the generated spans follows by linearity. The action formulas

$$Px_i = x_i, \quad (19.418)$$

$$Pd_i = 0, \quad (19.419)$$

$$Qx_i = \mu_i x_i + d_i, \quad (19.420)$$

$$Qd_i = \mu_i(1 - \mu_i)x_i + (1 - \mu_i)d_i \quad (19.421)$$

show that both projections preserve $B_i = \text{span}\{x_i, d_i\}$. Its dimension bound follows from its two generators. □

Definition 19.9.35 (Defect-block sum and complementary summand). Lean Lean ✓ Stmt For the compression defect blocks $(B_i)_i$, define

$$B = \bigvee_i B_i, \quad K = B^\perp. \quad (19.422)$$

The index i ranges over every vector in the chosen compression eigenbasis, including repeated eigenvalues and the endpoint eigenvalues 0 and 1.

Theorem 19.9.36 (Orthogonal defect-block sum and commuting remainder). Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt The blocks $(B_i)_i$ form an orthogonal internal direct sum in B , and

$$E = B \perp \oplus K. \quad (19.423)$$

Both B and K are invariant under P and Q . Moreover, $\text{ran } P \subseteq B$, so for every $z \in K$,

$$Pz = 0, \quad P(Qz) = 0, \quad Q(Pz) = 0. \quad (19.424)$$

Consequently the restrictions $P|_K$ and $Q|_K$ are orthogonal projections and commute. No commutation of P and Q on the ambient space is assumed.

Proof. ✓ Proof Pairwise orthogonality of the blocks implies independence, while their supremum is B by definition. Hence they form an internal direct sum in B . The standard orthogonal-complement decomposition gives $E = B \perp \oplus B^\perp$.

The local action formulas show that each B_i is invariant under both projections, hence so is B . Symmetry of each projection then makes $K = B^\perp$ invariant. The compression eigenvectors form a basis of $\text{ran } P$, and each lies in its block B_i ; therefore $\text{ran } P \subseteq B$. Thus P vanishes on K . Since $Qz \in K$ whenever $z \in K$, both $P(Qz)$ and $Q(Pz)$ vanish. The two restricted projections therefore commute. $\square$

Definition 19.9.37 (Reduced projection). [Lean](#) [✓ Stmt](#)

Let P and Q be orthogonal projections on a finite-dimensional complex inner-product space. Define $\widehat{P}$ as the orthogonal projection onto $\text{ran}(PQ)$.

Theorem 19.9.38 (Reduced-projection identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
✓ Stmt *The reduced projection satisfies*

$$\text{ran}(\widetilde{P}) = \text{ran}(PQ), \quad (19.425)$$

$$\text{ran}(PQP) = \text{ran}(PQ), \quad (19.426)$$

$$P\widetilde{P} = \widetilde{P}, \quad (19.427)$$

$$\widetilde{P}Q = PQ, \quad (19.428)$$

$$Q\widetilde{P} = QP, \quad (19.429)$$

$$\text{ran}(\widetilde{P}Q) = \text{ran}(\widetilde{P}). \quad (19.430)$$

The first two identities describe the canonical construction, while the next three establish the properties in source lines 767–769. The final range equality is the derived input to the right-factor theorem, not item (iv) of the source.

Proof. ✓ Proof Set $A = QP$. The standard range identity for $A^\dagger A$ gives

$$\text{ran}(PQP) = \text{ran}(A^\dagger A) = \text{ran}(A^\dagger) = \text{ran}(PQ). \quad (19.431)$$

By definition, $\text{ran}(\widetilde{P}) = \text{ran}(PQ)$, which also gives $P\widetilde{P} = \widetilde{P}$. For all x and z ,

$$\langle Qx - PQx, PQz \rangle = 0. \quad (19.432)$$

Hence $Qx - PQx \in \text{ran}(PQ)^\perp$, so orthogonal projection onto this range yields $\widetilde{P}Q = PQ$. Taking adjoints gives $Q\widetilde{P} = QP$. Finally,

$$\text{ran}(\widetilde{P}Q) = \text{ran}(PQ) = \text{ran}(\widetilde{P}). \quad (19.433)$$

☐

Theorem 19.9.39 (Transversality of the reduced range). [Lean] [Lean] [✓ Stmt] For orthogonal projections P and Q , the reduced range contains no nonzero vector annihilated by Q :

$$\ker Q \cap \operatorname{ran}(\widetilde{P}) = \{0\}. \quad (19.434)$$

Proof.  **Proof** If $x = PQy$ and $Qx = 0$, then $Px = x$ and symmetry of the two projections gives

$$\langle x, x \rangle = \langle PQy, PQy \rangle = \langle Qy, PQy \rangle = \langle y, QPQy \rangle = 0. \quad (19.435)$$

Hence $x = 0$.

☐

Set $T = \widetilde{P}$ and $\widetilde{W}^{ij} = TW^{ij}T$. Then, for every $N \geq 0$,

$$\text{MPO}_N(\widetilde{W}) = \text{MPO}_N(W), \quad (19.444)$$

$$E_{\widetilde{W}}(X) = \text{tr}((TLT)X)TRT, \quad (19.445)$$

$$\text{tr}((TLT)(TRT)) = 1. \quad (19.446)$$

Moreover, if $V^\dagger V = I$ and $VV^\dagger = T$, then

$$V^\dagger LV \succ 0, \quad V^\dagger RV \succ 0. \quad (19.447)$$

Proof. ✓ Proof For every nonempty word w , idempotence of T and (19.438) give

$$\widetilde{W}_w = TW_wT. \quad (19.448)$$

Cyclicity of the trace, together with $QW_w = W_w = W_wP$, therefore gives $\text{tr}(TW_wT) = \text{tr}(W_w)$. Length zero is immediate. Lemma 19.9.43, with $A = B = T$ and $T^\dagger = T$, gives

$$E_{\widetilde{W}}(X) = \text{tr}(LTX)TRT = \text{tr}((TLT)X)TRT. \quad (19.449)$$

Since P and Q are the support projections of L and R , respectively,

$$PL = LP = L, \quad QR = RQ = R. \quad (19.450)$$

Combining these absorptions with (19.438), $T^2 = T$, and cyclicity of the trace gives

$$\text{tr}((TLT)(TRT)) = \text{tr}(LTRT) = \text{tr}(LRT) = \text{tr}(RTL) = \text{tr}(RL) = \text{tr}(LR) = 1. \quad (19.451)$$

Let $x \neq 0$ and set $u = Vx$. Since $V^\dagger V = I$, one has $u \neq 0$; since $VV^\dagger = T$, one has $Tu = u$. Equation (19.438) gives $Pu = u$, and (19.434) gives $Qu \neq 0$. Thus

$$\text{supp}(L)u = Pu = u \neq 0, \quad \text{supp}(R)u = Qu \neq 0. \quad (19.452)$$

Applying Lemma 19.9.44 to L and R yields

$$x^\dagger(V^\dagger LV)x = u^\dagger Lu > 0, \quad x^\dagger(V^\dagger RV)x = u^\dagger Ru > 0. \quad (19.453)$$

□

Theorem 19.9.46 (Invertible factor for a reduced projection). Lean ✓ Stmt Let $L, R \succeq 0$, let P and Q be their support projections, and let $\widetilde{P}$ be the reduced projection of P and Q . There are invertible support-inverse extensions of L and R , together with an invertible matrix Y , such that $PQY = \widetilde{P}$.

Proof. ✓ Proof Extend the inverses on the supports by the identity on their orthogonal complements, and choose an invertible right factor for the reduced projection. □

Theorem 19.9.47 (Source-rank comparison of the hatted and reduced tensors). Lean Lean ✓ Stmt Let $W = (W^{ij})$ be an MPO tensor, let L, R, X, Z, Y be square matrices of the same size, let P be an orthogonal projection, let Q be a square matrix of the same size, and set $\widetilde{P}$ equal to their reduced projection. Assume, letter by letter, $W^{ij}P = W^{ij} = QW^{ij}$. Set $\widehat{W}^{ij} = LW^{ij}R$ and $\widetilde{W}^{ij} = \widetilde{P}W^{ij}\widetilde{P}$. If X, Z, Y satisfy $XL = P$, $RZ = Q$, and $PQY = \widetilde{P}$, then

$$X\widehat{W}ZY = PWQY \quad (19.454)$$

$$PWQY = PQWPQY \quad (19.455)$$

$$PQWPQY = \widetilde{P}W\widetilde{P} \quad (19.456)$$

$$\widetilde{P}W\widetilde{P} = \widetilde{W}. \quad (19.457)$$

If in addition X , Z , and Y are invertible, then

$$r[\widehat{W}] = r[\widetilde{W}], \quad (19.458)$$

$$\ell[\widehat{W}] = \ell[\widetilde{W}]. \quad (19.459)$$

If $L, R \succeq 0$ and $P = \text{supp}(L)$, $Q = \text{supp}(R)$, such invertible matrices may be chosen as $X = L^+ + (\mathbb{1} - P)$, $Z = R^+ + (\mathbb{1} - Q)$, together with an invertible right factor Y of the reduced projection.

Proof. ✓ Proof Expanding each virtual sandwich and using $XL = P$ and $RZ = Q$ gives (19.454). The absorptions $QW^{ij} = W^{ij} = W^{ij}P$ give, letter by letter,

$$PW^{ij}QY = P(QW^{ij})QY = PQ(W^{ij}P)QY = PQW^{ij}PQY,$$

which is (19.455). Since $\widetilde{P}Q = PQ$ and $PQY = \widetilde{P}$,

$$PQW^{ij}PQY = (\widetilde{P}Q)W^{ij}\widetilde{P} = \widetilde{P}W^{ij}\widetilde{P}. \quad (19.460)$$

This proves (19.456) and (19.457). Thus $\widetilde{W}$ is obtained from $\widehat{W}$ by left multiplication by X and right multiplication by ZY . Invertibility of these factors and the source ranks of a virtual sandwich (Theorem 19.9.56) give

$$\begin{aligned} r[\widehat{W}] &= r[X\widehat{W}(ZY)] = r[\widetilde{W}], \\ \ell[\widehat{W}] &= \ell[X\widehat{W}(ZY)] = \ell[\widetilde{W}], \end{aligned}$$

the first equality in each line being the invariance identity with invertible X and ZY , and the second rewriting $X\widehat{W}(ZY)$ through (19.454)–(19.456) and (19.457) to $\widetilde{W}$, which gives (19.458) and (19.459). For positive semidefinite L and R , their ambient support-inverse extensions supply X and Z ; the invertible right-factor identity for $\widetilde{P}Q$ supplies Y . $\square$

Theorem 19.9.48 (Lower semicontinuity of finite matrix rank). Lean Lean ✓ Stmt For finite index sets M, N , matrix rank defines a lower-semicontinuous map

$$\text{rank}: \text{Mat}_{M \times N}(\mathbb{C}) \longrightarrow \mathbb{N}. \quad (19.461)$$

Consequently, if $A: X \rightarrow \text{Mat}_{M \times N}(\mathbb{C})$ is continuous, then $x \mapsto \text{rank } A(x)$ is lower semicontinuous.

Proof. ✓ Proof For every $k \in \mathbb{N}$, the sublevel set is the closed determinantal set

$$\{A : \text{rank } A \leq k\}. \quad (19.462)$$

These closed sublevel sets characterize lower semicontinuity. Composition with a continuous map preserves lower semicontinuity. $\square$

Theorem 19.9.49 (Fixed positive product gives local constancy). Lean ✓ Stmt Let $r, \ell: X \rightarrow \mathbb{N}$ be lower semicontinuous. Suppose that $c > 0$ and

$$r(x)\ell(x) = c \quad (19.463)$$

for every $x \in X$. Then both r and ℓ are locally constant.

Proof. ✓ Proof Fix $x \in X$. Since $c > 0$, both $r(x)$ and $\ell(x)$ are positive. Lower semicontinuity gives a neighborhood U of x on which

$$r(y) \geq r(x), \quad \ell(y) \geq \ell(x). \quad (19.464)$$

For $y \in U$, monotonicity of multiplication and the fixed-product identity give

$$r(x)\ell(x) \leq r(y)\ell(x) \leq r(y)\ell(y) = r(x)\ell(x). \quad (19.465)$$

Positivity permits cancellation, first of $\ell(x)$ and then of $r(x)$, so $r(y) = r(x)$ and $\ell(y) = \ell(x)$. $\square$

Theorem 19.9.50 (Fixed positive product on a preconnected set). Lean Lean Lean ✓ Stmt Under the hypotheses of Theorem 19.9.49, if $S \subseteq X$ is preconnected, then for all $x, y \in S$,

$$r(x) = r(y), \quad \ell(x) = \ell(y). \quad (19.466)$$

In particular, both functions are constant when X is preconnected and when $X = [0, 1]$.

Proof. ✓ Proof A locally constant function has open fibers. A preconnected set cannot meet two disjoint nonempty open fibers, so each of r and ℓ has one value on S . The unit interval is connected, hence preconnected. $\square$

Definition 19.9.51 (Virtual sandwich). Lean Lean ✓ Stmt For an MPO tensor $\mathcal{U} = (U^{ij})_{0 \leq i, j < d}$ of bond dimension D and matrices $A, B \in \mathbb{C}^{D \times D}$, define

$$\widehat{\mathcal{U}}^{ij} = AU^{ij}B. \quad (19.467)$$

Theorem 19.9.52 (Continuity of the virtual sandwich). Lean ✓ Stmt If $x \mapsto A(x)$, $x \mapsto \mathcal{U}(x)$, and $x \mapsto B(x)$ are continuous families, then

$$x \mapsto A(x)\mathcal{U}(x)B(x) \quad (19.468)$$

is continuous.

Proof. ✓ Proof Every matrix entry is a finite sum of products of entries of $A(x)$, $\mathcal{U}(x)$, and $B(x)$, hence is continuous. $\square$

Definition 19.9.53 (Doubled-virtual contraction and factor-free sandwich). Lean Lean Lean ✓ Stmt Let $q : \{0, \dots, D-1\} \times \{0, \dots, D-1\} \rightarrow \{0, \dots, D^2-1\}$ be the standard enumeration of a pair. For an MPO tensor $\mathcal{W}$ and a doubled-virtual matrix $E \in \mathbb{C}^{D^2 \times D^2}$, define

$$C(\mathcal{W}, E)_{ab}^{ij} = \sum_{c, e=0}^{D-1} E_{q(b, e), q(c, a)} W_{ce}^{ij}. \quad (19.469)$$

If $E(\mathcal{W})$ denotes the normalized physical diagonal of the physical-adjoint double layer of $\mathcal{W}$, define

$$\widehat{\mathcal{W}}_{\text{ff}} = C(\mathcal{W}, E(\mathcal{W})). \quad (19.470)$$

This is the contraction shown in Figure IV_index4.png of [CPGSV17, Proposition IV.5].

Definition 19.9.57 (Ambient support-inverse extension). [Lean](#) [Lean](#) [✓ Stmt](#)

Let $A \succeq 0$ have support projection P , and let A^+ denote its inverse on the support. Define

$$X_A = A^+ + (\mathbb{1} - P). \quad (19.480)$$

Define also

$$Z_A = A + (\mathbb{1} - P). \quad (19.481)$$

This construction gives explicit witnesses for the invertible matrices required in Proposition IV.5 of [CPGSV17]; the paper requires those matrices but does not state these formulas.

Theorem 19.9.58 (Invertibility and support identities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#) The matrices X_A and Z_A are mutually inverse, and

$$X_A A = A X_A = P. \quad (19.482)$$

Consequently, X_A is a unit in the matrix algebra, with inverse Z_A , and $\det(X_A)$ is a unit in $\mathbb{C}$.

In the notation of Proposition IV.5 of [CPGSV17], let L and R be the positive semidefinite outer factors with support projections P and Q , respectively. The required witnesses are

$$X = X_L = L^+ + (\mathbb{1} - P), \quad (19.483)$$

$$Z = X_R = R^+ + (\mathbb{1} - Q). \quad (19.484)$$

Thus

$$X_L L = P, \quad (19.485)$$

$$R X_R = Q, \quad (19.486)$$

or equivalently $XL = P$ and $RZ = Q$. Here the paper's matrix Z is X_R , not the inverse-extension matrix Z_R from Definition 19.9.57.

Proof. [✓ Proof](#) The support identities give

$$A^+ A = A A^+ = P. \quad (19.487)$$

Their complementary forms are

$$A^+ (\mathbb{1} - P) = (\mathbb{1} - P) A^+ = 0, \quad (19.488)$$

$$A (\mathbb{1} - P) = (\mathbb{1} - P) A = 0, \quad (19.489)$$

and idempotence of P gives

$$(\mathbb{1} - P)^2 = \mathbb{1} - P. \quad (19.490)$$

Therefore

$$X_A Z_A = P + (\mathbb{1} - P) = \mathbb{1}. \quad (19.491)$$

Likewise,

$$Z_A X_A = P + (\mathbb{1} - P) = \mathbb{1}, \quad (19.492)$$

while multiplication by A gives $X_A A = A X_A = P$. $\square$

Proposition 19.9.59 (Constancy of the source ranks). Not ready Discussion: #6022 Let $[0, 1] \ni x \mapsto \mathcal{W}(x)$ be a continuous path of tensors generating MPUs, without requiring the path to be in canonical form. For each x , take the canonical form and the two source-cut ranks $r(x)$ and $\ell(x)$ of its associated simple tensor. Then $r(x)$ and $\ell(x)$ are constant on $[0, 1]$.

Proposition 19.9.60 (Admissible constancy of the source ranks). Not ready Discussion: #6022 Let $[0, 1] \ni x \mapsto \mathcal{W}(x)$ be a continuous path of tensors generating MPUs, without requiring the path to be in canonical form. Suppose that the path carries an admissible equivalence datum with blocking length m . Then the ranks $r(x)$ and $\ell(x)$ of the actual blocked path tensors $\mathcal{W}(x)_m$ are constant on $[0, 1]$.

Theorem 19.9.61 (Index Theorem). Not ready Discussion: #6023 For MPU tensors under the source's standing canonical-form convention, take the index defined from any simple blocking. This index has the following properties.

1. It does not change under blocking.
2. It is additive under tensor products and composition.
3. It is constant under continuous deformations through MPU tensors.
4. Two MPU tensors have the same index if and only if they are equivalent.

This is the unrestricted statement of [CPGSV17, Theorem IV.6].

Theorem 19.9.62 (Admissible Index Theorem). Not ready Discussion: #6023 Assume the following operation-specific admissibility hypotheses. Every pair of simple blockings compared for blocking independence carries reduced full-support source data. For a tensor product $\mathcal{U} \otimes \mathcal{V}$, there is one positive length k such that $\mathcal{U}_k$, $\mathcal{V}_k$, and $(\mathcal{U} \otimes \mathcal{V})_k$ are simple and carry their own data. For a composition tensor $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{U}'$, there is one positive length k such that

$$\mathcal{U}_k, \quad \mathcal{U}'_k, \quad \mathcal{W}_k, \quad \mathcal{U}_{2k}, \quad \mathcal{U}'_{2k}, \quad \mathcal{W}_{4k} \quad (19.493)$$

are simple and carry their own data. Continuous-deformation and equivalence claims use the path data of Definition 19.9.29. Under these hypotheses, the MPU index has the following properties.

1. It does not change under blocking.
2. It is additive under tensor products and composition.
3. It is constant under continuous deformations through MPU tensors.
4. Admissibly equivalent MPU tensors have the same index. Conversely, tensors with the same index are admissibly equivalent whenever the standard-form interpolation is supplied with an admissible path datum.

These are the conclusions of [CPGSV17, Theorem IV.6], restricted to the explicit admissibility hypotheses above.

Corollary 19.9.63 (Continuous choice of standard form). Not ready Discussion: #6024 For every continuous path $p \mapsto \mathcal{U}(p)$ of MPU tensors, not necessarily in canonical form, there are an integer $k_0 \leq D^4$ and continuous paths of unitaries $u(p)$ and $v(p)$ such that $U^{(2k_0 N)}(p)$ has standard form given by $u(p)$ and $v(p)$.

Corollary 19.9.64 (Admissible continuous choice of standard form). Not ready Discussion: #6024 Let k_0 be the common blocking length in an admissible equivalence datum for a continuous path $\mathcal{U}(p)$ of MPU tensors, and suppose that this supplied length satisfies $k_0 \leq D^4$. Then there are continuous paths of unitaries $u(p)$ and $v(p)$ such that the MPU $U^{(2k_0N)}(p)$ has standard form given by $u(p)$ and $v(p)$ for every $p \in [0, 1]$. This is the reduced-source version of [CPGSV17, Corollary IV.7].

19.9.1 Local characterization of symmetries

Proposition 19.9.1.1 (Conjugation symmetry). Not ready Discussion: #5713 An MPU in standard form is invariant under complex conjugation if and only if there are unitaries x and y such that either $x = x^T$ and $y = y^T$, or $x = -x^T$ and $y = -y^T$, and

$$\bar{u} = (x \otimes y)u, \quad \bar{v} = v(y^\dagger \otimes x^\dagger). \quad (19.494)$$

Proposition 19.9.1.2 (Admissible conjugation symmetry). Not ready Discussion: #5713 Let a tensor and its conjugate comparison tensor both be equipped with reduced full-support source data and written in admissible standard form. The generated MPU is invariant under complex conjugation if and only if there are unitaries x and y such that either $x = x^T$ and $y = y^T$, or $x = -x^T$ and $y = -y^T$, and

$$\bar{u} = (x \otimes y)u, \quad \bar{v} = v(y^\dagger \otimes x^\dagger). \quad (19.495)$$

This is the reduced-source version of the conjugation characterization in [CPGSV17, Proposition V.1].

Lemma 19.9.1.3 (Normal form of symmetric and skew-symmetric unitaries). Not ready Discussion: #5713 Let x be unitary and satisfy $x = x^T$ or $x = -x^T$. There are a symmetric unitary S and a matrix $\tilde{\Lambda}$ such that

$$x = S^T \tilde{\Lambda} S, \quad S = S^T. \quad (19.496)$$

In the symmetric case, $\tilde{\Lambda}$ is real and symmetric. In the skew-symmetric case, it is real and skew-symmetric. In both cases, $\det(\tilde{\Lambda}) = 1$.

Lemma 19.9.1.4 (Odd-dimensional skew-symmetric determinant). Lean ✓ Stmt Let I be a finite set of odd cardinality and let $A \in \text{Mat}_I(\mathbb{C})$ satisfy $A^T = -A$. Then

$$\det A = 0. \quad (19.497)$$

Proof. ✓ Proof Transpose invariance, skew symmetry, and the determinant of a negated matrix give

$$\det A = \det(A^T) = \det(-A) = (-1)^{|I|} \det A = -\det A. \quad (19.498)$$

Since the complex numbers have characteristic zero, $\det A = 0$. □

Lemma 19.9.1.5 (Parity of a nonsingular skew-symmetric matrix). Lean ✓ Stmt Let I be finite and let $A \in \text{Mat}_I(\mathbb{C})$ satisfy $A^T = -A$ and $\det A \neq 0$. Then $|I|$ is even.

Proof. ✓ Proof If $|I|$ were odd, Lemma 19.9.1.4 would give $\det A = 0$. Hence $|I|$ is not odd and is therefore even. □

Theorem 19.9.1.6 (Even dimension of a skew-symmetric unitary). [Lean](#) [✓ Stmt](#) Let $U \in \text{Mat}_d(\mathbb{C})$ be unitary and satisfy $U^T = -U$. Then d is even. This is the parity obstruction in [CPGSV17, Proposition V.2, Case II].

Proof. [✓ Proof](#) The determinant of a unitary matrix is a unit, so it is nonzero. Apply Lemma 19.9.1.5 to U . $\square$

Proposition 19.9.1.7 (Orthogonal and symplectic standard forms). [Not ready](#) [Discussion: #5713](#) Let an MPU in standard form be invariant under complex conjugation. Its standard-form unitaries can be chosen in exactly one of the following two cases. In Case I,

$$\overline{u'} = u', \quad \overline{v'} = v', \quad (19.499)$$

uniquely up to local orthogonal transformations. In Case II,

$$\overline{u'} = (\Sigma_\ell \otimes \Sigma_r)u', \quad \overline{v'} = v'(\Sigma_r \otimes \Sigma_\ell), \quad (19.500)$$

uniquely up to local symplectic transformations, where $\Sigma_x = \mathbb{1}_{x/2} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Case II can occur only when r and ℓ are even.

Proposition 19.9.1.8 (Admissible orthogonal and symplectic standard forms). [Not ready](#) [Discussion: #5713](#) Let an MPU in admissible standard form be invariant under complex conjugation, and suppose every standard-form tensor compared below carries reduced full-support source data. Its standard-form unitaries can be chosen in exactly one of the following two cases. In Case I,

$$\overline{u'} = u', \quad \overline{v'} = v', \quad (19.501)$$

and the choice is unique up to local orthogonal transformations. In Case II,

$$\overline{u'} = (\Sigma_\ell \otimes \Sigma_r)u', \quad \overline{v'} = v'(\Sigma_r \otimes \Sigma_\ell), \quad (19.502)$$

where

$$\Sigma_x = \mathbb{1}_{x/2} \otimes Y, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (19.503)$$

The choice is unique up to local transformations S_x satisfying $S_x^T \Sigma_x S_x = \Sigma_x$. Case II can occur only when r and ℓ are even.

Corollary 19.9.1.9 (Reduction of the symplectic case to orthogonal unitaries). [Not ready](#) [Discussion: #5713](#) In Case II of Proposition 19.9.1.7, one can write

$$u' = (\mathbb{1}_{r\ell/4} \otimes R)\tilde{u}, \quad v' = \tilde{v}(\mathbb{1}_{r\ell/4} \otimes R), \quad (19.504)$$

where $\tilde{u}$ and $\tilde{v}$ are orthogonal and

$$R = \frac{1-i}{2}[\mathbb{1}_4 + i(Y \otimes Y)]. \quad (19.505)$$

Corollary 19.9.1.10 (Admissible reduction of the symplectic case to orthogonal unitaries). [Not ready](#) [Discussion: #5713](#) In the admissible Case II of Proposition 19.9.1.8, one can write

$$u' = (\mathbb{1}_{r\ell/4} \otimes R)\tilde{u}, \quad v' = \tilde{v}(\mathbb{1}_{r\ell/4} \otimes R), \quad (19.506)$$

where $\tilde{u}$ and $\tilde{v}$ are orthogonal and

$$R = \frac{1-i}{2}[\mathbb{1}_4 + i(Y \otimes Y)]. \quad (19.507)$$

Proposition 19.9.1.11 (Local characterization of time reversal). Not ready Discussion: #5714 Let an MPU be in standard form, with one-site physical space $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_0}$. It satisfies $U^{(N)} = U^{(N)\dagger}$ for every even N if and only if there is a unitary x such that

$$(\mathbb{1}_1 \otimes v_{23} \otimes \mathbb{1}_4)(u_{12} \otimes u_{34}) = \pm(x_{12} \otimes x_{34}^\dagger)(\mathbb{1}_1 \otimes v_{23}^\dagger \otimes \mathbb{1}_4). \quad (19.508)$$

Proposition 19.9.1.12 (Admissible local characterization of time reversal). Not ready Discussion: #5714 Let an MPU be in admissible standard form. Suppose its one-site tensor $\mathcal{U}$, its actual two-site block $\mathcal{U}_2$, and the physical-adjoint comparison tensor $\mathcal{U}_2^\sharp$ are simple and each carries its own reduced full-support source datum. Write the one-site physical space as $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_0}$. Time-reversal symmetry then forces $\text{ind}(\mathcal{U}) = 0$. Under these hypotheses, $U^{(N)} = U^{(N)\dagger}$ for every even N if and only if there is a unitary x such that

$$(\mathbb{1}_1 \otimes v_{23} \otimes \mathbb{1}_4)(u_{12} \otimes u_{34}) = \pm(x_{12} \otimes x_{34}^\dagger)(\mathbb{1}_1 \otimes v_{23}^\dagger \otimes \mathbb{1}_4). \quad (19.509)$$

Proof. Under time reversal, $\mathcal{U}_2$ and $\mathcal{U}_2^\sharp$ generate the same MPU. The admissible fundamental theorem identifies their source spaces, so their corresponding source-cut ranks agree. Physical adjunction exchanges the two cuts. Hence

$$r(\mathcal{U}_2) = r(\mathcal{U}_2^\sharp) = \ell(\mathcal{U}_2). \quad (19.510)$$

Definition 19.9.17 therefore gives $\text{ind}(\mathcal{U}_2) = 0$, and blocking independence gives $\text{ind}(\mathcal{U}) = 0$. Apply the admissible fundamental theorem to the supplied standard forms of $\mathcal{U}_2$ and $\mathcal{U}_2^\sharp$ and follow the source proof. The converse follows by contracting the displayed local relation around every even periodic chain. $\square$

Proposition 19.9.1.13 (Local characterization of transposition). Not ready Discussion: #5714 Let an MPU be in standard form, with one-site physical space $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_0}$. It satisfies $U^{(N)} = U^{(N)T}$ for every even N if and only if there are a unitary x and a phase $e^{i\phi}$ such that

$$(\mathbb{1}_1 \otimes v_{23} \otimes \mathbb{1}_4)(u_{12} \otimes u_{34}) = e^{i\phi}(x_{12} \otimes x_{34}^T)(\mathbb{1}_1 \otimes v_{23}^T \otimes \mathbb{1}_4). \quad (19.511)$$

Proposition 19.9.1.14 (Admissible local characterization of transposition). Not ready Discussion: #5714 Let an MPU be in admissible standard form. Suppose its one-site tensor $\mathcal{U}$, its actual two-site block $\mathcal{U}_2$, and the transposed comparison tensor $\mathcal{U}_2^T$ are simple and each carries its own reduced full-support source datum. Write the one-site physical space as $\mathbb{C}^{d_0} \otimes \mathbb{C}^{d_0}$. Transposition symmetry then forces $\text{ind}(\mathcal{U}) = 0$. Under these hypotheses, $U^{(N)} = U^{(N)T}$ for every even N if and only if there are a unitary x and a phase $e^{i\phi}$ such that

$$(\mathbb{1}_1 \otimes v_{23} \otimes \mathbb{1}_4)(u_{12} \otimes u_{34}) = e^{i\phi}(x_{12} \otimes x_{34}^T)(\mathbb{1}_1 \otimes v_{23}^T \otimes \mathbb{1}_4). \quad (19.512)$$

Proof. Under transposition, $\mathcal{U}_2$ and $\mathcal{U}_2^T$ generate the same MPU. The admissible fundamental theorem identifies their source spaces, so their corresponding source-cut ranks agree. Transposition exchanges the two cuts. Hence

$$r(\mathcal{U}_2) = r(\mathcal{U}_2^T) = \ell(\mathcal{U}_2). \quad (19.513)$$

Thus $\text{ind}(\mathcal{U}_2) = 0$ by Definition 19.9.17, and blocking independence gives $\text{ind}(\mathcal{U}) = 0$. Apply the admissible fundamental theorem to the supplied standard forms of $\mathcal{U}_2$ and $\mathcal{U}_2^T$ and repeat the time-reversal argument, retaining the unrestricted phase $e^{i\phi}$. The converse follows by contracting the local relation around every even periodic chain. $\square$

19.9.2 Equivalence under symmetries

Definition 19.9.2.1 (Finite-chain operator symmetry). [Lean](#) [Lean](#) [✓ Stmt](#) A finite-chain operator symmetry $\mathcal{S}$ specifies a set A of applicable chain lengths and, for every physical dimension d and length N , a map

$$\mathcal{S}_{d,N}: \text{Mat}_{d^N}(\mathbb{C}) \longrightarrow \text{Mat}_{d^N}(\mathbb{C}). \quad (19.514)$$

We suppress the subscripts on $\mathcal{S}_{d,N}$ when d and N are understood. An MPO tensor $\mathcal{W}$ is invariant under $\mathcal{S}$ if

$$\mathcal{S}_{d,N}[W^{(N)}] = W^{(N)} \quad (19.515)$$

for every $N \in A$.

Definition 19.9.2.2 (Strict equivalence under a symmetry). [Lean](#) [✓ Stmt](#) [Discussion: #6948](#) Fix a virtual bond dimension D . Two canonical-form tensors $\mathcal{U}$ and $\mathcal{V}$ are strictly equivalent under a symmetry $\mathcal{S}$ if they have the same physical dimension and there is a continuous path $\mathcal{W}(p)$ of fixed-bond MPU tensors from $\mathcal{U}$ to $\mathcal{V}$, not necessarily in canonical form, such that

$$\mathcal{S}[W^{(N)}(p)] = W^{(N)}(p) \quad (19.516)$$

for every $p \in [0, 1]$ and every applicable chain length N .

Lemma 19.9.2.3 (Forgetting symmetry from strict equivalence). [Lean](#) [✓ Stmt](#) *Strict equivalence under a symmetry implies strict equivalence.*

Proof. [✓ Proof](#) The same path lies in the larger locus obtained by forgetting invariance under $\mathcal{S}$. □

Definition 19.9.2.4 (Admissible strict equivalence under a symmetry). [Not ready](#) [Discussion: #6017](#) Two tensors $\mathcal{U}$ and $\mathcal{V}$ are strictly equivalent under a symmetry $\mathcal{S}$ if they have the same physical dimension and there is a continuous path $\mathcal{W}(p)$ of MPU tensors from $\mathcal{U}$ to $\mathcal{V}$ such that

$$\mathcal{S}[W^{(N)}(p)] = W^{(N)}(p) \quad (19.517)$$

for every $p \in [0, 1]$ and every chain length N under consideration. The path need not be in canonical form. The equivalence is admissible if its witnessing path carries an admissible equivalence datum for the actual blocked path tensor and the fixed conjugate, physical-adjoint, transposed, and $\mathcal{S}$ -image comparison paths, and if the common blocking length m of that datum satisfies $m \leq D^4$.

Definition 19.9.2.5 (Equivalence under a symmetry). [Not ready](#) [Discussion: #6017](#) Let $\mathcal{U}$ and $\mathcal{V}$ have physical dimensions d_a and d_b . They are equivalent under a symmetry $\mathcal{S}$ if there are a positive blocking length k and coprime positive ancilla dimensions p_a, p_b with $p_a d_a = p_b d_b$ such that $(\mathcal{U}^{(p_a)})_k$ and $(\mathcal{V}^{(p_b)})_k$ are strictly equivalent under $\mathcal{S}$.

Definition 19.9.2.6 (Admissible equivalence under a symmetry). [Not ready](#) [Discussion: #6017](#) Let $\mathcal{U}$ and $\mathcal{V}$ have physical dimensions d_a and d_b . They are equivalent under a symmetry $\mathcal{S}$ if there are a positive blocking length k and coprime positive ancilla dimensions p_a, p_b with $p_a d_a = p_b d_b$ such that $(\mathcal{U}^{(p_a)})_k$ and $(\mathcal{V}^{(p_b)})_k$ are strictly equivalent under $\mathcal{S}$. Thus the ancillas are attached before the common blocking. The equivalence is admissible if this strict-equivalence witness carries an admissible equivalence datum for the actual blocked path and the fixed conjugate, physical-adjoint, transposed, and $\mathcal{S}$ -image comparison paths, with common blocking length at most D^4 .

Lemma 19.9.2.7 (Standard-form path criterion). Not ready Discussion: #6017 Two MPUs described by simple tensors are strictly equivalent under $\mathcal{S}$ if and only if their standard-form unitaries are joined by continuous families $\tilde{u}(p)$ and $\tilde{v}(p)$ whose generated MPU is invariant under $\mathcal{S}$ for every $p \in [0, 1]$.

Lemma 19.9.2.8 (Admissible standard-form path criterion). Not ready Discussion: #6017 Fix a positive integer $m \leq D^4$. The actual m -site blocked endpoint MPUs $U_1^{(mN)}$ and $U_2^{(mN)}$ are admissibly strictly equivalent under $\mathcal{S}$ if and only if there are continuous families of unitaries $\tilde{u}(p)$ and $\tilde{v}(p)$ whose endpoints generate $U_1^{(mN)}$ and $U_2^{(mN)}$, whose generated MPU is invariant under $\mathcal{S}$ for every $p \in [0, 1]$, and whose tensor path and fixed conjugate, physical-adjoint, transposed, and $\mathcal{S}$ -image comparison paths carry continuously varying reduced full-support source data. These families arise from the actual blocked path $\mathcal{W}(p)_m$ in an admissible equivalence datum of common length m ; the blocked path itself is the length-one admissible witness for the displayed endpoint families. When $m > 1$, this criterion does not assert strict equivalence of the unblocked families $U_1^{(N)}$ and $U_2^{(N)}$. This is the reduced-source form of [CPGSV17, Lemma following the symmetry-equivalence definitions].

Theorem 19.9.2.9 (Conjugation symmetry when one source rank is odd). Not ready Discussion: #5713 Suppose that either r or ℓ is odd. Two conjugation-invariant MPUs $\mathcal{U}_1$ and $\mathcal{U}_2$ are strictly equivalent under conjugation if and only if

$$\det(u_1 v_1) = \det(u_2 v_2). \quad (19.518)$$

Theorem 19.9.2.10 (Admissible conjugation symmetry when one source rank is odd). Not ready Discussion: #5713 For $i = 1, 2$, suppose the original one-site endpoint tensor $\mathcal{U}_i$ and its conjugate comparison tensor are simple and each carries reduced full-support source data. Suppose that either r or ℓ is odd and that the standard-form interpolation determined below carries an admissible path datum of blocking length one. Then $\mathcal{U}_1$ and $\mathcal{U}_2$ are admissibly strictly equivalent under conjugation symmetry if and only if

$$\det(u_1 v_1) = \det(u_2 v_2). \quad (19.519)$$

Corollary 19.9.2.11 (Equivalence of conjugation-symmetric MPUs after blocking). Not ready If either r or ℓ is odd, all conjugation-invariant MPUs with these source ranks are equivalent under conjugation symmetry.

Corollary 19.9.2.12 (Admissible equivalence of conjugation-symmetric MPUs after blocking). Not ready Discussion: #5713 For $i = 1, 2$, suppose the endpoint tensor $\mathcal{U}_i$, its conjugate comparison $\bar{\mathcal{U}}_i$, the actual two-site block $\mathcal{U}_{i,2}$, and its conjugate comparison $\bar{\mathcal{U}}_{i,2}$ are simple and each carries reduced full-support source data. Suppose that either r or ℓ is odd and that the resulting standard-form interpolation carries an admissible path datum of blocking length one. Then $\mathcal{U}_1$ and $\mathcal{U}_2$ are admissibly equivalent under conjugation symmetry.

Proof. If either r or ℓ is odd, both endpoints lie in Case I of Proposition 19.9.1.8. The full admissible classification in Corollary 19.9.2.16 handles both possible parities of the source ranks after blocking and gives the claimed admissible equivalence. $\square$

Theorem 19.9.2.13 (Strict conjugation equivalence for even source ranks). Not ready Discussion: #5713 Suppose r and ℓ are even. Two conjugation-invariant MPUs in the standard forms of Proposition 19.9.1.7 are strictly equivalent under conjugation if and only if they belong to the same case and

$$\det(u_1) = \det(u_2), \quad \det(v_1) = \det(v_2). \quad (19.520)$$

Theorem 19.9.2.14 (Admissible strict conjugation equivalence for even source ranks). Not ready Discussion: #5713 Suppose that r and ℓ are both even and that the standard-form interpolation determined below carries an admissible path datum of blocking length one. Choose admissible standard forms of two conjugation-invariant MPUs as in Proposition 19.9.1.8. They are admissibly strictly equivalent under conjugation symmetry if and only if they belong to the same case of Proposition 19.9.1.8 and

$$\det(u_1) = \det(u_2), \quad \det(v_1) = \det(v_2). \quad (19.521)$$

Corollary 19.9.2.15 (Equivalence under conjugation). Not ready Discussion: #5713 Two conjugation-invariant MPUs are equivalent under conjugation symmetry if and only if they belong to the same case of Proposition 19.9.1.7.

Corollary 19.9.2.16 (Admissible equivalence under conjugation). Not ready Discussion: #5713 For $i = 1, 2$, suppose the original one-site endpoint tensor $\mathcal{U}_i$, its conjugate comparison $\overline{\mathcal{U}}_i$, the actual two-site block $\mathcal{U}_{i,2}$, and its conjugate comparison $\overline{\mathcal{U}}_{i,2}$ are simple and each carries reduced full-support source data. Suppose also that the resulting standard-form interpolation carries an admissible path datum of blocking length one. Then $\mathcal{U}_1$ and $\mathcal{U}_2$ are admissibly equivalent under conjugation symmetry if and only if they belong to the same case of Proposition 19.9.1.8.

Proof. Suppose first that either r or ℓ is odd. Then both original endpoints lie in Case I. After one blocking, write $r^{(2)}$ and $\ell^{(2)}$ for the blocked source ranks. Admissible blocking independence, applied to the supplied one-site and two-site endpoint data, gives

$$r^{(2)} = dr, \quad (19.522)$$

and

$$\ell^{(2)} = d\ell. \quad (19.523)$$

For each endpoint,

$$\det(u_i^{(2)} v_i^{(2)}) = \det(u_i v_i)^{2r\ell} = 1. \quad (19.524)$$

Blocking preserves Case I. If either $r^{(2)}$ or $\ell^{(2)}$ is odd, the odd-rank theorem applied to the blocked endpoints gives admissible strict equivalence. If both blocked ranks are even, then d is even because one original source rank is odd. Hence $r\ell = d^2$ is even. Consequently,

$$\det(u_i^{(2)}) = (\det(u_i)^2 \det(v_i))^{r\ell} = 1, \quad (19.525)$$

and

$$\det(v_i^{(2)}) = \det(v_i)^{r\ell} = 1. \quad (19.526)$$

The even-rank theorem now applies to the preserved Case I endpoints. Thus the odd-source-rank branch has a single admissible equivalence class without imposing a parity condition on the blocked ranks.

If r and ℓ are both even, admissible blocking independence shows that the blocked ranks remain even. Since $r\ell$ is even, the same calculations give

$$\det(u_i^{(2)}) = (\det(u_i)^2 \det(v_i))^{r\ell} = 1, \quad (19.527)$$

and

$$\det(v_i^{(2)}) = \det(v_i)^{r_\ell} = 1. \quad (19.528)$$

Blocking preserves Case I and Case II. The even-rank theorem therefore shows that the blocked endpoints are admissibly strictly equivalent exactly when they belong to the same case. This is precisely admissible equivalence of the original endpoints. $\square$

Lemma 19.9.2.17 (SWAP trace contraction). [Lean](#) [Lean](#) [✓ Stmt](#) For matrices $A, B \in \mathbb{C}^{n \times n}$ and the exchange matrix $\mathbb{S}$ on $\mathbb{C}^n \otimes \mathbb{C}^n$,

$$\mathrm{tr}[\mathbb{S}(A \otimes B)] = \mathrm{tr}(AB). \quad (19.529)$$

In particular, if U is unitary, then

$$\mathrm{tr}[\mathbb{S}(U \otimes U^\dagger)] = n. \quad (19.530)$$

Proof. [✓ Proof](#) In product coordinates, the exchange matrix has entries $\mathbb{S}_{(i,j),(k,l)} = \delta_{i,l} \delta_{j,k}$. Therefore

$$\mathrm{tr}[\mathbb{S}(A \otimes B)] = \sum_{i,j} A_{j,i} B_{i,j} = \sum_{i,j} A_{i,j} B_{j,i} = \mathrm{tr}(AB). \quad (19.531)$$

Taking $A = U$ and $B = U^\dagger$ gives $\mathrm{tr}(UU^\dagger) = \mathrm{tr}(\mathbb{1}) = n$. $\square$

Lemma 19.9.2.18 (Trace formula for the time-reversal sign). [Not ready](#) [Discussion: #5714](#) For a time-reversal-invariant MPU in standard form, the sign σ is

$$\sigma = \frac{1}{d^2} \mathrm{tr}[\mathbb{S}_{12,34} v_{23} u_{12} u_{34} v_{23}], \quad (19.532)$$

where $\mathbb{S}_{12,34}$ exchanges the pairs of sites 12 and 34.

Lemma 19.9.2.19 (Admissible trace formula for the time-reversal sign). [Not ready](#) [Discussion: #5714](#) For a time-reversal-invariant MPU in admissible standard form, suppose its one-site tensor $\mathcal{U}$, its actual two-site block $\mathcal{U}_2$, and the physical-adjoint comparison tensor $\mathcal{U}_2^\sharp$ are simple and each carries its own reduced full-support source datum. Then the sign σ in Proposition 19.9.1.12 is

$$\sigma = \frac{1}{d^2} \mathrm{tr}[\mathbb{S}_{12,34} v_{23} u_{12} u_{34} v_{23}], \quad (19.533)$$

where $\mathbb{S}_{12,34}$ exchanges the pairs of sites 12 and 34.

Lemma 19.9.2.20 (Gauge invariance of the time-reversal sign). [Not ready](#) [Discussion: #5714](#) Under $u' = (x \otimes y)u$ and $v' = v(y^\dagger \otimes x^\dagger)$, the trace formula in Lemma 19.9.2.18 has the same value for (u', v') as for (u, v) . Thus σ is independent of the standard-form gauge.

Lemma 19.9.2.21 (Admissible gauge invariance of the time-reversal sign). [Not ready](#) [Discussion: #5714](#) Under the hypotheses of Lemma 19.9.2.19, let x and y be unitaries and change the standard-form gauge by

$$u' = (x \otimes y)u, \quad v' = v(y^\dagger \otimes x^\dagger). \quad (19.534)$$

Then the trace formula has the same value for (u', v') as for (u, v) . Consequently the sign σ is independent of the standard-form gauge.

Proof. Write x_i and y_i for the unitaries acting on site i . Then $u'_{12} = x_1 y_2 u_{12}$, $u'_{34} = x_3 y_4 u_{34}$, and $v'_{23} = v_{23} y_2^\dagger x_3^\dagger$. Substitution into Lemma 19.9.2.19, cancellation on the middle legs, and $\mathbb{S}_{12,34} x_1 y_4 = x_3 y_2 \mathbb{S}_{12,34}$ give

$$\mathrm{tr}[\mathbb{S}_{12,34} v'_{23} u'_{12} u'_{34} v'_{23}] = \mathrm{tr}[x_3 y_2 \mathbb{S}_{12,34} v_{23} u_{12} u_{34} v_{23} y_2^\dagger x_3^\dagger] \quad (19.535)$$

$$= \mathrm{tr}[\mathbb{S}_{12,34} v_{23} u_{12} u_{34} v_{23}]. \quad (19.536)$$

The last equality follows from cyclicity of the trace and unitarity of x and y . $\square$

Proposition 19.9.2.22 (Well-definedness of the time-reversal sign). Not ready Discussion: #5714
For a time-reversal-invariant MPU in standard form, the sign $\sigma = \pm 1$ is independent of the gauge and is constant under continuous deformations that preserve time-reversal symmetry.

Proposition 19.9.2.23 (Admissible well-definedness of the time-reversal sign). Not ready Discussion: #5714
Suppose a time-reversal-preserving path carries an admissible equivalence datum of blocking length one. At every path point, require the one-site tensor, its actual two-site block, and the physical-adjoint comparison of that two-site block to be simple and to carry continuously varying reduced full-support source data. Then the sign $\sigma = \pm 1$ is independent of the admissible standard-form gauge and is constant along the path.

Proposition 19.9.2.24 (Blocking parity of the time-reversal sign). Not ready Discussion: #5714 *Let $\mathcal{U}$ be a time-reversal-invariant MPU in standard form, and let $\sigma^{(k)}$ be the sign obtained from a standard form of its k -site block. Then*

$$\sigma^{(k)} = \begin{cases} \sigma^{(1)}, & k \text{ odd}, \\ \sigma^{(2)}, & k \text{ even}. \end{cases} \quad (19.537)$$

Proposition 19.9.2.25 (Admissible blocking parity of the time-reversal sign). Not ready Discussion: #5714
For every blocking length k under consideration, suppose the endpoint tensor $\mathcal{U}_k$, its actual two-site block $\mathcal{U}_{2k}$, and the physical-adjoint comparison tensor $\mathcal{U}_{2k}^\sharp$ are simple and each carries its own reduced full-support source datum. Let $\sigma^{(k)}$ be the sign of the resulting standard form. Then

$$\sigma^{(k)} = \begin{cases} \sigma^{(1)}, & k \text{ odd}, \\ \sigma^{(2)}, & k \text{ even}. \end{cases} \quad (19.538)$$

Proof. Gauge invariance permits the standard blocked representatives used in the source proof. Choose

$$u^{(2)} = v_{23} u_{12} u_{34}, \quad v^{(2)} = v_{45}, \quad (19.539)$$

and put $\zeta = \sigma^{(2)}/\sigma^{(1)}$. The one- and two-site local time-reversal relations give the unwinding identity

$$\zeta[(\mathbb{1} \otimes v \otimes \mathbb{1})(u \otimes x)]^\dagger = (\mathbb{1} \otimes v \otimes \mathbb{1})(x^\dagger \otimes u). \quad (19.540)$$

Applying this identity successively through a block of length k gives

$$\sigma^{(k)} = \sigma^{(1)} \zeta^{k-1}. \quad (19.541)$$

Since each sign is ± 1 , this equals $\sigma^{(1)}$ for odd k and $\sigma^{(2)}$ for even k . $\square$

Corollary 19.9.2.26 (Necessary time-reversal invariants). Not ready Discussion: #5714 Strictly equivalent time-reversal-invariant MPUs have the same $\sigma^{(1)}$ and $\sigma^{(2)}$. Equivalent time-reversal-invariant MPUs have the same $\sigma^{(2)}$.

Corollary 19.9.2.27 (Admissible time-reversal invariants of supplied witnesses). Not ready Discussion: #5714 Let two time-reversal-invariant MPUs be compared by an admissible symmetry path $p \mapsto \mathcal{W}(p)$ whose path datum has blocking length one. Suppose that the five path families $\mathcal{W}(p)$, $\mathcal{W}(p)_2$, $\mathcal{W}(p)_4$, $\mathcal{W}(p)_2^\sharp$, and $\mathcal{W}(p)_4^\sharp$ are simple and carry continuously varying reduced full-support source data. Then admissible strict equivalence requires equality of both $\sigma^{(1)}$ and $\sigma^{(2)}$.

For admissible equivalence after adjoining ancillas and applying the source blocking, let $p \mapsto \mathcal{W}_{\text{anc}}(p)$ be the actual ancilla-enlarged blocked witness path, not a reduced or equivalent representative, and regard it as a length-one admissible path. Require continuous reduced full-support source data along $\mathcal{W}_{\text{anc}}(p)$, its actual two- and four-site blocks, and the physical-adjoint comparisons of those two blocks. The conclusion is only

$$\sigma^{(2)}(\mathcal{W}_{\text{anc}}(0)) = \sigma^{(2)}(\mathcal{W}_{\text{anc}}(1)). \quad (19.542)$$

No equality between $\sigma^{(2)}$ of the original two MPUs is asserted. Such a conclusion requires preservation under adjoining identity ancillas, which is not established here.

Proposition 19.9.2.28 (Conjugation class and the MPS sign). Not ready Discussion: #5714 For a conjugation-invariant MPU, the MPS sign $G\overline{G} = \pm 1$ agrees with Case I and Case II, respectively, of Proposition 19.9.1.7.

Proposition 19.9.2.29 (Admissible conjugation class and the MPS sign). Not ready Discussion: #5714 Let $\mathcal{U}$ be conjugation invariant. Suppose $\mathcal{U}$ and its conjugate comparison tensor are simple and carry their own reduced full-support source data. Let $p \mapsto \mathcal{W}(p)$ be the actual standard-form interpolation from $\mathcal{U}$ to its Case I or Case II representative. Suppose this interpolation carries an admissible conjugation-symmetry path datum of blocking length one: the path $\mathcal{W}(p)$ and its fixed conjugate, physical-adjoint, transposed, and conjugation-image comparison paths are simple and carry continuously varying source data. Then the sign $G\overline{G} = \pm 1$ in the MPS classification agrees with Case I and Case II, respectively, of Proposition 19.9.1.8.

Proposition 19.9.2.30 (Time-reversal class and the MPS sign). Not ready Discussion: #5714 For a time-reversal-invariant MPU in standard form, the MPS sign $G\overline{G} = \pm 1$ equals the sign $\sigma^{(2)}$ obtained after blocking two sites.

Proposition 19.9.2.31 (Admissible time-reversal class and the MPS sign). Not ready Discussion: #5714 Let an MPU in admissible standard form be invariant under time reversal. Suppose $\mathcal{U}$, $\mathcal{U}_2$, and $\mathcal{U}_4$, together with the physical-adjoint comparison tensors $\mathcal{U}_2^\sharp$ and $\mathcal{U}_4^\sharp$, are simple and each carries its own reduced full-support source datum. Then the MPS sign $G\overline{G} = \pm 1$ equals the sign $\sigma^{(2)}$.

19.9.3 Symmetry examples

Proposition 19.9.3.1 (All phases under conjugation). Not ready Discussion: #5715 Let $u_+ = v_+ = \mathbb{1}_{r\ell}$ and $u_- = v_- = \text{diag}(-1, 1, \dots, 1)$. If at least one of r and ℓ is odd, (u_+, v_+) and (u_+, v_-) realize the two classes of Theorem 19.9.2.9. If both are even, the four pairs $(u_\pm, v_\pm)$ realize the four Case I classes. Multiplying by the matrix R of Corollary 19.9.1.9 realizes the four Case II classes.

Proposition 19.9.3.2 (Admissible phases under conjugation). Not ready Discussion: #5715 Assume the standard-form tensors displayed below carry reduced full-support source data and that the comparison paths used to classify them carry admissible path data of blocking length one. Let

$$u_+ = v_+ = \mathbb{1}_{r\ell}, \quad u_- = v_- = \text{diag}(-1, 1, \dots, 1). \quad (19.543)$$

If at least one of r and ℓ is odd, the pairs (u_+, v_+) and (u_+, v_-) represent the two classes of Theorem 19.9.2.10. If r and ℓ are both even, the four pairs $(u_\pm, v_\pm)$ represent the four Case I classes of Theorem 19.9.2.14. With

$$u'_\pm = (\mathbb{1}_{r\ell/4} \otimes R)u_\pm, \quad v'_\pm = v_\pm(\mathbb{1}_{r\ell/4} \otimes R), \quad (19.544)$$

the four pairs $(u'_\pm, v'_\pm)$ represent the four Case II classes.

Proposition 19.9.3.3 (CZX). Not ready Discussion: #5715 The CZX MPU is defined by

$$u = CZ(\sigma_x \otimes \sigma_x), \quad v = CZ, \quad (19.545)$$

where $CZ = \text{diag}(1, 1, 1, -1)$. Let

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (19.546)$$

be the Hadamard matrix. The corresponding tensor is

$$\mathcal{U}_{\ell r}^{ud} = (\sigma_x)_{ud} \delta_{u,\ell} H_{\ell,r}. \quad (19.547)$$

It has $\sigma^{(1)} = \sigma^{(2)} = -1$.

Proposition 19.9.3.4 (Admissible CZX data). Not ready Discussion: #5715 Suppose the CZX tensors $\mathcal{U}$, $\mathcal{U}_2$, and $\mathcal{U}_4$, together with the physical-adjoint comparison tensors $\mathcal{U}_2^\sharp$ and $\mathcal{U}_4^\sharp$, are simple and each carries its own reduced full-support source datum. The CZX MPU is defined by

$$u = CZ(\sigma_x \otimes \sigma_x), \quad v = CZ, \quad (19.548)$$

$$CZ = \text{diag}(1, 1, 1, -1). \quad (19.549)$$

Its tensor is

$$\mathcal{U}_{\ell r}^{ud} = (\sigma_x)_{ud} \delta_{u,\ell} H_{\ell,r}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (19.550)$$

It has $\sigma^{(1)} = \sigma^{(2)} = -1$.

Proposition 19.9.3.5 (The other three time-reversal classes). Not ready Discussion: #5715 The standard forms

$$u = v = \mathbb{1}, \quad (19.551)$$

$$u = i\mathbb{1}, \quad v = \mathbb{1}, \quad (19.552)$$

$$u = iCZ(\sigma_x \otimes \sigma_x), \quad v = CZ \quad (19.553)$$

realize, respectively,

$$(\sigma^{(1)}, \sigma^{(2)}) = (1, 1), \quad (-1, 1), \quad (1, -1). \quad (19.554)$$

Proposition 19.9.3.6 (Admissible remaining time-reversal classes). Not ready Discussion: #5715
For each displayed example, suppose the tensors $\mathcal{U}$, $\mathcal{U}_2$, and $\mathcal{U}_4$, together with the physical-adjoint comparison tensors $\mathcal{U}_2^\sharp$ and $\mathcal{U}_4^\sharp$, are simple and each carries its own reduced full-support source datum. The standard forms

$$u = v = \mathbb{1}, \quad (19.555)$$

$$u = i\mathbb{1}, \quad v = \mathbb{1}, \quad (19.556)$$

$$u = iCZ(\sigma_x \otimes \sigma_x), \quad v = CZ \quad (19.557)$$

realize, respectively,

$$(\sigma^{(1)}, \sigma^{(2)}) = (1, 1), \quad (-1, 1), \quad (1, -1). \quad (19.558)$$

Definition 19.9.3.7 (Right- and left-shift tensors). Lean Lean ✓ Stmt The right shift on $(\mathbb{C}^d)^{\otimes N}$ is

$$T^{(N)}|n_1, \dots, n_N\rangle = |n_N, n_1, \dots, n_{N-1}\rangle. \quad (19.559)$$

It has bond dimension d and local matrices $A^{ij} = |i\rangle\langle j|$. The physical adjoint of A defines the left-shift tensor.

Theorem 19.9.3.8 (Shift tensor formulas). Lean Lean Lean Lean Lean ✓ Stmt For every $N > 0$, the tensors in Definition 19.9.3.7 generate $T^{(N)}$ and $T^{(N)\dagger}$, respectively. Both tensors are matrix product unitaries.

Proof. ✓ Proof Closing the virtual indices of $A^{ij} = |i\rangle\langle j|$ leaves one nonzero virtual configuration. If $\tau_n = \sigma_{n+1}$ for every n modulo N , then

$$\langle \sigma | T^{(N)} | \tau \rangle = 1. \quad (19.560)$$

Otherwise the same matrix entry is zero. Hence the output configuration is $(\tau_{N-1}, \tau_0, \dots, \tau_{N-2})$, which fixes the permutation-matrix convention and gives the right shift. Its permutation matrix is unitary. Physical adjunction gives the conjugate transpose, hence the left shift and its unitarity. $\square$

Theorem 19.9.3.9 (Source cuts and ranks of the cyclic shifts). Lean Lean Lean Lean Lean
Lean Lean Lean ✓ Stmt Let A be the right-shift tensor and let $z_{(\alpha, i)} = \delta_{\alpha i}$ be the vectorized identity. Its raw source cuts are

$$M_1(A) = \mathbb{1}, \quad (19.561)$$

$$M_2(A) = zz^\top. \quad (19.562)$$

For the left-shift tensor $A^\sharp$ they are

$$M_1(A^\sharp) = zz^\top, \quad (19.563)$$

$$M_2(A^\sharp) = \mathbb{1}. \quad (19.564)$$

In every physical dimension, including $d = 0$,

$$r(A) = d^2, \quad (19.565)$$

$$\ell(A^\sharp) = d^2. \quad (19.566)$$

If $d > 0$, the remaining two ranks are

$$\ell(A) = 1, \quad (19.567)$$

$$r(A^\sharp) = 1. \quad (19.568)$$

These are the raw source-cut quantities from [CPGSV17, Section III]; no choice of source factors is involved.

Proof. ✓ Proof The entry formula $A_{\alpha\beta}^{ij} = \delta_{i\alpha}\delta_{j\beta}$ gives $M_1(A) = \mathbb{1}$ and $M_2(A) = zz^\top$. Hence the first rank is d^2 . For $d > 0$, the vector z has a nonzero diagonal coordinate, so its outer product has rank one. Physical adjunction exchanges the two cuts and their ranks, which gives all four left-shift formulas without a second rank calculation. $\square$

Definition 19.9.3.10 (Three topological-insulator shift MPUs). Lean Lean Lean Lean ✓ Stmt On two d -dimensional spins per site, define

$$U_1^{(N)} = (\mathbb{1} \otimes \mathbb{1})^{\otimes N}, \quad (19.569)$$

$$U_2^{(N)} = T^{(N)\dagger} \otimes T^{(N)}, \quad (19.570)$$

$$U_3^{(N)} = T^{(N)} \otimes T^{(N)\dagger}. \quad (19.571)$$

Theorem 19.9.3.11 (Finite-chain formulas for the three shift MPUs). Lean Lean Lean Lean Lean Lean ✓ Stmt The three tensors generate the operators displayed in Definition 19.9.3.10 on every nonempty chain, and each tensor is a matrix product unitary.

Proof. ✓ Proof The bond-one identity tensor generates the identity at every length. The finite-chain operator of an independent tensor product is the Kronecker product of the two finite-chain operators, after the sitewise identification $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes N} \cong (\mathbb{C}^d)^{\otimes N} \otimes (\mathbb{C}^d)^{\otimes N}$. Applying this identity to the right and left shifts gives the three displayed formulas. The identity tensor is unitary, and tensor products preserve unitarity, so all three tensors are matrix product unitaries. $\square$

Definition 19.9.3.12 (Supplied source factors for the three shift MPUs). Lean Lean Lean Lean Lean Lean ✓ Stmt Assume $d > 0$. Equip U_1 with the tensor product of the two bond-one identity source factorizations. Equip U_2 with the tensor product of the left-shift and right-shift source factorizations, in that order, and equip U_3 with the tensor product in the reverse order. In all three cases the virtual weight is the identity. These are explicit supplied source-factor witnesses; no identification with an arbitrary compact singular-value decomposition is asserted.

Theorem 19.9.3.13 (One-site supplied source tensors for U_1 and U_3). Lean Lean ✓ Stmt Assume $d > 0$. In the source coordinates of the supplied factorizations,

$$u_1 = \mathbb{1} \otimes \mathbb{1}, \quad v_1 = \mathbb{1} \otimes \mathbb{1}, \quad (19.572)$$

$$u_3 = \mathbb{S}, \quad v_3 = \mathbb{S}. \quad (19.573)$$

These are the formulas in [CPGSV17, lines 2009–2016]. The U_3 equalities use the source-coordinate presentation of that equation; they are distinct from the blocked four-spin presentation below.

Proof. ✓ Proof Write the source and physical indices as $((a, b), (c, e))$ and $((i, j), (k, l))$, respectively, and set $s = d^{-1/2}$. The identity factors give

$$[u_1]_{((a,b),(c,e)),((i,j),(k,l))} = [v_1]_{((i,j),(k,l)),((a,b),(c,e))} \quad (19.574)$$

$$= (\delta_{a,i}\delta_{c,k})(\delta_{b,j}\delta_{e,l}) \quad (19.575)$$

$$= \delta_{a,i}\delta_{b,j}\delta_{c,k}\delta_{e,l}. \quad (19.576)$$

For U_3 , the right-shift and left-shift factors instead give

$$[u_3]_{((a,b),(c,e)),((i,j),(k,l))} = (s\delta_{c,i}\delta_{a,k})(d s\delta_{e,j}\delta_{b,l}) \quad (19.577)$$

$$= \delta_{a,k}\delta_{b,l}\delta_{c,i}\delta_{e,j}, \quad (19.578)$$

$$[v_3]_{((i,j),(k,l)),((a,b),(c,e))} = (d s\delta_{a,k}\delta_{c,i})(s\delta_{b,l}\delta_{e,j}) \quad (19.579)$$

$$= \delta_{a,k}\delta_{b,l}\delta_{c,i}\delta_{e,j}, \quad (19.580)$$

where $d s^2 = 1$. The first display is the identity matrix, while the second interchanges the two composite indices and is therefore $\mathbb{S}$ in both orientations. $\square$

Theorem 19.9.3.14 (Central gates in the supplied two-site factorizations of U_2). [Lean](#) [Lean](#)

[✓ Stmt](#) Assume $d > 0$. The two open-leg factorizations of the blocked tensor have, as their respective central gates,

$$u_2^{(2)} = \mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}, \quad (19.581)$$

$$v_2^{(2)} = (\mathbb{S} \otimes \mathbb{S})(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}). \quad (19.582)$$

Here the two physical sites are decoded into the four-spin order of the paper, and the factors X_1, Y_2 and X_2, Y_1 remain as the two open boundaries of the respective factorizations. This is [CPGSV17, lines 2018–2026].

Proof. [✓ Proof](#) Write $B_2 = \mathcal{U}_{2,2}$ for the actual two-site block and decode $I = (I_1, I_2)$ and $J = (J_1, J_2)$. If π is the four-spin physical identification and ι_u, ι_v are the two source-rank identifications, the asserted open-leg equalities are

$$(B_2)_{I,J;\alpha,\gamma} = \sum_{r,l} (X_1)_{(\alpha,J_1),r} [\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}]_{\iota_u^{-1}(l,r),\pi^{-1}(I_1,I_2)} (Y_2)_{l,(J_2,\gamma)}, \quad (19.583)$$

$$(B_2)_{I,J;\alpha,\gamma} = \sum_{l,r} (X_2)_{(\alpha,I_1),l} [(\mathbb{S} \otimes \mathbb{S})(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})]_{\pi^{-1}(J_2,J_1),\iota_v^{-1}(r,l)} (Y_1)_{r,(I_2,\gamma)}. \quad (19.584)$$

Write $s = d^{-1/2}$, so that $d s^2 = 1$. In the product-rank coordinates, the entries of the two central gates are

$$[u_2^{(2)}]_{((a,b),(c,e)),((i,j),(k,l))} = (d s\delta_{a,i}\delta_{b,k})(s\delta_{c,j}\delta_{e,l}) \quad (19.585)$$

$$= \delta_{a,i}\delta_{b,k}\delta_{c,j}\delta_{e,l}, \quad (19.586)$$

$$[v_2^{(2)}]_{((i,j),(k,l)),((a,b),(c,e))} = (s\delta_{e,k}\delta_{c,i})(d s\delta_{b,l}\delta_{a,j}) \quad (19.587)$$

$$= \delta_{a,j}\delta_{b,l}\delta_{c,i}\delta_{e,k}. \quad (19.588)$$

These are respectively the entries of $\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}$ and $(\mathbb{S} \otimes \mathbb{S})(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})$; substituting them into the exact open two-site factorizations gives the asserted equalities. $\square$

Theorem 19.9.3.15 (Central gates in the supplied two-site factorizations of U_3). [Lean](#) [Lean](#)

[✓ Stmt](#) Assume $d > 0$. The two open-leg factorizations of the blocked tensor have, as their respective central gates,

$$u_3^{(2)} = (\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})(\mathbb{S} \otimes \mathbb{S}), \quad (19.589)$$

$$v_3^{(2)} = \mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}. \quad (19.590)$$

Again the four-spin order is the one displayed in the paper, and the source factors at the two open boundaries are retained explicitly. This is [CPGSV17, lines 2028–2034].

Proof. ✓ Proof With $B_3 = \mathcal{U}_{3,2}$ for the actual two-site block and the corresponding identifications π, ι_u, ι_v , the exact equalities are

$$(B_3)_{I,J;\alpha,\gamma} = \sum_{r,l} (X_1)_{(\alpha,J_1),r} [(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})(\mathbb{S} \otimes \mathbb{S})]_{\iota_u^{-1}(l,r), \pi^{-1}(I_1, I_2)} (Y_2)_{l, (J_2, \gamma)}, \quad (19.591)$$

$$(B_3)_{I,J;\alpha,\gamma} = \sum_{l,r} (X_2)_{(\alpha, I_1), l} [\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}]_{\pi^{-1}(J_2, J_1), \iota_v^{-1}(r, l)} (Y_1)_{r, (I_2, \gamma)}. \quad (19.592)$$

Again write $s = d^{-1/2}$, so that $d s^2 = 1$. Reversing the two supplied factors gives the entry identities

$$[u_3^{(2)}]_{((a,b),(c,e)),((i,j),(k,l))} = (s \delta_{c,i} \delta_{e,k}) (d s \delta_{a,j} \delta_{b,l}) \quad (19.593)$$

$$= \delta_{a,j} \delta_{b,l} \delta_{c,i} \delta_{e,k}, \quad (19.594)$$

$$[v_3^{(2)}]_{((i,j),(k,l)),((a,b),(c,e))} = (d s \delta_{b,k} \delta_{a,i}) (s \delta_{e,l} \delta_{c,j}) \quad (19.595)$$

$$= \delta_{a,i} \delta_{b,k} \delta_{c,j} \delta_{e,l}. \quad (19.596)$$

The first is the entry of $(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})(\mathbb{S} \otimes \mathbb{S})$, and the second is the entry of $\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}$. Substitution into the two exact open two-site factorizations proves the claim. $\square$

Proposition 19.9.3.16 (Standard forms of the three shift MPUs). Not ready Discussion: #5715
Assume $d > 0$. For the fixed standard-form source factors, the corresponding unitaries satisfy

$$u_1 = \mathbb{1} \otimes \mathbb{1}, \quad v_1 = \mathbb{1} \otimes \mathbb{1}, \quad (19.597)$$

$$u_3 = \mathbb{S}, \quad v_3 = \mathbb{S}, \quad (19.598)$$

$$u_2^{(2)} = \mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}, \quad v_2^{(2)} = (\mathbb{S} \otimes \mathbb{S})(\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}), \quad (19.599)$$

$$u_3^{(2)} = (\mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1})(\mathbb{S} \otimes \mathbb{S}), \quad v_3^{(2)} = \mathbb{1} \otimes \mathbb{S} \otimes \mathbb{1}. \quad (19.600)$$

The preceding theorems establish these matrices for explicit supplied source-factor witnesses. It remains to identify that supplied choice with the fixed standard-form choice.

Definition 19.9.3.17 (Transformation with swaps). Not ready Discussion: #5715 Define

$$\widetilde{U}_1^{(N)} = \mathbb{S}^{\otimes N}, \quad (19.601)$$

$$\widetilde{U}_2^{(N)} = \mathbb{S}^{\otimes N} (T^{(N)\dagger} \otimes T^{(N)}), \quad (19.602)$$

$$\widetilde{U}_3^{(N)} = \mathbb{S}^{\otimes N} (T^{(N)} \otimes T^{(N)\dagger}) = \mathbb{S}^{\otimes N} \widetilde{U}_2^{(N)} \mathbb{S}^{\otimes N}. \quad (19.603)$$

The first two standard forms are

$$\tilde{u}_1 = \mathbb{S}, \quad \tilde{v}_1 = \mathbb{1} \otimes \mathbb{1}, \quad (19.604)$$

$$\tilde{u}_2 = \mathbb{1} \otimes \mathbb{1}, \quad \tilde{v}_2 = \mathbb{S}. \quad (19.605)$$

Lemma 19.9.3.18 (Transformation with swaps and symmetry equivalence). Not ready Discussion: #5715

Two MPUs $U_i^{(N)}$ and $U_j^{(N)}$ are in the same phase under time reversal combined with the local swap, transposition combined with the local swap, or conjugation if and only if the corresponding $\widetilde{U}_i^{(N)}$ and $\widetilde{U}_j^{(N)}$ are in the same phase under time reversal, transposition, or conjugation, respectively.

Lemma 19.9.3.19 (Admissible transformation with swaps and symmetry equivalence). Not ready

Discussion: #5715 Fix one of the three symmetry comparisons and let $p \mapsto \mathcal{W}(p)$ be the actual source-constructed interpolation used in that comparison. Suppose it carries an admissible symmetry-path datum of blocking length one. Thus the path $\mathcal{W}(p)$ and the four fixed comparison paths $\mathcal{W}(p)$, $\mathcal{W}(p)^\sharp$, $\mathcal{W}(p)^\top$, and $\mathcal{S}[\mathcal{W}(p)]$ are simple and carry continuously varying reduced full-support source data. Suppose also that every one- and two-site endpoint tensor used in the local comparison carries its own datum. Then $U_i^{(N)}$ and $U_j^{(N)}$ are in the same admissible phase under time reversal combined with the local swap, transposition combined with the local swap, or conjugation if and only if the corresponding $\widetilde{U}_i^{(N)}$ and $\widetilde{U}_j^{(N)}$ are in the same admissible phase under time reversal, transposition, or conjugation, respectively.

Proposition 19.9.3.20 (Conjugation phases of U_2 and U_3). Not ready Discussion: #5715 After blocking two sites, $U_2^{(N)}$ and $U_3^{(N)}$ are both in the trivial phase under conjugation.

Proposition 19.9.3.21 (Admissible conjugation phases of U_2 and U_3). Not ready Discussion: #5715 Suppose the two-site blocked tensors carry reduced full-support source data and the standard-form comparison paths carry admissible path data of blocking length one. Then the actual two-site blocked tensors of U_2 and U_3 are both in the trivial admissible strict phase under conjugation. Consequently U_2 and U_3 lie in the same admissible equivalence class.

Proposition 19.9.3.22 (Time-reversal and transposition phases of U_2 and U_3). Not ready

Discussion: #5715 The MPUs $\widetilde{U}_2^{(N)}$ and $\widetilde{U}_3^{(N)}$ are in the same phase under both time reversal and transposition.

Proposition 19.9.3.23 (Admissible time-reversal and transposition phases of U_2 and U_3).

Not ready Discussion: #5715 For each of time reversal and transposition, let $p \mapsto \mathcal{W}_\mathcal{S}(p)$ be the actual source-constructed interpolation from $\widetilde{U}_2$ to $\widetilde{U}_3$. Suppose it carries an admissible symmetry-path datum of blocking length one. Thus the path $\mathcal{W}_\mathcal{S}(p)$ and its fixed conjugate, physical-adjoint, transposed, and $\mathcal{S}$ -image comparison paths are simple and carry continuously varying reduced full-support source data. Suppose also that the one- and two-site endpoint tensors used in the local comparison carry their own data. Then $\widetilde{U}_2^{(N)}$ and $\widetilde{U}_3^{(N)}$ are in the same admissible phase under both time reversal and transposition.

Proposition 19.9.3.24 (Conjugation phases of U_1 and U_2). Not ready Discussion: #5715 The MPUs $U_1^{(N)}$ and $U_2^{(N)}$ lie in different strict conjugation phases exactly when $d = 4k + 2$ for a nonnegative integer k . Otherwise they lie in the same strict phase. All these phases are in Case I, so the two MPUs are equivalent after blocking and adjoining ancillas.

Proposition 19.9.3.25 (Admissible conjugation phases of U_1 and U_2). Not ready Discussion: #5715

For each endpoint tensor $\mathcal{A}$ among U_1 , U_2 , $\widetilde{U}_1$, and $\widetilde{U}_2$, suppose that $\mathcal{A}$, its conjugate comparison $\overline{\mathcal{A}}$, its actual two-site block $\mathcal{A}_2$, and the conjugate comparison $\overline{\mathcal{A}_2}$ are simple and carry their own reduced full-support source data. For every strict comparison used below, let $p \mapsto \mathcal{W}(p)$ be the actual source-constructed interpolation between the corresponding endpoint tensors. Suppose it carries an admissible conjugation-symmetry path datum of blocking length one: the path and its fixed conjugate, physical-adjoint, transposed, and conjugation-image comparison paths are simple and carry continuously varying source data. Require a length-one admissible path datum also for the actual blocked interpolation used in the final equivalence claim. Under these hypotheses, $U_1^{(N)}$ and $U_2^{(N)}$ are in different admissible strict phases under conjugation exactly when k is a nonnegative integer and

$$d = 4k + 2. \quad (19.606)$$

Otherwise they are in the same admissible strict phase. All these standard forms lie in Case I, so $U_1^{(N)}$ and $U_2^{(N)}$ are admissibly equivalent after blocking and adding ancillas.

Proposition 19.9.3.26 (Ancilla equivalence of U_1 and U_2). Not ready Discussion: #5715 After adding one ancilla of dimension d per site, $\widetilde{U}_1^{(N)}$ and $\widetilde{U}_2^{(N)}$ are strictly equivalent under both time reversal and transposition.

Proposition 19.9.3.27 (Admissible ancilla equivalence of U_1 and U_2). Not ready Discussion: #5715 After adding one ancilla of dimension d per site and performing the source blocking, denote the actual enlarged blocked endpoints by $\widehat{U}_1$ and $\widehat{U}_2$. For either time reversal or transposition, let $p \mapsto \mathcal{W}_s(p)$ be the source-constructed interpolation from $\widehat{U}_1$ to $\widehat{U}_2$. Suppose this interpolation carries an admissible symmetry-path datum of blocking length one. Thus the path $\mathcal{W}_s(p)$ and its fixed conjugate, physical-adjoint, transposed, and $\mathcal{S}$ -image comparison paths are simple and carry continuously varying reduced full-support source data. Suppose also that the one- and two-site enlarged endpoint tensors used in the local comparison carry their own data. Then $\widehat{U}_1$ and $\widehat{U}_2$ are admissibly strictly equivalent under both time reversal and transposition. No equivalence of the original ancilla-free tensors is asserted, because the equal ancilla dimensions used here need not satisfy the coprimality condition in Definition 19.9.2.6.

19.10 Finite-region observable algebras

Definition 19.10.1 (Finite-region local algebra). Lean Lean ✓ Stmt Fix a nonnegative integer d . For a finite region $\Lambda \subset \mathbb{Z}$, the configuration space and local observable algebra are

$$\text{Config}_d(\Lambda) = \{0, \dots, d-1\}^\Lambda, \quad (19.607)$$

$$\mathcal{A}_\Lambda = M_{\text{Config}_d(\Lambda)}(\mathbb{C}) \cong M_d^{\otimes \Lambda}. \quad (19.608)$$

The algebra $\mathcal{A}_\Lambda$ carries its operator norm. This is the finite-region construction used in the QCA appendix of [CPGSV17]; no completion is taken here.

Definition 19.10.2 (Canonical inclusion of finite-region observables). Lean ✓ Stmt If $\Lambda \subseteq \Gamma$ are finite regions, define

$$\iota_{\Lambda, \Gamma}: \mathcal{A}_\Lambda \longrightarrow \mathcal{A}_\Gamma, \quad (19.609)$$

$$\iota_{\Lambda, \Gamma}(A) = A \otimes \mathbf{1}_{\Gamma \setminus \Lambda}. \quad (19.610)$$

Here product configurations are identified with configurations on Γ . This is a unital star-algebra homomorphism.

Lemma 19.10.3 (Injectivity of the finite-region inclusion). Lean ✓ Stmt For $d > 0$, the map $\iota_{\Lambda, \Gamma}$ is injective.

Proof. ✓ Proof Since $d > 0$, choose a configuration c on $\Gamma \setminus \Lambda$. For configurations x, y on Λ ,

$$\iota_{\Lambda, \Gamma}(A)((x, c), (y, c)) = A(x, y) \cdot \mathbf{1}_{\Gamma \setminus \Lambda}(c, c) = A(x, y). \quad (19.611)$$

Hence $\iota_{\Lambda, \Gamma}(A) = \iota_{\Lambda, \Gamma}(B)$ implies $A = B$. □

Lemma 19.10.4 (Operator-norm preservation). Lean ✓ Stmt If $d > 0$, then for every $A \in \mathcal{A}_\Lambda$,

$$\|\iota_{\Lambda, \Gamma}(A)\| = \|A\|. \quad (19.612)$$

Proof. ✓ Proof An injective star-algebra homomorphism between complex C^* -algebras is isometric. □

Lemma 19.10.5 (Isometry of the finite-region inclusion). Lean ✓ Stmt For $d > 0$, the map $\iota_{\Lambda, \Gamma}$ is an operator-norm isometry.

Proof. ✓ Proof Use injectivity and the general theorem that an injective star-algebra homomorphism between complex C^* -algebras is an isometry. □

Lemma 19.10.6 (Identity inclusion). Lean ✓ Stmt For every finite region Λ , $\iota_{\Lambda, \Lambda}$ is the identity on $\mathcal{A}_\Lambda$.

Proof. ✓ Proof When the two regions agree, their complement is empty. Hence

$$\iota_{\Lambda, \Lambda}(A) = A \otimes \mathbf{1}_\emptyset = A \otimes 1 = A. \quad (19.613)$$

□

Lemma 19.10.7 (Composition of finite-region inclusions). Lean ✓ Stmt If $\Lambda \subseteq \Gamma \subseteq \Delta$, then

$$\iota_{\Gamma, \Delta} \circ \iota_{\Lambda, \Gamma} = \iota_{\Lambda, \Delta}. \quad (19.614)$$

Proof. ✓ Proof The complement decomposes as the disjoint union

$$\Delta \setminus \Lambda = (\Gamma \setminus \Lambda) \sqcup (\Delta \setminus \Gamma), \quad (19.615)$$

$$\mathbf{1}_{\Delta \setminus \Lambda} = \mathbf{1}_{\Gamma \setminus \Lambda} \otimes \mathbf{1}_{\Delta \setminus \Gamma}. \quad (19.616)$$

Therefore

$$\iota_{\Gamma, \Delta}(\iota_{\Lambda, \Gamma}(A)) = (A \otimes \mathbf{1}_{\Gamma \setminus \Lambda}) \otimes \mathbf{1}_{\Delta \setminus \Gamma} \quad (19.617)$$

$$= A \otimes \mathbf{1}_{\Delta \setminus \Lambda} \quad (19.618)$$

$$= \iota_{\Lambda, \Delta}(A). \quad (19.619)$$

□

Theorem 19.10.8 (Disjoint finite-region observables commute). Lean ✓ Stmt If $\Lambda \cap \Gamma = \emptyset$, $A \in \mathcal{A}_\Lambda$, and $B \in \mathcal{A}_\Gamma$, then their images in the algebra of the union satisfy

$$\iota_{\Lambda, \Lambda \cup \Gamma}(A) \iota_{\Gamma, \Lambda \cup \Gamma}(B) = \iota_{\Gamma, \Lambda \cup \Gamma}(B) \iota_{\Lambda, \Lambda \cup \Gamma}(A). \quad (19.620)$$

This tensor-factor identity is local algebra infrastructure for the quasi-local algebra used in [CPGSV17, lines 2292–2300]; the appendix does not state it as a separate theorem.

Proof. ✓ Proof For configurations x, y on $\Lambda \cup \Gamma$, split each intermediate configuration z into $(z|_\Lambda, z|_\Gamma)$. The matrix entry of the left-hand side is

$$\sum_{z_\Lambda, z_\Gamma} A(x|_\Lambda, z_\Lambda) \delta_{x|_\Gamma, z_\Gamma} B(z_\Gamma, y|_\Gamma) \delta_{z_\Lambda, y|_\Lambda} = A(x|_\Lambda, y|_\Lambda) B(x|_\Gamma, y|_\Gamma). \quad (19.621)$$

Reversing the two factors gives the same scalar product. □

Theorem 19.10.9 (Finite relative commutant). [Lean](#) [✓ Stmt](#) *Let $\Lambda \subseteq \Gamma$ be finite regions and $A \in \mathcal{A}_\Gamma$. Then A acts trivially outside Λ if and only if it commutes with the full observable algebra on the complement:*

$$A \in \iota_{\Lambda, \Gamma}(\mathcal{A}_\Lambda) \iff \forall B \in \mathcal{A}_{\Gamma \setminus \Lambda}, \quad A \iota_{\Gamma \setminus \Lambda, \Gamma}(B) = \iota_{\Gamma \setminus \Lambda, \Gamma}(B)A. \quad (19.622)$$

This is a finite-dimensional tensor-factor identity. It is not a statement of Schumacher–Werner or the QCA appendix of [CPGSV17].

Proof. [✓ Proof](#) Splitting configurations gives $\mathcal{A}_\Gamma \cong \mathcal{A}_\Lambda \otimes \mathcal{A}_{\Gamma \setminus \Lambda}$. Under this identification, the two inclusions are $C \mapsto C \otimes \mathbf{1}$ and $B \mapsto \mathbf{1} \otimes B$. Write an arbitrary matrix in blocks indexed by configurations on Λ . If it commutes with every $\mathbf{1} \otimes B$, each block commutes with the full matrix algebra on $\Gamma \setminus \Lambda$ and is therefore scalar. Thus the matrix is $C \otimes \mathbf{1}$ for some $C \in \mathcal{A}_\Lambda$. The converse follows from

$$(C \otimes \mathbf{1})(\mathbf{1} \otimes B) = C \otimes B = (\mathbf{1} \otimes B)(C \otimes \mathbf{1}). \quad (19.623)$$

□

19.10.1 Bipartite support algebras

Let I and J be finite sets. For $X \in M_{I \times J}(\mathbb{C})$, the existing left coefficient slice and right operator block are the matrices

$$X_{rs}^L(i, j) = X((i, r), (j, s)), \quad X_{ij}^R(r, s) = X((i, r), (j, s)). \quad (19.624)$$

For a star-subalgebra $\mathcal{D} \subseteq M_K(\mathbb{C})$, write $\mathcal{D}' = \{Y \in M_K(\mathbb{C}) \mid [Y, Z] = 0 \text{ for every } Z \in \mathcal{D}\}$ for its commutant. The construction below is the full complex matrix-algebra specialization of the coefficient construction in [SW04, Section 4.3] and of the leastness and commutant statements in [GNVW12, Lemma 7].

Local fix (star-closed input): The printed statement of Lemma 7 says only that $\mathcal{A}$ is a subalgebra. Its coefficient-generated support and leastness clauses remain valid for arbitrary algebra input, but its commutant characterization fails unless $\mathcal{A}$ is closed under the involution. For instance, the algebra $\text{span}\{\mathbf{1}, E_{12}\} \subset M_2(\mathbb{C})$ gives a counterexample to the printed commutant clause if involution closure is omitted. Schumacher and Werner first define the ordinary algebra generated by the coefficient support and then derive its adjoint closure from adjoint closure of the observable-algebra input [SW04, lines 1161–1171]. Accordingly, the statements below use a star-subalgebra. This is a necessary correction to the printed hypothesis. In the QCA application the input is the image of a local observable algebra under a star-automorphism [GNVW12, Section 7.2]. The MPU application likewise takes the image of a local star-algebra under a star-automorphism [CPGSV17, lines 2300–2306]. This correction, including the counterexample without involution closure and the exact application ranges, is recorded in [con26w].

Scope restriction (full matrix factors): After this correction, Lemma 7 permits arbitrary finite-dimensional C^* -algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, whereas the statements below take $\mathcal{B}_1 = M_I(\mathbb{C})$ and $\mathcal{B}_2 = M_J(\mathbb{C})$. Full matrix factors suffice for the MPU spin chain because every one-site algebra is isomorphic to $M_d(\mathbb{C})$ [CPGSV17, lines 2320–2334]. The general-factor result is not claimed here; see [con26k].

No full-matrix structure theorem, dimension formula, Margolus gate, depth-two circuit, MPU standard form, or index comparison is asserted here.

Theorem 19.10.1.1 (Adjoint of a right coefficient block). [Lean](#) [✓ Stmt](#) *For every $X \in M_{I \times J}(\mathbb{C})$ and $i, j \in I$,*

$$(X^\dagger)_{ij}^R = (X_{ji}^R)^\dagger. \quad (19.625)$$

Proof. [✓ Proof](#) Compare the (r, s) entries. Both sides equal $\overline{X((j, s), (i, r))}$. $\square$

Definition 19.10.1.2 (Left coefficient set). [Lean](#) [✓ Stmt](#) Let $\mathcal{A} \subseteq M_{I \times J}(\mathbb{C})$ be a star-subalgebra. Its set of left coefficients is

$$\text{Coeff}_L(\mathcal{A}) = \{X_{rs}^L \mid X \in \mathcal{A}, r, s \in J\} \subseteq M_I(\mathbb{C}). \quad (19.626)$$

Definition 19.10.1.3 (Right coefficient set). [Lean](#) [✓ Stmt](#) Let $\mathcal{A} \subseteq M_{I \times J}(\mathbb{C})$ be a star-subalgebra. Its set of right coefficients is

$$\text{Coeff}_R(\mathcal{A}) = \{X_{ij}^R \mid X \in \mathcal{A}, i, j \in I\} \subseteq M_J(\mathbb{C}). \quad (19.627)$$

Definition 19.10.1.4 (Left support algebra). [Lean](#) [✓ Stmt](#) The left support algebra of $\mathcal{A}$ is the star-subalgebra

$$\text{Spp}_L(\mathcal{A}) = \text{alg}^*(\text{Coeff}_L(\mathcal{A})) \subseteq M_I(\mathbb{C}) \quad (19.628)$$

generated by all left coefficients.

Definition 19.10.1.5 (Right support algebra). [Lean](#) [✓ Stmt](#) The right support algebra of $\mathcal{A}$ is the star-subalgebra

$$\text{Spp}_R(\mathcal{A}) = \text{alg}^*(\text{Coeff}_R(\mathcal{A})) \subseteq M_J(\mathbb{C}) \quad (19.629)$$

generated by all right coefficients.

Theorem 19.10.1.6 (Left coefficient generators belong to the support algebra). [Lean](#) [✓ Stmt](#) If $X \in \mathcal{A}$ and $r, s \in J$, then

$$X_{rs}^L \in \text{Spp}_L(\mathcal{A}). \quad (19.630)$$

Proof. [✓ Proof](#) The matrix X_{rs}^L is one of the generators of the defining star-algebra closure. $\square$

Theorem 19.10.1.7 (Right coefficient generators belong to the support algebra). [Lean](#) [✓ Stmt](#) If $X \in \mathcal{A}$ and $i, j \in I$, then

$$X_{ij}^R \in \text{Spp}_R(\mathcal{A}). \quad (19.631)$$

Proof. [✓ Proof](#) The matrix X_{ij}^R is one of the generators of the defining star-algebra closure. $\square$

Theorem 19.10.1.8 (Reconstruction from left coefficients). [Lean](#) [✓ Stmt](#) For every $X \in M_{I \times J}(\mathbb{C})$,

$$X = \sum_{r, s \in J} X_{rs}^L \otimes E_{rs}. \quad (19.632)$$

Proof. [✓ Proof](#) At the entry $((i, r), (j, s))$, only the matrix unit E_{rs} contributes, and its coefficient is $X_{rs}^L(i, j) = X((i, r), (j, s))$. $\square$

Theorem 19.10.1.9 (Reconstruction from right coefficients). [Lean](#) [✓ Stmt](#) For every $X \in M_{I \times J}(\mathbb{C})$,

$$X = \sum_{i, j \in I} E_{ij} \otimes X_{ij}^R. \quad (19.633)$$

Proof. ✓ Proof At the entry $((i, r), (j, s))$, only the matrix unit E_{ij} contributes, and its coefficient is $X_{ij}^R(r, s) = X((i, r), (j, s))$. $\square$

Theorem 19.10.1.10 (Minimality of the left support algebra). Lean ✓ Stmt For every star-subalgebra $\mathcal{B} \subseteq M_I(\mathbb{C})$,

$$\text{Spp}_L(\mathcal{A}) \subseteq \mathcal{B} \iff X_{rs}^L \in \mathcal{B} \text{ for every } X \in \mathcal{A} \text{ and } r, s \in J. \quad (19.634)$$

Proof. ✓ Proof Use the universal property of the star-subalgebra generated by $\text{Coeff}_L(\mathcal{A})$. $\square$

Theorem 19.10.1.11 (Minimality of the right support algebra). Lean ✓ Stmt For every star-subalgebra $\mathcal{C} \subseteq M_J(\mathbb{C})$,

$$\text{Spp}_R(\mathcal{A}) \subseteq \mathcal{C} \iff X_{ij}^R \in \mathcal{C} \text{ for every } X \in \mathcal{A} \text{ and } i, j \in I. \quad (19.635)$$

Proof. ✓ Proof Use the universal property of the star-subalgebra generated by $\text{Coeff}_R(\mathcal{A})$. $\square$

Theorem 19.10.1.12 (Tensor-submodule criterion for left support). Lean ✓ Stmt Let $I = \{0, \dots, p-1\}$ and $J = \{0, \dots, q-1\}$. For every star-subalgebra $\mathcal{B} \subseteq M_p(\mathbb{C})$,

$$\text{Spp}_L(\mathcal{A}) \subseteq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B} \otimes M_q(\mathbb{C}), \quad (19.636)$$

where the right-hand side is the tensor-product submodule consisting of matrices whose left coefficient slices lie in $\mathcal{B}$.

Proof. ✓ Proof Membership in the tensor-product submodule is equivalent to membership of every left coefficient in $\mathcal{B}$. Apply Theorem 19.10.1.10. $\square$

Definition 19.10.1.13 (Kronecker-product submodule). Lean ✓ Stmt For submodules $S \subseteq M_I(\mathbb{C})$ and $T \subseteq M_J(\mathbb{C})$, define

$$S \boxtimes T = \text{span}_{\mathbb{C}}\{A \otimes B \mid A \in S, B \in T\} \subseteq M_{I \times J}(\mathbb{C}). \quad (19.637)$$

This is the tensor-product subspace used below, rather than a conjunction of two one-sided containment conditions.

Theorem 19.10.1.14 (Coefficient criterion for the Kronecker-product submodule). Lean ✓ Stmt For $X \in M_{I \times J}(\mathbb{C})$,

$$X \in S \boxtimes T \iff X_{rs}^L \in S \text{ for every } r, s \in J, \quad (19.638)$$

$$\text{and } X_{ij}^R \in T \text{ for every } i, j \in I. \quad (19.639)$$

Proof. ✓ Proof For $A \in S$ and $B \in T$, direct evaluation gives

$$(A \otimes B)_{rs}^L = B_{rs} A, \quad (A \otimes B)_{ij}^R = A_{ij} B. \quad (19.640)$$

Thus the forward implication extends by linearity. Conversely, expand the right blocks in a basis of T . Applying the dual coordinate functionals to those blocks produces matrices in S from the left-slice hypothesis, and the resulting finite sum reconstructs X as a sum of Kronecker products from S and T . $\square$

Theorem 19.10.1.15 (The one-sided tensor submodule as a Kronecker span). [Lean](#) [✓ Stmt](#) Let $I = \{0, \dots, p-1\}$ and $J = \{0, \dots, q-1\}$. For every submodule $S \subseteq M_p(\mathbb{C})$,

$$S \boxtimes M_q(\mathbb{C}) = \{X \in M_{p \times q}(\mathbb{C}) \mid X_{rs}^L \in S \text{ for all } r, s\}. \quad (19.641)$$

Thus the two-sided Kronecker span extends the existing left-oriented tensor-submodule construction.

Proof. [✓ Proof](#) In Theorem 19.10.1.14, take the right submodule to be the full matrix space. The right-block condition is then automatic, leaving precisely the left-slice criterion. $\square$

Theorem 19.10.1.16 (Kronecker-submodule criterion for right support). [Lean](#) [✓ Stmt](#) For every star-subalgebra $\mathcal{C} \subseteq M_J(\mathbb{C})$,

$$\text{Spp}_R(\mathcal{A}) \subseteq \mathcal{C} \iff \mathcal{A} \subseteq M_I(\mathbb{C}) \boxtimes \mathcal{C}. \quad (19.642)$$

Proof. [✓ Proof](#) The coefficient criterion reduces the right-hand containment to membership of every right block in $\mathcal{C}$. This is equivalent to the left-hand inclusion by Theorem 19.10.1.11. $\square$

Theorem 19.10.1.17 (Two-sided support-algebra containment). [Lean](#) [✓ Stmt](#) Every bipartite star-subalgebra satisfies the literal two-sided containment

$$\mathcal{A} \subseteq \text{Spp}_L(\mathcal{A}) \boxtimes \text{Spp}_R(\mathcal{A}). \quad (19.643)$$

This is the full complex matrix-algebra specialization of the support-algebra containment in Schumacher–Werner Section 4.3 and of Gross–Nesme–Vogts–Werner equation (22). The latter follows after applying Lemma 7 to both tensor factors. The displayed assertion is an inclusion in an actual tensor-product span, not merely the conjunction of the two one-sided inclusions. The MPU appendix applies this construction to $\mathcal{R}_{2x}$ in [CPGSV17, lines 2320–2334].

Proof. [✓ Proof](#) If $X \in \mathcal{A}$, every left coefficient of X lies in the left support algebra and every right coefficient lies in the right support algebra. Theorem 19.10.1.14 places X in their Kronecker-product submodule. $\square$

Theorem 19.10.1.18 (Left Kronecker commutation is coefficientwise). [Lean](#) [✓ Stmt](#) For $B \in M_I(\mathbb{C})$ and $X \in M_{I \times J}(\mathbb{C})$,

$$[B \otimes \mathbf{1}_J, X] = 0 \iff [B, X_{rs}^L] = 0 \text{ for every } r, s \in J. \quad (19.644)$$

Proof. [✓ Proof](#) For $i, j \in I$ and $r, s \in J$, the two product entries are

$$[(B \otimes \mathbf{1}_J)X]_{(i,r),(j,s)} = \sum_{k \in I} B_{ik} X_{(k,r),(j,s)} = (BX_{rs}^L)_{ij}, \quad (19.645)$$

$$[X(B \otimes \mathbf{1}_J)]_{(i,r),(j,s)} = \sum_{k \in I} X_{(i,r),(k,s)} B_{kj} = (X_{rs}^L B)_{ij}. \quad (19.646)$$

Equality of the two products is therefore equivalent to $[B, X_{rs}^L] = 0$ for every $r, s \in J$. $\square$

Theorem 19.10.1.19 (Right Kronecker commutation is coefficientwise). [Lean](#) [✓ Stmt](#) For $C \in M_J(\mathbb{C})$ and $X \in M_{I \times J}(\mathbb{C})$,

$$[\mathbf{1}_I \otimes C, X] = 0 \iff [C, X_{ij}^R] = 0 \text{ for every } i, j \in I. \quad (19.647)$$

Proof. ✓ Proof For $i, j \in I$ and $r, s \in J$, the two product entries are

$$[(\mathbf{1}_I \otimes C)X]_{(i,r),(j,s)} = \sum_{t \in J} C_{rt} X_{(i,t),(j,s)} = (CX_{ij}^R)_{rs}, \quad (19.648)$$

$$[X(\mathbf{1}_I \otimes C)]_{(i,r),(j,s)} = \sum_{t \in J} X_{(i,r),(j,t)} C_{ts} = (X_{ij}^R C)_{rs}. \quad (19.649)$$

Equality of the two products is therefore equivalent to $[C, X_{ij}^R] = 0$ for every $i, j \in I$. □

Theorem 19.10.1.20 (Left support-algebra centralizer). Lean ✓ Stmt For $B \in M_I(\mathbb{C})$,

$$B \otimes \mathbf{1}_J \in \mathcal{A}' \iff B \in \text{Spp}_L(\mathcal{A})'. \quad (19.650)$$

Equivalently,

$$\text{Spp}_L(\mathcal{A})' = \{B \in M_I(\mathbb{C}) \mid B \otimes \mathbf{1}_J \in \mathcal{A}'\}. \quad (19.651)$$

Proof. ✓ Proof By Theorem 19.10.1.18, commutation of $B \otimes \mathbf{1}_J$ with every $X \in \mathcal{A}$ is equivalent to commutation of B with every left coefficient. The coefficient set is closed under adjoints because $\mathcal{A}$ is a star-subalgebra. The universal property of the generated star-subalgebra then extends this commutation relation to all of $\text{Spp}_L(\mathcal{A})$; the converse follows from generator membership. □

Theorem 19.10.1.21 (Right support-algebra centralizer). Lean ✓ Stmt For $C \in M_J(\mathbb{C})$,

$$\mathbf{1}_I \otimes C \in \mathcal{A}' \iff C \in \text{Spp}_R(\mathcal{A})'. \quad (19.652)$$

Equivalently,

$$\text{Spp}_R(\mathcal{A})' = \{C \in M_J(\mathbb{C}) \mid \mathbf{1}_I \otimes C \in \mathcal{A}'\}. \quad (19.653)$$

Proof. ✓ Proof By Theorem 19.10.1.19, commutation of $\mathbf{1}_I \otimes C$ with every $X \in \mathcal{A}$ is equivalent to commutation of C with every right coefficient. Theorem 19.10.1.1 shows that the right coefficient set is closed under adjoints. The universal property of the generated star-subalgebra extends commutation to all of $\text{Spp}_R(\mathcal{A})$; the converse follows from generator membership. □

Theorem 19.10.1.22 (Commutation of overlapping support algebras). Lean ✓ Stmt Let I, J , and K be finite sets, and let

$$\mathcal{A}_1 \subseteq M_{I \times J}(\mathbb{C}), \quad \mathcal{A}_2 \subseteq M_{J \times K}(\mathbb{C}) \quad (19.654)$$

be star-subalgebras. Suppose that, for every $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$,

$$(X \otimes \mathbf{1}_K)(\mathbf{1}_I \otimes Y) = (\mathbf{1}_I \otimes Y)(X \otimes \mathbf{1}_K). \quad (19.655)$$

Then the two support algebras on the common factor commute:

$$RL = LR \quad \text{for every } R \in \text{Spp}_R(\mathcal{A}_1), \quad L \in \text{Spp}_L(\mathcal{A}_2). \quad (19.656)$$

This is the full complex matrix-algebra specialization of Schumacher–Werner Lemma *sppcomm* [SW04, lines 1174–1194] and GNVW Lemma *sppcomm* [GNVW12, Section 7.1, lines 1221–1246].

Local fixes. For the generated C^* -algebras, the input must be closed under the involution; see [con26w]. In both printed coefficient proofs, the basis element e'_ν belongs to the third factor, rather than the second factor as printed; see [con26o].

Scope restriction (full matrix factors). The cited lemmas permit arbitrary finite-dimensional C^* -algebra factors, whereas the statement above treats full complex matrix algebras. This restriction suffices for the MPU application in [CPGSV17, lines 2300–2306 and 2313–2334]; the general-factor statement remains open as recorded in [con26k].

Proof. ✓ Proof For $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, $i, i' \in I$, and $k, k' \in K$, applying Theorem [LTC26, “Middle-factor coefficients of commuting overlapping operators”] to (19.655) gives

$$X_{ii'}^R Y_{kk'}^L = Y_{kk'}^L X_{ii'}^R. \quad (19.657)$$

Applying the same identity to $Y^\dagger \in \mathcal{A}_2$, with k and k' exchanged, also gives commutation with $(Y_{kk'}^L)^\dagger$. Hence every generator of $\text{Spp}_R(\mathcal{A}_1)$ lies in the star-subalgebra commuting with all left coefficients of $\mathcal{A}_2$. Passing to the generated star-subalgebra yields

$$LR = RL, \quad L^\dagger R = RL^\dagger \quad (19.658)$$

for every $R \in \text{Spp}_R(\mathcal{A}_1)$ and every left coefficient L of $\mathcal{A}_2$.

Since $\text{Spp}_R(\mathcal{A}_1)$ is closed under adjoints, applying the first equality in (19.658) to $R^\dagger$ shows that each left coefficient commutes with both R and $R^\dagger$. Passing to the star-subalgebra generated by the left coefficients gives (19.656). $\square$

Definition 19.10.1.23 (Translation of a finite region). Lean Lean Lean Lean Lean Lean Lean

✓ Stmt For a finite region $\Lambda \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, define

$$\Lambda + a = \{i + a \mid i \in \Lambda\}. \quad (19.659)$$

Thus

$$j \in \Lambda + a \iff j - a \in \Lambda, \quad (19.660)$$

$$\Lambda + 0 = \Lambda, \quad (19.661)$$

$$(\Lambda + a) + b = \Lambda + (a + b). \quad (19.662)$$

The map $\Lambda \mapsto \Lambda + a$ is injective. If $\Lambda \subseteq \Gamma$, then $\Lambda + a \subseteq \Gamma + a$.

Definition 19.10.1.24 (Sumset of finite regions). Lean Lean Lean Lean Lean Lean Lean Lean

Lean Lean Lean Lean Lean ✓ Stmt For finite regions $\Lambda, \mathcal{N} \subset \mathbb{Z}$, define their sumset by

$$\Lambda + \mathcal{N} = \{i + a \mid i \in \Lambda, a \in \mathcal{N}\}. \quad (19.663)$$

Hence $j \in \Lambda + \mathcal{N}$ exactly when $j = i + a$ for some $i \in \Lambda$ and $a \in \mathcal{N}$. The sumset is monotone in both arguments, either empty argument gives the empty region, and the singleton $\{0\}$ is an identity. A singleton displacement recovers translation,

$$\Lambda + \{a\} = \Lambda + a. \quad (19.664)$$

Translation commutes with either argument, and sumsets are associative:

$$(\Lambda + a) + \mathcal{N} = (\Lambda + \mathcal{N}) + a, \quad (19.665)$$

$$\Lambda + (\mathcal{N} + a) = (\Lambda + \mathcal{N}) + a, \quad (19.666)$$

$$(\Lambda + \mathcal{N}) + \mathcal{M} = \Lambda + (\mathcal{N} + \mathcal{M}). \quad (19.667)$$

These identities support composition of propagation bounds. This is the finite region $\Lambda + \mathcal{N}$ in the propagation condition of the QCA appendix of [CPGSV17].

Definition 19.10.1.25 (Translation of sites and configurations). Lean Lean Lean Lean Lean

✓ Stmt Translation gives bijections

$$t_{a, \Lambda}: \Lambda \longrightarrow \Lambda + a, \quad t_{a, \Lambda}(i) = i + a, \quad (19.668)$$

$$T_{a, \Lambda}: \text{Config}_d(\Lambda) \longrightarrow \text{Config}_d(\Lambda + a), \quad (T_{a, \Lambda}x)(j) = x(j - a). \quad (19.669)$$

Their inverses translate sites and configurations by $-a$.

Definition 19.10.1.26 (Finite-region translation of local observables). [Lean](#) [Lean](#) [✓ Stmt](#)
Relabelling configurations defines a star-algebra equivalence

$$\tau_{a,\Lambda}: \mathcal{A}_\Lambda \longrightarrow \mathcal{A}_{\Lambda+a}, \quad (19.670)$$

$$\tau_{a,\Lambda}(A)(x, y) = A(T_{a,\Lambda}^{-1}x, T_{a,\Lambda}^{-1}y). \quad (19.671)$$

This is the finite-region translation associated with the lattice translation $i \mapsto i + a$.

Lemma 19.10.1.27 (Norm preservation of finite-region translation). [\[Learn\]](#) [\[Learn\]](#) [✓ Stmt](#) Let d be a nonnegative integer, let $\Lambda \subset \mathbb{Z}$ be finite, and let $a \in \mathbb{Z}$. For every $A \in \mathcal{A}_\Lambda$,

$$\|\tau_{a,\Lambda}(A)\| = \|A\|. \quad (19.672)$$

Consequently, $\tau_{a,\Lambda}$ is an isometry from $\mathcal{A}_\Lambda$ onto $\mathcal{A}_{\Lambda+a}$.

Proof. ✔ Proof Translation bijectively relabels the configuration basis. The resulting star-algebra equivalence between the two finite matrix algebras preserves the operator norm. Applying this identity to $A - B$ and using linearity gives

$$\|\tau_{a,\Lambda}(A) - \tau_{a,\Lambda}(B)\| = \|\tau_{a,\Lambda}(A - B)\| = \|A - B\|, \quad (19.673)$$

so distances are preserved.

Lemma 19.10.1.28 (Finite-region translation action laws). [Lean] [Lean] [Lean] [Lean] [Lean] [Lean]
[Lean] [Lean] [Stmnt] *Translation by zero acts trivially on sites, configurations, and finite-region observables. Successive translations by a and b agree with translation by $a + b$. Pointwise, for $i \in \Lambda$, $x, y \in \text{Config}_i(\Lambda)$, and $A \in \mathcal{A}_\Lambda$,*

$$t_{0,\Lambda}(i) = i, \quad (19.674)$$

$$t_{b,\Lambda+a}(t_{a,\Lambda}(i)) = t_{a+b,\Lambda}(i), \quad (19.675)$$

$$(T_{a,\Lambda}x)(t_{a,\Lambda}(i)) = x(i), \quad (19.676)$$

$$(T_{b,\Lambda+a}T_{a,\Lambda}x)(t_{b,\Lambda+a}(t_{a,\Lambda}(i))) = (T_{a+b,\Lambda}x)(t_{a+b,\Lambda}(i)), \quad (19.677)$$

$$(\tau_{a,\Lambda}(A))(T_{a,\Lambda}x, T_{a,\Lambda}y) = A(x, y), \quad (19.678)$$

$$(\tau_{b,\Lambda+a}\tau_{a,\Lambda}(A))(T_{b,\Lambda+a}T_{a,\Lambda}x, T_{b,\Lambda+a}T_{a,\Lambda}y) = (\tau_{a+b,\Lambda}(A))(T_{a+b,\Lambda}x, T_{a+b,\Lambda}y). \quad (19.679)$$

These identities express the additive action without identifying the distinct configuration spaces before applying their canonical translation bijections.

Proof. ✓ Proof The site identities follow from

$$(i + a) + b = i + (a + b). \quad (19.680)$$

Relabelling a configuration is precomposition by the inverse site translation, so evaluation at the corresponding translated site gives

$$(T_{a \wedge} x)(t_{a \wedge}(i)) = x(i). \quad (19.681)$$

Applying this identity successively proves the configuration action laws. Finally, the local observable translation relabels both matrix indices by $T_{a,\Lambda}^{-1}$; evaluating on translated configurations therefore gives $A(x, y)$, and applying this entrywise identity successively proves the local observable action laws. $\square$

Lemma 19.10.1.29 (Naturality of finite-region translation). [Lean](#) [Lean](#) [✓ Stmt](#) If $\Lambda \subseteq \Gamma$, then for every $A \in \mathcal{A}_\Lambda$,

$$\tau_{a,\Gamma}(\iota_{\Lambda,\Gamma}(A)) = \iota_{\Lambda+a,\Gamma+a}(\tau_{a,\Lambda}(A)). \quad (19.682)$$

Proof. [✓ Proof](#) The lattice translation identifies $\Gamma \setminus \Lambda$ with $(\Gamma + a) \setminus (\Lambda + a)$. Hence relabelling configurations carries the identity on the first complement to the identity on the second complement, and

$$\tau_{a,\Gamma}(A \otimes \mathbf{1}_{\Gamma \setminus \Lambda}) = \tau_{a,\Lambda}(A) \otimes \mathbf{1}_{(\Gamma+a) \setminus (\Lambda+a)}. \quad (19.683)$$

□

Definition 19.10.1.30 (Algebraic local algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The finite-region local algebras form a directed system under the maps $\iota_{\Lambda,\Gamma}$. Its algebraic direct limit is the algebraic local algebra

$$\mathcal{A}_{\text{loc}} = \lim_{\Lambda \in \mathbb{Z}} \mathcal{A}_\Lambda. \quad (19.684)$$

For every finite region Λ , write $\iota_\Lambda: \mathcal{A}_\Lambda \rightarrow \mathcal{A}_{\text{loc}}$ for the canonical inclusion. Whenever $\Lambda \subseteq \Gamma$, these inclusions satisfy

$$\iota_\Gamma \circ \iota_{\Lambda,\Gamma} = \iota_\Lambda. \quad (19.685)$$

This is the union $\bigcup_{\Lambda \in \mathbb{Z}} \mathcal{A}_\Lambda$ in the Appendix of [\[CPGSV17\]](#). No norm completion is taken.

Definition 19.10.1.31 (Finite-chain realization of a periodic MPU). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For $a \in \mathbb{Z}$ and $N \in \mathbb{N}$, let $I_{a,N} = [a, a + N) \cap \mathbb{Z}$. The increasing bijection $\{0, \dots, N-1\} \simeq I_{a,N}$ identifies the periodic configuration basis with $\text{Config}_d(I_{a,N})$. In this basis, write $U_a^{(N)}$ for the matrix obtained by simultaneously reindexing the rows and columns of $U^{(N)}$.

Definition 19.10.1.32 (Finite-chain MPU conjugation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $\mathcal{U}$ is an MPU and $N > 1$, then $U_a^{(N)}$ is unitary and defines the star-algebra automorphism

$$\text{Ad}_{U_a^{(N)}}(B) = U_a^{(N)} B U_a^{(N)\dagger}. \quad (19.686)$$

The factor order is the one in [\[CPGSV17, lines 2300–2306\]](#).

Proof. [✓ Proof](#) Simultaneous reindexing preserves both unitarity equations for $U^{(N)}$. Conjugation by the resulting unitary preserves multiplication, the unit, complex scalars, and the adjoint. □

Definition 19.10.1.33 (Finite-chain term on a local observable). [Lean](#) [Lean](#) [✓ Stmt](#) Let $\Lambda \subseteq I_{a,N}$ and $A \in \mathcal{A}_\Lambda$. Set $A_L = \iota_{\Lambda, I_{a,N}}(A)$. For $N > 1$, the finite-chain term in [\[CPGSV17, lines 2300–2306\]](#) is the star-algebra map

$$\mathcal{A}_\Lambda \longrightarrow \mathcal{A}_{\text{loc}}, \quad (19.687)$$

$$A \longmapsto \iota_{I_{a,N}}(U_a^{(N)} A_L U_a^{(N)\dagger}). \quad (19.688)$$

No stabilization in N is asserted.

Definition 19.10.1.34 (Operator norm on the local algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
 Assume $d > 0$. If a local observable $A \in \mathcal{A}_{\text{loc}}$ is represented by $A_\Lambda \in \mathcal{A}_\Lambda$ on a finite region Λ , define its operator norm by

$$\|A\| = \|A_\Lambda\|. \quad (19.689)$$

This does not depend on the finite region or representative. It is the norm whose topology is completed in the Appendix of [CPGSV17] to obtain the quasi-local algebra.

Proof. [✓ Proof](#) If A_Λ and A_Γ represent the same local observable, then the defining relation for the inductive limit gives a finite region $\Delta \supseteq \Lambda \cup \Gamma$ such that

$$\iota_{\Lambda, \Delta}(A_\Lambda) = \iota_{\Gamma, \Delta}(A_\Gamma). \quad (19.690)$$

Both inclusions preserve the operator norm, so $\|A_\Lambda\| = \|A_\Gamma\|$. □

Definition 19.10.1.35 (Normed star algebra of local observables). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
 For $d > 0$, the local algebra with the operator norm is a normed complex star algebra. No completeness assertion is made.

Proof. [✓ Proof](#) Choose finite-region representatives of the local observables and enlarge their regions to a common finite region. The norm axioms, the product inequality, and compatibility with complex scalar multiplication then follow from the corresponding properties of the finite-region matrix algebra. □

Lemma 19.10.1.36 (C-star identity for local observables). [Lean](#) [✓ Stmt](#) For $d > 0$ and every local observable A ,

$$\|A^*A\| = \|A\|^2. \quad (19.691)$$

Proof. [✓ Proof](#) Choose a finite region Λ and a representative $A_\Lambda \in \mathcal{A}_\Lambda$. Then

$$\|A^*A\| = \|A_\Lambda^*A_\Lambda\| = \|A_\Lambda\|^2 = \|A\|^2. \quad (19.692)$$

□

Definition 19.10.1.37 (C-star axiom structure on local observables). [Lean](#) [✓ Stmt](#) The normed star algebra of local observables satisfies the C-star axiom. This structure does not include completeness.

Proof. [✓ Proof](#) The C-star identity gives the required norm inequality in the reverse direction; submultiplicativity gives the other direction. □

Lemma 19.10.1.38 (Local observables embed isometrically). [Lean](#) [✓ Stmt](#) For $d > 0$ and every finite region Λ , the canonical map

$$\iota_\Lambda : \mathcal{A}_\Lambda \longrightarrow \mathcal{A}_{\text{loc}} \quad (19.693)$$

is an operator-norm isometry.

Proof. [✓ Proof](#) The norm of $\iota_\Lambda(A)$ is defined to be the operator norm of its representative $A \in \mathcal{A}_\Lambda$. Hence $\|\iota_\Lambda(A)\| = \|A\|$; applying this equality to differences proves preservation of distance. □

Theorem 19.10.1.43 (Algebraic-local relative commutant). [Lean](#) [✓ Stmt](#) Assume $d > 0$. A local observable $X \in \mathcal{A}_{\text{loc}}$ is supported in a finite region Λ if and only if it commutes with every local observable having a finite support disjoint from Λ :

$$X \in \iota_{\Lambda}(\mathcal{A}_{\Lambda}) \iff \forall \Gamma, \forall Y, (\Gamma \cap \Lambda = \emptyset \wedge Y \in \iota_{\Gamma}(\mathcal{A}_{\Gamma})) \implies XY = YX. \quad (19.699)$$

This is an algebraic-local statement. It does not assert the corresponding characterization in the norm completion and is not stated by Schumacher–Werner. Their Theorem 6 gives the generalized Margolus structure, and Corollary 7 says that the inverse of a nearest-neighbor QCA exists and is a nearest-neighbor QCA.

Proof. [✓ Proof](#) The forward implication is commutation of disjointly supported observables. Conversely, choose a finite support Δ of X and represent X by $A \in \mathcal{A}_{\Lambda \cup \Delta}$. The hypothesis says that A commutes with every observable on $(\Lambda \cup \Delta) \setminus \Lambda$. Since $d > 0$, the canonical map from $\mathcal{A}_{\Lambda \cup \Delta}$ into $\mathcal{A}_{\text{loc}}$ is injective, so this commutation already holds in the finite-region algebra. The finite relative-commutant theorem gives $A = \iota_{\Lambda, \Lambda \cup \Delta}(A_{\Lambda})$ for some $A_{\Lambda} \in \mathcal{A}_{\Lambda}$. Hence $X = \iota_{\Lambda}(A_{\Lambda})$. $\square$

Definition 19.10.1.44 (Algebraic-local translation homomorphism). [Lean](#) [Lean](#) [✓ Stmt](#) For $a \in \mathbb{Z}$, finite-region translations induce a star-algebra homomorphism

$$\tau_a : \mathcal{A}_{\text{loc}} \longrightarrow \mathcal{A}_{\text{loc}} \quad (19.700)$$

determined on every finite-region representative by

$$\tau_a(\iota_{\Lambda}(A)) = \iota_{\Lambda+a}(\tau_{a,\Lambda}(A)). \quad (19.701)$$

Proof. [✓ Proof](#) Naturality gives, for $\Lambda \subseteq \Gamma$,

$$\iota_{\Gamma+a}(\tau_{a,\Gamma}(\iota_{\Lambda,\Gamma}(A))) = \iota_{\Gamma+a}(\iota_{\Lambda+a,\Gamma+a}(\tau_{a,\Lambda}(A))) \quad (19.702)$$

$$= \iota_{\Lambda+a}(\tau_{a,\Lambda}(A)). \quad (19.703)$$

Hence the finite-region maps form a compatible family and induce the stated homomorphism on the algebraic direct limit. $\square$

Lemma 19.10.1.45 (Action laws for algebraic-local translation homomorphisms). [Lean](#) [Lean](#) [✓ Stmt](#) The homomorphisms satisfy

$$\tau_0 = \text{id}, \quad (19.704)$$

$$\tau_b \circ \tau_a = \tau_{a+b}. \quad (19.705)$$

Proof. [✓ Proof](#) On a representative $A \in \mathcal{A}_{\Lambda}$, the two sides of the composition law are represented on the equal finite regions $(\Lambda + a) + b$ and $\Lambda + (a + b)$. The finite-region action identity therefore gives

$$\tau_b(\tau_a(\iota_{\Lambda}(A))) = \iota_{\Lambda+(a+b)}(\tau_{a+b,\Lambda}(A)) = \tau_{a+b}(\iota_{\Lambda}(A)). \quad (19.706)$$

The zero law is proved in the same way. Equality on all finite-region representatives determines equality on the algebraic direct limit. $\square$

Definition 19.10.1.46 (Translation of algebraic-local observables). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every $a \in \mathbb{Z}$, the homomorphism τ_a is a star-algebra automorphism of $\mathcal{A}_{\text{loc}}$, and on finite-region representatives

$$\tau_a(\iota_{\Lambda}(A)) = \iota_{\Lambda+a}(\tau_{a,\Lambda}(A)). \quad (19.707)$$

Proof. ✓ Proof The homomorphism action laws give

$$\tau_{-a} \circ \tau_a = \tau_0 = \text{id}, \quad (19.708)$$

$$\tau_a \circ \tau_{-a} = \tau_0 = \text{id}. \quad (19.709)$$

Thus τ_a is bijective with inverse τ_{-a} . The formula on finite-region representatives is the defining formula for the homomorphism. $\square$

Lemma 19.10.1.47 (Algebraic-local translation action laws). Lean Lean Lean ✓ Stmt *The automorphisms form an additive action of $\mathbb{Z}$:*

$$\tau_0 = \text{id}, \quad (19.710)$$

$$\tau_b \circ \tau_a = \tau_{a+b}, \quad (19.711)$$

$$\tau_a^{-1} = \tau_{-a}. \quad (19.712)$$

Proof. ✓ Proof The first two identities are the homomorphism action laws expressed as identities of star-algebra automorphisms. Taking $b = -a$ in the composition law gives

$$\tau_{-a} \circ \tau_a = \tau_0 = \text{id}, \quad (19.713)$$

$$\tau_a \circ \tau_{-a} = \tau_0 = \text{id}, \quad (19.714)$$

which is the inverse identity. $\square$

Lemma 19.10.1.48 (Exact translation of algebraic-local supports). Lean Lean ✓ Stmt *A local observable A is supported in a finite region Λ if and only if $\tau_a(A)$ is supported in $\Lambda + a$.*

Proof. ✓ Proof If $A = \iota_\Lambda(A_\Lambda)$, then

$$\tau_a(A) = \iota_{\Lambda+a}(\tau_{a,\Lambda}(A_\Lambda)), \quad (19.715)$$

which proves the forward implication. Apply the same argument to $\tau_a(A)$ with displacement $-a$, and use $\tau_{-a}\tau_a = \text{id}$ and $(\Lambda + a) - a = \Lambda$, for the reverse implication. $\square$

Lemma 19.10.1.49 (Norm isometry of algebraic-local translation). Lean Lean ✓ Stmt *If $d > 0$, then every translation preserves the operator norm and distance:*

$$\|\tau_a(A)\| = \|A\|, \quad (19.716)$$

$$\|\tau_a(A) - \tau_a(B)\| = \|A - B\|. \quad (19.717)$$

Proof. ✓ Proof Choose $A_\Lambda \in \mathcal{A}_\Lambda$ with $A = \iota_\Lambda(A_\Lambda)$. Finite-region translation is a star-algebra equivalence of matrix algebras, hence preserves the operator norm. Therefore

$$\|\tau_a(A)\| = \|\tau_{a,\Lambda}(A_\Lambda)\| = \|A_\Lambda\| = \|A\|. \quad (19.718)$$

Applying this equality to $A - B$ gives distance preservation. $\square$

Definition 19.10.1.50 (C-star structure on a completion). Lean Lean Lean Lean Lean Lean ✓ Stmt The norm completion of a complex normed star algebra satisfying the C-star identity carries the continuously extended involution and a C-star algebra structure. The canonical map into the completion is a star-algebra homomorphism.

Proof. ✓ Proof Extend the isometric involution continuously. The algebraic identities and the C-star identity hold on the dense canonical image and hence on the completion. $\square$

Definition 19.10.1.51 (Quasi-local observable algebra). [Lean](#) [Lean](#) [✓ Stmt](#) For $d > 0$, the quasi-local observable algebra is the operator-norm completion

$$\mathcal{A} = \overline{\mathcal{A}_{\text{loc}}}^{\|\cdot\|}. \quad (19.719)$$

The involution on $\mathcal{A}$ is the unique continuous extension of the involution on $\mathcal{A}_{\text{loc}}$. It makes $\mathcal{A}$ a complex C-star algebra, and the canonical star-algebra homomorphism

$$j: \mathcal{A}_{\text{loc}} \longrightarrow \mathcal{A} \quad (19.720)$$

is the completion map.

Proof. [✓ Proof](#) The involution is isometric on $\mathcal{A}_{\text{loc}}$, hence extends continuously to the completion. The identities $(A + B)^* = A^* + B^*$, $(AB)^* = B^*A^*$, and $(A^*)^* = A$ hold on the dense subalgebra $j(\mathcal{A}_{\text{loc}})$ and therefore on $\mathcal{A}$. The conjugate-linear identity $(cA)^* = \bar{c}A^*$ extends by the same argument. Density also extends

$$\|A^*A\| = \|A\|^2, \quad (19.721)$$

so the completed algebra satisfies the C-star identity. $\square$

Definition 19.10.1.52 (Finite-region quasi-local observable). [Lean](#) [✓ Stmt](#) For every finite region Λ , define

$$j_\Lambda: \mathcal{A}_\Lambda \longrightarrow \mathcal{A}, \quad j_\Lambda = j \circ \iota_\Lambda. \quad (19.722)$$

Lemma 19.10.1.53 (Finite-region observables in the quasi-local algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every finite region Λ , the map j_Λ is an injective isometry. If $\Lambda \subseteq \Gamma$, then

$$j_\Gamma \circ \iota_{\Lambda, \Gamma} = j_\Lambda. \quad (19.723)$$

Proof. [✓ Proof](#) The algebraic inclusion ι_Λ and the completion map j both preserve the norm. Their composite is therefore an isometry and hence injective. Compatibility follows from $\iota_\Gamma \circ \iota_{\Lambda, \Gamma} = \iota_\Lambda$. $\square$

Theorem 19.10.1.54 (Density of the canonical completion map). [Lean](#) [✓ Stmt](#) The canonical star-algebra homomorphism from a normed star algebra into its completion has dense range.

Proof. [✓ Proof](#) This is the defining density property of the completion. $\square$

Theorem 19.10.1.55 (Density of finite-region observables). [Lean](#) [Lean](#) [✓ Stmt](#) Algebraic local observables are dense in the quasi-local algebra. Equivalently,

$$\overline{j(\mathcal{A}_{\text{loc}})} = \overline{\bigcup_{\Lambda \in \mathbb{Z}} j_\Lambda(\mathcal{A}_\Lambda)} = \mathcal{A}. \quad (19.724)$$

Proof. [✓ Proof](#) Density of $j(\mathcal{A}_{\text{loc}})$ is the defining density property of the completion. Every algebraic local observable has a representative in some finite region, so

$$j(\mathcal{A}_{\text{loc}}) = \bigcup_{\Lambda \in \mathbb{Z}} j_\Lambda(\mathcal{A}_\Lambda). \quad (19.725)$$

Taking closures proves the second equality. This theorem concerns the union over all finite regions and makes no claim about the image of a fixed region. $\square$

The appendix introduces the quasi-local algebra and then invokes lattice translation in the definition of a QCA and in its covariance condition [CPGSV17, lines 2292–2306]. The following completion result and translation laws make this background structure explicit; they are not stated as separate results in the appendix.

Definition 19.10.1.56 (Extension of star-algebra equivalences to completions). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A and B be complex normed star algebras, and let $f: A \rightarrow B$ be a star-algebra equivalence. If f and f^{-1} are continuous, then f extends to a star-algebra equivalence

$$\hat{f}: \hat{A} \longrightarrow \hat{B} \quad (19.726)$$

between their norm completions. Its underlying map is the continuous extension of f . Writing j_A and j_B for the canonical maps into the completions, the extension satisfies

$$\hat{f}(j_A(x)) = j_B(f(x)). \quad (19.727)$$

Proof. [✓ Proof](#) Apply the completion functor to f and f^{-1} . On the dense canonical images their composites satisfy

$$\widehat{f^{-1}}(\hat{f}(j_A(x))) = j_A(x), \quad \widehat{f}(f^{-1}(j_B(y))) = j_B(y). \quad (19.728)$$

Continuity gives the two inverse identities on the completions. Multiplication, addition, scalar multiplication, and the star operation commute with the extension because the corresponding identities hold on the canonical images and both sides are continuous. $\square$

Definition 19.10.1.57 (Translation of quasi-local observables). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For $d > 0$ and every $a \in \mathbb{Z}$, algebraic-local translation extends to a star-algebra automorphism

$$\tau_a: \mathcal{A} \longrightarrow \mathcal{A} \quad (19.729)$$

of the quasi-local algebra. On the dense algebraic-local subalgebra and on finite-region observables it satisfies

$$\tau_a(j(A)) = j(\tau_a(A)), \quad (19.730)$$

$$\tau_a(j_\Lambda(A_\Lambda)) = j_{\Lambda+a}(\tau_{a,\Lambda}(A_\Lambda)). \quad (19.731)$$

Proof. [✓ Proof](#) Algebraic-local translation and its inverse translation by $-a$ are operator-norm isometries, hence continuous. Extend this equivalence to the norm completion. The extension identity gives

$$\tau_a(j(A)) = j(\tau_a(A)). \quad (19.732)$$

Substituting $A = \iota_\Lambda(A_\Lambda)$ and using the finite-region translation formula gives the second identity. $\square$

Lemma 19.10.1.58 (Quasi-local translation action laws). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For $d > 0$, the quasi-local automorphisms form an additive action of $\mathbb{Z}$:

$$\tau_0 = \text{id}, \quad (19.733)$$

$$\tau_b \circ \tau_a = \tau_{a+b}, \quad (19.734)$$

$$\tau_a^{-1} = \tau_{-a}. \quad (19.735)$$

Proof. [✓ Proof](#) On the canonical image of the algebraic-local algebra,

$$\tau_b(\tau_a(j(A))) = j(\tau_b(\tau_a(A))) = j(\tau_{a+b}(A)), \quad (19.736)$$

$$\tau_0(j(A)) = j(A). \quad (19.737)$$

Continuity extends these identities to the completion. Taking $(a, b) = (a, -a)$ and $(-a, a)$ identifies the inverse of τ_a with τ_{-a} . $\square$

Definition 19.10.1.59 (Translation covariance). [Lean](#) [✓ Stmt](#) For $d > 0$, a star-algebra automorphism ω of the quasi-local algebra $\mathcal{A}$ is translation covariant if

$$\omega \circ \tau_a = \tau_a \circ \omega \quad (19.738)$$

for every $a \in \mathbb{Z}$. This is the covariance condition in the definition of a one-dimensional QCA [CPGSV17, line 2298].

Theorem 19.10.1.60 (Pointwise characterization of translation covariance). [Lean](#) [✓ Stmt](#) For $d > 0$, a star-algebra automorphism ω of $\mathcal{A}$ is translation covariant if and only if

$$\omega(\tau_a(A)) = \tau_a(\omega(A)) \quad (19.739)$$

for every $a \in \mathbb{Z}$ and every $A \in \mathcal{A}$.

Proof. [✓ Proof](#) Two automorphisms are equal if and only if they agree on every observable. Apply this observation to the two composites in the definition. $\square$

Theorem 19.10.1.61 (Translation covariance from unit translation). [Lean](#) [✓ Stmt](#) For $d > 0$, a star-algebra automorphism ω of $\mathcal{A}$ commutes with every integer translation if and only if it commutes with unit translation:

$$\omega \circ \tau_1 = \tau_1 \circ \omega. \quad (19.740)$$

Proof. [✓ Proof](#) The forward implication follows by taking the translation step to be 1. Conversely, use two-sided integer induction. The zero case follows from $\tau_0 = \text{id}$. For the successor step, the additive action law gives $\tau_{i+1} = \tau_1 \circ \tau_i$, and

$$\omega \circ \tau_{i+1} = \omega \circ \tau_1 \circ \tau_i = \tau_1 \circ \omega \circ \tau_i \quad (19.741)$$

$$= \tau_1 \circ \tau_i \circ \omega = \tau_{i+1} \circ \omega. \quad (19.742)$$

For the predecessor step, $\tau_{-1} = \tau_1^{-1}$ transfers commutation to τ_{-1} , while $\tau_{i-1} = \tau_{-1} \circ \tau_i$, so

$$\omega \circ \tau_{i-1} = \omega \circ \tau_{-1} \circ \tau_i = \tau_{-1} \circ \omega \circ \tau_i \quad (19.743)$$

$$= \tau_{-1} \circ \tau_i \circ \omega = \tau_{i-1} \circ \omega. \quad (19.744)$$

$\square$

Theorem 19.10.1.62 (Identity and composition of translation-covariant automorphisms). [Lean](#) [Lean](#) [✓ Stmt](#) For $d > 0$, the identity automorphism of $\mathcal{A}$ is translation covariant. If ω and η are translation covariant, then so is the composite

$$A \mapsto \eta(\omega(A)), \quad (19.745)$$

which first applies ω and then η .

Proof. ✓ Proof The identity commutes pointwise with every translation. For the composite, translation covariance gives

$$\eta(\omega(\tau_a(A))) = \eta(\tau_a(\omega(A))) = \tau_a(\eta(\omega(A))). \quad (19.746)$$

□

Lemma 19.10.1.63 (Norm isometry of quasi-local translation). Lean Lean ✓ Stmt For $d > 0$, every quasi-local translation preserves the operator norm and distance:

$$\|\tau_a(A)\| = \|A\|, \quad (19.747)$$

$$\|\tau_a(A) - \tau_a(B)\| = \|A - B\|. \quad (19.748)$$

Proof. ✓ Proof A star-algebra equivalence between complex C-star algebras preserves the C-star norm. Since τ_a is linear,

$$\|\tau_a(A) - \tau_a(B)\| = \|\tau_a(A - B)\| = \|A - B\|. \quad (19.749)$$

□

Definition 19.10.1.64 (Finite-region support in the quasi-local algebra). Lean Lean Lean ✓ Stmt For $d > 0$, a quasi-local observable $x \in \mathcal{A}$ is supported in a finite region $\Lambda \subset \mathbb{Z}$ if it belongs to the image of the canonical embedding $j_\Lambda: \mathcal{A}_\Lambda \rightarrow \mathcal{A}$. Equivalently,

$$x \text{ is supported in } \Lambda \iff x \in j_\Lambda(\mathcal{A}_\Lambda) \quad (19.750)$$

$$\iff \text{there exists } A \in \mathcal{A}_\Lambda \text{ such that } j_\Lambda(A) = x. \quad (19.751)$$

In particular, $j_\Lambda(A)$ is supported in Λ for every $A \in \mathcal{A}_\Lambda$.

Proof. ✓ Proof These assertions follow directly from membership in the range of j_Λ . □

Lemma 19.10.1.65 (Monotonicity of quasi-local support). Lean ✓ Stmt Let $d > 0$. If $x \in \mathcal{A}$ is supported in Λ and $\Lambda \subseteq \Gamma$, then x is supported in Γ .

Proof. ✓ Proof Write $x = j_\Lambda(A)$. Enlarge A to $\mathcal{A}_\Gamma$ by tensoring with the identity on $\Gamma \setminus \Lambda$. Compatibility of the canonical embeddings identifies the image of this enlargement with x . □

Theorem 19.10.1.66 (Disjointly supported quasi-local observables commute). Lean Lean ✓ Stmt Let $d > 0$. If $x, y \in \mathcal{A}$ are supported in finite regions Λ and Γ with $\Lambda \cap \Gamma = \emptyset$, then

$$xy = yx. \quad (19.752)$$

Equivalently, for $A \in \mathcal{A}_\Lambda$ and $B \in \mathcal{A}_\Gamma$,

$$j_\Lambda(A)j_\Gamma(B) = j_\Gamma(B)j_\Lambda(A). \quad (19.753)$$

This is finite-support infrastructure for the quasi-local algebra in [CPGSV17, lines 2292–2300]; it is not stated there as a separate theorem.

Proof. ✓ Proof The completion map is a star-algebra homomorphism. Applying it to the algebraic-local identity gives

$$j(XY) = j(YX), \quad (19.754)$$

and finite-region support provides representatives $x = j_\Lambda(A)$ and $y = j_\Gamma(B)$. □

The following finite-dimensional averaging fact is standard and is not a claim from [CPGSV17]. It will be used to characterize exact local support by commutation with observables on the complementary tensor factor.

Definition 19.10.1.67 (Complementary-factor Weyl twirl). [Lean](#) [Lean](#) [✓ Stmt](#) Let S and C be finite sets, let $n = |C| > 0$, and choose an equivalence $e: C \simeq \mathbb{Z}/n\mathbb{Z}$ and a primitive n -th root of unity ζ . If $W_{a,b}$ is the corresponding reindexed Weyl operator on $\mathbb{C}^C$, define

$$\mathcal{E}_C(M) = \frac{1}{n^2} \sum_{a,b \in \mathbb{Z}/n\mathbb{Z}} (\mathbf{1}_S \otimes W_{a,b}) M (\mathbf{1}_S \otimes W_{a,b})^\dagger. \quad (19.755)$$

Lemma 19.10.1.68 (Complementary-factor Weyl average). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every $M \in \text{Mat}_{S \times C}(\mathbb{C})$,

$$\mathcal{E}_C(M) = \text{tr}_C(M) \otimes \frac{\mathbf{1}_C}{n} \quad (19.756)$$

$$= \left(\frac{1}{n} \text{tr}_C(M) \right) \otimes \mathbf{1}_C. \quad (19.757)$$

Proof. [✓ Proof](#) Split M into $C \times C$ blocks indexed by S . The Weyl average sends each block B to $(\text{tr } B/n) \mathbf{1}_C$, so its block trace is the corresponding entry of $\text{tr}_C(M)$. $\square$

Lemma 19.10.1.69 (Norm estimate for the complementary-factor twirl). [Lean](#) [Lean](#) [✓ Stmt](#) Let S be a nonempty finite set. The average is contractive in the operator norm. Moreover, if every complementary Weyl conjugation fixes X , then

$$\|\mathcal{E}_C(M) - X\| \leq \|M - X\|. \quad (19.758)$$

Proof. [✓ Proof](#) Unitary conjugation preserves the operator norm, and the coefficients of the finite average sum to one. Apply this contraction to $M - X$. $\square$

Theorem 19.10.1.70 (Exact support from outside commutation). [Lean](#) [✓ Stmt](#) Let $d > 0$, let $\Lambda \subset \mathbb{Z}$ be finite, and let $x \in \mathcal{A}$. Then

$$x \in \mathcal{A}_\Lambda \iff [x, y] = 0 \text{ for every } y \in \mathcal{A}_\Gamma \text{ and every finite } \Gamma \cap \Lambda = \emptyset. \quad (19.759)$$

This is standard infrastructure for the UHF algebra used in *TNLean*. It is not a theorem stated in [CPGSV17] or by Schumacher–Werner, *quant-ph/0405174*.

Proof. [✓ Proof](#) The forward implication follows from commutation of disjoint tensor factors. Conversely, approximate x by $A \in \mathcal{A}_\Delta$, where $\Lambda \subseteq \Delta$. Split $\Delta = \Lambda \sqcup (\Delta \setminus \Lambda)$ and average A over Weyl unitaries on the second factor. Every such conjugation fixes x by the commutation hypothesis, so unitary invariance and convexity give

$$\|\mathcal{E}_{\Delta \setminus \Lambda}(A) - x\| \leq \|A - x\|. \quad (19.760)$$

The Weyl identity writes the average as $B \otimes \mathbf{1}$ for some $B \in \mathcal{A}_\Lambda$. Thus elements of $\mathcal{A}_\Lambda$ approximate x arbitrarily well. The embedding of $\mathcal{A}_\Lambda$ is isometric, hence has closed range, and therefore $x \in \mathcal{A}_\Lambda$. $\square$

The following theorem gives the scalar center of the homogeneous UHF algebra. GNVW invoke this conclusion for the quasi-local algebra at [GNVW12, lines 1276–1282], specifically line 1279, but do not prove it there. The homogeneous construction in [CPGSV17, lines 2292–2298] likewise defines the norm completion without stating this conclusion separately. Schumacher–Werner give the corresponding homogeneous norm-completed construction in [SW04, lines 263–285], again without a separate scalar-center theorem.

Theorem 19.10.1.71 (Scalar center of the homogeneous quasi-local algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let $d > 0$ and let $z \in \mathcal{A}$. If

$$\forall \Gamma \subset \mathbb{Z} \text{ finite}, \quad \forall A \in \mathcal{A}_\Gamma, \quad z j_\Gamma(A) = j_\Gamma(A) z, \quad (19.761)$$

then there is a unique $c \in \mathbb{C}$ such that

$$z = c \mathbf{1}. \quad (19.762)$$

Consequently,

$$\{z \in \mathcal{A} \mid za = az \text{ for every } a \in \mathcal{A}\} = \mathbb{C} \mathbf{1}. \quad (19.763)$$

This statement concerns the fixed on-site dimension d ; it does not assert the site-dependent extension used in GNVW.

Proof. [Lean](#) [Stmnt](#) Apply Theorem 19.10.1.70 with $\Lambda = \emptyset$. The commutation hypothesis gives $z \in j_\emptyset(\mathcal{A}_\emptyset)$, so there is an $A \in \mathcal{A}_\emptyset$ with $z = j_\emptyset(A)$. Let $e: \mathbb{C} \xrightarrow{\sim} \mathcal{A}_\emptyset$ be the canonical isomorphism and write $A = e(c)$. If also $z = j_\emptyset(e(c'))$, then injectivity of $j_\emptyset$ and of e give

$$j_\emptyset(e(c)) = j_\emptyset(e(c')) \implies e(c) = e(c') \implies c = c'. \quad (19.764)$$

Thus c is the unique scalar for which $z = c \mathbf{1}$.

Write $Z(\mathcal{A})$ for the center. Since every $j_\Gamma(B)$ belongs to $\mathcal{A}$, the two inclusions are

$$z \in Z(\mathcal{A}) \implies (\forall \Gamma \subset \mathbb{Z} \text{ finite}, \forall B \in \mathcal{A}_\Gamma, z j_\Gamma(B) = j_\Gamma(B) z) \implies z \in \mathbb{C} \mathbf{1}, \quad (19.765)$$

$$z = c \mathbf{1} \quad (c \in \mathbb{C}) \implies (\forall a \in \mathcal{A}, (c \mathbf{1})a = a(c \mathbf{1})) \implies z \in Z(\mathcal{A}). \quad (19.766)$$

Hence $Z(\mathcal{A}) = \mathbb{C} \mathbf{1}$. □

The appendix requires that “there exists a finite subset $\mathcal{N} \subset \mathbb{Z}$ ” such that, for every finite $\Lambda \subset \mathbb{Z}$,

$$\omega(\mathcal{A}_\Lambda) \subset \mathcal{A}_{\Lambda + \mathcal{N}}. \quad (19.767)$$

This is the propagation condition stated at line 2298 of [CPGSV17]. The definitions below isolate this forward inclusion. They do not impose translation covariance or a propagation bound for ω^{-1} .

Definition 19.10.1.72 (Propagation within a fixed neighborhood). [Lean](#) [Stmnt](#) Let $d > 0$, let $\omega: \mathcal{A} \rightarrow \mathcal{A}$ be a star-algebra automorphism, and let $\mathcal{N} \subset \mathbb{Z}$ be finite. The automorphism ω propagates within $\mathcal{N}$ if, for every finite region Λ and every $x \in \mathcal{A}$,

$$x \text{ supported in } \Lambda \implies \omega(x) \text{ supported in } \Lambda + \mathcal{N}. \quad (19.768)$$

Definition 19.10.1.73 (Finite forward propagation). [Lean](#) [Stmnt](#) Let $d > 0$ and let $\omega: \mathcal{A} \rightarrow \mathcal{A}$ be a star-algebra automorphism. The automorphism ω has finite forward propagation if there exists a finite set $\mathcal{N} \subset \mathbb{Z}$ such that ω propagates within $\mathcal{N}$.

Lemma 19.10.1.74 (Equivalent formulations of forward propagation). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let $d > 0$, let $\omega: \mathcal{A} \rightarrow \mathcal{A}$ be a star-algebra automorphism, and let $\mathcal{N} \subset \mathbb{Z}$ be finite. The following conditions are equivalent:

$$\omega \text{ propagates within } \mathcal{N}; \quad (a)$$

$$\forall \Lambda \in \mathbb{Z}, \forall A \in \mathcal{A}_\Lambda, \quad \omega(j_\Lambda(A)) \text{ is supported in } \Lambda + \mathcal{N}; \quad (b)$$

$$\forall \Lambda \in \mathbb{Z}, \quad \omega(j_\Lambda(\mathcal{A}_\Lambda)) \subseteq j_{\Lambda + \mathcal{N}}(\mathcal{A}_{\Lambda + \mathcal{N}}); \quad (c)$$

$$\forall \Lambda \in \mathbb{Z}, \forall A \in \mathcal{A}_\Lambda, \exists B \in \mathcal{A}_{\Lambda + \mathcal{N}}, \quad \omega(j_\Lambda(A)) = j_{\Lambda + \mathcal{N}}(B). \quad (d)$$

Thus condition (c) is precisely the range-inclusion formula from line 2298 of [CPGSV17], expressed inside the quasi-local algebra, and condition (d) is its finite-region witness form.

Proof. ✓ Proof Every observable supported in Λ has the form $j_\Lambda(A)$, so (a) and (b) are equivalent. Replacing support by membership in the range of the canonical embedding gives (c). Finally, membership in $j_{\Lambda+\mathcal{N}}(\mathcal{A}_{\Lambda+\mathcal{N}})$ is equivalent to the existence of a representative B in that finite-region algebra, which gives (d). $\square$

Lemma 19.10.1.75 (Enlargement of propagation neighborhoods). Lean Lean ✓ Stmt Let $d > 0$ and let $\omega: \mathcal{A} \rightarrow \mathcal{A}$ be a star-algebra automorphism. If $\mathcal{N} \subseteq \mathcal{M}$ and ω propagates within $\mathcal{N}$, then ω propagates within $\mathcal{M}$. More explicitly, for every finite region Λ ,

$$\Lambda + \mathcal{N} \subseteq \Lambda + \mathcal{M}, \quad (19.769)$$

$$x \text{ supported in } \Lambda \implies \omega(x) \text{ supported in } \Lambda + \mathcal{N} \implies \omega(x) \text{ supported in } \Lambda + \mathcal{M}. \quad (19.770)$$

Consequently, if ω has finite forward propagation, then for every prescribed finite set $\mathcal{N} \subset \mathbb{Z}$ there is a finite set $\mathcal{M} \subset \mathbb{Z}$ such that

$$\mathcal{N} \subseteq \mathcal{M}, \quad (19.771)$$

$$\forall \Lambda \in \mathbb{Z}, \quad \omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+\mathcal{M}}. \quad (19.772)$$

This is the neighborhood-enlargement consequence of the propagation condition in Appendix line 2298 of [CPGSV17].

Proof. ✓ Proof Monotonicity of the sumset in its second argument gives $\Lambda + \mathcal{N} \subseteq \Lambda + \mathcal{M}$. Monotonicity of quasi-local support then enlarges the support of $\omega(x)$ from $\Lambda + \mathcal{N}$ to $\Lambda + \mathcal{M}$. For the final assertion, choose a finite propagation neighborhood $\mathcal{K}$ and take $\mathcal{M} = \mathcal{K} \cup \mathcal{N}$. $\square$

Lemma 19.10.1.76 (Composition of forward propagation bounds). Lean Lean ✓ Stmt Let $d > 0$, let $\omega, \eta: \mathcal{A} \rightarrow \mathcal{A}$ be star-algebra automorphisms, and let $\mathcal{N}, \mathcal{M} \subset \mathbb{Z}$ be finite. Suppose that ω propagates within $\mathcal{N}$ and η propagates within $\mathcal{M}$. The composite $\eta \circ \omega$ first applies ω and then η , so

$$(\eta \circ \omega)(x) = \eta(\omega(x)), \quad (19.773)$$

$$x \text{ supported in } \Lambda \implies \omega(x) \text{ supported in } \Lambda + \mathcal{N} \quad (19.774)$$

$$\implies \eta(\omega(x)) \text{ supported in } (\Lambda + \mathcal{N}) + \mathcal{M} = \Lambda + (\mathcal{N} + \mathcal{M}). \quad (19.775)$$

Hence $\eta \circ \omega$ propagates within $\mathcal{N} + \mathcal{M}$. In particular, if ω and η have finite forward propagation, then $\eta \circ \omega$ has finite forward propagation. This is the composition closure of the finite-neighborhood condition in Appendix line 2298 of [CPGSV17].

Proof. ✓ Proof Apply the two propagation bounds successively and use associativity of finite region sumsets. For the existential assertion, choose one finite propagation neighborhood for each automorphism and use their sumset. $\square$

Lemma 19.10.1.77 (Symmetric-interval normalization of finite propagation). Lean Lean Lean ✓ Stmt Every finite set $\mathcal{N} \subset \mathbb{Z}$ is contained in $[-R, R] \cap \mathbb{Z}$ for some $R \in \mathbb{N}$. This includes $\mathcal{N} = \emptyset$, for which one may take $R = 0$. If ω propagates within a fixed set $\mathcal{N}$, then R may be chosen so that

$$\mathcal{N} \subseteq [-R, R] \cap \mathbb{Z}, \quad (19.776)$$

$$\omega \text{ propagates within } [-R, R] \cap \mathbb{Z}. \quad (19.777)$$

Consequently, finite forward propagation admits a symmetric-interval witness:

$$\exists R \in \mathbb{N}, \quad \omega \text{ propagates within } [-R, R] \cap \mathbb{Z}. \quad (19.778)$$

Proof. ✓ Proof Let R be the maximum of $|n|$ over $n \in \mathcal{N}$, with value 0 when $\mathcal{N}$ is empty. Then $-R \leq n \leq R$ for every $n \in \mathcal{N}$, which gives $\mathcal{N} \subseteq [-R, R] \cap \mathbb{Z}$. Enlargement of propagation neighborhoods gives the fixed-neighborhood assertion. Finally, choose a finite propagation neighborhood and apply this enlargement to its symmetric interval. $\square$

Definition 19.10.1.78 (Two-sided finite-propagation predicates). Lean Lean ✓ Stmt Let $d > 0$, and let ω be a star-algebra automorphism of $\mathcal{A}$. A finite set $\mathcal{N} \subset \mathbb{Z}$ is a two-sided propagation neighborhood when

$$\omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+\mathcal{N}} \quad (19.779)$$

and

$$\omega^{-1}(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+\mathcal{N}} \quad (19.780)$$

for every finite $\Lambda \subset \mathbb{Z}$. The automorphism has two-sided finite propagation when ω and ω^{-1} each have a finite forward propagation neighborhood; the two neighborhoods need not initially agree.

These are derived convenience predicates. The definition in [CPGSV17, line 2298] and Schumacher–Werner, quant-ph/0405174, Definition 1, both define locality by the forward inclusion for ω alone. In particular, these predicates do not assert that forward locality implies locality of the inverse.

Lemma 19.10.1.79 (Common neighborhoods for two-sided propagation). Lean Lean Lean Lean Lean ✓ Stmt For a finite set $\mathcal{N} \subset \mathbb{Z}$, write

$$-\mathcal{N} = \{-a \mid a \in \mathcal{N}\}. \quad (19.781)$$

Reflection fixes every symmetric interval:

$$-([-R, R] \cap \mathbb{Z}) = [-R, R] \cap \mathbb{Z}. \quad (19.782)$$

If $\mathcal{N} \subseteq \mathcal{M}$, then every common two-sided bound by $\mathcal{N}$ is also a common bound by $\mathcal{M}$. More generally, if ω propagates within $\mathcal{N}$ and ω^{-1} propagates within $-\mathcal{N}$, then both propagate within

$$\mathcal{N} \cup (-\mathcal{N}). \quad (19.783)$$

Finally, any two-sided finite-propagation pair admits a common finite neighborhood and therefore a common symmetric-interval neighborhood. Thus there is an $R \in \mathbb{N}$ such that

$$\omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+([-R, R] \cap \mathbb{Z})} \quad (19.784)$$

and

$$\omega^{-1}(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+([-R, R] \cap \mathbb{Z})} \quad (19.785)$$

for every finite $\Lambda \subset \mathbb{Z}$.

Proof. ✓ Proof Negation reverses the two endpoint inequalities, so it preserves $[-R, R] \cap \mathbb{Z}$. Enlarge each supplied neighborhood using monotonicity. For reflected bounds, both inclusions into $\mathcal{N} \cup (-\mathcal{N})$ are immediate. For arbitrary finite forward and inverse neighborhoods, first take their union and then enlarge that common neighborhood to a symmetric interval. $\square$

Lemma 19.10.1.80 (Disjointness for reflected neighborhoods). [Lean](#) [✓ Stmt](#) Let $\Gamma, \Lambda, \mathcal{N} \subset \mathbb{Z}$ be finite. Then

$$\Gamma \cap (\Lambda + (-\mathcal{N})) = \emptyset \iff (\Gamma + \mathcal{N}) \cap \Lambda = \emptyset. \quad (19.786)$$

Proof. [✓ Proof](#) If $g + n = \lambda$ for $g \in \Gamma$, $n \in \mathcal{N}$, and $\lambda \in \Lambda$, then $g = \lambda + (-n) \in \Lambda + (-\mathcal{N})$, proving the forward implication. Conversely, if $g \in \Gamma$ and $g = \lambda + (-n)$ with $\lambda \in \Lambda$ and $n \in \mathcal{N}$, then $\lambda = g + n \in \Gamma + \mathcal{N}$. $\square$

Theorem 19.10.1.81 (Inverse propagation and inverse covariance). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $d > 0$, let ω be a star-algebra automorphism of $\mathcal{A}$, and let $\mathcal{N} \subset \mathbb{Z}$ be finite. If ω propagates within $\mathcal{N}$, then ω^{-1} propagates within $-\mathcal{N}$. Therefore finite forward propagation of ω implies finite forward propagation of ω^{-1} . Moreover, if ω is translation covariant, then so is ω^{-1} .

This is a consequence of the forward-only locality definition, not part of that definition. Schumacher–Werner, quant-ph/0405174, derive inverse locality through Lemma 5, Theorem 6, and Corollary 7 using the generalized Margolus structure. The proof here instead uses TNLear’s exact outside-commutant infrastructure.

Proof. [✓ Proof](#) Let $x \in \mathcal{A}_\Lambda$ and let $y \in \mathcal{A}_\Gamma$, where $\Gamma \cap (\Lambda + (-\mathcal{N})) = \emptyset$. Forward locality gives $\omega(y) \in \mathcal{A}_{\Gamma+\mathcal{N}}$. The regions $\Gamma + \mathcal{N}$ and Λ are disjoint, so x commutes with $\omega(y)$. Applying ω^{-1} shows that $\omega^{-1}(x)$ commutes with y . The exact outside-commutant characterization now gives

$$\omega^{-1}(x) \in \mathcal{A}_{\Lambda+(-\mathcal{N})}. \quad (19.787)$$

Choosing the reflected finite neighborhood proves the existential statement. For each lattice translation τ_a , multiplying the covariance identity on the left and right by ω^{-1} gives

$$\omega \circ \tau_a = \tau_a \circ \omega \implies \omega^{-1} \circ \tau_a = \tau_a \circ \omega^{-1}. \quad (19.788)$$

$\square$

Theorem 19.10.1.82 (Forward and two-sided finite propagation). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $d > 0$ and let ω be a star-algebra automorphism of $\mathcal{A}$. Then

$$\omega \text{ has finite forward propagation} \iff \omega \text{ has two-sided finite propagation}. \quad (19.789)$$

More precisely, a forward bound by a finite set $\mathcal{N}$ gives an inverse bound by $-\mathcal{N}$, so both maps propagate within the enlarged common neighborhood

$$\mathcal{N} \cup (-\mathcal{N}). \quad (19.790)$$

This statement remains valid when $\mathcal{N} = \emptyset$. For an arbitrary finite forward-propagation witness, there is therefore a common finite neighborhood for ω and ω^{-1} , and it may be enlarged to a symmetric interval $[-R, R] \cap \mathbb{Z}$.

This equivalence is a derived convenience theorem. It does not change the forward-only locality definition at line 2298 of [CPGSV17] or in Schumacher–Werner, Definition 1.

Proof. [✓ Proof](#) Derived inverse locality sends a forward bound by $\mathcal{N}$ to an inverse bound by $-\mathcal{N}$. Enlarging both bounds to $\mathcal{N} \cup (-\mathcal{N})$ gives the fixed-neighborhood statement. At the existential level, forward finite propagation gives finite propagation of the inverse, hence the two-sided predicate. The converse is its first conjunct. Finally, apply the common-neighborhood and symmetric-interval enlargement results to the resulting two-sided pair. $\square$

Definition 19.10.1.83 (One-dimensional quantum cellular automaton). [Lean](#) [✓ Stmt](#) Let $d > 0$ and let $\omega: \mathcal{A} \rightarrow \mathcal{A}$ be a star-algebra automorphism. The automorphism ω is a one-dimensional quantum cellular automaton if it is translation covariant and has finite forward propagation. Thus there exists a finite set $\mathcal{N} \subset \mathbb{Z}$ such that, for every finite $\Lambda \subset \mathbb{Z}$,

$$\omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+\mathcal{N}}. \quad (19.791)$$

This is the definition at line 2298 of [\[CPGSV17\]](#). It imposes only the displayed forward propagation condition and makes no locality claim for ω^{-1} .

Theorem 19.10.1.84 (Exact interface for quantum cellular automata). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For $d > 0$, a star-algebra automorphism ω of $\mathcal{A}$ is a quantum cellular automaton if and only if it is translation covariant and has finite forward propagation. These two properties therefore construct an automaton, and every quantum cellular automaton satisfies each of them.

Proof. [✓ Proof](#) This is the conjunction defining a quantum cellular automaton, together with its introduction and elimination rules. $\square$

Theorem 19.10.1.85 (Identity quantum cellular automaton). [Lean](#) [✓ Stmt](#) For $d > 0$, the identity automorphism of $\mathcal{A}$ is a quantum cellular automaton.

Proof. [✓ Proof](#) The identity is translation covariant. For finite forward propagation, choose the neighborhood $\{0\}$. An observable supported in Λ remains unchanged, and $\Lambda + \{0\} = \Lambda$. $\square$

Theorem 19.10.1.86 (Composition of quantum cellular automata). [Lean](#) [✓ Stmt](#) Let $d > 0$. If ω and η are quantum cellular automata on $\mathcal{A}$, then $\eta \circ \omega$ is a quantum cellular automaton. This composite first applies ω and then η . No propagation condition for either inverse is required.

Proof. [✓ Proof](#) Both defining properties are closed under composition in the stated order. Translation covariance of ω and η gives translation covariance of $\eta \circ \omega$. If $\mathcal{N}$ and $\mathcal{M}$ are forward propagation neighborhoods for ω and η , respectively, then $\mathcal{N} + \mathcal{M}$ is a forward propagation neighborhood for $\eta \circ \omega$. Hence the composite is a quantum cellular automaton. $\square$

Theorem 19.10.1.87 (Inverse quantum cellular automaton). [Lean](#) [✓ Stmt](#) Let $d > 0$. If ω is a quantum cellular automaton on $\mathcal{A}$, then ω^{-1} is also a quantum cellular automaton.

Proof. [✓ Proof](#) Translation covariance and finite forward propagation are both preserved by inversion. These are the two defining properties of a quantum cellular automaton. $\square$

Definition 19.10.1.88 (Expansion of a blocked region). [Lean](#) [Lean](#) [✓ Stmt](#) Let $L > 0$ be a block length and let $\Lambda \subset \mathbb{Z}$ be finite. The expansion of Λ is

$$\widehat{\Lambda}_L = \{Lx + r : x \in \Lambda, 0 \leq r < L\}. \quad (19.792)$$

Every $z \in \widehat{\Lambda}_L$ has unique block coordinates (x, r) with $x \in \Lambda$ and $0 \leq r < L$. Increasing r gives the consecutive-site order within the block. This is the finite-region site grouping used in [\[CPGSV17\]](#), lines 2308 and 2313–2320; it does not assert the QCA-to-MPU converse stated at line 2308.

Theorem 19.10.1.89 (Coordinates and monotonicity of expanded regions). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) An integer z belongs to $\widehat{\Lambda}_L$ exactly when its quotient by L belongs to Λ . The inverse block-coordinate map sends (x, r) to $Lx + r$, and $\Lambda \subseteq \Gamma$ implies $\widehat{\Lambda}_L \subseteq \widehat{\Gamma}_L$.

Proof. ✓ Proof Apply Euclidean division by the positive integer L . Its quotient and residue are unique, the residue lies in $\{0, \dots, L-1\}$, and taking the inverse coordinates gives $Lx + r$. Monotonicity follows by retaining the same quotient and residue. $\square$

Definition 19.10.1.90 (Blocking equivalence for finite configurations). Lean ✓ Stmt For on-site dimension d , configurations of d^L -level blocked spins on Λ are equivalent to configurations of d -level spins on $\widehat{\Lambda}_L$. The blocked label at x is decoded into its ordered length- L word, and its r th letter is placed at $Lx + r$.

Theorem 19.10.1.91 (Evaluation and restriction of blocked configurations). Lean Lean Lean ✓ Stmt Evaluation at $Lx + r$ reads the r th letter of the blocked spin at x . Conversely, encoding collects these letters in increasing residue order. At the configuration level, decoding commutes with restriction from Γ to $\Lambda \subseteq \Gamma$.

Proof. ✓ Proof The two evaluation formulas are the inverse laws for the ordered word encoding. Restriction preserves the quotient block coordinate and residue, so both sides have the same value at every expanded site. $\square$

Definition 19.10.1.92 (Finite local-algebra blocking equivalence). Lean ✓ Stmt Reindexing matrix rows and columns by the configuration blocking equivalence gives a star-algebra equivalence

$$\mathcal{A}_{\Lambda}^{(d^L)} \simeq \mathcal{A}_{\widehat{\Lambda}_L}^{(d)}. \quad (19.793)$$

This is the finite-region equivalence underlying the algebraic-local blocking construction below. No quasi-local completion, compatibility with translations, propagation statement, circuit representation, or transport of quantum cellular automata is asserted here.

Theorem 19.10.1.93 (Finite local-algebra blocking formulas). Lean Lean Lean ✓ Stmt Matrix entries are transported by applying the inverse configuration equivalence to both indices. The blocking equivalence preserves the operator norm and is an isometry.

Proof. ✓ Proof The entry formula is matrix reindexing. A star-algebra equivalence between finite matrix algebras preserves the C-star norm and hence distances. $\square$

Definition 19.10.1.94 (Finite block hull). Lean ✓ Stmt For $L > 0$ and a finite original region $\Delta \subset \mathbb{Z}$, define its block hull by

$$\text{Hull}_L(\Delta) = \{\lfloor z/L \rfloor : z \in \Delta\}. \quad (19.794)$$

This is the finite set of length- L blocks that meet Δ .

Lemma 19.10.1.95 (Block-hull containment and monotonicity). Lean Lean Lean ✓ Stmt Let $L > 0$. Every finite original region is contained in the expansion of its block hull,

$$\Delta \subseteq \widehat{\text{Hull}_L(\Delta)}_L. \quad (19.795)$$

If $\Delta \subseteq \Theta$, then $\text{Hull}_L(\Delta) \subseteq \text{Hull}_L(\Theta)$. Expansion and block hull are exact in the blocked direction:

$$\text{Hull}_L(\widehat{\Lambda}_L) = \Lambda. \quad (19.796)$$

Proof. ✓ Proof The quotient coordinate of each $z \in \Delta$ belongs to the image defining the hull, which proves containment after expansion. Monotonicity follows by retaining the same witness z in the larger region. Finally, every expanded site has quotient in Λ , while for each $x \in \Lambda$ the site Lx belongs to $\widehat{\Lambda}_L$ and has quotient x . Hence $\text{Hull}_L(\widehat{\Lambda}_L) = \Lambda$. $\square$

Lemma 19.10.1.96 (Blocking commutes with finite local inclusions). [Lean](#) [✓ Stmt](#) Let $L > 0$. For finite blocked regions $\Lambda \subseteq \Gamma$, the square formed by finite blocking and the two canonical local inclusions commutes:

$$\mathcal{B}_{L,\Gamma} \iota_{\Lambda,\Gamma} = \iota_{\widehat{\Lambda}_L, \widehat{\Gamma}_L} \mathcal{B}_{L,\Lambda}. \quad (19.797)$$

Proof. [✓ Proof](#) For configurations x, y on $\widehat{\Gamma}_L$, let $\tilde{x}$ and $\tilde{y}$ be their block encodings on Γ . The two matrix entries in the commuting square are

$$[\mathcal{B}_{L,\Gamma}(\iota_{\Lambda,\Gamma}(A))]_{x,y} = A_{\tilde{x}|_{\Lambda}, \tilde{y}|_{\Lambda}} \mathbf{1}_{\{\tilde{x}|_{\Gamma \setminus \Lambda} = \tilde{y}|_{\Gamma \setminus \Lambda}\}}, \quad (19.798)$$

$$[\iota_{\widehat{\Lambda}_L, \widehat{\Gamma}_L}(\mathcal{B}_{L,\Lambda}(A))]_{x,y} = A_{\tilde{x}|_{\Lambda}, \tilde{y}|_{\Lambda}} \mathbf{1}_{\{x|_{\widehat{\Gamma}_L \setminus \widehat{\Lambda}_L} = y|_{\widehat{\Gamma}_L \setminus \widehat{\Lambda}_L}\}}. \quad (19.799)$$

Restriction commutes with block encoding, and decoding identifies $\Gamma \setminus \Lambda$ with the original sites in $\widehat{\Gamma}_L \setminus \widehat{\Lambda}_L$. Hence the two indicator conditions are equivalent and the entries agree. $\square$

Definition 19.10.1.97 (Forward and reverse algebraic-local blocking maps). [Lean](#) [Lean](#) [✓ Stmt](#) Let $L > 0$. The compatible finite blocking maps induce a star-algebra homomorphism

$$\mathcal{B}_L : \mathcal{A}_{\text{loc}}^{(d^L)} \longrightarrow \mathcal{A}_{\text{loc}}^{(d)}. \quad (19.800)$$

In the reverse direction there is a star-algebra homomorphism

$$\mathcal{U}_L : \mathcal{A}_{\text{loc}}^{(d)} \longrightarrow \mathcal{A}_{\text{loc}}^{(d^L)}. \quad (19.801)$$

A representative on an arbitrary original region Δ is first included into $\widehat{\text{Hull}_L(\Delta)}_L$ and then finite blocking is inverted. Thus the reverse construction does not require Δ itself to be an expanded region.

Proof. [✓ Proof](#) For $\Lambda \subseteq \Gamma$, the finite commuting square gives

$$\iota_{\widehat{\Gamma}_L} \mathcal{B}_{L,\Gamma}(\iota_{\Lambda,\Gamma}(A)) = \iota_{\widehat{\Lambda}_L} \mathcal{B}_{L,\Lambda}(A), \quad (19.802)$$

so the forward finite-region maps form a compatible family. For the reverse family, block-hull monotonicity sends $\Delta \subseteq \Theta$ to $\text{Hull}_L(\Delta) \subseteq \text{Hull}_L(\Theta)$. Transitivity of the canonical inclusions and the inverse commuting square then give

$$\iota_{\text{Hull}_L(\Theta)} \mathcal{U}_{L,\Theta}(\iota_{\Delta,\Theta}(A)) = \iota_{\text{Hull}_L(\Delta)} \mathcal{U}_{L,\Delta}(A). \quad (19.803)$$

The universal property of the algebraic direct limit therefore induces both star-algebra homomorphisms. $\square$

Lemma 19.10.1.98 (Evaluation of the algebraic-local blocking maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $L > 0$. The forward map applies finite blocking to a representative on Λ and regards the result as supported on $\widehat{\Lambda}_L$. The reverse map enlarges a representative on Δ to the expansion of its block hull, applies the inverse finite blocking equivalence, and regards the result as supported on $\text{Hull}_L(\Delta)$. Their composites have the orientations

$$\mathcal{B}_L \mathcal{U}_L = \text{id}_{\mathcal{A}_{\text{loc}}^{(d)}}, \quad (19.804)$$

$$\mathcal{U}_L \mathcal{B}_L = \text{id}_{\mathcal{A}_{\text{loc}}^{(d^L)}}. \quad (19.805)$$

The first identity holds for every on-site dimension d ; the second also assumes $d > 0$.

Proof. ✓ Proof These are the evaluation formulas for the two compatible-family lifts. Forward compatibility is the commuting square. Reverse compatibility follows by monotonicity of block hulls, composition of local inclusions, and the same square applied in the inverse direction. Forward blocking after the reverse map reduces to the canonical inclusion of Δ into its expanded block hull. The other composite is the identity by injectivity of the forward map. $\square$

Definition 19.10.1.99 (Algebraic-local site-blocking equivalence). Lean ✓ Stmt For $d > 0$ and $L > 0$, forward blocking gives a star-algebra equivalence

$$\mathcal{A}_{\text{loc}}^{(d^L)} \simeq \mathcal{A}_{\text{loc}}^{(d)}. \quad (19.806)$$

This formalizes only the algebraic-local change of site grouping used at lines 2308 and 2313–2320 of [CPGSV17]. It does not assert the QCA-to-MPU converse at line 2308, extend blocking to the quasi-local completion, intertwine translations, provide a propagation estimate or a circuit representation, or transport a quantum cellular automaton.

Proof. ✓ Proof The identity $\mathcal{B}_L \mathcal{U}_L = \text{id}$ makes $\mathcal{B}_L$ surjective. For injectivity, represent two blocked observables on finite regions Λ and Γ , enlarge both to $\Omega = \Lambda \cup \Gamma$, and suppose their images under $\mathcal{B}_L$ agree. Injectivity of the canonical map from the finite algebra on $\widehat{\Omega}_L$, followed by injectivity of $\mathcal{B}_{L,\Omega}$ and the finite commuting square, gives equality of the two representatives in the algebra on Ω . Thus $\mathcal{B}_L$ is bijective, with inverse $\mathcal{U}_L$, and defines the stated star-algebra equivalence. $\square$

Lemma 19.10.1.100 (Algebraic-local blocking formulas and norm). Lean Lean Lean Lean Lean Lean ✓ Stmt Let $d > 0$ and $L > 0$. The equivalence agrees with the forward lift, and its inverse agrees with the block-hull unblocking map. For $A \in \mathcal{A}_{\Lambda}^{(d^L)}$ and $C \in \mathcal{A}_{\Delta}^{(d)}$,

$$\mathcal{B}_L(\iota_{\Lambda}(A)) = \iota_{\widehat{\Lambda}_L}(\mathcal{B}_{L,\Lambda}(A)), \quad (19.807)$$

$$\mathcal{B}_L^{-1}(\iota_{\Delta}(C)) = \iota_{\text{Hull}_L(\Delta)} \left(\mathcal{B}_{L,\text{Hull}_L(\Delta)}^{-1}(\iota_{\Delta, \widehat{\text{Hull}_L(\Delta)}_L}(C)) \right). \quad (19.808)$$

In particular, if $C \in \mathcal{A}_{\widehat{\Lambda}_L}^{(d)}$, then

$$\mathcal{B}_L^{-1}(\iota_{\widehat{\Lambda}_L}(C)) = \iota_{\Lambda}(\mathcal{B}_{L,\Lambda}^{-1}(C)). \quad (19.809)$$

The forward and inverse maps are operator-norm isometries. In particular, each preserves the norm.

Proof. ✓ Proof The first two formulas are the evaluation identities for the forward and reverse maps. For an expanded region, the block-hull identity removes the enlargement and leaves the inverse finite blocking equivalence. On a finite representative, forward norm preservation reduces to norm preservation by finite blocking and by the canonical local inclusion; distance preservation follows by applying the norm identity to differences. The inverse isometry follows from the forward isometry and the right-inverse identity, rather than repeating the norm calculation. $\square$

Definition 19.10.1.101 (Quasi-local site-blocking equivalence). Lean ✓ Stmt For $d > 0$ and $L > 0$, site blocking extends uniquely by continuity to a star-algebra equivalence

$$\overline{\mathcal{B}}_L : \mathcal{A}^{(d^L)} \simeq \mathcal{A}^{(d)}. \quad (19.810)$$

This is only the completion of the change of site grouping used at [CPGSV17, lines 2308 and 2313–2320]. This slice does not define scaled translations or carry-aware neighborhoods, prove

propagation estimates, construct blocked automorphisms, transport the property of being a quantum cellular automaton, realize a circuit, or reconstruct an MPU from a quantum cellular automaton.

Proof. ✓ Proof The algebraic blocking equivalence and its inverse are isometries, hence continuous. Extending both maps to the operator-norm completions gives mutually inverse star-algebra homomorphisms. $\square$

Lemma 19.10.1.102 (Quasi-local blocking evaluation and norm). Lean Lean Lean Lean Lean Lean ✓ Stmt Let $d > 0$ and $L > 0$. Write $j^{(d)}$ for the canonical map from the algebraic-local algebra into its completion. The underlying map of $\overline{\mathcal{B}}_L$ is the continuous extension of $\mathcal{B}_L$, and on the two dense algebraic-local subalgebras,

$$\overline{\mathcal{B}}_L(j^{(d^L)}(A)) = j^{(d)}(\mathcal{B}_L(A)), \quad (19.811)$$

$$\overline{\mathcal{B}}_L^{-1}(j^{(d)}(C)) = j^{(d^L)}(\mathcal{B}_L^{-1}(C)). \quad (19.812)$$

For $A \in \mathcal{A}_\Lambda^{(d^L)}$ and $C \in \mathcal{A}_\Delta^{(d)}$, these identities become

$$\overline{\mathcal{B}}_L(j_\Lambda^{(d^L)}(A)) = j_{\widehat{\Lambda}_L}^{(d)}(\mathcal{B}_{L,\Lambda}(A)), \quad (19.813)$$

$$\overline{\mathcal{B}}_L^{-1}(j_\Delta^{(d)}(C)) = j_{\text{Hull}_L(\Delta)}^{(d^L)}\left(\mathcal{B}_{L,\text{Hull}_L(\Delta)}^{-1}(\iota_{\Delta,\widehat{\text{Hull}_L(\Delta)}_L}(C))\right). \quad (19.814)$$

If $C \in \mathcal{A}_{\widehat{\Lambda}_L}^{(d)}$, the inverse formula reduces exactly to

$$\overline{\mathcal{B}}_L^{-1}(j_{\widehat{\Lambda}_L}^{(d)}(C)) = j_\Lambda^{(d^L)}(\mathcal{B}_{L,\Lambda}^{-1}(C)). \quad (19.815)$$

Finally, for all quasi-local observables X and Y on the blocked chain,

$$\|\overline{\mathcal{B}}_L(X)\| = \|X\|, \quad (19.816)$$

$$\|\overline{\mathcal{B}}_L(X) - \overline{\mathcal{B}}_L(Y)\| = \|X - Y\|. \quad (19.817)$$

Proof. ✓ Proof The first identity is the defining completion map, and the two dense formulas are its canonical-image formulas for the algebraic equivalence and its inverse. Composing them with the finite-region canonical maps gives the two finite formulas. The exact expanded-region formula uses $\text{Hull}_L(\widehat{\Lambda}_L) = \Lambda$. A star-algebra equivalence of C-star algebras preserves the norm; applying this to $X - Y$ gives distance preservation. $\square$

Lemma 19.10.1.103 (Support transport under site blocking). Lean Lean Lean ✓ Stmt Let $d > 0$ and $L > 0$. If a blocked-chain observable X is supported in Λ , then $\overline{\mathcal{B}}_L(X)$ is supported in $\widehat{\Lambda}_L$. If an original-chain observable Y is supported in Δ , then $\overline{\mathcal{B}}_L^{-1}(Y)$ is supported in $\text{Hull}_L(\Delta)$. In particular, support transport is exact on expanded regions:

$$\overline{\mathcal{B}}_L(X) \text{ is supported in } \widehat{\Lambda}_L \iff X \text{ is supported in } \Lambda. \quad (19.818)$$

Proof. ✓ Proof Choose a finite-region representative and apply the corresponding forward or inverse finite evaluation formula. For the reverse implication in the exact statement, apply inverse blocking and simplify the inverse composite and the block hull of the expanded region. $\square$

Lemma 19.10.1.104 (Expansion of a translated blocked region). Lean ✓ Stmt Let $L > 0$, let $a \in \mathbb{Z}$, and let $\Lambda \subset \mathbb{Z}$ be a finite blocked region. Then

$$\widehat{\Lambda + aL} = \widehat{\Lambda}_L + aL. \quad (19.819)$$

This is the translation rule for the site grouping used in [CPGSV17, lines 2308 and 2313–2320].

Proof. ✓ Proof Write each original site uniquely as $xL + r$, with $0 \leq r < L$. Translation of the blocked coordinate by a sends this site to $(x+a)L + r = xL + r + aL$. The argument is unchanged when x or a is negative. $\square$

Definition 19.10.1.105 (Carry-aware blocked neighborhood). Lean ✓ Stmt Let $L > 0$ and let $\mathcal{N} \subset \mathbb{Z}$ be a finite original-lattice neighborhood. Write $\widehat{\{0\}}_L = \{0, \dots, L-1\}$ for the sites in the reference block. Define the induced blocked-lattice neighborhood by

$$\mathcal{C}_L(\mathcal{N}) = \text{Hull}_L(\widehat{\{0\}}_L + \mathcal{N}). \quad (19.820)$$

The full reference block is included because the block reached after a displacement can depend on the starting residue. This is the finite-region geometry implicit in the grouping of sites in [CPGSV17, lines 2308 and 2313–2320].

Lemma 19.10.1.106 (Residue description of the blocked neighborhood). Lean Lean ✓ Stmt Let $L > 0$, let $\mathcal{N} \subset \mathbb{Z}$ be finite, and let $b \in \mathbb{Z}$. Then

$$b \in \mathcal{C}_L(\mathcal{N}) \iff \exists r \in \{0, \dots, L-1\}, \exists n \in \mathcal{N}, \left\lfloor \frac{r+n}{L} \right\rfloor = b, \quad (19.821)$$

$$\iff \exists r, s \in \{0, \dots, L-1\}, \exists n \in \mathcal{N}, r + n = bL + s. \quad (19.822)$$

The quotient is the Euclidean quotient by the positive integer L . Consequently these formulas include negative displacements and both positive and negative block carries. In general, $\text{Hull}_L(\mathcal{N})$ alone omits some carries caused by the starting residue. This makes explicit the boundary geometry implicit in [CPGSV17, lines 2313–2320].

Proof. ✓ Proof An element of $\widehat{\{0\}}_L + \mathcal{N}$ has the form $r + n$ with $0 \leq r < L$ and $n \in \mathcal{N}$. Taking its block coordinate gives the first equivalence. Euclidean division writes $r + n = bL + s$ with the unique remainder $0 \leq s < L$, which gives the second equivalence without any sign restriction on n . $\square$

Lemma 19.10.1.107 (Exact block hull after adding a neighborhood). Lean ✓ Stmt Let $L > 0$. For every finite blocked region Λ and every finite original-lattice neighborhood $\mathcal{N}$,

$$\text{Hull}_L(\widehat{\Lambda}_L + \mathcal{N}) = \Lambda + \mathcal{C}_L(\mathcal{N}). \quad (19.823)$$

This is the exact finite-region carry identity associated with the site grouping in [CPGSV17, lines 2313–2320].

Proof. ✓ Proof Write an expanded site as $xL + r$, where $x \in \Lambda$ and $0 \leq r < L$. For $n \in \mathcal{N}$, Euclidean division gives

$$\left\lfloor \frac{xL + r + n}{L} \right\rfloor = x + \left\lfloor \frac{r + n}{L} \right\rfloor. \quad (19.824)$$

The second summand ranges over exactly $\mathcal{C}_L(\mathcal{N})$ by Lemma 19.10.1.106. This proves both inclusions, including for negative x and n . $\square$

Lemma 19.10.1.108 (Carry-aware expansion containment). [Lean](#) [✓ Stmt](#) Let $L > 0$. For every finite blocked region Λ and every finite original-lattice neighborhood $\mathcal{N}$,

$$\widehat{\Lambda}_L + \mathcal{N} \subseteq \Lambda + \widehat{\mathcal{C}_L(\mathcal{N})}_L. \quad (19.825)$$

Proof. [✓ Proof](#) Every finite original region is contained in the expansion of its block hull. Apply this containment to $\widehat{\Lambda}_L + \mathcal{N}$ and use Lemma 19.10.1.107 to identify that hull. $\square$

Lemma 19.10.1.109 (Algebraic-local blocking intertwines scaled translations). [Lean](#) [✓ Stmt](#) Let $d, L > 0$. If $\mathcal{B}_L$ denotes the algebraic-local blocking equivalence, then for every $a \in \mathbb{Z}$,

$$\mathcal{B}_L \tau_a^{(d^L)} = \tau_{aL}^{(d)} \mathcal{B}_L. \quad (19.826)$$

This is the translation rule induced by the blocking of sites used in [CPGSV17, lines 2308 and 2313–2320].

Proof. [✓ Proof](#) On a representative supported in a finite blocked region Λ , both sides apply the same matrix reindexing. Their supports agree because

$$\widehat{\Lambda + aL} = \widehat{\Lambda}_L + aL. \quad (19.827)$$

Equality on all finite-region representatives proves the identity on the algebraic direct limit. $\square$

Lemma 19.10.1.110 (Quasi-local blocking intertwines scaled translations). [Lean](#) [✓ Stmt](#) Let $d, L > 0$. If $\overline{\mathcal{B}}_L$ denotes the quasi-local blocking equivalence, then for every $a \in \mathbb{Z}$,

$$\overline{\mathcal{B}}_L \overline{\tau}_a^{(d^L)} = \overline{\tau}_{aL}^{(d)} \overline{\mathcal{B}}_L. \quad (19.828)$$

This is the completion of the translation rule induced by the site grouping in [CPGSV17, lines 2308 and 2313–2320].

Proof. [✓ Proof](#) The two sides are continuous maps on the quasi-local algebra. On the dense algebraic-local subalgebra their values agree by Lemma 19.10.1.109. They therefore agree on the completion. $\square$

Line 2308 of [CPGSV17] invokes a converse representation theorem: after blocking finitely many sites, the restriction of every one-dimensional QCA to the local algebra is represented by a depth-two circuit of nearest-neighbor unitaries, and hence by an MPU in standard form. The results below concern the preceding change of grouping. A given QCA remains a QCA under any prescribed finite blocking. This one-way transport alone does not produce the circuit or MPU representation asserted by the converse theorem.

Definition 19.10.1.111 (Blocked automorphism). [Lean](#) [✓ Stmt](#) Let $d, L > 0$, let $\overline{\mathcal{B}}_L: \mathcal{A}^{(d^L)} \simeq \mathcal{A}^{(d)}$ be the quasi-local blocking equivalence, and let ω be a star-algebra automorphism of $\mathcal{A}^{(d)}$. The induced automorphism of the blocked chain is

$$\omega^{[L]} = \overline{\mathcal{B}}_L^{-1} \circ \omega \circ \overline{\mathcal{B}}_L. \quad (19.829)$$

Lemma 19.10.1.112 (Evaluation of the blocked automorphism). [Lean](#) [✓ Stmt](#) For every $X \in \mathcal{A}^{(d^L)}$,

$$\omega^{[L]}(X) = \overline{\mathcal{B}}_L^{-1}(\omega(\overline{\mathcal{B}}_L(X))). \quad (19.830)$$

Proof. ✓ Proof This is the evaluation formula for the defining conjugation. □

Lemma 19.10.1.113 (Inverse of the blocked automorphism). Lean ✓ Stmt *The inverse of the blocked automorphism is the blocked inverse:*

$$(\omega^{[L]})^{-1} = (\omega^{-1})^{[L]}. \quad (19.831)$$

Proof. ✓ Proof Evaluate both sides on a blocked-chain observable. The blocking equivalence and its inverse cancel on the two sides of ω^{-1} . □

Lemma 19.10.1.114 (Propagation neighborhood after site blocking). Lean ✓ Stmt *Let $\mathcal{N} \subset \mathbb{Z}$ be finite. If ω propagates within $\mathcal{N}$, then $\omega^{[L]}$ propagates within the carry-aware blocked neighborhood $\mathcal{C}_L(\mathcal{N})$. Explicitly, for every finite blocked region Λ and every $X \in \mathcal{A}^{(d^L)}$ supported in Λ , the observable $\omega^{[L]}(X)$ is supported in $\Lambda + \mathcal{C}_L(\mathcal{N})$.*

Proof. ✓ Proof Forward blocking sends support in Λ to support in $\widehat{\Lambda}_L$. Apply the propagation bound for ω , then apply inverse blocking. The resulting support is

$$\text{Hull}_L(\widehat{\Lambda}_L + \mathcal{N}) = \Lambda + \mathcal{C}_L(\mathcal{N}) \quad (19.832)$$

by the exact carry-aware block-hull identity. □

Lemma 19.10.1.115 (Finite propagation after site blocking). Lean ✓ Stmt *If ω has finite forward propagation, then $\omega^{[L]}$ has finite forward propagation.*

Proof. ✓ Proof Choose a finite propagation neighborhood $\mathcal{N}$ for ω . Lemma 19.10.1.114 gives the finite neighborhood $\mathcal{C}_L(\mathcal{N})$ for $\omega^{[L]}$. □

Theorem 19.10.1.116 (Translation covariance of the blocked automorphism). Lean ✓ Stmt *The blocked automorphism is translation covariant if and only if the original automorphism commutes with translation by one whole block:*

$$\omega^{[L]} \text{ is translation covariant} \iff \omega \circ \bar{\tau}_L^{(d)} = \bar{\tau}_L^{(d)} \circ \omega. \quad (19.833)$$

Thus blocked unit-translation covariance records commutation of ω with translation by L , not necessarily commutation with original unit translation. For $L \neq 1$, the latter condition can be strictly stronger.

Proof. ✓ Proof By Theorem 19.10.1.61, translation covariance on the blocked chain is equivalent to commutation with blocked unit translation. Under $\bar{\mathcal{B}}_L$, blocked unit translation corresponds to original translation by L . Conjugating the commutation identity by the blocking equivalence proves both implications. □

Lemma 19.10.1.117 (Translation covariance after site blocking). Lean ✓ Stmt *If ω is translation covariant, then $\omega^{[L]}$ is translation covariant.*

Proof. ✓ Proof Translation covariance of ω gives commutation with $\bar{\tau}_L^{(d)}$. Apply Theorem 19.10.1.116. □

Lemma 19.10.1.118 (QCA transport through site blocking). Lean ✓ Stmt *Let $d, L > 0$. If ω is a one-dimensional QCA on $\mathcal{A}^{(d)}$, then*

$$\omega^{[L]} = \bar{\mathcal{B}}_L^{-1} \circ \omega \circ \bar{\mathcal{B}}_L \quad (19.834)$$

is a one-dimensional QCA on $\mathcal{A}^{(d^L)}$.

Proof. ✓ Proof The two defining properties of a one-dimensional QCA are finite forward propagation and translation covariance. They are preserved by Lemma 19.10.1.115 and Lemma 19.10.1.117, respectively. $\square$

Line 2308 of [CPGSV17] invokes the external Schumacher–Werner structure theorem to obtain support algebras, a depth-two nearest-neighbor circuit, and an MPU in standard form. The next statements establish only the elementary normalization of a finite propagation neighborhood to $[-1, 1] \cap \mathbb{Z}$ by regrouping sites. They do not prove any part of that external structure theorem beyond this change of grouping.

Lemma 19.10.1.119 (Nearest-neighbor carries from a strict displacement bound). Lean ✓ Stmt *Let $L > 0$ and let $\mathcal{N} \subset \mathbb{Z}$ be finite. If $|n| < L$ for every $n \in \mathcal{N}$, then*

$$\mathcal{C}_L(\mathcal{N}) \subseteq [-1, 1] \cap \mathbb{Z}. \quad (19.835)$$

Proof. ✓ Proof If $\mathcal{N} = \emptyset$, membership in $\mathcal{C}_L(\mathcal{N})$ is impossible. In general, a point $b \in \mathcal{C}_L(\mathcal{N})$ supplies residues $0 \leq r, s < L$ and $n \in \mathcal{N}$ such that

$$r + n = bL + s. \quad (19.836)$$

Since $-L < n < L$, the equality is incompatible with $b \leq -2$ and with $b \geq 2$. Thus $-1 \leq b \leq 1$, including for negative displacements. $\square$

Lemma 19.10.1.120 (Supremum criterion for nearest-neighbor carries). Lean ✓ Stmt *Let $L > 0$ and let $\mathcal{N} \subset \mathbb{Z}$ be finite. If*

$$\sup_{n \in \mathcal{N}} |n| < L, \quad (19.837)$$

then

$$\mathcal{C}_L(\mathcal{N}) \subseteq [-1, 1] \cap \mathbb{Z}. \quad (19.838)$$

Proof. ✓ Proof Every $|n|$ is bounded above by the finite supremum, so the strict supremum bound supplies the pointwise hypothesis. When $\mathcal{N} = \emptyset$, the supremum is 0 and the conclusion is vacuous. $\square$

Lemma 19.10.1.121 (Canonical nearest-neighbor block length). Lean ✓ Stmt *For every finite $\mathcal{N} \subset \mathbb{Z}$, define*

$$L_{\mathcal{N}} = 1 + \sup_{n \in \mathcal{N}} |n|. \quad (19.839)$$

Then $L_{\mathcal{N}} > 0$ and

$$\mathcal{C}_{L_{\mathcal{N}}}(\mathcal{N}) \subseteq [-1, 1] \cap \mathbb{Z}. \quad (19.840)$$

With the finite-supremum convention, $L_{\emptyset} = 1$.

Proof. ✓ Proof The finite supremum is strictly smaller than its successor, which is positive even when $\mathcal{N}$ is empty. $\square$

Lemma 19.10.1.122 (Nearest-neighbor propagation after a bounded blocking). Lean ✓ Stmt *Let $d, L > 0$. Suppose that ω propagates within the finite neighborhood $\mathcal{N}$ and that $|n| < L$ for every $n \in \mathcal{N}$. For every finite blocked region Λ and every observable $X \in \mathcal{A}^{(d^L)}$ supported in Λ , the observable $\omega^{[L]}(X)$ is supported in*

$$\Lambda + ([-1, 1] \cap \mathbb{Z}). \quad (19.841)$$

Proof. ✓ Proof Blocking first gives propagation within $\mathcal{C}_L(\mathcal{N})$. The carry bound places this neighborhood inside $[-1, 1] \cap \mathbb{Z}$, and enlargement of propagation neighborhoods gives the claim. $\square$

Theorem 19.10.1.123 (Finite propagation becomes nearest-neighbor). Lean ✓ Stmt Let $d > 0$. If ω has finite forward propagation, then there exists a positive integer L such that, for every finite blocked region Λ and every observable $X \in \mathcal{A}^{(d^L)}$ supported in Λ , the observable $\omega^{[L]}(X)$ is supported in

$$\Lambda + ([-1, 1] \cap \mathbb{Z}). \quad (19.842)$$

Proof. ✓ Proof Choose a finite propagation neighborhood $\mathcal{N}$ and take $L = 1 + \sup_{n \in \mathcal{N}} |n|$. This L is positive, including when $\mathcal{N}$ is empty. Blocking gives propagation within $\mathcal{C}_L(\mathcal{N})$; the canonical carry inclusion and propagation monotonicity then give the nearest-neighbor bound. $\square$

Theorem 19.10.1.124 (A QCA becomes nearest-neighbor after blocking). Lean ✓ Stmt Let $d > 0$ and let ω be a one-dimensional QCA on $\mathcal{A}^{(d)}$. There exists a positive integer L such that $\omega^{[L]}$ is a QCA on $\mathcal{A}^{(d^L)}$ and, for every finite blocked region Λ and every observable $X \in \mathcal{A}^{(d^L)}$ supported in Λ , the observable $\omega^{[L]}(X)$ is supported in

$$\Lambda + ([-1, 1] \cap \mathbb{Z}). \quad (19.843)$$

Proof. ✓ Proof Apply the preceding theorem to the finite propagation of ω , using the canonical positive block length. At the same length, site blocking preserves both finite propagation and translation covariance, hence the QCA property. $\square$

Lines 2292–2298 of [CPGSV17] specify the finite local algebras, their canonical inclusions, translation covariance, and the locality inclusion

$$\omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda+\mathcal{N}}. \quad (19.844)$$

The following statements record the finite-dimensional maps determined by this inclusion. They are consequences and coordinate formulations of the cited locality relation, not separately named theorems of the paper. The left/right order below agrees with the adjacent-pair convention in [GNVW12].

Definition 19.10.1.125 (Finite-region restriction). Lean ✓ Stmt Let $d > 0$, let ω be a star-algebra automorphism of $\mathcal{A}^{(d)}$, and suppose that ω propagates within the finite neighborhood $\mathcal{N}$. For every finite region Λ , define the finite-region restriction

$$\omega_{\Lambda, \mathcal{N}}: \mathcal{A}_\Lambda \longrightarrow \mathcal{A}_{\Lambda+\mathcal{N}} \quad (19.845)$$

to be the unique star-algebra homomorphism satisfying

$$\iota_{\Lambda+\mathcal{N}}(\omega_{\Lambda, \mathcal{N}}(A)) = \omega(\iota_\Lambda(A)) \quad (A \in \mathcal{A}_\Lambda). \quad (19.846)$$

Theorem 19.10.1.126 (Finite-region restriction is an embedding). Lean Lean ✓ Stmt The finite-region restriction is injective, and for every $A \in \mathcal{A}_\Lambda$ one has

$$\iota_{\Lambda+\mathcal{N}}(\omega_{\Lambda, \mathcal{N}}(A)) = \omega(\iota_\Lambda(A)). \quad (19.847)$$

Proof. ✓ Proof The defining equality follows from the locality inclusion and the injectivity of $\iota_{\Lambda+\mathcal{N}}$. If two local observables have the same image under $\omega_{\Lambda, \mathcal{N}}$, their canonical images become equal after applying ω . Injectivity of ω and of ι_Λ then gives equality of the two local observables. $\square$

Theorem 19.10.1.127 (Enlargement of finite-region restrictions). [Lean](#) [✓ Stmt](#) If $\Lambda \subseteq \Gamma$, then for every $A \in \mathcal{A}_\Lambda$,

$$\omega_{\Gamma, \mathcal{N}}(\iota_{\Lambda, \Gamma}(A)) = \iota_{\Lambda + \mathcal{N}, \Gamma + \mathcal{N}}(\omega_{\Lambda, \mathcal{N}}(A)). \quad (19.848)$$

Proof. [✓ Proof](#) Apply the canonical inclusion into the quasi-local algebra to both sides. Both expressions become $\omega(\iota_\Lambda(A))$; injectivity of the target inclusion gives the equality. $\square$

Theorem 19.10.1.128 (Translation of finite-region restrictions). [Lean](#) [✓ Stmt](#) Suppose in addition that ω is translation covariant. For $a \in \mathbb{Z}$ and $A \in \mathcal{A}_\Lambda$,

$$\iota_{(\Lambda+a)+\mathcal{N}}(\omega_{\Lambda+a, \mathcal{N}}(\tau_{a, \Lambda}(A))) = \iota_{(\Lambda+\mathcal{N})+a}(\tau_{a, \Lambda+\mathcal{N}}(\omega_{\Lambda, \mathcal{N}}(A))). \quad (19.849)$$

Here $(\Lambda + a) + \mathcal{N} = (\Lambda + \mathcal{N}) + a$.

Proof. [✓ Proof](#) The left-hand side is $\omega(\tau_a(\iota_\Lambda(A)))$. Translation covariance moves τ_a past ω , and the finite-region translation formula gives the right-hand side. $\square$

Definition 19.10.1.129 (Bipartite coordinates for disjoint local algebras). [Lean](#) [Lean](#) [✓ Stmt](#) Let $\Gamma, \Delta \subset \mathbb{Z}$ be disjoint finite regions. Writing $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Delta$ for their configuration spaces, restriction to the two regions gives

$$\mathcal{C}_{\Gamma \cup \Delta} \simeq \mathcal{C}_\Gamma \times \mathcal{C}_\Delta, \quad (19.850)$$

with the Γ -configuration first. Reindexing matrix rows and columns gives a star-algebra equivalence

$$\beta_{\Gamma, \Delta} : \mathcal{A}_{\Gamma \cup \Delta} \simeq M_{\mathcal{C}_\Gamma \times \mathcal{C}_\Delta}(\mathbb{C}). \quad (19.851)$$

Theorem 19.10.1.130 (Canonical inclusions in bipartite coordinates). [Lean](#) [Lean](#) [✓ Stmt](#) For $A \in \mathcal{A}_\Gamma$ and $B \in \mathcal{A}_\Delta$,

$$\beta_{\Gamma, \Delta}(\iota_{\Gamma, \Gamma \cup \Delta}(A)) = A \otimes \mathbf{1}, \quad (19.852)$$

$$\beta_{\Gamma, \Delta}(\iota_{\Delta, \Gamma \cup \Delta}(B)) = \mathbf{1} \otimes B. \quad (19.853)$$

Proof. [✓ Proof](#) Evaluate both matrices on pairs of configurations. For the first local inclusion, restriction to Γ gives the matrix coefficient of A , while equality of the complementary Δ -configurations gives the matrix coefficient of the identity. This is $A \otimes \mathbf{1}$. Interchanging the two regions gives $\mathbf{1} \otimes B$. $\square$

Definition 19.10.1.131 (Finite local image in bipartite coordinates). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Define the finite local image by

$$\mathcal{R}_{\Lambda, \mathcal{N}} = \omega_{\Lambda, \mathcal{N}}(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda + \mathcal{N}}. \quad (19.854)$$

It is a star-subalgebra star-isomorphic to $\mathcal{A}_\Lambda$. If $\Lambda + \mathcal{N} = \Gamma \cup \Delta$ with $\Gamma \cap \Delta = \emptyset$, then

$$\beta_{\Gamma, \Delta}(\mathcal{R}_{\Lambda, \mathcal{N}}) \subseteq M_{\mathcal{C}_\Gamma \times \mathcal{C}_\Delta}(\mathbb{C}) \quad (19.855)$$

is the bipartite matrix star-subalgebra to which the left and right support-algebra constructions apply. No support algebra is defined in this statement.

Proof. [✓ Proof](#) Apply Theorem 19.10.1.17 to S_x in the ordered coordinates of (19.859). Its left support is $\mathcal{R}_{2x}$ and its right support is $\mathcal{R}_{2x+1}$ by (19.861), which gives (19.863). $\square$

Theorem 19.10.1.136 (Physical support of the site-indexed support algebras). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $\widehat{\mathcal{R}}_y$ denotes the canonical quasi-local copy of $\mathcal{R}_y$, then for every $x \in \mathbb{Z}$,

$$\widehat{\mathcal{R}}_{2x} \subseteq \mathcal{A}_{L_x}, \quad \widehat{\mathcal{R}}_{2x+1} \subseteq \mathcal{A}_{P_x}. \quad (19.864)$$

More generally, the canonical image of any matrix on a finite region Λ is supported in Λ . This is the homogeneous specialization of the localization accompanying equation RR2x in [GNVW12, lines 1261–1274]; see [con26l].

Proof. [✓ Proof](#) An element of $\widehat{\mathcal{R}}_{2x}$ is the canonical image of a matrix in $\mathcal{R}_{2x} \subseteq \mathcal{A}_{L_x}$ and is therefore supported in L_x . The same argument places $\widehat{\mathcal{R}}_{2x+1}$ in $\mathcal{A}_{P_x}$. $\square$

Equation (A1) in lines 2300–2306 of [CPGSV17] defines the local action associated with an MPU by eventual finite-chain conjugation. In lines 2300–2306, eventual constancy is used to obtain a norm-preserving star-algebra automorphism of $\mathcal{A}_{\text{loc}}$, together with a translation formula and the support bound $\mathcal{N} = [-4D^4, 4D^4] \cap \mathbb{Z}$, and then to extend this automorphism to $\mathcal{A}$. The results below isolate the direct-limit and completion argument. They begin with compatible forward and inverse finite-local actions and assume a forward norm formula, a support bound, and a unit-translation formula. The inverse norm formula follows from forward isometry and mutual invertibility. The numerical radius above belongs to the MPU-specific stabilization argument: the generic result below neither proves that radius nor constructs the finite-local actions from an MPU. It also does not provide the blocked circuit or MPU standard form discussed at line 2308.

Definition 19.10.1.137 (Compatible finite-local automorphism data). [Lean](#) [✓ Stmt](#) For each finite region Λ , let

$$f_\Lambda, g_\Lambda : \mathcal{A}_\Lambda \longrightarrow \mathcal{A}_{\text{loc}} \quad (19.865)$$

be star-algebra homomorphisms such that, whenever $\Lambda \subseteq \Gamma$,

$$f_\Gamma \circ \iota_{\Lambda, \Gamma} = f_\Lambda, \quad g_\Gamma \circ \iota_{\Lambda, \Gamma} = g_\Lambda. \quad (19.866)$$

Let $f, g : \mathcal{A}_{\text{loc}} \rightarrow \mathcal{A}_{\text{loc}}$ be the maps induced by these compatible families. Compatible finite-local automorphism data require

$$g(f_\Lambda(A)) = \iota_\Lambda(A), \quad f(g_\Lambda(A)) = \iota_\Lambda(A) \quad (19.867)$$

for every $A \in \mathcal{A}_\Lambda$.

Definition 19.10.1.138 (Induced algebraic-local homomorphisms). [Lean](#) [Lean](#) [✓ Stmt](#) The compatible families induce star-algebra homomorphisms

$$f, g : \mathcal{A}_{\text{loc}} \longrightarrow \mathcal{A}_{\text{loc}}. \quad (19.868)$$

Proof. [✓ Proof](#) Apply the star-algebra universal property to each compatible family. $\square$

Lemma 19.10.1.139 (Evaluation of the induced homomorphisms). [Lean](#) [Lean](#) [✓ Stmt](#) For every finite region Λ and $A \in \mathcal{A}_\Lambda$,

$$f(\iota_\Lambda(A)) = f_\Lambda(A), \quad g(\iota_\Lambda(A)) = g_\Lambda(A). \quad (19.869)$$

Proof. ✓ Proof These are the evaluation identities supplied by the universal property. □

Definition 19.10.1.140 (Induced algebraic-local equivalence). Lean ✓ Stmt Compatible finite-local automorphism data determine a star-algebra equivalence

$$F: \mathcal{A}_{\text{loc}} \simeq \mathcal{A}_{\text{loc}} \quad (19.870)$$

whose forward and inverse maps are f and g , respectively.

Proof. ✓ Proof Induct on finite-region representatives in the algebraic direct limit. The two inverse assumptions give

$$g(f(\iota_\Lambda(A))) = g(f_\Lambda(A)) = \iota_\Lambda(A), \quad (19.871)$$

$$f(g(\iota_\Lambda(A))) = f(g_\Lambda(A)) = \iota_\Lambda(A). \quad (19.872)$$

Thus f and g are mutually inverse. □

Lemma 19.10.1.141 (Evaluation of the algebraic equivalence). Lean Lean ✓ Stmt For every $A \in \mathcal{A}_{\text{loc}}$,

$$F(A) = f(A), \quad F^{-1}(A) = g(A). \quad (19.873)$$

Proof. ✓ Proof Both identities follow from the defining forward and inverse maps of F . □

Definition 19.10.1.142 (Forward local norm hypothesis). Lean ✓ Stmt Assume $d > 0$. The forward local norm hypothesis is

$$\|f_\Lambda(A)\| = \|A\| \quad (19.874)$$

for every finite region Λ and every $A \in \mathcal{A}_\Lambda$.

Lemma 19.10.1.143 (Forward algebraic norm formula). Lean ✓ Stmt Under the forward local norm hypothesis, every $A \in \mathcal{A}_{\text{loc}}$ satisfies

$$\|F(A)\| = \|A\|. \quad (19.875)$$

Proof. ✓ Proof Induct on finite-region representatives. For $A = \iota_\Lambda(A_\Lambda)$,

$$\|F(A)\| = \|f_\Lambda(A_\Lambda)\| = \|A_\Lambda\| = \|A\|. \quad (19.876)$$

□

Lemma 19.10.1.144 (Forward and inverse algebraic isometries). Lean Lean ✓ Stmt Under the forward local norm hypothesis, for all $A, B \in \mathcal{A}_{\text{loc}}$,

$$\|F(A) - F(B)\| = \|A - B\|, \quad \|F^{-1}(A) - F^{-1}(B)\| = \|A - B\|. \quad (19.877)$$

Proof. ✓ Proof Apply the forward norm identity to $A - B$ to prove that F is an isometry. Since $F \circ F^{-1}$ is the identity, the right-inverse law then makes F^{-1} an isometry as well. □

Lemma 19.10.1.145 (Inverse algebraic norm formula). Lean ✓ Stmt Under the forward local norm hypothesis, every $A \in \mathcal{A}_{\text{loc}}$ satisfies

$$\|F^{-1}(A)\| = \|A\|. \quad (19.878)$$

Proof. ✓ Proof Apply the inverse isometry to the distance from A to 0 and use $F^{-1}(0) = 0$. $\square$

Definition 19.10.1.146 (Completed compatible local automorphism). Lean ✓ Stmt Under the forward local norm hypothesis, the algebraic equivalence extends to a star-algebra equivalence

$$\overline{F}: \mathcal{A} \simeq \mathcal{A} \quad (19.879)$$

of the quasi-local completion.

Proof. ✓ Proof The forward norm formula makes F an isometry, and the inverse law makes F^{-1} an isometry. Apply the completion extension for star-algebra equivalences to these two continuous maps. $\square$

Lemma 19.10.1.147 (Dense evaluation of the completed automorphism). Lean Lean ✓ Stmt Let $j: \mathcal{A}_{\text{loc}} \rightarrow \mathcal{A}$ be the canonical completion map. For every $A \in \mathcal{A}_{\text{loc}}$,

$$\overline{F}(j(A)) = j(F(A)), \quad \overline{F}^{-1}(j(A)) = j(F^{-1}(A)). \quad (19.880)$$

Proof. ✓ Proof These are the canonical-image formulas for the continuous extension of a star-algebra equivalence and for its inverse. $\square$

Lemma 19.10.1.148 (Finite evaluation of the completed automorphism). Lean Lean ✓ Stmt For every finite region Λ and $A \in \mathcal{A}_\Lambda$,

$$\overline{F}(j_\Lambda(A)) = j(f_\Lambda(A)), \quad \overline{F}^{-1}(j_\Lambda(A)) = j(g_\Lambda(A)). \quad (19.881)$$

Proof. ✓ Proof Substitute $j_\Lambda = j \circ \iota_\Lambda$ into the dense evaluation formulas and use $F(\iota_\Lambda(A)) = f_\Lambda(A)$ and $F^{-1}(\iota_\Lambda(A)) = g_\Lambda(A)$. $\square$

Definition 19.10.1.149 (Finite-local support bound). Lean ✓ Stmt For a finite neighborhood $\mathcal{N} \subset \mathbb{Z}$, the forward family has support bound $\mathcal{N}$ if

$$f_\Lambda(A) \in \iota_{\Lambda+\mathcal{N}}(\mathcal{A}_{\Lambda+\mathcal{N}}) \quad (19.882)$$

for every finite region Λ and every $A \in \mathcal{A}_\Lambda$.

Theorem 19.10.1.150 (Propagation from a finite-local support bound). Lean ✓ Stmt If the forward family has support bound $\mathcal{N}$, then the completed automorphism $\overline{F}$ propagates within $\mathcal{N}$.

Proof. ✓ Proof Let $X = j_\Lambda(A)$ be supported in Λ . Choose $B \in \mathcal{A}_{\Lambda+\mathcal{N}}$ such that $f_\Lambda(A) = \iota_{\Lambda+\mathcal{N}}(B)$. Then

$$\overline{F}(X) = j(f_\Lambda(A)) = j_{\Lambda+\mathcal{N}}(B), \quad (19.883)$$

so the image is supported in $\Lambda + \mathcal{N}$. $\square$

Definition 19.10.1.151 (Finite-local unit-translation formula). Lean ✓ Stmt The forward family commutes with unit translation if, for every finite region Λ and every $A \in \mathcal{A}_\Lambda$,

$$f_{\Lambda+1}(\tau_{1,\Lambda}(A)) = \tau_1(f_\Lambda(A)). \quad (19.884)$$

Lemma 19.10.1.152 (Algebraic commutation with unit translation). Lean ✓ Stmt If the finite-local unit-translation formula holds, then every $A \in \mathcal{A}_{\text{loc}}$ satisfies

$$F(\tau_1(A)) = \tau_1(F(A)). \quad (19.885)$$

Proof. ✓ Proof Induct on finite-region representatives and write $A = \iota_\Lambda(A_\Lambda)$. Then

$$F(\tau_1(A)) = f_{\Lambda+1}(\tau_{1,\Lambda}(A_\Lambda)) = \tau_1(f_\Lambda(A_\Lambda)) = \tau_1(F(A)). \quad (19.886)$$

□

Theorem 19.10.1.153 (Translation covariance from unit translation). Lean ✓ Stmt *If the finite-local unit-translation formula holds, then the completed automorphism $\overline{F}$ is translation covariant.*

Proof. ✓ Proof By Theorem 19.10.1.61, it suffices to prove that $\overline{F}$ commutes with translation by one site. Induct on the completion. On every $A \in \mathcal{A}_{\text{loc}}$,

$$\overline{F}(\overline{\tau}_1(j(A))) = j(F(\tau_1(A))) = j(\tau_1(F(A))) = \overline{\tau}_1(\overline{F}(j(A))). \quad (19.887)$$

The equality set is closed because both composites are continuous, so the identity extends from the canonical dense subalgebra to the completion. □

Theorem 19.10.1.154 (QCA assembled from compatible finite-local actions). Lean ✓ Stmt *Let $d > 0$. Suppose the compatible forward family preserves the local operator norm, has a finite support bound, and satisfies the finite-local unit-translation formula. Then the completed automorphism $\overline{F}$ is a one-dimensional QCA.*

Proof. ✓ Proof The support hypothesis supplies a finite forward propagation neighborhood, while the unit-translation formula supplies translation covariance. These are exactly the two defining properties of a one-dimensional QCA. □

Chapter 20

Mixed States: Renormalization of Matrix Product Operators

Starting from the MPO, MPDO, and LPDO notation of Chapter 17, this chapter studies renormalization fixed points for density-operator chains. Following the order of the mixed-state analysis in [CPGSV16, Section 4], the discussion first states the local renormalization condition and then records its comparison with the pure-state formalism. It next passes to zero correlation length, purification fixed points, mutual information, and saturation of the area law. The source definitions of simple tensors and Gibbs states of nearest-neighbor commuting Hamiltonians, together with the cyclic coordinates used for local embeddings, precede the simple local and commuting structures. The boundary-theory interlude of the source is outside the present MPO development. The general case and its algebraic characterization continue in Chapter 21.

20.1 Preliminaries for physical renormalization

We first collect the transfer-map, channel, and physical-closure notions used to state the mixed-state renormalization condition. The renormalization fixed point of [CPGSV16, Definition 4.1] is stated in the next section.

Theorem 20.1.1 (Tensor extension preserves trace-preserving complete positivity). [Lean](#) [Lean](#)

[✓ Stmt](#)

Let $\mathcal{S} : \text{End}_{\mathbb{C}}(H) \rightarrow \text{End}_{\mathbb{C}}(K)$ be trace-preserving and completely positive, and let R be another finite-dimensional space. Then

$$\begin{aligned}\mathcal{S} \otimes \text{id}_R &: \text{End}_{\mathbb{C}}(H \otimes R) \longrightarrow \text{End}_{\mathbb{C}}(K \otimes R), \\ \text{id}_R \otimes \mathcal{S} &: \text{End}_{\mathbb{C}}(R \otimes H) \longrightarrow \text{End}_{\mathbb{C}}(R \otimes K)\end{aligned}$$

are trace-preserving and completely positive.

Proof. [✓ Proof](#) The first assertion is the tensor-extension closure theorem. For the second, let $\tau_H : R \otimes H \simeq H \otimes R$ and $\tau_K : K \otimes R \simeq R \otimes K$ be the canonical factor swaps. Then

$$\text{id}_R \otimes \mathcal{S} = \mathcal{R}_{\tau_K} \circ (\mathcal{S} \otimes \text{id}_R) \circ \mathcal{R}_{\tau_H},$$

where $\mathcal{R}_{\tau}$ denotes reindexing both matrix coordinates along τ . Both reindexing maps are trace-preserving and completely positive, so closure under composition proves the claim. No choice of Kraus family is needed for this preservation statement. $\square$

Lemma 20.1.2 (Compressed sums of completely positive maps). [Lean](#) [✓ Stmt](#)

Let $(P_i)_i$ be finitely many operators on H and let each $\Phi_i : \text{End}_{\mathbb{C}}(H) \rightarrow \text{End}_{\mathbb{C}}(K)$ be completely positive. Then

$$X \mapsto \sum_i \Phi_i(P_i X P_i^\dagger)$$

is completely positive. No completeness or orthogonality condition on the P_i is required.

Proof. [✓ Proof](#)

Composing each Kraus family of Φ_i with the corresponding P_i gives one combined rectangular Kraus family representing the sum. $\square$

Lemma 20.1.3 (Trace selection by projector-controlled sums). [Lean](#) [✓ Stmt](#)

Let $(P_i)_i$ be finitely many orthogonal projections on H , not necessarily resolving the identity, and let each $\Phi_i : \text{End}_{\mathbb{C}}(H) \rightarrow \text{End}_{\mathbb{C}}(K)$ be trace-preserving and completely positive. Then, for every X ,

$$\text{tr}\left(\sum_i \Phi_i(P_i X P_i)\right) = \text{tr}\left(\left(\sum_i P_i\right)X\right).$$

Proof. [✓ Proof](#) Trace preservation of each Φ_i reduces the left side to $\sum_i \text{tr}(P_i X P_i)$; cyclicity of the trace and idempotence of each projection give $\text{tr}(P_i X P_i) = \text{tr}(P_i X)$. $\square$

The state-preparation map $\mathcal{P}_\rho(X) = X \otimes \rho$ from matrices on A to matrices on $A \otimes B$, the elementary preparation operation used in the maps $\mathcal{T}_1$ and $\mathcal{S}_1$ of [CPGSV16, Appendix C.2, lines 1527–1533 and 1551–1555], is a fully generic construction with no MPDO-specific content, stated as Definition [LTC26, “State-preparation map”] in the companion quantum-channel volume.

Definition 20.1.4 (Physical closure maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For an MPO tensor M , a length N , and a virtual operator X , define the physical operator $M_N(X)$ by

$$M_N(X)_{i,j} = \text{tr}(M^{i_0 j_0} M^{i_1 j_1} \dots M^{i_{N-1} j_{N-1}} X).$$

This is linear in X . The displayed order chooses the cut of the cyclic virtual contraction immediately before the first site, so that X follows the last site. The one- and two-site operators are constructed in [CPGSV16, lines 638–654 and Definition 4.1]. When $M = \mathcal{K}$, write these operators as $\mathcal{K}_N(X)$; the cases $N = 2, 3$ occur in [CPGSV16, Proposition C.7, lines 1510–1516].

Lemma 20.1.5 (Periodic closure). [Lean](#) [✓ Stmt](#) Closing the virtual legs with the identity gives the periodic MPO operator: $M_N(1) = \rho^{(N)}(M)$.

Proof. [✓ Proof](#) For every pair of physical words σ, τ ,

$$M_N(1)_{\sigma, \tau} = \text{tr}(M^{\sigma, \tau} \cdot 1) = \text{tr}(M^{\sigma, \tau}) = \rho^{(N)}(M)_{\sigma, \tau}.$$

$\square$

Theorem 20.1.6 (One- and two-site physical closures). [Lean](#) [Lean](#) [✓ Stmt](#) Under the canonical identifications of one-site configurations with physical indices and two-site configurations with pairs of physical indices,

$$\begin{aligned} M_1(X)_{i,j} &= \text{tr}(M^{ij} X), \\ M_2(X)_{(i_0, i_1), (j_0, j_1)} &= \text{tr}(M^{i_0 j_0} M^{i_1 j_1} X). \end{aligned}$$

Proof. ✓ Proof Under these identifications, the length-one and length-two word evaluations are M^{ij} and $M^{i_0 j_0} M^{i_1 j_1}$, respectively. $\square$

Lemma 20.1.7 (Three-site physical closure). Lean Lean Lean ✓ Stmt *The general three-site physical closure has coefficients*

$$M_3(X)_{(i_0, (i_1, i_2)), (j_0, (j_1, j_2))} = \text{tr}(M^{i_0 j_0} M^{i_1 j_1} M^{i_2 j_2} X).$$

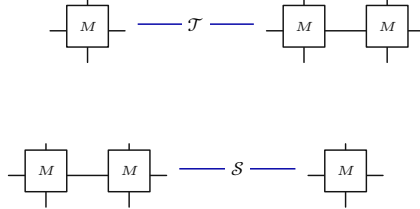
For $M = \mathcal{K}$, this is the operator $\mathcal{K}_3(X)$ in [CPGSV16, Proposition C.7, lines 1510–1516].

Proof. ✓ Proof The length-three word evaluation is $M^{i_0 j_0} M^{i_1 j_1} M^{i_2 j_2}$. $\square$

20.2 Renormalization fixed points

The following is the mixed-state fixed-point condition of [CPGSV16, Definition 4.1]. Its two local channels extend to every longer periodic chain by acting on the first one or two sites and leaving the remaining sites unchanged.

Definition 20.2.1 (Renormalization fixed point via trace-preserving maps). Lean ✓ Stmt An MPO tensor M is a *renormalization fixed point* if there exist two trace-preserving completely positive maps $\mathcal{S}, \mathcal{T}$ on the physical indices such that $\mathcal{S}[M_2(X)] = M_1(X)$ and $\mathcal{T}[M_1(X)] = M_2(X)$ for every virtual operator X , where $M_1(X)_{ij} = \text{tr}(M^{ij} X)$ and $M_2(X)_{(i_1 i_2)(j_1 j_2)} = \text{tr}(M^{i_1 j_1} M^{i_2 j_2} X)$ are the one- and two-site physical operators obtained by closing one or two tensors with X . This condition is distinct from idempotence of the doubled-index transfer map for general mixed states. Its source zero-correlation-length consequence concerns the physical-trace transfer and is proved below. In [CPGSV16, Definition 4.1, lines 638–660], this local condition is imposed on a tensor already in canonical form and generating matrix product density operators. The definition above isolates the two channel equations; results that use the source’s standing hypotheses state them separately. On the open coefficient tensors the two maps have types:



Theorem 20.2.2 (Pure-state specialization of the physical-map fixed point). Lean Lean ✓ Stmt Let $A = (A^i)_{i=0}^{d-1}$ be an MPS tensor, and let $M(A)^{ij} = A^i \otimes \overline{A^j}$ be its doubled MPO tensor. Then

$$\begin{aligned} M(A) \text{ satisfies Definition 20.2.1} &\iff \mathcal{E}_A^2 = \mathcal{E}_A \\ &\iff A \text{ has a physical blocking isometry.} \end{aligned}$$

More explicitly, if $V: \mathbb{C}^d \rightarrow \mathbb{C}^{d^2}$ is the blocking isometry, the refinement channel has the orientation

$$\mathcal{T}(X) = V X V^\dagger.$$

If $d > 0$, put $P = V V^\dagger$ and choose a density matrix ρ_0 on the one-site space. A trace-preserving coarse-graining channel is

$$\mathcal{S}(Y) = V^\dagger Y V + \text{tr}((\mathbb{1} - P)Y(\mathbb{1} - P))\rho_0.$$

On the range of V , the second term vanishes and this agrees with the source construction $\mathcal{S}(Y) = \text{tr}_2(U^\dagger Y U)$ after extending V to a unitary with a fixed pure ancilla. If $d = 0$, both matrix spaces are trivial and the unique zero maps give $\mathcal{S}$ and $\mathcal{T}$. This is the pure-state specialization stated at [CPGSV16, line 668, labels *Def-equiv-pure* and *T-and-S-mixed-equiv-pure*, lines 675–687].

Proof. ✓ Proof The blocking relation $A^{i_1} A^{i_2} = \sum_j V_{(i_1, i_2), j} A^j$ implies

$$M_2(X) = V M_1(X) V^\dagger.$$

For $d > 0$, the refinement identity follows directly:

$$\mathcal{T}(M_1(X)) = V M_1(X) V^\dagger = M_2(X).$$

For the coarse-graining identity, put $P = V V^\dagger$. Since $V^\dagger V = \mathbb{1}$ and $(\mathbb{1} - P)V = 0$, substitution gives

$$\begin{aligned} \mathcal{S}(M_2(X)) &= V^\dagger (V M_1(X) V^\dagger) V + \text{tr}((\mathbb{1} - P)(V M_1(X) V^\dagger)(\mathbb{1} - P)) \rho_0 \\ &= M_1(X) + 0 = M_1(X). \end{aligned}$$

If $d = 0$, the unique zero channels satisfy both identities. Conversely, trace preservation of $\mathcal{T}$ gives idempotence of the physical-trace transfer of $M(A)$, as in [CPGSV16, Appendix C, lines 1333–1340]. After exchanging the two tensor factors, this matrix is the transfer matrix of $\mathcal{E}_A$. Faithfulness of the transfer-matrix representation therefore gives $\mathcal{E}_A^2 = \mathcal{E}_A$. The physical-isometry formulation follows from Theorem 18.1.5. $\square$

Definition 20.2.3 (Fibonacci vacuum-sector support tensor). Lean Lean Lean Lean Lean Lean

✓ Stmt This is the concrete tensor in [CPGSV16, Appendix D, lines 2116–2121]. Its physical labels are triples $i, j, k \in \{0, 1\}$ and its virtual labels are $\alpha, \beta \in \{0, 1\}$. The fusion weights satisfy

$$N_{ijk} = 0 \iff (i, j, k) \in \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}. \quad (20.1)$$

Every weight outside this zero set is positive, and

$$A_{\alpha\beta}^{ijk} = \delta_{i,\alpha} \delta_{k,\beta} N_{ijk}.$$

The canonical support tensor takes every nonzero N_{ijk} to be one.

Definition 20.2.4 (Fibonacci transition matrix). Lean ✓ Stmt The transition matrix from [CPGSV16, Appendix D, lines 2146–2149] is

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Definition 20.2.5 (Fibonacci boundary transition counts). Lean ✓ Stmt For $n \in \mathbb{N}$ and $\alpha, \beta \in \{0, 1\}$, define

$$x_{\alpha\beta}^n = (B^n)_{\alpha\beta}. \quad (20.2)$$

These are the transition counts governed by the recurrence in [CPGSV16, Appendix D, lines 2131–2149].

Theorem 20.2.6 (Fibonacci boundary transition recurrence). Lean Lean ✓ Stmt In the order $(00, 01, 10, 11)$, one has $x^1 = (1, 1, 1, 2)$ and

$$x_{\alpha 0}^{n+1} = x_{\alpha 0}^n + x_{\alpha 1}^n, \quad x_{\alpha 1}^{n+1} = x_{\alpha 0}^n + 2x_{\alpha 1}^n.$$

This is the recurrence in [CPGSV16, Appendix D, lines 2131–2145].

Proof. [✓ Proof](#) Multiply the row indexed by α in B^n by the two columns of B . □

Definition 20.2.7 (Periodic Fibonacci transition count). [Lean](#) [✓ Stmt](#) Closing the two boundary labels gives the transition count

$$a_N = x_{00}^N + x_{11}^N = \text{tr}(B^N).$$

This is the boundary-label closure used in [CPGSV16, Appendix D, line 2157].

Theorem 20.2.8 (Characteristic recurrence for periodic Fibonacci transition counts). [Lean](#) [✓ Stmt](#) For every $N \in \mathbb{N}$,

$$a_{N+2} + a_N = 3a_{N+1}.$$

Proof. [✓ Proof](#) Expand $B^{N+2} = B^{N+1}B$ and $B^{N+1} = B^N B$. Substitution of $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ into the diagonal sums gives $a_{N+2} + a_N = 3a_{N+1}$. □

Theorem 20.2.9 (First periodic Fibonacci transition counts). [Lean](#) [Lean](#) [✓ Stmt](#) The first two periodic transition counts are $a_1 = 3$ and $a_2 = 7$.

Proof. [✓ Proof](#) Substitute $x^1 = (1, 1, 1, 2)$ into the recurrence once and close the two diagonal boundary labels. □

Theorem 20.2.10 (Golden-ratio formula for periodic Fibonacci transition counts). [Lean](#) [✓ Stmt](#) Set $\tau_+ = (1 + \sqrt{5})/2$ and $\tau_- = (1 - \sqrt{5})/2$. For every $N \in \mathbb{N}$,

$$a_N = \tau_+^{2N} + \tau_-^{2N}.$$

This is the transition-count formula in [CPGSV16, Appendix D, lines 2146–2176].

Proof. [✓ Proof](#) Use two-step induction on N . The cases $N = 0$ and $N = 1$ follow by direct calculation. Since $\tau_\pm^2 = \tau_\pm + 1$, each sequence $\tau_\pm^{2N}$ satisfies $\tau_\pm^{2(N+2)} + \tau_\pm^{2N} = 3\tau_\pm^{2(N+1)}$. The induction step now follows from the same recurrence for a_N . □

Theorem 20.2.11 (Periodic Fibonacci transition counts are not geometric). [Lean](#) [✓ Stmt](#) There are no $r, s \in \mathbb{N}$ such that $a_N = rs^{N-1}$ for every $N > 0$. This is the non-geometricity assertion at [CPGSV16, Appendix D, line 2125] for the transition-count sequence.

Proof. [✓ Proof](#) Such a formula would give $r = a_1 = 3$ and $7 = a_2 = 3s$, which is impossible in $\mathbb{N}$. □

Definition 20.2.12 (Open-boundary Fibonacci operator). [Lean](#) [Lean](#) [✓ Stmt](#) For $n > 0$ and $\alpha, \beta \in \{0, 1\}$, define the matrix $\rho_{\alpha\beta}^{(n)}(A)$, indexed by physical words σ and τ , by

$$(\rho_{\alpha\beta}^{(n)}(A))_{\sigma,\tau} = \begin{cases} (\alpha | A^{\sigma_1} \cdots A^{\sigma_n} | \beta), & \sigma = \tau, \\ 0, & \sigma \neq \tau. \end{cases} \quad (20.3)$$

Theorem 20.2.13 (Open-boundary Fibonacci operator ranks). [Lean](#) [✓ Stmt](#) For $n > 0$ and $\alpha, \beta \in \{0, 1\}$,

$$\text{rank } \rho_{\alpha\beta}^{(n)}(A) = x_{\alpha\beta}^n. \quad (20.4)$$

This is [CPGSV16, Appendix D, lines 2127–2130].

Proof. ✓ Proof A diagonal entry is nonzero exactly when its physical labels form an allowed path from α to β . Splitting a path at its first physical label shows that the numbers of supported words obey the transition rule determined by B , with the same length-zero initial values as $x_{\alpha\beta}^n$. Induction therefore identifies the number of nonzero diagonal entries with $x_{\alpha\beta}^n$. The rank of a diagonal matrix is the number of its nonzero diagonal entries. $\square$

Theorem 20.2.14 (Fibonacci periodic operator-rank formula). Lean Lean ✓ Stmt Let A be obtained from any positive choice of fusion weights with the support prescribed in Definition 20.2.3. Set $\tau_{\pm} = (1 \pm \sqrt{5})/2$, as in [CPGSV16, Appendix D, line 2125]. For every $N > 0$,

$$\text{rank } \rho^{(N)}(A) = a_N = x_{00}^N + x_{11}^N = \tau_+^{2N} + \tau_-^{2N}. \quad (20.5)$$

This is [CPGSV16, Appendix D, lines 2122–2125 and 2157–2176].

Proof. ✓ Proof The diagonal MPO rank counts physical words with nonzero periodic amplitude. For $N > 0$, the supports obtained by closing the virtual endpoint at 0 and at 1 are disjoint, since the first physical bit determines the initial endpoint. Their cardinalities are x_{00}^N and x_{11}^N by the open-boundary formula. The golden-ratio identity for a_N gives the final equality. $\square$

Theorem 20.2.15 (Fibonacci periodic operator ranks are not geometric). Lean ✓ Stmt Let A be obtained from any positive choice of fusion weights with the support prescribed in Definition 20.2.3. There are no $r, s \in \mathbb{N}$ such that $\text{rank } \rho^{(N)}(A) = rs^{N-1}$ for every $N > 0$. This is the operator-level conclusion in [CPGSV16, Appendix D, line 2125].

Proof. ✓ Proof The operator rank equals a_N at every positive length, while $a_1 = 3$ and $a_2 = 7$. A geometric formula would therefore give $r = 3$ and $7 = 3s$, which is impossible in $\mathbb{N}$. $\square$

Theorem 20.2.16 (The Fibonacci boundary tensor is not a strong fixed point). Lean ✓ Stmt Let A be obtained from any positive choice of fusion weights with the support prescribed in Definition 20.2.3. Then its canonical diagonal MPO tensor is not a strong renormalization fixed point.

Proof. ✓ Proof Suppose otherwise, and let P be the positive witness for the strong fixed-point relation. Then, for every $N > 0$, the periodic ranks satisfy

$$\text{rank } \rho^{(N)}(A) = \text{rank } \rho^{(1)}(A) (\text{rank } P)^{N-1}. \quad (20.6)$$

(20.6) contradicts the non-geometricity of the Fibonacci periodic ranks. $\square$

Definition 20.2.17 (Initial-coordinate regroupings). Lean Lean ✓ Stmt Let R_1 identify an $(N+1)$ -tuple with its first coordinate and the remaining N -tuple. Let R_2 identify an $(N+2)$ -tuple with its first two coordinates and the remaining N -tuple.

Definition 20.2.18 (Renormalization maps on a longer ring). Lean Lean ✓ Stmt Let N count the sites that are left unchanged. For physical maps $\mathcal{T} : M_d \rightarrow M_{d^2}$ and $\mathcal{S} : M_{d^2} \rightarrow M_d$, define

$$\begin{aligned} \widehat{\mathcal{T}}_N &= \mathcal{T} \otimes \text{id}_{d^N}, \\ \widehat{\mathcal{S}}_N &= \mathcal{S} \otimes \text{id}_{d^N}, \end{aligned}$$

using R_1 and R_2 . Thus $N = 0$ leaves no spectator site, but the maps still act on a one-site or two-site operator; no empty ring occurs.

Lemma 20.2.19 (Regrouping the first physical sites). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a physical word σ of length $N + 1$,

$$R_1(\sigma) = (\sigma_0, (\sigma_1, \dots, \sigma_N)),$$

$$R_1^{-1}(i, (a_1, \dots, a_N)) = (i, a_1, \dots, a_N).$$

For a physical word σ of length $N + 2$,

$$R_2(\sigma) = ((\sigma_0, \sigma_1), (\sigma_2, \dots, \sigma_{N+1})),$$

$$R_2^{-1}((i, j), (a_1, \dots, a_N)) = (i, j, a_1, \dots, a_N).$$

Proof. [✓ Proof](#) All four identities hold by definition of the product-index equivalences. $\square$

Lemma 20.2.20 (Localized renormalization maps are channels). [Lean](#) [Lean](#) [✓ Stmt](#) If $\mathcal{S}$ and $\mathcal{T}$ are trace-preserving completely positive, then so are $\hat{\mathcal{S}}_N$ and $\widehat{\mathcal{T}}_N$ for every N .

Proof. [✓ Proof](#) If $(K_j)_j$ are Kraus operators for $\mathcal{T}$ with $\sum_j K_j^\dagger K_j = I_d$, then the localized map has Kraus operators $K_j \otimes I_{d^N}$, and

$$\sum_j (K_j \otimes I_{d^N})^\dagger (K_j \otimes I_{d^N}) = \left(\sum_j K_j^\dagger K_j \right) \otimes I_{d^N} = I_d \otimes I_{d^N}.$$

The two regroupings are unitary conjugations and preserve complete positivity and the trace. The argument for $\mathcal{S}$ is identical. $\square$

Theorem 20.2.21 (Global action of the renormalization maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that $\mathcal{S}[M_2(X)] = M_1(X)$ and $\mathcal{T}[M_1(X)] = M_2(X)$ for every virtual operator X . Then, for every N and X ,

$$\hat{\mathcal{S}}_N[M_{N+2}(X)] = M_{N+1}(X),$$

$$\widehat{\mathcal{T}}_N[M_{N+1}(X)] = M_{N+2}(X).$$

In particular, these identities hold for every periodic MPO operator. Hence the two maps supplied by a renormalization fixed point act on rings of every length as required in the proof of [\[CPGSV16, Appendix C, lines 1333–1341\]](#).

Proof. [✓ Proof](#) For words u, v on the N spectator sites, write $M^{u,v}$ for their virtual word evaluation and write $[Y]_{u,v}^{(r)}$ for the block of an operator Y on the first r sites. Regrouping the first site gives

$$[M_{N+1}(X)]_{u,v}^{(1)} = M_1(M^{u,v}X).$$

Therefore

$$\mathcal{T}([M_{N+1}(X)]_{u,v}^{(1)}) = M_2(M^{u,v}X) = [M_{N+2}(X)]_{u,v}^{(2)}.$$

Equality of all blocks proves the refinement identity. Interchanging M_1, M_2 and using $\mathcal{S}[M_2(Y)] = M_1(Y)$ proves the coarsening identity. Setting $X = 1$ and using $M_N(1) = \rho^{(N)}(M)$ (Lemma [20.1.5](#)) gives the periodic-ring identities

$$\widehat{\mathcal{T}}_N[\rho^{(N+1)}(M)] = \rho^{(N+2)}(M),$$

$$\hat{\mathcal{S}}_N[\rho^{(N+2)}(M)] = \rho^{(N+1)}(M).$$

$\square$

20.3 Pure-state recovery inside the MPO formalism

Definition 20.3.1 (Diagonal pure-state embedding as an MPO). [Lean](#) [✓ Stmt](#) An MPS tensor $A = \{A^i\}_{i=0}^{d-1}$ determines an MPO tensor by placing A^i on the diagonal in the bra and ket indices:

$$M^{ij} = \delta_{ij} A^i.$$

Equivalently, $M^{ii} = A^i$ and $M^{ij} = 0$ for $i \neq j$.

Theorem 20.3.2 (Operator and rank of the diagonal pure-state embedding). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every N , the embedded MPO is diagonal with diagonal $V^{(N)}(A)$, and hence

$$\rho^{(N)}(A^{\text{MPO}}) = \text{diag } V^{(N)}(A), \quad \text{rank } \rho^{(N)}(A^{\text{MPO}}) = \#\{\sigma : V^{(N)}(A)_\sigma \neq 0\}.$$

Proof. [✓ Proof](#) A ket–bra word evaluation vanishes at the first unequal pair of physical labels. Equal words recover the original MPS word evaluation. The rank formula follows by counting the nonzero entries of a diagonal matrix. $\square$

Theorem 20.3.3 (Transfer map of the diagonal pure-state embedding). [Lean](#) [✓ Stmt](#) The transfer map of the diagonal MPO associated to A agrees with the original MPS transfer map: $\mathcal{E}_{A^{\text{MPO}}} = \mathcal{E}_A$.

Proof. [✓ Proof](#) In the double sum defining $\mathcal{E}_{A^{\text{MPO}}}$, all off-diagonal terms vanish because $M^{ij} = 0$ for $i \neq j$, while the diagonal terms are exactly $A^i X(A^i)^\dagger$. Thus only the sum over i remains, which is the defining formula for $\mathcal{E}_A$. $\square$

Theorem 20.3.4 (Pure-state recovery of transfer-map idempotence). [Lean](#) [✓ Stmt](#) For an MPS tensor A , the diagonal MPO embedding has idempotent transfer map if and only if the MPS transfer map is idempotent:

$$\mathcal{E}_{A^{\text{MPO}}} \circ \mathcal{E}_{A^{\text{MPO}}} = \mathcal{E}_{A^{\text{MPO}}} \iff \mathcal{E}_A \circ \mathcal{E}_A = \mathcal{E}_A.$$

Proof. [✓ Proof](#) Both conditions assert idempotence of the corresponding transfer map. Theorem 20.3.3 identifies those transfer maps, so the two predicates are equivalent. $\square$

Theorem 20.3.5 (Pure-state recovery of zero correlation length). [Lean](#) [✓ Stmt](#) For an MPS tensor A , the diagonal MPO embedding has idempotent transfer map if and only if A has zero correlation length.

Proof. [✓ Proof](#) Combine Theorem 20.3.4 with the characterization of zero correlation length by idempotence of the MPS transfer map (Theorem 18.5.21). $\square$

20.4 The unnormalized strong fixed-point relation

Appendix D of [CPGSV16, lines 2109–2116] considers a stronger local relation in which the two-site tensor is obtained by adjoining a positive physical matrix to one copy of the tensor and applying a unitary change of basis. The matrix is not assumed to have trace one.

Definition 20.4.1 (Strong renormalization fixed-point relation). [Lean](#) [✓ Stmt](#) An MPO tensor M satisfies the strong renormalization fixed-point relation if it admits a matrix $P \geq 0$ on the

physical space and a unitary U on the two-site physical space such that, for all two-site physical indices $i = (i_1, i_2)$ and $j = (j_1, j_2)$,

$$M^{i_1 j_1} M^{i_2 j_2} = \sum_{a,b} U_{i,a} P_{a_2 b_2} \overline{U_{j,b}} M^{a_1 b_1}. \quad (20.7)$$

Equivalently, the literal orientation of the source diagram is $M_2 = U(M_1 \otimes P)U^\dagger$, with M on the left physical site and P on the right. There is no virtual change of basis and no scalar proportionality. This is the relation labelled *Strong-RFP* in [CPGSV16, Appendix D, lines 2109–2113].

Theorem 20.4.2 (Physical closures of the strong fixed-point relation). [Lean] [Lean] [Stmt] *For fixed matrices P and U , the tensor identity (20.7) holds if and only if every virtual matrix X satisfies*

$$M_2(X) = U(M_1(X) \otimes P)U^\dagger. \quad (20.8)$$

Hence the strong renormalization fixed-point relation is equivalent to the existence of $P \geq 0$ and a two-site unitary U satisfying (20.8) for every X .

Proof. [Proof] Multiplying (20.7) on the right by X and taking the virtual trace gives, entrywise,

$$\text{tr}(M^{i_1 j_1} M^{i_2 j_2} X) = \sum_{a,b} U_{i,a} P_{a_2 b_2} \overline{U_{j,b}} \text{tr}(M^{a_1 b_1} X),$$

which is (20.8). Conversely, subtract the right-hand side of (20.7). Equation (20.8) says that its product with every X has trace zero. Nondegeneracy of the matrix trace pairing makes the difference zero. $\square$

Definition 20.4.3 (First-site preparation shuffle). [Lean] [Stmt] After R_1 separates the first physical coordinate from the N spectator coordinates, define

$$\Sigma_N : ((i, u), p) \mapsto ((i, p), u), \quad (20.9)$$

where $i, p \in \{0, \dots, d-1\}$ and $u \in \{0, \dots, d-1\}^N$. Thus, for a matrix Z indexed by $((i, u), p)$,

$$(\Sigma_N[Z])_{((i,p),u),((j,q),v)} = Z_{((i,u),p),((j,v),q)}. \quad (20.10)$$

Lemma 20.4.4 (Localized first-site preparation). [Lean] [Stmt] *Localizing the preparation $Y \mapsto Y \otimes P$ at the first site gives*

$$(\text{prep}_P \otimes \text{id})[Y] = \Sigma_N[Y \otimes P]. \quad (20.11)$$

Proof. [Proof] At row $((i, p), u)$ and column $((j, q), v)$, both sides equal $Y_{(i,u),(j,v)} P_{p,q}$. $\square$

Theorem 20.4.5 (Global operator identity for fixed strong fixed-point witnesses). [Lean] [Lean] [Stmt] *Fix witnesses P and U in (20.8). For every $N \geq 0$ and every virtual matrix X , write R_1 and R_2 for the initial-coordinate regroupings of lengths $N+1$ and $N+2$. Then*

$$R_2[M_{N+2}(X)] = (U \otimes \mathbb{1}) \Sigma_N(R_1[M_{N+1}(X)] \otimes P) (U^\dagger \otimes \mathbb{1}). \quad (20.12)$$

Equivalently, applying R_2^{-1} to the right-hand side gives the $(N+2)$ -site closure in its original physical coordinates. The same formula with $M_k(X)$ replaced by the ordinary periodic operator $\rho^{(k)}(M)$ holds after closing the virtual indices with the identity.

Proof. [✓ Proof](#) Apply Theorem 20.2.21 to the local map $Y \mapsto U(Y \otimes P)U^\dagger$ and the identity (20.8). This gives the localized closure equality. Expanding the regroupings R_1 and R_2 , localized unitary conjugation is conjugation by $U \otimes 1$, while Lemma 20.4.4 gives the middle factor $\Sigma_N(R_1[M_{N+1}(X)] \otimes P)$. The identity closure from Lemma 20.1.5 gives the periodic formula. $\square$

Theorem 20.4.6 (Rank invariance under unitary conjugation). [Lean](#) [✓ Stmt](#) *If V is unitary, then*

$$\text{rank}(VAV^\dagger) = \text{rank } A. \quad (20.13)$$

Proof. [✓ Proof](#) Left and right multiplication by invertible matrices preserve rank. $\square$

Theorem 20.4.7 (Geometric periodic rank growth of a strong fixed point). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Fix witnesses $P \geq 0$ and U for the strong fixed-point relation. For every $N \geq 0$, the ordinary full periodic MPO matrix ranks satisfy*

$$\text{rank } \rho^{(N+2)}(M) = \text{rank } \rho^{(N+1)}(M) \text{ rank } P, \quad (20.14)$$

$$\text{rank } \rho^{(N+1)}(M) = \text{rank } \rho^{(1)}(M) (\text{rank } P)^N. \quad (20.15)$$

Consequently every strong fixed point admits a single positive witness P whose matrix rank is the factor in both formulas. No normalization of P and no positivity or normalization assumption on the periodic operators is required.

Proof. [✓ Proof](#) Unitary conjugation and relabelling of matrix coordinates preserve rank, while rank is multiplicative under Kronecker products. Applying these facts to (20.12) gives the recurrence. Induction from chain length one gives the geometric formula. Finally, Theorem 20.4.2 exposes the same positive witness P for any tensor satisfying the existential strong fixed-point relation. $\square$

20.5 Zero correlation length

Definition 20.5.1 (Physical-trace transfer). [Lean](#) [Lean](#) [✓ Stmt](#) The physical-trace transfer of an MPO tensor M is the virtual matrix obtained by contracting the ket and bra physical legs of one tensor:

$$\mathcal{T}_M = \sum_i M^{ii}.$$

This is the transfer object appearing in the zero-correlation-length condition of [CPGSV16, Definition 4.2, lines 735–739].

Definition 20.5.2 (Literal physical-trace idempotence). [Lean](#) [✓ Stmt](#) An MPO tensor M has zero correlation length when its physical-trace transfer is idempotent:

$$\mathcal{T}_M^2 = \mathcal{T}_M.$$

This is the literal fixed-tensor condition of [CPGSV16, Definition 4.2, lines 735–739].

Theorem 20.5.3 (Equation for literal physical-trace idempotence). [Lean](#) [✓ Stmt](#) *An MPO tensor satisfies literal physical-trace idempotence if and only if $\mathcal{T}_M^2 = \mathcal{T}_M$.*

Proof. [✓ Proof](#) This is the defining equation. $\square$

Definition 20.5.4 (Nonzero up-to-scalar physical-trace relation). [Lean](#) [✓ Stmt](#) An MPO tensor satisfies the nonzero up-to-scalar physical-trace relation when, for some real number $\lambda > 0$, the following conditions hold:

$$\begin{aligned}\mathcal{T}_M &\neq 0, \\ \mathcal{T}_M^2 &= \lambda \mathcal{T}_M.\end{aligned}$$

The nonzero condition excludes the degenerate zero transfer, and the positive scalar makes the condition invariant under positive real rescaling of M . This is a scale-invariant relation at the level of the generated state family. It is broader than the paper’s literal diagram, which fixes $\lambda = 1$ for the displayed tensor.

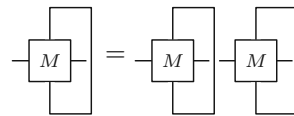
Theorem 20.5.5 (The nonzero physical-trace relation requires a nonzero bond space). [Lean](#) [✓ Stmt](#) *If an MPO tensor satisfies the nonzero up-to-scalar physical-trace relation, then its bond dimension is positive.*

Proof. [✓ Proof](#) On a zero-dimensional bond space every endomorphism vanishes, contrary to the nonzero physical-trace transfer in Definition 20.5.4. $\square$

Theorem 20.5.6 (Literal idempotence gives the up-to-scalar physical-trace relation). [Lean](#) [✓ Stmt](#) *If $\mathcal{T}_M \neq 0$ and M has literal physical-trace idempotence, then M satisfies the nonzero up-to-scalar physical-trace relation.*

Proof. [✓ Proof](#) This is the preceding definition with $\lambda = 1$. $\square$

$$\mathcal{T}_M = \mathcal{T}_M^2.$$



This is the literal identity in [CPGSV16, Definition 4.2, lines 735–739]; Definition 20.5.4 permits the positive factor λ .

Lemma 20.5.7 (Physical-trace normalization gives an idempotent transfer). [Lean](#) [✓ Stmt](#) *If M satisfies the nonzero up-to-scalar physical-trace relation, then there exists $\lambda > 0$ such that $\lambda^{-1}\mathcal{T}_M$ is idempotent.*

Proof. [✓ Proof](#) Choose $\lambda > 0$ from the up-to-scalar relation, so that $\mathcal{T}_M^2 = \lambda \mathcal{T}_M$. Then

$$(\lambda^{-1}\mathcal{T}_M)^2 = \lambda^{-2}\mathcal{T}_M^2 = \lambda^{-1}\mathcal{T}_M.$$

$\square$

Definition 20.5.8 (Doubled-index transfer idempotence for MPO tensors). [Lean](#) [✓ Stmt](#) An MPO tensor M satisfies the doubled-index transfer condition when its completely positive transfer map is idempotent:

$$\mathcal{E}_M \circ \mathcal{E}_M = \mathcal{E}_M.$$

This is the condition obtained from the doubled-index MPS view. It is not the physical-trace condition of Definition 20.5.2.

Proof. ✓ Proof Since the two virtual sectors are diagonal, set $z_0 = 1$ and $z_1 = -1$. A physical configuration σ has weight

$$2^{-N} + 4^{-N} \prod_{k=0}^{N-1} z_{\sigma_k}.$$

This proves the closed-operator formula. Each diagonal entry is either $2^{-N} + 4^{-N}$ or $2^{-N} - 4^{-N}$ and is nonnegative. Contracting the physical indices gives $\mathcal{T}_K = \text{diag}(1, 0)$, whence $\text{tr } \rho_N(K) = \text{tr}(\mathcal{T}_K^N) = 1$ for $N \geq 1$ and $\mathcal{T}_K^2 = \mathcal{T}_K$.

Direct contraction with a virtual matrix X gives the two closure formulas. Set $X_* = \text{diag}(2, 8)$. Then

$$\begin{aligned} K_2(X_*) &= \frac{1}{2}(I \otimes I + Z \otimes Z) \geq 0, \\ K_1(X_*) &= I + 2Z = \text{diag}(3, -1) \not\geq 0. \end{aligned}$$

If the coarse-graining channel $\mathcal{S}$ of Definition 20.2.1 existed, complete positivity would give $\mathcal{S}[K_2(X_*)] \geq 0$, whereas its intertwining identity would give $\mathcal{S}[K_2(X_*)] = K_1(X_*)$. This is a contradiction. $\square$

Theorem 20.5.12 (A canonical-form, area-law-saturating MPDO without fixed-tensor renormalization maps). Lean Lean Lean ✓ Stmt *Let K be the bond-two MPO tensor of Theorem 20.5.11, and let A_K be its doubled-index MPS tensor, an MPS tensor with four physical symbols. Then A_K is in literal CPSV canonical form: every letter is the weighted direct sum $A_K^{(i,j)} = \mu_0 A_0^{(i,j)} \oplus \mu_1 A_1^{(i,j)}$ for $i, j \in \{0, 1\}$, with weights and bond-one normal tensors*

$$\begin{aligned} \mu_0 &= \frac{1}{\sqrt{2}}, & \mu_1 &= \frac{1}{2\sqrt{2}}, \\ A_0^{(i,j)} &= \frac{\delta_{ij}}{\sqrt{2}}, & A_1^{(i,j)} &= (-1)^i \frac{\delta_{ij}}{\sqrt{2}}. \end{aligned}$$

Moreover, K saturates the area law: for every $N \geq 2$ and every $1 \leq L < N$, the reduced state of the first L sites is maximally mixed,

$$\rho_L^{(N)}(K) = 2^{-L} I, \quad S_L^{(N)}(K) = L \log 2.$$

Hence the mutual information $I_L^{(N)} = S_L^{(N)}(K) + S_{N-L}^{(N)}(K) - S_N^{(N)}(K) = N \log 2 - S_N^{(N)}(K)$ is constant in L , giving $I_1 = I_2 = \dots = I_{\lfloor N/2 \rfloor}$. Consequently, there is a bond-two MPO tensor whose doubled-index MPS tensor is in literal CPSV canonical form, which generates an MPDO, whose physical-trace transfer is idempotent, which saturates the area law, and which is not a renormalization fixed point in the sense of Definition 20.2.1.

The canonical-form decomposition follows the construction of [CPGSV16, Section 2.3, lines 214–245]. The decomposition and the entropy evaluation are project derivations; neither is asserted for this tensor in [Kat24] or in [CPGSV16].

Proof. ✓ Proof Every letter of A_K is diagonal and the block letters are scalars, so the weighted direct sum is the entrywise check

$$\mu_0 A_0^{(i,i)} = \frac{1}{2}, \quad \mu_1 A_1^{(i,i)} = \frac{(-1)^i}{4},$$

whose right-hand sides are the two virtual diagonal entries of K^{ii} ; the letters with $i \neq j$ vanish on both sides. Each block satisfies $\sum_{i,j} A_k^{(i,j)} \overline{A_k^{(i,j)}} = 1$, hence has identity transfer map and is a normal tensor by Lemma E.1.3. The weighted direct sum of positive-dimensional normal blocks is in literal CPSV canonical form by Definition 8.6.7.

Since $\text{tr } \rho_N(K) = 1$, the normalized state is $\rho_N(K)$ itself. For $1 \leq L < N$, the partial trace over the complementary $N - L$ sites annihilates the summand $4^{-N} Z^{\otimes N}$ of $\rho_N(K) = 2^{-N} I + 4^{-N} Z^{\otimes N}$, since $\text{tr } Z = 0$, and contracts the remaining summand to $2^{-L} I$. Hence $\rho_L^{(N)}(K) = 2^{-L} I$ and $S_L^{(N)}(K) = L \log 2$. Substitution in $I_L^{(N)} = S_L^{(N)}(K) + S_{N-L}^{(N)}(K) - S_N^{(N)}(K)$ gives $I_L^{(N)} = N \log 2 - S_N^{(N)}(K)$, which is independent of L ; in particular $I_L = I_{L+1}$ whenever $1 \leq L < \lfloor N/2 \rfloor$, so K saturates the area law by Definition 20.7.4.

Theorem 20.5.11 supplies the remaining three properties: K generates an MPDO, its physical-trace transfer is idempotent, and it is not a renormalization fixed point. Together with the two properties above, this gives the existential. $\square$

Theorem 20.5.13 (Renormalization fixed points satisfy the up-to-scalar physical-trace relation).

[Lean](#) [✓ Stmt](#) *If M is a renormalization fixed point via trace-preserving maps and $\mathcal{T}_M \neq 0$, then M satisfies the nonzero up-to-scalar physical-trace relation.*

Proof. [✓ Proof](#) Apply the preceding idempotence theorem and take the positive scalar in Definition 20.5.4 to be 1. $\square$

20.6 Purification fixed points

Definition 20.6.1 (Purifying spin–ancilla tensor). [Lean](#) [✓ Stmt](#) Given matrices $A^{(i,k)}$ with a spin index i and an ancillary index k , the associated MPS tensor on the product physical space is $\widehat{A}^{(i,k)} = A^{(i,k)}$. This is the tensor whose finite-chain vector appears in the purification formula of [CPGSV16, lines 744–751].

Definition 20.6.2 (Ancillary trace of a purifying MPS). [Lean](#) [✓ Stmt](#) For a purifying spin–ancilla tensor A , the reduced spin density matrix is defined by

$$[\rho_p^{(N)}(A)]_{\sigma,\tau} = \sum_{\kappa} \psi_A(\sigma, \kappa) \overline{\psi_A(\tau, \kappa)},$$

$$\psi_A(\sigma, \kappa) = \text{tr}(A^{(\sigma_0, \kappa_0)} A^{(\sigma_1, \kappa_1)} \dots A^{(\sigma_{N-1}, \kappa_{N-1})}).$$

This is the coefficient form of the ancillary trace in [CPGSV16, eq. (4.7), line 751].

Definition 20.6.3 (Single-site ancillary trace map). [Lean](#) [✓ Stmt](#)

For a matrix X on the spin–ancilla space, the ancillary trace map is

$$[\text{tr}_a(X)]_{ij} = \sum_k X_{(i,k),(j,k)}.$$

This is the one-site trace over ancillary degrees of freedom appearing in [CPGSV16, lines 751 and 761–764].

Theorem 20.6.4 (The ancillary trace is trace-preserving completely positive). [Lean](#) [✓ Stmt](#) *For every ancillary dimension, the one-site ancillary trace map is trace-preserving and completely positive.*

Proof. ✓ Proof

This is Lemma [LTC26, “The right partial trace is trace-preserving completely positive”] with A the spin space and B the ancillary space. $\square$

Definition 20.6.5 (Global purification equation). Lean ✓ Stmt An MPO tensor M satisfies the global purification equation with a purifying tensor A if $\rho^{(N)}(M) = \rho_p^{(N)}(A)$ for every positive length N . Equivalently, for every positive length N ,

$$\rho^{(N)}(M) = \text{tr}_a(|\Psi^{(N)}(A)\rangle\langle\Psi^{(N)}(A)|),$$

as in [CPGSV16, line 751].

Definition 20.6.6 (Trace-preserving spin reduction). Lean ✓ Stmt The spin reduction obtained by tracing the ancillary index is trace-preserving completely positive: the ancillary trace map tr_a satisfies the corresponding definition in the companion quantum-channel volume [LTC26, “Trace-preserving completely positive map”]. This is the structure described immediately after Definition 4.3 in [CPGSV16, lines 761–764].

Corollary 20.6.7 (Trace-preserving spin reduction of the ancillary trace). Lean ✓ Stmt *For every ancillary dimension, the spin reduction obtained by tracing the ancillary index is trace-preserving and completely positive.*

Proof. ✓ Proof This is exactly Theorem 20.6.4, unpacked through Definition 20.6.6. $\square$

Definition 20.6.8 (Bare global purification RFP witness). Lean ✓ Stmt An MPO tensor M has a bare global purification RFP witness if it satisfies the global purification equation for some purifying spin–ancilla tensor A , and $\hat{A}$ is a pure-state renormalization fixed point: $\mathcal{E}_{\hat{A}} \circ \mathcal{E}_{\hat{A}} = \mathcal{E}_{\hat{A}}$. This records the displayed conditions of Definition 4.3 in [CPGSV16, lines 756–764]. It does not include the nondegeneracy implicit when the subsequent theorem speaks of density operators.

Definition 20.6.9 (Bare global purification RFP). Lean ✓ Stmt An MPO tensor satisfies the bare global purification RFP predicate if it has a positive-length global purification RFP witness. Equivalently, it admits a purifying tensor A such that, for every positive length N ,

$$\rho^{(N)}(M) = \text{tr}_a(|\Psi^{(N)}(A)\rangle\langle\Psi^{(N)}(A)|),$$

and $\hat{A}$ is a pure-state renormalization fixed point. These are the displayed conditions of Definition 4.3 in [CPGSV16, lines 756–764]; the nondegenerate tensor-level predicate is Definition 20.6.13.

Definition 20.6.10 (Trace-preserving spin reduction for a purification RFP). Lean ✓ Stmt A purification RFP is viewed together with trace-preserving spin reduction by applying Corollary 20.6.7 to the ancillary trace map appearing in its purification equation. This records the structure obtained after tracing the ancillary indices in [CPGSV16, lines 761–764], without adding a separate hypothesis to Definition 4.3.

Lemma 20.6.11 (Forgetting the recorded spin reduction). Lean Lean ✓ Stmt *Recording the trace-preserving ancillary reduction does not change the purification RFP condition: $\text{PRFP}_{\text{tp}}(M) \iff \text{PRFP}(M)$.*

Proof. ✓ Proof Immediate from the equivalence in Definition 20.6.10. $\square$

Definition 20.6.12 (Local purification RFP condition). [Lean](#) [✓ Stmt](#) An MPO tensor M satisfies the local purification RFP condition if it is an LPDO whose purifying tensor A , viewed as a matrix product state tensor on the combined spin–ancilla index, is a pure-state renormalization fixed point. Thus

$$M^{ij} = \left(\sum_k A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_{e,e}.$$

This local condition is motivated by the purification tensor formula [CPGSV16, lines 744–747], but it is a one-site tensor-level condition rather than the global finite-chain definition in Definition 20.6.5.

Definition 20.6.13 (Nondegenerate purification RFP). [Lean](#) [✓ Stmt](#) An MPO tensor is a nondegenerate purification renormalization fixed point if it is the local ancillary contraction of a pure-state renormalization fixed point and $\mathcal{T}_M = \sum_i M^{ii} \neq 0$. The nonzero condition records the nondegeneracy implicit in the density operators of [CPGSV16, lines 744–786] and excludes the zero purifying tensor.

Theorem 20.6.14 (A local purification RFP generates MPDOs). [Lean](#) [✓ Stmt](#) Every tensor satisfying the local purification RFP condition generates matrix product density operators.

Proof. [✓ Proof](#) Dropping the renormalization-fixed-point condition on the purifying tensor leaves the local purification structure, so M is an LPDO; the conclusion follows from Theorem 17.3.3. $\square$

Theorem 20.6.15 (Local purification RFP does not imply doubled-index idempotence). [Lean](#) [✓ Stmt](#) There is an MPO tensor satisfying the local purification RFP condition for which the literal transfer-map idempotence $\mathcal{E}_M \circ \mathcal{E}_M = \mathcal{E}_M$ fails.

Proof. [✓ Proof](#) Take the diagonal purification at $d = d_K = 2$ and $D = D' = 1$ with amplitudes $A = [\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}]$. The purifying tensor is a pure-state renormalization fixed point, since $\sum |A|^2 = 1$ makes its transfer map the identity. The ancilla contraction $M^{ij} = \sum_k A^{(i,k)} \overline{A^{(j,k)}}$ gives the scalar entries $M^{00} = M^{11} = \frac{1}{2}$ (off-diagonal 0), so the induced transfer map is $\frac{1}{2} \cdot \text{id}$ and $\mathcal{E}_M \circ \mathcal{E}_M = \frac{1}{4} \cdot \text{id} \neq \mathcal{E}_M$. The trace contraction has dropped the leading eigenvalue from 1 to $\frac{1}{2}$. By contrast, the physical-trace transfer is 1 and hence is literally idempotent. Thus this example separates doubled-index idempotence from the paper’s physical-trace condition. $\square$

Lemma 20.6.16 (Physical-trace transfer of the maximally mixed witness). [Lean](#) [✓ Stmt](#) For the maximally mixed witness M_w , closing the ket and bra physical legs gives $\mathcal{T}_{M_w} = M_w^{00} + M_w^{11} = \frac{1}{2} + \frac{1}{2} = 1$.

Proof. [✓ Proof](#) This is the direct computation of the two nonzero diagonal entries of the witness tensor. $\square$

Theorem 20.6.17 (The maximally mixed witness satisfies the nonzero physical-trace relation). [Lean](#) [✓ Stmt](#) The maximally mixed witness M_w satisfies the nonzero up-to-scalar physical-trace relation.

Proof. [✓ Proof](#) Lemma 20.6.16 gives $\mathcal{T}_{M_w} = 1$. The identity is nonzero and idempotent, so the relation holds with $\lambda = 1$. $\square$

Theorem 20.6.18 (Fixed local purification: pure RFP iff normalized physical-trace idempotence). [Lean](#) [✓ Stmt](#) Suppose that M is the ancillary contraction of a fixed spin-ancilla tensor A through a bond identification e :

$$M^{ij} = \left(\sum_k A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_{e,e}.$$

Then the purification tensor $\hat{A}$ is a pure-state renormalization fixed point if and only if $\mathcal{T}_M^2 = \mathcal{T}_M$. This is the normalized tensor-level equivalence underlying the purification theorem in [CPGSV16, lines 744–786].

Proof. [✓ Proof](#) Let K' be the transfer matrix of $\hat{A}$, let s interchange the two Kronecker factors, and let K denote the same Kronecker sum in the physical ordering. The local contraction gives

$$\begin{aligned} \mathcal{T}_M &= K_e, \\ K &= (K')_s. \end{aligned}$$

Since both reindexings are bijective and commute with matrix multiplication,

$$(K')^2 = K' \iff \mathcal{T}_M^2 = \mathcal{T}_M.$$

Injectivity of the transfer-matrix representation gives

$$(K')^2 = K' \iff \mathcal{E}_{\hat{A}} \circ \mathcal{E}_{\hat{A}} = \mathcal{E}_{\hat{A}}.$$

□

Theorem 20.6.19 (Physical-trace transfer is idempotent under the local purification RFP condition). [Lean](#) [✓ Stmt](#) If an MPO tensor M is the ancilla contraction of a pure-state renormalization fixed point, then its physical-trace transfer $\mathcal{T}_M = \sum_i M^{ii}$ is idempotent: $\mathcal{T}_M \mathcal{T}_M = \mathcal{T}_M$.

Proof. [✓ Proof](#) Apply the forward implication of Theorem 20.6.18 to the local purifying tensor and bond identification. □

Theorem 20.6.20 (Local purification form and transfer idempotence characterize a local purification RFP). [Lean](#) [✓ Stmt](#) An MPO tensor satisfies the local purification RFP condition if and only if it has a local purification and its physical-trace transfer is idempotent:

$$\text{LocalPRFP}(M) \iff \text{LPDO}(M) \text{ and } \mathcal{T}_M^2 = \mathcal{T}_M.$$

Proof. [✓ Proof](#) Choose the local purifying tensor and bond identification. The equivalence is Theorem 20.6.18 for that fixed purification. □

Theorem 20.6.21 (Nondegenerate PRFP equivalence). [Lean](#) [✓ Stmt](#) An MPO tensor is a nondegenerate purification renormalization fixed point if and only if it is a local purification density operator (Definition 17.3.2) and

$$\begin{aligned} \mathcal{T}_M &\neq 0, \\ \mathcal{T}_M^2 &= \mathcal{T}_M. \end{aligned}$$

This is the normalized, nonzero structural characterization used in the forward PRFP-to-ZCL implication of [CPGSV16, lines 744–786].

Proof. ✓ Proof Combine the preceding equivalence with the nonzero clause in Definition 20.6.13. □

Theorem 20.6.22 (Local purification RFP with nonzero physical-trace transfer satisfies the up-to-scalar relation). Lean ✓ Stmt *Let M be the ancilla contraction of a pure-state renormalization fixed point, satisfying the local purification renormalization fixed point condition. If the physical-trace transfer $\mathcal{T}_M = \sum_i M^{ii}$ is nonzero, then M satisfies the nonzero up-to-scalar physical-trace relation.*

Proof. ✓ Proof Theorem 20.6.19 gives idempotence of the physical-trace transfer, and the nonzero hypothesis lets Theorem 20.5.6 conclude the up-to-scalar relation with $\lambda = 1$. □

Theorem 20.6.23 (A nondegenerate purification RFP satisfies the nonzero physical-trace relation). Lean ✓ Stmt *Every nondegenerate purification renormalization fixed point satisfies the nonzero up-to-scalar physical-trace relation.*

Proof. ✓ Proof Definition 20.6.13 supplies both the local purification RFP condition and the nonzero physical-trace transfer required by Theorem 20.6.22. □

Theorem 20.6.24 (The tensor purification identity yields the global purification equation). Lean ✓ Stmt *Let M be the ancilla contraction of a purifying spin-ancilla tensor A through a bond identification e , so that*

$$M^{ij} = \left(\sum_k A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_{e,e}.$$

Then at every length N the matrix product operator equals the ancillary trace of the pure spin-ancilla matrix product state, $\rho^{(N)}(M) = \rho_p^{(N)}(A)$. This is the coefficient form of the purification equation of [CPGSV16, line 751].

Proof. ✓ Proof Applying the mixed-product property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ repeatedly, the length- N product of the contracted tensor entries distributes over the ancillary index sum:

$$\prod_{l=0}^{N-1} \left(\sum_k A^{(\sigma_l, k)} \otimes \overline{A^{(\tau_l, k)}} \right) = \sum_{\kappa: [N] \rightarrow [d_K]} \left(\prod_l A^{(\sigma_l, \kappa_l)} \right) \otimes \overline{\left(\prod_l A^{(\tau_l, \kappa_l)} \right)}.$$

Tracing this sum and using $\text{tr}(X \otimes \overline{Y}) = \text{tr}(X)\text{tr}(\overline{Y})$ gives the coefficient of the ancillary trace at every σ, τ . The equation holds at every length, so no positive-length restriction is needed. □

Theorem 20.6.25 (The local purification RFP condition implies the purification RFP). Lean ✓ Stmt *Every MPO tensor satisfying the local purification renormalization fixed point condition is a purification renormalization fixed point in the sense of Definition 4.3 of [CPGSV16, lines 756–764].*

Proof. ✓ Proof The local condition supplies a purifying tensor A that is a pure-state renormalization fixed point and whose ancilla contraction is M . By Theorem 20.6.24 this contraction satisfies the global purification equation at every positive length, so the same A is a purification RFP witness. □

Corollary 20.6.26 (A nondegenerate purification RFP satisfies the global PRFP predicate). Lean ✓ Stmt *Every nondegenerate purification renormalization fixed point satisfies the positive-length global purification predicate of Definition 4.3 in [CPGSV16, lines 744–758].*

Proof. ✓ Proof Forget the nonzero condition and apply Theorem 20.6.25. □

Theorem 20.6.27 (Printed purification fixed-point equivalence). Not ready *Let an MPO tensor admit the global purification equation of Definition 20.6.5. According to [CPGSV16, Theorem 4.4], being a purification renormalization fixed point, having zero correlation length, and admitting the displayed trace-preserving purification form are equivalent.*

This equivalence is false under the printed hypotheses. Positive-length equations do not detect nilpotent hidden bond sectors. Theorem 20.6.30 gives a nonzero matrix product density operator that is a purification fixed point but fails the literal physical-trace idempotence equation of Definition 20.5.2. The corrected local purification equivalences above require a tensor-level purification identity and do not recover the printed global statement.

Theorem 20.6.28 (The bare global PRFP predicate does not imply the nonzero physical-trace relation). Lean ✓ Stmt *There is an MPO tensor satisfying the bare positive-length global purification RFP predicate which does not satisfy the nonzero up-to-scalar physical-trace relation.*

Proof. ✓ Proof Take the zero tensor at spin, ancillary, and bond dimensions one, purified by the zero spin-ancilla tensor. Its transfer map is idempotent and every positive-length density operator is zero, so the global purification equation holds. Its physical-trace transfer is zero, which is excluded by the nonzero up-to-scalar physical-trace relation. □

Theorem 20.6.29 (A nonzero global PRFP need not satisfy the up-to-scalar physical-trace relation). Lean ✓ Stmt *There is an MPO tensor which is an MPDO, satisfies the positive-length global purification RFP predicate, and has nonzero physical-trace transfer, but does not satisfy the nonzero up-to-scalar physical-trace relation.*

Proof. ✓ Proof Take one physical state and bond dimension three, with the only tensor entry

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\text{tr}(Q^N) = 1$ for every $N > 0$, all positive-length MPOs equal the scalar density operator 1 and are purified by the scalar pure-state RFP tensor $A = 1$. At length zero the MPO is the positive scalar 3, so the tensor is an MPDO at every length. Its physical-trace transfer is the nonzero matrix Q . However, $Q^2 = \text{diag}(1, 0, 0)$ is not a positive scalar multiple of Q , because the $(2, 3)$ entry is lost on squaring. Thus the up-to-scalar physical-trace relation fails. The global equation cannot detect this nilpotent bond sector. □

Theorem 20.6.30 (A nonzero global PRFP need not satisfy literal physical-trace idempotence). Lean ✓ Stmt *There is an MPO tensor which is an MPDO, satisfies the positive-length global purification RFP predicate, and has nonzero physical-trace transfer, but its physical-trace transfer is not idempotent.*

Proof. ✓ Proof Use the tensor from Theorem 20.6.29. If its nonzero physical-trace transfer $\mathcal{T}_M$ were idempotent, then

$$\mathcal{T}_M^2 = \mathcal{T}_M = 1 \cdot \mathcal{T}_M,$$

which is the scalar-one case of the up-to-scalar relation refuted by that theorem. □

20.7 Saturation of the area law

Definition 20.7.1 (Block entropy of an MPDO). [Lean](#) [✓ Stmt](#) For a fixed system size N , a block length $L \leq N$, and an MPO tensor M such that $\rho^{(N)}(M)$ is positive semidefinite, let

$$\sigma^{(N)}(M) = \frac{\rho^{(N)}(M)}{\text{tr}[\rho^{(N)}(M)]},$$

$$\rho_L^{(N)}(M) = \text{tr}_{N \setminus L}[\sigma^{(N)}(M)]$$

denote the normalized state and the reduced state of the first L of N spins. The L -block entropy is $S_L^{(N)}(M) = S(\rho_L^{(N)}(M))$.

Theorem 20.7.2 (Block entropy from characteristic roots). [Lean](#) [✓ Stmt](#) If the characteristic roots of $\rho_L^{(N)}(M)$ are the real multiset $\{r_j\}_j$, then

$$S_L^{(N)}(M) = \sum_j -r_j \log r_j.$$

Proof. [✓ Proof](#) The spectral decomposition of the Hermitian reduced state identifies the characteristic roots with its eigenvalues. The von Neumann entropy is the sum of $-r \log r$ over these eigenvalues, counted with multiplicity. $\square$

Definition 20.7.3 (Mutual information of an MPDO). [Lean](#) [Lean](#) [✓ Stmt](#) For the same fixed system size N , block length $L \leq N$, and positive semidefiniteness of $\rho^{(N)}(M)$, the *mutual information* between a block of L spins and the rest is $I_L = S_L^{(N)}(M) + S_{N-L}^{(N)}(M) - S_N^{(N)}(M)$.

Definition 20.7.4 (Saturation of the area law). [Lean](#) [✓ Stmt](#) An MPO tensor M that generates an MPDO and whose positive-length finite-chain operators $\rho^{(N)}(M)$ all have nonzero trace (so the normalized states are well-defined) *saturates the area law* if $I_L = I_{L+1}$ for all $1 \leq L < \lfloor N/2 \rfloor$ and all N ; that is, the mutual information is constant, $I_1 = I_2 = \dots = I_{\lfloor N/2 \rfloor}$. This is [CPGSV16, Definition 4.6, line 811].

Lemma 20.7.5 (Constancy on a finite interval). [Lean](#) [✓ Stmt](#) Let $a, n \in \mathbb{N}$, and let f be defined on the integer interval $[0, n]$. If $f(m) = f(m+1)$ for every $a \leq m < n$, then $f(i) = f(j)$ for all $i, j \in [a, n]$.

Proof. [✓ Proof](#) Compose the successive equalities from the smaller index to the larger. $\square$

Lemma 20.7.6 (Saturation telescopes to a constant mutual information). [Lean](#) [✓ Stmt](#) If an MPO tensor M saturates the area law, then the mutual informations coincide, $I_L = I_{L'}$, for all $1 \leq L, L' \leq \lfloor N/2 \rfloor$.

Proof. [✓ Proof](#) The defining condition supplies $I_m = I_{m+1}$ throughout the interval $1 \leq m < \lfloor N/2 \rfloor$. Apply Lemma 20.7.5. $\square$

Theorem 20.7.7 (Tripartite marginals as block entropies). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let the N -site operator generated by M be positive semidefinite, and let three consecutive regions have lengths a , b , and c , with $a + b + c \leq N$. After reindexing the reduced state of the first $a + b + c$ sites as a tripartite state on $A \otimes B \otimes C$, its four entropies in strong subadditivity are

$$\begin{aligned} S(ABC) &= S_{a+b+c}, & S(AB) &= S_{a+b}, \\ S(B) &= S_b, & S(BC) &= S_{b+c}. \end{aligned}$$

Proof. ✓ Proof Write $\rho^{(k)}$ for the first- k -site reduced state, with $\cong$ denoting equality up to a permutation of basis indices. Reindexing a matrix does not change its von Neumann entropy, giving the first equality $S(ABC) = S_{a+b+c}$ immediately. Tracing out C leaves the first $a + b$ sites, $\text{tr}_C(\rho_{ABC}) \cong \rho^{(a+b)}$, so $S(AB) = S_{a+b}$. Tracing out A and C leaves the middle b sites, while tracing out A leaves the final $b + c$ sites; translation invariance gives

$$\begin{aligned}\text{tr}_{A,C}(\rho_{ABC}) &\cong \rho^{(b)}, \\ \text{tr}_A(\rho_{ABC}) &\cong \rho^{(b+c)}.\end{aligned}$$

Hence $S(B) = S_b$ and $S(BC) = S_{b+c}$. □

Theorem 20.7.8 (Printed mutual-information proposition). Not ready According to [CPGSV16, Proposition 4.5], one has $I_L \leq I_{L+1}$ for $L < \lfloor N/2 \rfloor$, the mutual information is bounded in terms of the bond dimension, and the indicated iterated thermodynamic limit exists and is finite.

The monotonicity and finite-chain bound hold. The limit assertion is false without an additional convergence hypothesis. For the parity-sensitive positive MPDO of Theorem 20.7.13, the mutual information is 0 at odd chain lengths and $h_2(1/3) > 0$ at even chain lengths, so the inner limit does not exist.

Proposition 20.7.9 (Monotonicity of the mutual information). Lean ✓ Stmt Let M generate an MPDO whose finite-chain operator $\rho^{(N)}(M)$ is positive semidefinite with nonzero trace. For every block length L with $2L + 1 \leq N$, one has $I_L \leq I_{L+1}$. The implication $1 \leq L < \lfloor N/2 \rfloor \implies 2L + 1 \leq N$ shows that this contains the range of [CPGSV16, Proposition 4.5, line 801].

Proof. ✓ Proof Apply strong subadditivity to the reduced state of the first $N - L$ spins, split into three contiguous segments of lengths 1, L , and $N - 2L - 1$. Tracing the appropriate segments yields the four block entropies S_{N-L} , S_L , S_{L+1} , and S_{N-L-1} , where the cyclic invariance of $\rho^{(N)}(M)$ identifies the entropy of a contiguous block with that of any block of the same length. Strong subadditivity then gives $S_{N-L} + S_L \leq S_{L+1} + S_{N-L-1}$, which rearranges to $I_L \leq I_{L+1}$ after subtracting S_N . □

Definition 20.7.10 (Parity-sensitive classical MPDO). Lean ✓ Stmt Let M have physical dimension two and bond dimension three, with

$$\begin{aligned}M^{00} &= \text{diag}(1, 0, 0), & M^{11} &= \text{diag}(0, 1, -1), \\ M^{01} &= 0, & M^{10} &= 0.\end{aligned}$$

Theorem 20.7.11 (A periodic MPDO without a thermodynamic limit). Lean Lean Lean Lean ✓ Stmt The tensor in Definition 20.7.10 generates the positive operators

$$\rho^{(N)} = |0^N\rangle\langle 0^N| + (1 + (-1)^N)|1^N\rangle\langle 1^N|.$$

Their traces are $2 + (-1)^N$. The all-zero diagonal entry of the normalized one-site reduced state is therefore one for odd N and $1/3$ for positive even N . In particular, the normalized periodic one-site reduced states do not converge as $N \rightarrow \infty$.

Thus normalized periodic marginals need not converge under the hypotheses of [CPGSV16, Proposition 4.5, lines 801–806]. The mutual information of this same family is computed in [con26b].

Proof. ✓ Proof A contraction around the virtual loop vanishes unless the bra and ket configurations agree and are both constant. The all-zero configuration receives weight one. The all-one configuration receives the sum of the two virtual-sector weights, namely $1 + (-1)^N$. This gives

the displayed diagonal form. Its coefficients are non-negative: the second coefficient is zero for odd N and two for even N . Hence every positive-length operator is positive semidefinite, and summing its two possible diagonal entries gives the trace $2 + (-1)^N$.

Theorem 20.7.12 (Mutual information without a thermodynamic limit). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Fix $L \geq 1$. For every $N > L$, the tensor in Definition 20.7.10 satisfies

Consequently the sequence $K \mapsto I_L^{(L+K+1)}$ does not converge, giving the counterexample described in [CPGSV16, Proposition 4.5, lines 801–806].

Theorem 20.7.13 (Counterexample to the Proposition 4.5 limit clause). Lean ✔ Stmt *The parity-sensitive tensor is an MPDO, and for every fixed $L \geq 1$ the sequence $K \mapsto I_L^{(L+K+1)}$ has no real limit. Thus the unrestricted thermodynamic-limit clause of [CPGSV16, Proposition 4.5] is false. An aperiodicity or convergence condition that excludes the parity oscillation is a genuine boundary condition, not a hypothesis of the printed proposition.*

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Map on the second tensor factor”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Data processing on the second tensor factor”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Data processing for bipartite mutual information”].

Definition 20.7.14 (Normalized state across a consecutive bipartition). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) For $N = L + K$, split a physical configuration into consecutive intervals of lengths L and K , flatten each interval to a finite index, and denote the resulting bipartite form of the normalized periodic MPO by $\sigma_{L,K}^{(N)}$. The flattened local maps on the two factors are denoted by $\widehat{\mathcal{F}}_L$ and $\widehat{\mathcal{S}}_K$.

Lemma 20.7.15 (Chain mutual information across a consecutive cut). [\[Lean\]](#) [\[Lean\]](#) [\[Lean\]](#) [\[Lean\]](#)
 *Stm1* Across the consecutive cut after the first L sites, the two marginals of $\sigma_{L:N-L}^{(N)}$ are the reduced states on L and $N-L$ sites, up to the standard finite-index identifications. Consequently, $I_L(M)$ is the ordinary bipartite mutual information of $\sigma_{L:N-L}^{(N)}$.

Proof. ✓ Proof The first marginal is the defining partial trace over the last $N - L$ sites. Periodic translation invariance identifies the second marginal with the reduced state on the first $N - L$ sites. Entropy is invariant under the finite-index identifications, and retaining the full chain leaves the normalized state unchanged. $\square$

Lemma 20.7.16 (Renormalization maps transfer one site across a cut). [\[Lean\]](#) [\[Lean\]](#) [\[✓ Stmt\]](#)
Suppose that $N = (a + 1) + (b + 2) = (a + 2) + (b + 1)$. The local closure equations imply

$$\begin{aligned}(\widehat{\mathcal{T}}_a \otimes \widehat{\mathcal{S}}_b)(\sigma_{a+1:b+2}^{(N)}) &= \sigma_{a+2:b+1}^{(N)}, \\(\widehat{\mathcal{S}}_a \otimes \widehat{\mathcal{T}}_b)(\sigma_{a+2:b+1}^{(N)}) &= \sigma_{a+1:b+2}^{(N)}.\end{aligned}$$

Proof.  **Proof** Fixing the word on the second interval leaves a physical closure on the first interval, whose virtual boundary is the fixed word evaluation. Apply the appropriate global closure identity to this slice. Fixing the resulting word on the first interval gives the closure on the second interval. Cyclicity of the virtual trace identifies the two cuts of the periodic contraction. The reverse identity is the same argument with $\mathcal{S}$ and $\mathcal{T}$ interchanged. $\square$

Theorem 20.7.17 (Conditional RFP mutual-information equality). [Lean] [Lean] [Stm] Let M be a renormalization fixed point. If its N -site operator is positive semidefinite and has nonzero trace, then $I(\sigma_{a+1:b+2}^{(N)}) = I(\sigma_{a+2:b+1}^{(N)})$. This is the channel calculation in [CPGSV16, Appendix C, lines 1333–1341]. It depends only on positivity, nonzero trace, and the local renormalization equations; no horizontal decomposition enters the calculation.

Proof. ✓ **Proof** Apply mutual-information data processing to the forward pair of local channels and to the reverse pair. The two transfer identities turn the resulting inequalities into opposite inequalities between the displayed mutual informations. $\square$

Lemma 20.7.18 (Nonvanishing of positive-length BNT-refined RFP rings). [\[Lean\]](#) [\[Lean\]](#) [\[Stm\]](#)
Let M be in normalized BNT-refined horizontal form, generate positive semidefinite ring operators, and satisfy the local renormalization fixed-point equations. This horizontal hypothesis is stronger than literal CPSV canonical form. Then $\mathrm{tr} \rho^{(N)}(M) \neq 0$ for every $N \geq 1$.

Proof.  **Proof** Normalized BNT-refined horizontal form gives a nonzero sector compression for one positive length. Positivity makes its trace nonzero. The fixed-point equations make the physical-trace transfer idempotent, so the trace of its positive powers is independent of the positive exponent. $\square$

Theorem 20.7.19 (Trace-normalized renormalization fixed points saturate the area law). [Lean](#) [✓ Stmt](#) *Let M generate positive semidefinite ring operators with $\text{tr}[\rho^{(N)}(M)] \neq 0$ for every $N \geq 1$. If M satisfies the local renormalization fixed-point equations of Definition 4.1, then M saturates the area law: $I_L(M) = I_{L+1}(M)$ whenever $1 \leq L < \lfloor N/2 \rfloor$. Nonvanishing is the normalization boundary implicit in the density-operator language of [CPGSV16]. Positive semidefiniteness alone admits the zero family.*

Proof. [✓ Proof](#) Write the cut as $(a+1) : (b+2)$. The two channel identities and data processing give $I(\sigma_{a+2:b+1}^{(N)}) \leq I(\sigma_{a+1:b+2}^{(N)})$ and $I(\sigma_{a+1:b+2}^{(N)}) \leq I(\sigma_{a+2:b+1}^{(N)})$. Hence the two quantities are equal. The consecutive-cut identification gives $I_L(M) = I_{L+1}(M)$. $\square$

Corollary 20.7.20 (Horizontal renormalization fixed points saturate the area law). [Lean](#) [✓ Stmt](#) *Let M be in normalized BNT-refined horizontal form, generate positive semidefinite ring operators, and satisfy the local renormalization fixed-point equations of Definition 4.1. Then M saturates the area law.*

Proof. [✓ Proof](#) Lemma 20.7.18 gives nonzero trace at every positive length. Apply Theorem 20.7.19. $\square$

Theorem 20.7.21 (Trace-normalized renormalization fixed points have literal ZCL and SAL). [Lean](#) [Lean](#) [✓ Stmt](#) *Let M generate positive semidefinite ring operators with $\text{tr}[\rho^{(N)}(M)] \neq 0$ for every $N \geq 1$. If M satisfies the local renormalization fixed-point equations of Definition 4.1, then $\mathcal{T}_M^2 = \mathcal{T}_M$ and M saturates the area law. This is the literal ZCL and SAL forward implication proved in [CPGSV16, Appendix C, lines 1333–1341], at the normalization boundary implicit in its density-operator language. Positive semidefiniteness alone admits the zero family.*

Proof. [✓ Proof](#) Trace preservation in the local renormalization equations gives $\mathcal{T}_M^2 = \mathcal{T}_M$ for the physical-trace transfer. If $\mathcal{T}_M = 0$, then $\text{tr}[\rho^{(1)}(M)] = \text{tr}(\mathcal{T}_M) = 0$, contradicting nonvanishing at length one. Thus the literal idempotence is nontrivial, and Theorem 20.7.19 gives saturation of the area law. $\square$

Corollary 20.7.22 (Horizontal renormalization fixed points have literal ZCL and SAL). [Lean](#) [Lean](#) [✓ Stmt](#) *Let M be in normalized BNT-refined horizontal form, generate positive semidefinite ring operators, and satisfy the local renormalization fixed-point equations of Definition 4.1. Then $\mathcal{T}_M^2 = \mathcal{T}_M$ and M saturates the area law.*

Proof. [✓ Proof](#) Lemma 20.7.18 gives nonzero trace at every positive length. Apply Theorem 20.7.21. $\square$

Lemma 20.7.23 (Nonnegativity of block entropies). [Lean](#) [✓ Stmt](#) *For a normalized positive semidefinite finite-chain operator, every block entropy is non-negative: $0 \leq S_m$.*

Proof. [✓ Proof](#) The reduced state of a block is a density matrix, and the von Neumann entropy of a density matrix is non-negative. $\square$

Lemma 20.7.24 (Nonnegativity of the mutual information). [Lean](#) [✓ Stmt](#) *Let M generate an MPDO whose finite-chain operator $\rho^{(N)}(M)$ is positive semidefinite with nonzero trace. For every block length $L \leq N$, one has $0 \leq I_L$.*

Proof. [✓ Proof](#) Apply strong subadditivity with a trivial middle subsystem (equivalently, subadditivity) to the bipartition of the chain into the first L and last $N-L$ spins: $S_N + S_0 \leq S_L + S_{N-L}$. Since $S_0 \geq 0$ (Lemma 20.7.23), rearranging gives $I_L = S_L + S_{N-L} - S_N \geq S_0 \geq 0$. $\square$

Lemma 20.7.25 (Symmetry of the mutual information across the cut). [Lean](#) [✓ Stmt](#) For every block length $L \leq N$, the mutual information of the first L spins equals that of their complement: $I_L = I_{N-L}$.

Proof. [✓ Proof](#) Both sides equal $S_L + S_{N-L} - S_N$ by the symmetric definition $I_L = S_L + S_{N-L} - S_N$, using $N - (N - L) = L$. $\square$

Lemma 20.7.26 (Positive rank of a normalized state). [Lean](#) [✓ Stmt](#) A positive semidefinite matrix of trace one has positive rank.

Proof. [✓ Proof](#) If its rank were zero, every eigenvalue would vanish, contradicting that their sum is the trace, which equals one. $\square$

Lemma 20.7.27 (Zero entropy in rank one). [Lean](#) [✓ Stmt](#) A positive semidefinite matrix of trace one and rank at most one has zero von Neumann entropy.

Proof. [✓ Proof](#) Its rank is positive by Lemma 20.7.26, hence it is exactly one. The rank bound gives $S \leq \log 1 = 0$, while entropy is non-negative. $\square$

Lemma 20.7.28 (Entropy of the empty block). [Lean](#) [✓ Stmt](#) For a normalized positive semidefinite finite-chain operator, the block entropy of the empty block vanishes: $S_0 = 0$.

Proof. [✓ Proof](#) The reduced state of zero spins is one-dimensional, so its rank is at most one. Lemma 20.7.27 gives $S_0 = 0$. $\square$

Lemma 20.7.29 (Empty-block mutual information vanishes). [Lean](#) [✓ Stmt](#) For a normalized positive semidefinite finite-chain operator, the mutual information of the empty block vanishes: $I_0 = 0$.

Proof. [✓ Proof](#) By definition, $I_0 = S_0 + S_N - S_N = S_0$, which is zero by Lemma 20.7.28. $\square$

Definition 20.7.30 (Ancillary trace on a block). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a block of L spin-ancilla pairs, regroup

$$(\mathbb{C}^d \otimes \mathbb{C}^{d_K})^{\otimes L} \cong (\mathbb{C}^d)^{\otimes L} \otimes (\mathbb{C}^{d_K})^{\otimes L}$$

and trace the ancillary factor. Denote this map by $\text{Tr}_{\text{anc},L}$. It is the blockwise analog of the one-site ancillary trace in Definition 20.6.3.

Theorem 20.7.31 (Block ancillary trace is a channel). [Lean](#) [✓ Stmt](#) The map $\text{Tr}_{\text{anc},L}$ is trace-preserving and completely positive.

Proof. [✓ Proof](#) The reindexing equivalence is a channel, as is the partial trace over the ancillary factor. Their composition is therefore trace-preserving and completely positive. $\square$

Theorem 20.7.32 (Coefficients of the ancillary traces). [Lean](#) [Lean](#) [✓ Stmt](#) If X is an operator on a spin-ancilla block, then for $i, j \in [d]^L$,

$$\text{Tr}_{\text{anc},L}(X)_{i,j} = \sum_{\kappa \in [d_K]^L} X_{(i,\kappa),(j,\kappa)}.$$

For $i, j \in [d]^L$ and $x, y \in [d]^K$, applying the ancillary traces to both blocks across the cut $L \mid K$ gives

$$[(\text{Tr}_{\text{anc},L} \otimes \text{Tr}_{\text{anc},K})(X)]_{(i,x),(j,y)} = \sum_{\kappa \in [d_K]^L} \sum_{\lambda \in [d_K]^K} X_{((i,\kappa),(x,\lambda)),((j,\kappa),(y,\lambda))}.$$

Proof. ✓ Proof After regrouping the spin and ancillary factors, the definition of the partial trace gives the first coefficient identity. Applying this identity to both tensor factors gives the displayed iterated sum; the two finite sums may be interchanged. $\square$

Theorem 20.7.33 (Local purification as a block-channel image). Lean Lean Lean ✓ Stmt Fix a local-purification representation

$$M^{ij} = \left(\sum_{k=0}^{d_K-1} A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_{e,e},$$

with purifying MPS tensor $\tilde{A}$ of physical dimension dd_K and bond dimension D' . For every $N = L + K$, tracing all ancillary degrees of freedom gives

$$\begin{aligned} \rho^{(N)}(M) &= \text{Tr}_{\text{anc},N} (|V^{(N)}(\tilde{A})\rangle\langle V^{(N)}(\tilde{A})|), \\ \text{tr } \rho^{(N)}(M) &= \|V^{(N)}(\tilde{A})\|^2. \end{aligned}$$

Define the normalized-form purifying operator by

$$\pi_{\tilde{A}}^{(N)} := \|V^{(N)}(\tilde{A})\|^{-2} |V^{(N)}(\tilde{A})\rangle\langle V^{(N)}(\tilde{A})|,$$

where the inverse scalar is taken with the convention $0^{-1} = 0$. After reindexing across the cut $L \mid K$,

$$\sigma_{L:K}^{(N)}(M) = (\text{Tr}_{\text{anc},L} \otimes \text{Tr}_{\text{anc},K}) \left(\left(\pi_{\tilde{A}}^{(N)} \right)_{L:K} \right).$$

This identity also holds when the common trace vanishes, in which case both normalized-form operators are zero.

This is the finite-chain channel identity from the purification preceding equation (4) of [WVHC08], under the explicit local purification of [CPGSV16, Section 4.3].

Proof. ✓ Proof The coefficient formula for one block gives

$$\begin{aligned} &\text{Tr}_{\text{anc},N} (|V^{(N)}(\tilde{A})\rangle\langle V^{(N)}(\tilde{A})|)_{\sigma,\tau} \\ &= \sum_{\kappa \in [d_K]^N} V^{(N)}(\tilde{A})_{(\sigma,\kappa)} \overline{V^{(N)}(\tilde{A})_{(\tau,\kappa)}} = \rho^{(N)}(M)_{\sigma,\tau}. \end{aligned}$$

Trace preservation gives equality of the two normalizing traces. Splitting κ into its left and right restrictions changes the last sum into the two independent sums in Theorem 20.7.32, which proves the normalized identity across $L \mid K$. $\square$

Theorem 20.7.34 (LPDOs admit a block-channel representation). Lean ✓ Stmt Let M be an LPDO. For every decomposition $N = L + K$, there exist an ancillary dimension d_K , a purifying bond dimension D' , and a local-purification family A . Let $\tilde{A}$ be its associated purifying MPS tensor. Then

$$\sigma_{L:K}^{(N)}(M) = (\text{Tr}_{\text{anc},L} \otimes \text{Tr}_{\text{anc},K}) \left(\left(\pi_{\tilde{A}}^{(N)} \right)_{L:K} \right).$$

Proof. ✓ Proof Choose a local-purification representation of M and apply Theorem 20.7.33 to its purifying tensor. $\square$

Theorem 20.7.35 (LPDOs admit a cut-uniform block-channel representation). [Lean](#) [✓ Stmt](#) Let M be an LPDO. There exist an ancillary dimension d_K , a purifying bond dimension D' , and a local-purification family A , all independent of the cut, such that for every decomposition $N = L + K$,

$$\sigma_{L:K}^{(N)}(M) = (\text{Tr}_{\text{anc},L} \otimes \text{Tr}_{\text{anc},K}) \left(\left(\pi_{\tilde{A}}^{(N)} \right)_{L:K} \right),$$

where $\tilde{A}$ is the purifying MPS tensor associated with A .

Proof. [✓ Proof](#) Choose one local-purification representation of M . For an arbitrary decomposition $N = L + K$, apply Theorem 20.7.33 to the same purifying tensor. $\square$

Theorem 20.7.36 (Mutual-information bound for a local-channel image). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor whose generated N -site operator is positive semidefinite. Let A be an MPS tensor of bond dimension $D' \geq 1$ whose N -site vector is nonzero. Suppose that, across the cut $L \mid (N - L)$, the normalized state generated by M is obtained from the normalized pure state of A by trace-preserving completely positive maps Φ_L and Φ_R on the two blocks:

$$\sigma_{L:N-L}^{(N)}(M) = (\Phi_L \otimes \Phi_R) \left(\frac{|V^{(N)}(A)\rangle\langle V^{(N)}(A)|}{\|V^{(N)}(A)\|^2} \right)_{L:N-L}.$$

Then $I_L(M) \leq 4 \log D'$.

Proof. [✓ Proof](#) Write $\sigma_A^{(N)}$ for the normalized pure state of A in the $L \mid (N - L)$ bipartition. Then

$$I_L(M) = I((\Phi_L \otimes \Phi_R)(\sigma_A^{(N)})) \leq I(\sigma_A^{(N)}) \leq 4 \log D'.$$

The equality is the channel-image hypothesis, the first inequality is data processing, and the last inequality is Proposition 20.9.23. $\square$

Theorem 20.7.37 (Mutual-information bound for a locally purified tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Fix a local-purification representation with purifying MPS bond dimension D' . If the N -site operator is positive semidefinite and its purifying pure state is nonzero, then $I_L(M) \leq 4 \log D'$ for every $L \leq N$.

Consequently, if M is an LPDO, $N \geq 1$, and $\text{tr} \rho^{(N)}(M) \neq 0$, then its local-purification representation supplies a bond dimension D' satisfying this estimate. The quantity D' is the bond dimension of the purifying MPS, not the MPO bond dimension D .

Proof. [✓ Proof](#) By Theorem 20.7.33,

$$\begin{aligned} I_L(M) &= I\left(\left(\text{Tr}_{\text{anc},L} \otimes \text{Tr}_{\text{anc},N-L}\right)\left(\left(\pi_{\tilde{A}}^{(N)}\right)_{L:N-L}\right)\right) \\ &\leq I\left(\left(\pi_{\tilde{A}}^{(N)}\right)_{L:N-L}\right) \leq 4 \log D'. \end{aligned}$$

The first inequality is data processing and the second is the pure-state boundary estimate. Nonvanishing of the purifying pure state implies $D' > 0$. For an LPDO, positivity follows from Theorem 17.3.3, while the trace identity in Theorem 20.7.33 transfers the nonzero-trace hypothesis to the purifying pure state. $\square$

Theorem 20.7.38 (Rank of diagonal coefficients across a periodic cut). [Lean](#) [Lean](#) [✓ Stmt](#) Split a periodic chain into consecutive blocks of lengths L and R . For words $x \in \{0, \dots, d-1\}^L$ and $y \in \{0, \dots, d-1\}^R$, set

$$P_{x,y} = \text{tr}(M^{x_0 x_0} \dots M^{x_{L-1} x_{L-1}} M^{y_0 y_0} \dots M^{y_{R-1} y_{R-1}}).$$

Then the ordinary complex rank of P satisfies $\text{rank}_{\mathbb{C}} P \leq D^2$.

Proof. [✓ Proof](#) For each word, write $A_x = M^{x_0 x_0} \dots M^{x_{L-1} x_{L-1}}$ and $B_y = M^{y_0 y_0} \dots M^{y_{R-1} y_{R-1}}$. The identity

$$P_{x,y} = \text{tr}(A_x B_y) = \sum_{a,b=0}^{D-1} (A_x)_{ab} (B_y)_{ba}$$

factors P through the D^2 -dimensional space indexed by pairs (a, b) . □

Theorem 20.7.39 (Rank under faithful scalar extension). [Lean](#) [✓ Stmt](#) Let $K \subseteq L$ be a faithful field extension, let $\iota : K \hookrightarrow L$ be the inclusion, and let A be a finite matrix over K . Then $\text{rank}_L((\iota(A_{ij}))_{ij}) = \text{rank}_K A$.

Proof. [✓ Proof](#) For every finite selection of columns, the selected columns are linearly independent over K if and only if their scalar extensions are linearly independent over L . Selecting a basis from each column space proves the two rank inequalities. □

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Joint probability distribution and marginals”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Entropy and classical mutual information”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Entropy is bounded by the logarithm of the support cardinality”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Classical mutual information is bounded by ordinary rank”].

Theorem 20.7.40 (Classical estimate for normalized diagonal cut coefficients). [Lean](#) [✓ Stmt](#) Let $D \geq 1$, and let W be a real diagonal coefficient matrix across a periodic cut of an MPO with bond dimension D . Suppose its scalar extension to $\mathbb{C}$ is the cut matrix $P^{\mathbb{C}}$ from Theorem 20.7.38. If $Z > 0$ and $P = Z^{-1}W$ is a joint probability distribution, then $I(X : Y)_P \leq 2 \log D$.

Proof. [✓ Proof](#) Normalization by the nonzero scalar Z^{-1} does not change real rank. Faithful scalar extension does not change rank by Theorem 20.7.39; hence Theorem 20.7.38 gives $\text{rank}_{\mathbb{R}} P \leq D^2$. Theorem [LTC26, “Classical mutual information is bounded by ordinary rank”] gives $I(X : Y)_P \leq \log \text{rank}_{\mathbb{R}} P$, and monotonicity of the logarithm gives the result. □

Definition 20.7.41 (Normalized diagonal distribution at fixed length). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a chain of length $N = L + R$, call $\rho^{(N)}(M)$ diagonal if $\rho^{(N)}(M)_{u,v} = 0$ whenever $u \neq v$. Call M globally diagonal if $\rho^{(N)}(M)$ is diagonal for every $N > 0$. For configurations x and y on the two consecutive blocks, define

$$\begin{aligned} W_{x,y} &= \text{Re } \rho^{(L+R)}(M)_{(x,y),(x,y)}, \\ Z &= \sum_{x,y} W_{x,y}, \\ \widehat{P} &= Z^{-1}W. \end{aligned}$$

Lemma 20.7.42 (Diagonal coefficient mass and trace). [Lean](#) [✓ Stmt](#) For every MPO tensor M and cut lengths L, R , the total diagonal coefficient mass satisfies $Z = \text{Re tr } \rho^{(L+R)}(M)$.

Proof. [✓ Proof](#) Pairing each configuration σ on $L + R$ sites with its unique block split (x, y) on L and R sites gives

$$Z = \sum_{x,y} W_{x,y} = \sum_{\sigma} \text{Re } \rho^{(L+R)}(M)_{\sigma,\sigma} = \text{Re tr } \rho^{(L+R)}(M).$$

□

Theorem 20.7.43 (Classical estimate for a positive diagonal finite-chain operator). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $D \geq 1$. Suppose that $\rho^{(L+R)}(M)$ is diagonal and positive semidefinite, and that $Z > 0$. Then $W_{x,y} \geq 0$, the scalar extension of W to $\mathbb{C}$ is the diagonal cut matrix of Theorem 20.7.38, and $\hat{P}$ is a joint probability distribution. Its classical mutual information satisfies $I(X : Y)_{\hat{P}} \leq 2 \log D$.

Proof. [✓ Proof](#) Positive semidefiniteness makes every diagonal entry real and non-negative, so $W_{x,y} \geq 0$ and its scalar extension recovers the complex diagonal cut matrix. Since Z is the sum of the entries of W , $\sum_{x,y} \hat{P}_{x,y} = Z^{-1} \sum_{x,y} W_{x,y} = 1$. Thus $\hat{P}$ is a joint probability distribution. Diagonality says that these coefficients constitute the full classical finite-chain state, and Theorem 20.7.40 gives the bound. □

Corollary 20.7.44 (Classical estimate for a diagonal matrix product density operator). [Lean](#) [✓ Stmt](#) Let $D \geq 1$, and let M generate diagonal positive semidefinite operators at every positive chain length. If $L + R > 0$ and the trace at length $L + R$ is positive, then the classical mutual information of its normalized diagonal coefficients satisfies $I(X : Y) \leq 2 \log D$.

Proof. [✓ Proof](#) Global diagonality and positivity give the fixed-length hypotheses of Theorem 20.7.43. Since the chain operator is diagonal,

$$Z = \sum_{x,y} W_{x,y} = \text{Re tr } \rho^{(L+R)}(M),$$

so the trace positivity hypothesis supplies the required normalization. □

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Positivity of the operator-Schmidt rank”].

Theorem 20.7.45 (Two virtual boundaries bound the operator-Schmidt rank). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M be an MPO tensor of operator bond dimension D , and let $N = L + K$. Opening the two virtual bonds at the consecutive cut gives

$$\sigma_{L:K}^{(N)} = \sum_{a,b=0}^{D-1} X_{ab} \otimes Y_{ab},$$

and

$$\text{OSR}(\sigma_{L:K}^{(N)}) \leq D^2.$$

For left-block words x, x' and right-block words y, y' , the factors are

$$(X_{ab})_{x,x'} = \text{tr}[\rho^{(N)}(M)]^{-1} (M^{x_0 x'_0} \dots M^{x_{L-1} x'_{L-1}})_{ab},$$

and

$$(Y_{ab})_{y,y'} = (M^{y_0 y'_0} \dots M^{y_{K-1} y'_{K-1}})_{ba}.$$

This is the identity $\text{tr}(AB) = \sum_{a,b} A_{ab} B_{ba}$, with the normalization scalar absorbed into the left factor. The algebraic decomposition does not use positivity. Since $0^{-1} = 0$, it also remains valid when the trace vanishes; a nonzero trace is needed only to interpret $\sigma_{L:K}^{(N)}$ as the normalized physical state.

This is the two-virtual-boundary algebraic step toward the finite mutual-information estimate in [CPGSV16, Proposition 4.5]. It does not by itself prove $I_L \leq 4 \log D$: such a conclusion would require an entropy bound in terms of the operator-Schmidt rank.

Proof. ✓ Proof Split each closed virtual word into its left and right products. Expanding the trace over the two exposed virtual indices gives the displayed sum of D^2 product matrices. The minimality property of the operator-Schmidt rank then bounds it by the number of displayed terms. $\square$

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Coefficient-space characterization of operator-Schmidt rank”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Operator-Schmidt rank under local isometries”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Support of a bipartite positive operator”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Compression to both marginal supports”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Partial traces under product isometries”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Faithful marginals after simultaneous support compression”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Channel of a faithful bipartite marginal”].

20.8 Support compression for entropy functionals

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Isometric conjugation into a larger matrix algebra”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Functional calculus under rectangular isometric conjugation”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Trace preservation under isometric inclusion”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Compression cancels isometric expansion”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Reconstruction from a reference support”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Relative entropy under support compression”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Functional calculus under transpose”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Logarithm of a positive square root”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Logarithm of a real power”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Support-aware vector-state Jensen inequality for the logarithm”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Order-two sandwiched trace functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Nonnegativity of the order-two functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Faithful cyclic form of the order-two functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Relative entropy bounded by the faithful order-two functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Order-two sandwiched functional under support compression”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Reduction of the singular order-two comparison to the faithful case”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Relative entropy bounded by the support-restricted order-two functional”].

20.9 Rectangular Choi matrices and a range-dimension estimate

The following construction is the unnormalized, different-dimension form of the Choi correspondence [Wol12, Proposition 2.1]. The resulting estimate bounds the order-two Choi functional of a Hilbert–Schmidt contraction by the dimension of its range. It is used below to bound mutual information in terms of operator-Schmidt rank.

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Matrix of a linear map in matrix-unit bases”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Unnormalized rectangular Choi matrix”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Frobenius norm under rectangular Choi reshaping”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Matrix-unit Parseval identity for a rectangular Choi matrix”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Hilbert–Schmidt contraction”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Rectangular Choi norm bounded by the range dimension”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Full-support eigenbasis whitening as a Choi matrix”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Full-support eigenbasis whitened Choi estimate”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Real powers under unitary conjugation”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Real powers of positive tensor products”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Inverse quarter-power from iterated square roots”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Unitary invariance of the order-two functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Reindexing invariance of the order-two functional”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Faithful marginal whitening bound”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Marginal-support whitening bound”].

The sandwiched Rényi comparison also uses the following unnormalized trace term. For trace-one ρ and $\alpha > 1$, it is the quantity inside the logarithm of the sandwiched Rényi divergence [Bei13, Equation (3)] [MLDS⁺13, Definition 2] whenever $\rho, \omega \geq 0$ and $\ker \omega \subseteq \ker \rho$. This includes singular ω , with negative powers interpreted as the generalized inverse on its support. In this regime, finiteness of the divergence requires the support inclusion. The present application uses $1 < \alpha \leq 2$; the order-one case is the Umegaki endpoint. The statements below concern only this algebraic expression. They are not assertions of [CPGSV16, Proposition 4.5], and they do not establish continuity or monotonicity in the order, interpolation between the endpoints, differentiability at order one, a limit to Umegaki relative entropy, a logarithmic divergence, or a comparison with relative entropy.

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Total sandwiched Rényi trace”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Nonnegativity on the faithful domain”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Order-one sandwiched trace”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Order-two sandwiched trace”].

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Mutual information from operator-Schmidt rank”].

Theorem 20.9.1 (Finite-chain mutual information from the operator bond). Lean ✓ Stmt *Let M be an MPO tensor of operator bond dimension D . Fix a chain of length N and a cut with $0 \leq L \leq N$. If the corresponding finite-chain operator is positive semidefinite and has nonzero trace, then*

$$I_L^{(N)} \leq 2 \log D.$$

No global positivity, local purification, canonical form, or separate positive-bond-dimension hypothesis is required. The cuts $L = 0$ and $L = N$ are included.

Proof. ✓ Proof Reindex the normalized chain operator across the consecutive cut. Its trace is one, and its positivity is preserved by this reindexing. Hence

$$I_L^{(N)} \leq \log \text{OSR}(\sigma_{L:N-L}^{(N)}) \leq \log D^2 = 2 \log D.$$

Nonzero trace makes the normalized operator nonzero, so its operator-Schmidt rank is positive. If $D = 0$, then every finite-chain trace vanishes. Hence $D > 0$, which justifies the last logarithmic identity. □

Theorem 20.9.2 (Finite-chain mutual-information estimate for matrix product density operators). Lean ✓ Stmt *Let M be a matrix product density operator of operator bond dimension D .*

For every positive chain length N , every cut $0 \leq L \leq N$, and every nonzero finite-chain trace,

$$I_L^{(N)} \leq 4 \log D.$$

This is the finite-chain estimate invoked in the proof of Proposition 4.5 of [CPGSV16]. The nonzero-trace hypothesis states explicitly the normalization convention preceding that proposition. This theorem does not assert the proposition’s thermodynamic-limit clause.

Proof. ✓ Proof Global positivity supplies positivity at the chosen positive chain length. Apply Theorem 20.9.1. The same nonzero-trace argument gives $D > 0$, hence $0 \leq \log D$ and $2 \log D \leq 4 \log D$. $\square$

Remark 20.9.3 (Boundedness of the mutual information). The proof of Proposition 4.5 of [CPGSV16] invokes the uniform estimate $I_L \leq 4 \log D$ for every MPDO. Its cited argument, however, treats mixed tensor-network states obtained from local completely positive maps [WVHC08]. After purifying those maps, data processing and the pure-state boundary estimate give

$$I(A : B)_\rho \leq I(A : B)_{|\Psi\rangle\langle\Psi|} = 2S(A)_{|\Psi\rangle\langle\Psi|} \leq 2|\partial A| \log D.$$

A periodic one-dimensional interval has two boundary bonds, and hence the last expression is $4 \log D$. A general positive MPO need not admit this local purification.

By Theorem [LTC26, “Coefficient-space characterization of operator-Schmidt rank”], the two virtual bonds give an operator-Schmidt decomposition with at most D^2 terms, but they do not bound the ordinary rank of either marginal. For example, a bond-one tensor can generate $q^{\otimes N}$ for a full-rank one-site density matrix q . Thus its L -site marginal has rank $\text{rank}(q)^L$, although its mutual information vanishes.

Theorem 20.9.1 gives the stronger estimate

$$I_L^{(N)} \leq \log D^2 = 2 \log D$$

whenever the finite-chain operator is positive and has positive trace. Thus no classical or genuinely quantum counterexample exists. The diagonal argument gives an independent special-case proof. Indeed, Theorem 20.7.43 bounds by $2 \log D$ the classical mutual information of the normalized diagonal coefficients whenever the finite-chain operator is diagonal and positive semidefinite and has positive total mass.

The same example as Theorem 20.7.11 makes the iterated mutual-information limit false; see [con26b]. The channel-image implication is Theorem 20.7.36, and the pure-state instance is Proposition 20.9.23 below. The locally purified case is Theorem 20.7.37.

Definition 20.9.4 (Block entropy and area-law saturation for pure states). Lean Lean ✓ Stmt For an MPS tensor A , a system size N , and a block length $L \leq N$, let

$$\begin{aligned} \sigma^{(N)}(A) &= \frac{|V^{(N)}(A)\rangle\langle V^{(N)}(A)|}{\|V^{(N)}(A)\|^2}, \\ \rho_L^{(N)}(A) &= \text{tr}_{N \setminus L}[\sigma^{(N)}(A)] \end{aligned}$$

denote the normalized state and the reduced state of the first L spins. The L -block entropy is $S_L^{(N)}(A) = S(\rho_L^{(N)}(A))$. The tensor saturates the area law if $S_L^{(N)}(A) = S_{L+1}^{(N)}(A)$ for all $1 \leq L < \lfloor N/2 \rfloor$ and all N . This is [CPGSV16, Definition 3.13, line 600].

Definition 20.9.5 (Purification tensor). Lean ✓ Stmt For an MPS tensor A on bond space $\mathbb{C}^D$, define the associated MPO tensor M on bond space $\mathbb{C}^{D^2}$ by $M^{ij} = A^i \otimes \overline{A^j}$. This is the single-ancilla purification tensor of [CPGSV16].

Lemma 20.9.6 (Pure state as a purification MPDO). [Lean](#) [✓ Stmt](#) For an MPS tensor A , the pure-state operator $|V^{(N)}(A)\rangle\langle V^{(N)}(A)|$ equals the MPDO generated by the tensor $M^{ij} = A^i \otimes \bar{A}^j$, acting on a bond space of dimension D^2 :

$$\rho^{(N)}(A \otimes \bar{A}) = |V^{(N)}(A)\rangle\langle V^{(N)}(A)|.$$

This is the single-ancilla case of the purification picture of [CPGSV16]; it lets the MPDO block-entropy theory apply to pure-state block entropies.

Proof. [✓ Proof](#) The word evaluation of the doubled tensor is the Kronecker product of the word evaluation of A and its conjugate, $M^{\sigma,\tau} = A^\sigma \otimes \bar{A}^\tau$, so its trace factors as $\text{tr}(M^{\sigma,\tau}) = \text{tr}(A^\sigma) \text{tr}(\bar{A}^\tau) = \overline{V^{(N)}(A)}_\tau V^{(N)}(A)_\sigma$, the corresponding matrix element of $|V^{(N)}(A)\rangle\langle V^{(N)}(A)|$. $\square$

Lemma 20.9.7 (Block entropy of the purification tensor). [Lean](#) [✓ Stmt](#) For an MPS tensor A , system size N , and block length $L \leq N$, the MPDO block entropy of the tensor $M^{ij} = A^i \otimes \bar{A}^j$ agrees with the pure-state block entropy: $S_L^{(N)}(M) = S_L^{(N)}(A)$.

Proof. [✓ Proof](#) Lemma 20.9.6 gives $\rho^{(N)}(M) = |V^{(N)}(A)\rangle\langle V^{(N)}(A)|$, hence

$$\sigma^{(N)}(M) = \frac{\rho^{(N)}(M)}{\text{tr}[\rho^{(N)}(M)]} = \frac{|V^{(N)}(A)\rangle\langle V^{(N)}(A)|}{\|V^{(N)}(A)\|^2} = \sigma^{(N)}(A).$$

Therefore the reduced block states agree:

$$\rho_L^{(N)}(M) = \text{tr}_{N \setminus L}[\rho^{(N)}(M)] = \text{tr}_{N \setminus L}[\sigma^{(N)}(A)] = \rho_L^{(N)}(A).$$

Thus $S_L^{(N)}(M) = S(\rho_L^{(N)}(M)) = S(\rho_L^{(N)}(A)) = S_L^{(N)}(A)$. $\square$

Lemma 20.9.8 (Schmidt symmetry of the block entropy). [Lean](#) [✓ Stmt](#) For an MPS tensor A and a block length $L \leq N$, the block entropies of a block and its complement coincide: $S_L^{(N)}(A) = S_{N-L}^{(N)}(A)$.

Proof. [✓ Proof](#) The reduced state of the first L spins is $\rho_L \propto WW^\dagger$, where W is the matrix of amplitudes $\langle u w | V^{(N)}(A) \rangle$ with u the first L spins and w the remaining $N - L$. By cyclic invariance of the amplitudes, the reduced state of the complement is $\propto \bar{W}^\dagger \bar{W}$. Cyclic invariance of the entropy, $S(WW^\dagger) = S(W^\dagger W)$, together with invariance of the entropy under entrywise conjugation of a Hermitian matrix, $S(\bar{W}^\dagger \bar{W}) = S(W^\dagger W)$, gives $S_L^{(N)}(A) = S_{N-L}^{(N)}(A)$. $\square$

Proposition 20.9.9 (Pure-state area-law monotonicity). [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with $V^{(N)}(A) \neq 0$. For every block length L with $2L + 1 \leq N$, covering $1 \leq L < \lfloor N/2 \rfloor$, $S_L^{(N)}(A) \leq S_{L+1}^{(N)}(A)$. This is the pure-state area law of [CPGSV16, Section 3, line 599].

Proof. [✓ Proof](#) Apply the block-entropy form of strong subadditivity, the inequality underlying Proposition 20.7.9, to the purification MPDO of A , giving $S_{N-L} + S_L \leq S_{L+1} + S_{N-L-1}$. By the Schmidt symmetry $S_{N-L} = S_L$ and $S_{N-L-1} = S_{L+1}$, this reduces to $S_L \leq S_{L+1}$. $\square$

Lemma 20.9.10 (Operator-Schmidt Gram is a transfer-map power). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor of bond dimension D . For a length- ℓ configuration σ and a length- m configuration τ , write the operator-Schmidt factors as

$$(L_\ell)_{\sigma, (a_1, a_2)} = (A^\sigma)_{a_1 a_2},$$

$$(R_m)_{(p_1, p_2), \tau} = (A^\tau)_{p_2 p_1}.$$

The $D^2 \times D^2$ Gram matrix of the left factor equals the ℓ -fold transfer map evaluated on a matrix unit:

$$\begin{aligned} (L_\ell^\dagger L_\ell)_{(a_1, a_2), (b_1, b_2)} &= \sum_{\sigma} \overline{(A^\sigma)_{a_1 a_2}} (A^\sigma)_{b_1 b_2} \\ &= (\mathcal{E}_A^\ell(|b_2\rangle\langle a_2|))_{b_1, a_1}, \end{aligned}$$

where the sum runs over length- ℓ configurations σ . The Gram of the right factor R_m of an m -spin complement satisfies the bond-swapped identity

$$\begin{aligned} (R_m R_m^\dagger)_{(p_1, p_2), (q_1, q_2)} &= \sum_{\tau} (A^\tau)_{p_2 p_1} \overline{(A^\tau)_{q_2 q_1}} \\ &= (\mathcal{E}_A^m(|p_1\rangle\langle q_1|))_{p_2, q_2}, \end{aligned}$$

where the sum runs over length- m configurations τ .

Proof. ✓ Proof The left Gram entry is $\sum_{\sigma} \overline{(A^\sigma)_{a_1 a_2}} (A^\sigma)_{b_1 b_2}$ by definition of L_ℓ . Expanding

$$\mathcal{E}_A^\ell(|b_2\rangle\langle a_2|) = \sum_{\sigma} A^\sigma |b_2\rangle\langle a_2| (A^\sigma)^\dagger$$

and reading off the (b_1, a_1) entry gives the same sum, since

$$(A^\sigma |b_2\rangle\langle a_2| (A^\sigma)^\dagger)_{b_1, a_1} = (A^\sigma)_{b_1 b_2} \overline{(A^\sigma)_{a_1 a_2}}.$$

The right-factor formula is the same expansion with the two boundary bonds interchanged: the (p, q) entry of $R_m R_m^\dagger$ is $\sum_{\tau} (A^\tau)_{p_2 p_1} \overline{(A^\tau)_{q_2 q_1}}$, which is the (p_2, q_2) entry of $\mathcal{E}_A^m(|p_1\rangle\langle q_1|)$. $\square$

Lemma 20.9.11 (RFP collapse of the operator-Schmidt Grams). Lean Lean ✓ Stmt Let A be a renormalization fixed point. For every $L \geq 1$ and $m \geq 1$, the operator-Schmidt Grams of the L -block and of the m -site complement are the one-site Grams:

$$\begin{aligned} L_L^\dagger L_L &= L_1^\dagger L_1, \\ R_m R_m^\dagger &= R_1 R_1^\dagger. \end{aligned}$$

Proof. ✓ Proof By definition, a renormalization fixed point satisfies $\mathcal{E}_A^2 = \mathcal{E}_A$, so $\mathcal{E}_A^k = \mathcal{E}_A$ for every $k \geq 1$. Applying Lemma 20.9.10 to the left and right Gram entries gives the two displayed identities. $\square$

Lemma 20.9.12 (Pure block entropy as a bond-environment charpoly sum). Lean ✓ Stmt Let A be an MPS tensor, let $L \leq N$, put $m = N - L$, and write $c = (\text{tr } \rho^{(N)})^{-1}$. Then the pure block entropy is the charpoly-root entropy sum of the $D^2 \times D^2$ bond environment:

$$S_L^{(N)}(A) = \sum_{\lambda \in \text{roots } \chi_{c(R_m R_m^\dagger)(L_L^\dagger L_L)}} -\text{Re}(\lambda) \log \text{Re}(\lambda).$$

The roots are counted with algebraic multiplicity.

Proof. ✓ Proof The reduced block state has the exact factorization

$$\rho_L^{(N)}(A) = c L_L (R_m R_m^\dagger) L_L^\dagger.$$

Cyclic invariance of the charpoly-root entropy sum gives

$$S(c L_L (R_m R_m^\dagger) L_L^\dagger) = S(c (R_m R_m^\dagger) (L_L^\dagger L_L)),$$

which is the displayed formula. $\square$

Theorem 20.9.13 (A renormalization fixed point saturates the area law). [Lean](#) [✓ Stmt](#) *If an MPS tensor A is a renormalization fixed point, then it saturates the area law: for every chain length N and every block length $1 \leq L < \lfloor N/2 \rfloor$, the pure-state block entropy is constant in the block size, $S_L^{(N)}(A) = S_{L+1}^{(N)}(A)$. The source statement is [CPGSV16, Proposition 3.14, lines 606–608]. It assumes canonical form; the theorem above is the stronger pure-state implication obtained from transfer-map idempotence alone.*

Proof. [✓ Proof](#) Write $c = (\text{tr } \rho^{(N)})^{-1}$. Lemma 20.9.12 computes the entropy from the bond environment:

$$S_L^{(N)}(A) = \sum_{\lambda \in \text{roots } \chi_{c(R_{N-L} R_{N-L}^\dagger)(L_L^\dagger L_L)}} -\text{Re}(\lambda) \log \text{Re}(\lambda).$$

Since $1 \leq L < \lfloor N/2 \rfloor$, the four lengths L , $L+1$, $N-L$, and $N-(L+1)$ are positive. The RFP Gram-collapse lemma gives

$$\begin{aligned} L_L^\dagger L_L &= L_1^\dagger L_1, \\ L_{L+1}^\dagger L_{L+1} &= L_1^\dagger L_1, \\ R_{N-L} R_{N-L}^\dagger &= R_1 R_1^\dagger, \\ R_{N-(L+1)} R_{N-(L+1)}^\dagger &= R_1 R_1^\dagger. \end{aligned}$$

Therefore the two bond environments agree:

$$\begin{aligned} (R_{N-L} R_{N-L}^\dagger)(L_L^\dagger L_L) &= (R_1 R_1^\dagger)(L_1^\dagger L_1) \\ &= (R_{N-(L+1)} R_{N-(L+1)}^\dagger)(L_{L+1}^\dagger L_{L+1}). \end{aligned}$$

The scalar c depends only on N , so the two charpoly-root entropy sums are equal. Hence $S_L^{(N)}(A) = S_{L+1}^{(N)}(A)$. $\square$

Theorem 20.9.14 (A renormalization fixed point has a constant block-entropy chain). [Lean](#) [✓ Stmt](#) *If an MPS tensor A is a renormalization fixed point, then all block entropies in the range $1 \leq L \leq \lfloor N/2 \rfloor$ coincide:*

$$S_1^{(N)}(A) = S_2^{(N)}(A) = \dots = S_{\lfloor N/2 \rfloor}^{(N)}(A).$$

This is the explicit constant-entropy chain of the area-law saturation [CPGSV16, Definition 3.13, line 600], under the fixed-point hypothesis.

Proof. [✓ Proof](#) The fixed point saturates the area law (Theorem 20.9.13), so by the telescoping of saturation (Lemma 20.9.21) the block entropies at any two lengths $1 \leq L, L' \leq \lfloor N/2 \rfloor$ agree: $S_L^{(N)}(A) = S_{L'}^{(N)}(A)$. $\square$

Lemma 20.9.15 (Operator-Schmidt rank bound). [Lean](#) [✓ Stmt](#) *For an MPS tensor A of bond dimension D and a block length $L \leq N$, the reduced state $\rho_L^{(N)}(A)$ of the first L spins has rank at most D^2 .*

Proof. [✓ Proof](#) The reduced state is $\rho_L \propto WW^\dagger$, where the amplitude matrix

$$W(u, w) = \langle u | w | V^{(N)}(A) \rangle = \text{tr}(A^{u_1} \dots A^{u_L} A^{w_1} \dots A^{w_{N-L}})$$

factors through the two bond pairs cut by the block boundary. Writing the trace as a sum over the bond index pair $(a, b) \in \{1, \dots, D\}^2$ exhibits $W = L'R'$ with L' having D^2 columns, so $\text{rank } \rho_L \leq \text{rank } W \leq D^2$. $\square$

Lemma 20.9.16 (The reduced pure block state is a density matrix). [Lean](#) [Lean](#) [✓ Stmt](#) For an MPS tensor A with $V^{(N)}(A) \neq 0$, the reduced state $\rho_L^{(N)}(A)$ of the first L spins is a density matrix: it is positive semidefinite and has unit trace.

Proof. [✓ Proof](#) Positive semidefiniteness is preserved by the partial trace of the positive semidefinite normalized pure state, and the partial trace preserves the trace, which the normalization fixes to one. $\square$

Proposition 20.9.17 (Pure-state area-law upper bound). [Lean](#) [✓ Stmt](#) Let A be an MPS tensor of bond dimension $D \geq 1$ with $V^{(N)}(A) \neq 0$. For every block length $L \leq N$, $S_L^{(N)}(A) \leq 2 \log D$, uniformly in L and N . This is the area law of [CPGSV16, Section 3, line 599]: the block entropy is bounded by a constant independent of the block size.

Proof. [✓ Proof](#) The reduced state $\rho_L^{(N)}(A)$ is a density matrix (Lemma 20.9.16), so its rank is positive by Lemma 20.7.26. Its rank is at most D^2 by Lemma 20.9.15. The rank bound on the entropy gives $S_L^{(N)}(A) \leq \log(D^2) = 2 \log D$. $\square$

Lemma 20.9.18 (Nonnegativity of the pure-state block entropy). [Lean](#) [✓ Stmt](#) For an MPS tensor A with $V^{(N)}(A) \neq 0$ and every block length $L \leq N$, the pure-state block entropy is non-negative: $0 \leq S_L^{(N)}(A)$.

Proof. [✓ Proof](#) The reduced state $\rho_L^{(N)}(A)$ is a density matrix (Lemma 20.9.16), and the von Neumann entropy of a density matrix is non-negative. $\square$

Lemma 20.9.19 (Full-block pure-state entropy vanishes). [Lean](#) [✓ Stmt](#) For an MPS tensor A with $V^{(N)}(A) \neq 0$, the entropy of the whole chain vanishes: $S_N^{(N)}(A) = 0$.

Proof. [✓ Proof](#) After reindexing the full block, the normalized operator is the rank-one pure state

$$\frac{|V^{(N)}(A)\rangle\langle V^{(N)}(A)|}{\langle V^{(N)}(A)|V^{(N)}(A)\rangle}.$$

A pure state has von Neumann entropy zero. $\square$

Lemma 20.9.20 (Empty-block pure-state entropy vanishes). [Lean](#) [✓ Stmt](#) For an MPS tensor A with $V^{(N)}(A) \neq 0$, the pure-state block entropy of the empty block vanishes: $S_0^{(N)}(A) = 0$.

Proof. [✓ Proof](#) By the Schmidt symmetry $S_0^{(N)}(A) = S_N^{(N)}(A)$, and the entropy of the full system vanishes for a pure state (Lemma 20.9.19). $\square$

Lemma 20.9.21 (Saturation telescopes to a constant block entropy). [Lean](#) [✓ Stmt](#) If an MPS tensor A saturates the area law, then the block entropies coincide for all $1 \leq L, L' \leq \lfloor N/2 \rfloor$: $S_L^{(N)}(A) = S_{L'}^{(N)}(A)$.

Proof. [✓ Proof](#) The defining condition supplies $S_m^{(N)}(A) = S_{m+1}^{(N)}(A)$ throughout the interval $1 \leq m < \lfloor N/2 \rfloor$. Apply Lemma 20.7.5. $\square$

Lemma 20.9.22 (Pure-state mutual information is twice the block entropy). [Lean](#) [✓ Stmt](#) For an MPS tensor A with $V^{(N)}(A) \neq 0$, the mutual information of the purification MPDO equals twice the pure-state block entropy: $I_L = 2S_L^{(N)}(A)$.

in (20.18). Contracting the two physical indices and using $V^\dagger V = \mathbb{1}$ gives the equality of the matrices $\mathcal{T}$.

Finally, let $Z_{\mathcal{K}}$ stack all columns of all physical slices. Directly from the definition, $Z_{\mathcal{K}_V} = VZ_{\mathcal{K}}R$, where R is the blockwise copy of $V^\dagger$ and $RR^\dagger = \mathbb{1}$. Hence $Z_{\mathcal{K}_V}Z_{\mathcal{K}_V}^\dagger = VZ_{\mathcal{K}}Z_{\mathcal{K}}^\dagger V^\dagger$, and taking column-support projections proves the last identity in (20.18). A common virtual similarity distributes termwise through the physical sums, proving the gauge assertion. $\square$

20.11 Simple tensors

Definition 20.11.1 (Physical-trace transfer of a doubled-index tensor). [Lean](#) [✓ Stmt](#) For a doubled-index tensor B with physical dimension d^2 , the physical-trace transfer is the bond matrix obtained by closing the ket leg against the bra leg of one tensor:

$$\mathcal{T}_B = \sum_i B^{ii}. \quad (20.19)$$

This is the transfer object of the zero-correlation-length condition of [CPGSV16, Definition 4.2, lines 735–739], applied to one element of a basis of normal tensors.

Lemma 20.11.2 (Identification with the tensor-level transfer). [Lean](#) [✓ Stmt](#) The physical-trace transfer of the doubled-index view of an MPO tensor is the physical-trace transfer of the tensor itself.

Proof. [✓ Proof](#) If M^{dbl} is the doubled-index view of M , then $(M^{\text{dbl}})^{(i,j)} = M^{ij}$, and hence $\mathcal{T}_{M^{\text{dbl}}} = \sum_i (M^{\text{dbl}})^{ii} = \sum_i M^{ii} = \mathcal{T}_M$. $\square$

Lemma 20.11.3 (Blockwise form of the physical-trace transfer). [Lean](#) [✓ Stmt](#) For a sector decomposition with basis elements A_j and weights $\mu_{j,q}$, the physical-trace transfer of the assembled tensor is the block direct sum of the weighted transfers of the copies:

$$\mathcal{T} = \bigoplus_j \bigoplus_q \mu_{j,q} \mathcal{T}_{A_j}. \quad (20.20)$$

Proof. [✓ Proof](#) The diagonal physical sum distributes over the block direct sum:

$$\begin{aligned} \mathcal{T} &= \sum_i S^{ii} = \sum_i \bigoplus_{j,q} \mu_{j,q} A_j^{ii} \\ &= \bigoplus_{j,q} \mu_{j,q} \sum_i A_j^{ii} = \bigoplus_{j,q} \mu_{j,q} \mathcal{T}_{A_j}. \end{aligned}$$

$\square$

Theorem 20.11.4 (Gauge-phase invariance of nilpotent physical trace). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If two doubled-index tensors are related by an invertible gauge and a nonzero phase, then the physical-trace transfer of one is nilpotent if and only if that of the other is nilpotent. Identifying equal bond dimensions does not alter this condition.

Proof. [✓ Proof](#) The physical-trace transfer transforms by the same nonzero phase and simultaneous similarity. Powers therefore transform by the corresponding nonzero scalar power and similarity, which preserves whether a power vanishes. $\square$

Theorem 20.11.11 (Scaling law for closed matrix-product operators). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let $c \in \mathbb{C}$. For every chain length N , the closed operator generated by cM is

$$\rho^{(N)}(cM) = c^N \rho^{(N)}(M).$$

Proof. [✓ Proof](#) Apply Lemma 20.11.8 to the two length- N physical words, unfold the closed MPO family, and use linearity of the virtual trace. $\square$

Theorem 20.11.12 (Normalization is invariant under nonzero rescaling). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let $c \in \mathbb{C}$ be nonzero. For every chain length N , the normalized density operators agree:

$$\text{tr}[\rho^{(N)}(cM)]^{-1} \rho^{(N)}(cM) = \text{tr}[\rho^{(N)}(M)]^{-1} \rho^{(N)}(M),$$

with the inverse of zero understood to be zero.

Proof. [✓ Proof](#) By Theorem 20.11.11, the closed operator and its trace are both multiplied by c^N , and the two nonzero factors cancel in the normalization. When the trace vanishes, both sides are zero by the convention on the inverse of zero. $\square$

Definition 20.11.13 (Word encoded by a physical-block index). [Lean](#) [✓ Stmt](#) A physical index for an L -site block determines a word by decoding the index into its L original physical letters and listing them in site order.

Lemma 20.11.14 (Length of an encoded physical-block word). [Lean](#) [✓ Stmt](#) The word encoded by an L -site physical-block index has length L .

Proof. [✓ Proof](#) This follows from the fixed-length block-word encoding. $\square$

Theorem 20.11.15 (Scaling law for physically blocked MPO tensors). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let $c \in \mathbb{C}$. For every blocking length L , one has

$$(cM)^{[L]} = c^L M^{[L]}.$$

Proof. [✓ Proof](#) Unfold physical blocking and apply Lemma 20.11.8 to the two encoded words; Lemma 20.11.14 gives their length L . $\square$

Theorem 20.11.16 (Nonnegative rescaling preserves the MPDO condition). [Lean](#) [✓ Stmt](#) Let M generate positive semidefinite closed operators at every positive length, and let $r \geq 0$ be real. Then rM has the same property.

Proof. [✓ Proof](#) The length- N operator is multiplied by the nonnegative scalar r^N . $\square$

Theorem 20.11.17 (Positive rescaling invariance of the MPDO condition). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let $r > 0$ be real. Then rM generates positive semidefinite closed operators at every positive chain length if and only if M does.

Proof. [✓ Proof](#) Apply Theorem 20.11.16 first with r and then with r^{-1} . $\square$

Theorem 20.11.18 (Positive rescaling preserves simplicity). [Lean](#) [✓ Stmt](#) Let M be simple and let $r > 0$ be real. Then rM is simple.

Proof. [✓ Proof](#) Positive rescaling preserves the MPDO condition. For a blocking length L , it multiplies the blocked tensor by the nonzero scalar r^L . Multiply every copy weight by r^L and apply Theorem 20.11.9; the representatives and their non-nilpotent physical-trace transfers are unchanged. $\square$

Theorem 20.11.19 (Positive rescaling invariance of simplicity). [Lean](#) [✓ Stmt](#) *Let M be an MPO tensor and let $r > 0$ be real. Then rM is simple if and only if M is simple.*

Proof. [✓ Proof](#) Apply the preceding implication first with r and then with r^{-1} . $\square$

Definition 20.11.20 (Normalized simplicity in blocked canonical form). [Lean](#) [✓ Stmt](#) A tensor generating MPDOs satisfies this predicate if it has a normalized horizontal basis-of-normal-tensors decomposition whose basis elements all have non-nilpotent physical-trace transfer. This records the horizontal-canonical-form data of the already-blocked tensor K fixed in [CPGSV16, Appendix C.2, line 1628].

The definition concerns this fixed representative. Its witness includes the global unit-weight convention of [CPGSV16, line 246], whereas the non-nilpotency condition itself is unchanged by nonzero scalar rescaling. It supplies the Appendix C.2 hypothesis on the already-blocked tensor and is not the simplicity predicate of Definition 20.11.6, which quantifies over blocking lengths and presentations. No invariance under rescaling or replacement of the canonical presentation is asserted.

Lemma 20.11.21 (Normalized simple canonical forms are horizontal canonical). [Lean](#) [✓ Stmt](#) *A tensor satisfying normalized simplicity in the chosen blocked canonical-form setting is in horizontal canonical form.*

Proof. [✓ Proof](#) The canonical-form data in the simplicity hypothesis—the sector decomposition, its basis-of-normal-tensors property, and the block-diagonal gauge—give a horizontal canonical form. Positivity and non-nilpotency are not needed for this conclusion. $\square$

20.12 Cyclic local-embedding coordinates

Definition 20.12.1 (Cyclic window-complement coordinates). [Lean](#) [✓ Stmt](#) A periodic configuration is identified with the ordered pair consisting of its restriction to the cyclic window W_i^L and its restriction to the complementary cyclic interval.

Theorem 20.12.2 (Local embeddings in window-complement coordinates). [Lean](#) [✓ Stmt](#) *In cyclic window-complement coordinates, every local embedding has the form $E_i^L(B) = B \otimes \mathbb{1}_{\text{comp}}$.*

Proof. [✓ Proof](#) The matrix entry between (x, u) and (y, v) vanishes unless the complementary configurations agree, $u = v$. In that case it equals $B_{x,y}$, which is the corresponding entry of $B \otimes \mathbb{1}_{\text{comp}}$. $\square$

Theorem 20.12.3 (Two-site local embeddings in pair coordinates). [Lean](#) [Lean](#) [✓ Stmt](#) *Under the canonical identification of two-site configurations with ordered pairs, embedding an arbitrary two-site operator B beginning at site zero gives B , while embedding it beginning at site one gives SBS^{-1} , where $S(x \otimes y) = y \otimes x$ exchanges the two factors.*

Proof. [✓ Proof](#) The cyclic windows beginning at sites zero and one read the physical sites in the orders $(0, 1)$ and $(1, 0)$, respectively. Reindexing by ordered pairs therefore preserves B in the first case and conjugates it by S in the second. $\square$

20.13 Gibbs states of nearest-neighbor commuting Hamiltonians

Definition 20.13.1 (GSNNCH). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) A density operator on a periodic chain is a Gibbs state of a nearest-neighbor commuting Hamiltonian if

$$\rho^{(N)} \propto \bigoplus_x n_x \exp\left(-\sum_{j=1}^N \tau_j(h^{(x)})\right),$$

$$n_x \in \mathbb{N},$$

where the one-site spaces belonging to distinct labels x are orthogonal and $[h^{(x)}, \tau_1(h^{(x)})] = 0$. With the projector-limit convention of [CPGSV16, Definition 4.8, lines 829–850], this is equivalently

$$\rho^{(N)} \propto \bigoplus_x n_x \prod_{j=1}^N \tau_j(B^{(x)}),$$

$$B^{(x)} \geq 0,$$

with commuting neighboring translates. The orthogonal sum and the multiplicities are part of the definition. Definition 4.8 decomposes the global Hilbert space into the uniform product sectors and requires the associated one-site spaces to be mutually orthogonal; it does not impose a resolution of a larger one-site identity by the sector projections.

The following auxiliary single-bond presentation does not make the sector labels or multiplicities explicit. Its bond acts on the full two-site space. It gives the one-sector case of Definition 20.13.1; the converse comparison is not used.

Definition 20.13.2 (Single-bond commuting-form data for an MPDO). [Lean](#) [Stmnt](#) For a fixed chain length $N \geq 2$, this consists of one positive semidefinite two-site operator $B \geq 0$ whose translated copies commute pairwise on the periodic N -site chain: $[B_{i,i+1}, B_{j,j+1}] = 0$.

Definition 20.13.3 (Single-bond commuting form). [Lean](#) [Stmnt](#) An MPO tensor M has the single-bond commuting-form property if for every $N \geq 2$ there is a two-site operator B as in Definition 20.13.2 and a constant $c > 0$ such that

$$\rho^{(N)}(M) = c \prod_{i=0}^{N-1} B_{i,i+1}.$$

The bond in this chainwise condition may depend on N . It is therefore weaker than the single translation-invariant bond family used in the proposition at [CPGSV16, Appendix C.2, lines 1597–1619].

Definition 20.13.4 (Single-bond product predicate). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) This condition asks, for every chain length $N \geq 2$, for one positive bond on the full two-site space and one positive normalization such that

$$\rho^{(N)}(M) = c \prod_{i=0}^{N-1} B_{i,i+1},$$

$$c > 0,$$

$$B \geq 0,$$

$$[B_{i,i+1}, B_{j,j+1}] = 0.$$

This auxiliary condition contains no explicit sector labels or multiplicities.

Theorem 20.13.5 (Single-bond predicate iff single-bond commuting form). [Lean](#) [✓ Stmt](#) *The single-bond product predicate is equivalent to the existence of single-bond commuting-form data on every finite chain. This is an equivalence between two presentations of the same condition.*

Proof. [✓ Proof](#) Expanding the two definitions gives the same identity $\rho^{(N)}(M) = c \prod_{i=0}^{N-1} B_{i,i+1}$ with $c > 0$. $\square$

Definition 20.13.6 (Single-bond predicate with doubled-index idempotence). [Lean](#) [✓ Stmt](#) This is the conjunction of the single-bond condition with the doubled-index transfer identity $\mathcal{E}_M \circ \mathcal{E}_M = \mathcal{E}_M$. The source physical-trace condition is Definition 20.5.4.

Theorem 20.13.7 (Single-bond predicate with idempotence iff single-bond form with it). [Lean](#) [✓ Stmt](#) *The single-bond condition with doubled-index transfer idempotence is equivalent to the conjunction of the single-bond commuting-form property with that same idempotence condition.*

Proof. [✓ Proof](#) Expand the single-bond conjunction and apply Theorem 20.13.5. $\square$

Theorem 20.13.8 (A single bond gives one GSNNCH sector). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *A single positive commuting bond is the one-sector instance of the explicit GSNNCH decomposition, with sector projection equal to the identity and multiplicity one.*

Proof. [✓ Proof](#) Take one sector, its one-site projection to be the identity, its multiplicity to be one, and its bond to be the given bond. $\square$

Definition 20.13.9 (Restriction to an orthogonal physical sector). [Lean](#) [✓ Stmt](#) Let P be an orthogonal one-site projection and let K be an MPO tensor fixed by compression with P on its two physical indices. A physical support restriction consists of a dimension e and an isometry $V : \mathbb{C}^e \rightarrow \mathbb{C}^d$ such that

$$\begin{aligned} V^*V &= \mathbb{1}, \\ VV^* &= P, \end{aligned}$$

for which the restricted tensor $K_P = V^*KV$ is injective and its inclusion into the ambient physical space recovers K exactly.

Theorem 20.13.10 (Injective restriction to a BNT projector range). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Suppose that the source BNT projectors have been constructed after the common physical blocking, and let $\mathcal{K}_s$ be the normal representative with its common copy weight absorbed. Then $\mathcal{K}_s$ admits a physical support restriction to the range of P_s . In particular, for some isometry V_s ,*

$$\begin{aligned} V_s V_s^* &= P_s, \\ \mathcal{K}_s &= V_s (V_s^* \mathcal{K}_s V_s) V_s^*, \end{aligned}$$

and the restricted tensor $V_s^* \mathcal{K}_s V_s$ is injective.

Proof. [✓ Proof](#) The projector identity $P_s \mathcal{K}_s P_s = \mathcal{K}_s$ gives an isometric inclusion of its range. The simultaneous one-site spanning condition makes each normal representative injective, and multiplication by its nonzero common weight preserves injectivity. Write $K_{P_s} = V_s^* \mathcal{K}_s V_s$. Each ambient matrix slice satisfies

$$(\mathcal{K}_s)_{ij} = \sum_{p,q} (V_s)_{ip} \overline{(V_s)_{jq}} (K_{P_s})_{pq}.$$

Hence

$$\text{span}_{\mathbb{C}}\{(\mathcal{K}_s)_{ij}\}_{i,j} \subseteq \text{span}_{\mathbb{C}}\{(K_{P_s})_{pq}\}_{p,q}.$$

The left-hand side is the full virtual matrix algebra, so K_{P_s} is injective. The two isometry identities give the exact reconstruction. $\square$

Theorem 20.13.11 (Normalized operators under an isometric physical inclusion). [Lean](#) [✓ Stmt](#) *Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ satisfy $V^*V = I$, and let K^V be obtained from an MPO tensor K by conjugating each physical matrix by V . The normalized periodic operators satisfy*

$$\sigma^{(N)}(K^V) = V^{\otimes N} \sigma^{(N)}(K) (V^{\otimes N})^*.$$

Proof. [✓ Proof](#) [Lean](#) Since $(V^{\otimes N})^* V^{\otimes N} = I$, cyclicity of the trace gives

$$\text{tr}[V^{\otimes N} \rho^{(N)}(K) (V^{\otimes N})^*] = \text{tr}[\rho^{(N)}(K)].$$

Therefore the scalar normalization commutes with the congruence:

$$\sigma^{(N)}(K^V) = \frac{V^{\otimes N} \rho^{(N)}(K) (V^{\otimes N})^*}{\text{tr}[\rho^{(N)}(K)]} = V^{\otimes N} \sigma^{(N)}(K) (V^{\otimes N})^*.$$

$\square$

Theorem 20.13.12 (Prefix marginal of a sitewise isometric image). [Lean](#) [✓ Stmt](#) *Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ satisfy $V^*V = I$. If an operator on $L + K$ sites is conjugated by $V^{\otimes(L+K)}$, then its first- L -site marginal is conjugated by $V^{\otimes L}$.*

Proof. [✓ Proof](#) [Lean](#) Let X denote the operator before conjugation. Reindex the chain into its first L and final K sites. The sitewise matrix becomes $V^{\otimes L} \otimes V^{\otimes K}$. Since $(V^{\otimes K})^* V^{\otimes K} = I$, the partial trace satisfies

$$\begin{aligned} \text{tr}_K[(V^{\otimes L} \otimes V^{\otimes K}) X (V^{\otimes L} \otimes V^{\otimes K})^*] \\ = V^{\otimes L} \text{tr}_K[X] (V^{\otimes L})^*. \end{aligned}$$

$\square$

Lemma 20.13.13 (Rectangular characteristic-polynomial cyclicity). [✓ ML stmt](#) *Let A be an $e \times d$ matrix and B a $d \times e$ matrix, with $d \leq e$. Then*

$$\chi_{AB}(X) = X^{e-d} \chi_{BA}(X).$$

Theorem 20.13.14 (Characteristic polynomial under an isometric inclusion). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ satisfy $V^*V = I_d$, and let A be a $d \times d$ matrix. Then $d \leq e$ and*

$$\chi_{VAV^*}(X) = X^{e-d} \chi_A(X).$$

Equivalently, the roots of χ_{VAV^*} are the roots of χ_A together with zero of multiplicity $e - d$.

Proof. [✓ Proof](#) Write $VAV^* = V(AV^*)$. Characteristic-polynomial cyclicity for rectangular factors gives

$$\chi_{V(AV^*)}(X) = X^{e-d} \chi_{(AV^*)V}(X) = X^{e-d} \chi_A(X),$$

where the last equality uses $V^*V = I_d$. Taking roots of the product gives

$$\text{roots}(\chi_{VAV^*}) = \text{roots}(X^{e-d}\chi_A) \quad (20.21)$$

$$\begin{aligned} &= \text{roots}(X^{e-d}) + \text{roots}(\chi_A) \\ &= (e-d) \cdot \{0\} + \text{roots}(\chi_A). \end{aligned} \quad (20.22)$$

The middle equality applies because X^{e-d} is nonzero and the monic polynomial χ_A is nonzero. $\square$

Theorem 20.13.15 (Reduced states under an isometric physical inclusion). [Lean](#) [✓ Stmt](#) *Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ be an isometry, and let K^V be obtained from an MPO tensor K by conjugating each physical matrix by V . For every $0 \leq L \leq N$, the contiguous reduced states satisfy*

$$\rho_L^{(N)}(K^V) = V^{\otimes L} \rho_L^{(N)}(K) (V^{\otimes L})^*.$$

Proof. [✓ Proof](#) The normalized periodic operators are related by the sitewise isometric congruence. Splitting the chain into the retained interval and its complement gives

$$\begin{aligned} &\text{tr}_{N \setminus L} \left[(V^{\otimes L} \otimes V^{\otimes(N-L)}) \sigma^{(N)}(K) (V^{\otimes L} \otimes V^{\otimes(N-L)})^* \right] \\ &= V^{\otimes L} \rho_L^{(N)}(K) (V^{\otimes L})^*. \end{aligned} \quad (20.23)$$

Here $(V^{\otimes(N-L)})^* V^{\otimes(N-L)} = I$, so the isometry on the traced factor disappears under the partial trace. $\square$

Theorem 20.13.16 (SAL descends through an isometric physical inclusion). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ be an isometry, and let K^V be obtained from an MPO tensor K by conjugating each physical matrix by V . For every $N \geq 1$,*

$$\text{tr}(\sigma^{(N)}(K^V)) = \text{tr}(\sigma^{(N)}(K)).$$

If the periodic operators of K^V are positive semidefinite, then those of K are positive semidefinite. Their mutual informations agree at every contiguous cut. Consequently, if K^V satisfies saturation of the area law, then so does K .

Proof. [✓ Proof](#) At length N , Lemma 21.3.47 identifies the changed periodic operator with conjugation by $V^{\otimes N}$. Since $(V^{\otimes N})^* V^{\otimes N} = \mathbb{1}$, cyclicity of the trace gives the displayed equality, and compression by $(V^{\otimes N})^*$ preserves positivity.

The same trace equality shows that normalized periodic operators obey the same conjugation identity. After a chain is divided into a retained interval and its complement, the isometry on the traced-out factor disappears under the partial trace. The retained marginal is therefore conjugated by the corresponding tensor power of V , so its nonzero spectrum and entropy are unchanged. Thus the mutual informations, and hence the consecutive equalities defining SAL, agree. $\square$

Theorem 20.13.17 (SAL passes to the physical support restriction). [Lean](#) [✓ Stmt](#) *If K satisfies saturation of the area law and $K = VK_P V^*$ is a physical support restriction, then the injective restricted tensor K_P also satisfies saturation of the area law.*

Proof. [✓ Proof](#) On a chain of every positive length, the sitewise inclusion $V^{\otimes N}$ is an isometry and

$$\rho^{(N)}(K) = V^{\otimes N} \rho^{(N)}(K_P) (V^{\otimes N})^*.$$

The trace is therefore unchanged. After a contiguous marginal is taken, the isometry on the traced-out sites disappears, while the isometry on the retained sites remains. Thus every block marginal of K is an isometric conjugate of the corresponding marginal of K_P . Their nonzero spectra, and hence their entropies and mutual informations, coincide. The consecutive equalities defining SAL for K consequently give those for K_P . $\square$

Theorem 20.13.18 (The restricted commuting product has a supported ambient realization).

[Lean](#) [✓ Stmt](#) Let $K = VK_P V^*$ be a physical support restriction of an MPO tensor satisfying SAL. Proposition C.8 applied to the injective tensor K_P gives a positive two-site bond B_P whose cyclic translates commute pairwise on every periodic chain of length $N \geq 2$. Its lift $B = V^{\otimes 2} B_P (V^{\otimes 2})^*$ is positive, its cyclic translates commute pairwise on every such ambient chain, and $(P \otimes P)B(P \otimes P) = B$. Moreover, for every $N \geq 2$ one has the exact identity

$$\rho^{(N)}(K) = \prod_{i=0}^{N-1} B_{i,i+1}.$$

Proof. [✓ Proof](#) The preceding theorem gives SAL for the injective restricted tensor, so Proposition C.8 supplies its positive bond. Since $V^*V = \mathbb{1}$, positivity is preserved under conjugation by $V^{\otimes 2}$. Also, $V^{\otimes 2}(V^{\otimes 2})^* = P \otimes P$; multiplying the lifted bond by this projection on either side therefore leaves it unchanged.

It remains to prove commutativity after the lift. On a three-site chain, isometric conjugation carries the product of two adjacent restricted bonds to the product of their lifted bonds. The restricted product is Hermitian because its positive factors commute. Its conjugate is therefore Hermitian; since the lifted factors are Hermitian, their product equals the product in the reverse order. The two-site periodic chain is handled by the same argument with the two oppositely oriented bonds. Locality and cyclic translation then give pairwise commutativity for every $N \geq 2$.

Finally, isometric conjugation preserves every nonempty product of restricted bonds. The image of each restricted bond is the corresponding lifted bond multiplied by the projection onto the range of $V^{\otimes N}$. The support projections contributed by the first $N - 1$ bonds cover all sites and leave the full commuting product fixed. Hence the range projection disappears, and the restricted finite-chain product identity becomes the asserted ambient identity, still with normalization scalar one. $\square$

Definition 20.13.19 (Proportionally realized orthogonal commuting sector products). [Lean](#)

[✓ Stmt](#) Let $\{K_x\}_{x=0}^{g-1}$ be MPO tensors on a common one-site space. A proportionally realized orthogonal commuting sector family consists of pairwise orthogonal one-site projections P_x and positive two-site operators $B^{(x)}$ such that

$$\begin{aligned} (P_x \otimes P_x)B^{(x)}(P_x \otimes P_x) &= B^{(x)}, \\ [\tau_i(B^{(x)}), \tau_j(B^{(x)})] &= 0. \end{aligned}$$

For every sector x and every $N \geq 2$, there is a number $a_{x,N} > 0$ such that

$$\rho^{(N)}(K_x) = a_{x,N} \prod_{i=0}^{N-1} \tau_i(B^{(x)}).$$

Theorem 20.13.20 (One-site positivity completes proportional sector positivity). [Lean](#) [Lean](#)

[✓ Stmt](#) Every sector operator of length $N \geq 2$ in a proportionally realized orthogonal commuting sector family is positive semidefinite. Consequently, if the one-site operator $\rho^{(1)}(K_x)$ is positive semidefinite for every sector x , then every K_x generates matrix product density operators at all positive lengths.

Proof. [✓ Proof](#) For $N \geq 2$, the sector operator is a positive scalar multiple of a product of commuting positive bonds, hence is positive semidefinite. The one-site hypothesis supplies the only remaining positive chain length. $\square$

Definition 20.13.21 (Exactly realized orthogonal commuting sector products). [Lean](#) [Lean](#)

[✓ Stmt](#) Let $\{K_x\}_{x=0}^{g-1}$ be MPO tensors on a common one-site space. An orthogonally supported family of commuting sector products consists of pairwise orthogonal one-site projections P_x and positive two-site operators $B^{(x)}$ such that

$$(P_x \otimes P_x)B^{(x)}(P_x \otimes P_x) = B^{(x)},$$

$$[\tau_i(B^{(x)}), \tau_j(B^{(x)})] = 0,$$

and, for every $N \geq 2$,

$$\rho^{(N)}(K_x) = \prod_{i=0}^{N-1} \tau_i(B^{(x)}).$$

Together with natural multiplicities n_x , these objects determine the corresponding finite-chain sector decomposition in Definition 20.13.1.

Definition 20.13.22 (Exact sector products as proportionally realized products). [Lean](#) [✓ Stmt](#)

Every exactly realized orthogonal commuting sector family determines a proportionally realized orthogonal commuting sector family with the same one-site projections and two-site bonds.

Proof. [✓ Proof](#) Take $a_{x,N} = 1$ for every sector x and every $N \geq 2$. $\square$

Theorem 20.13.23 (Positive factorizations on orthogonal physical restrictions). [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) Let $\{\mathcal{K}_x\}_{x=0}^{g-1}$ be injective MPO tensors and let $\{P_x\}_{x=0}^{g-1}$ be pairwise orthogonal projections. Suppose that every physical slice is absorbed on both sides:

$$P_x \mathcal{K}_x[\beta, \alpha] = \mathcal{K}_x[\beta, \alpha] = \mathcal{K}_x[\beta, \alpha] P_x.$$

Write $\mathcal{K}_x = V_x \mathcal{K}_{x,P_x} V_x^*$ for the restriction to the range of P_x . If every $\mathcal{K}_{x,P_x}$ admits a physical-sector factorization whose neighboring operators are positive semidefinite, then there are positive bonds $B^{(x)}$ such that

$$(P_x \otimes P_x)B^{(x)}(P_x \otimes P_x) = B^{(x)}$$

and all translates of each $B^{(x)}$ commute pairwise. For every $N \geq 2$,

$$\rho^{(N)}(\mathcal{K}_x) = \prod_{i=0}^{N-1} \tau_i(B^{(x)}).$$

Thus these bonds form an orthogonally supported family of commuting sector products.

Proof. [✓ Proof](#) The two-sided equations identify each tensor with its compression to $\text{ran}(P_x)$ and give an injective restricted tensor. The positive neighboring operators determine the positive commuting bond on this restricted space. Conjugating by $V_x^{\otimes 2}$ preserves positivity and commutativity, gives the displayed support equation, and transports every nonempty periodic product exactly. $\square$

Theorem 20.13.24 (SAL supplies orthogonally supported BNT sector products). [Lean](#) [✓ Stmt](#)
 Suppose that the BNT physical projectors P_x have been selected and that every common-weight-absorbed representative $\mathcal{K}_x$ satisfies SAL. Then there are positive two-site bonds $B^{(x)}$ such that the projections P_x are pairwise orthogonal,

$$(P_x \otimes P_x)B^{(x)}(P_x \otimes P_x) = B^{(x)},$$

all cyclic translates of each $B^{(x)}$ commute pairwise, and, for every $N \geq 2$,

$$\rho^{(N)}(\mathcal{K}_x) = \prod_{i=0}^{N-1} \tau_i(B^{(x)}).$$

Thus the absorbed BNT representatives have an orthogonally supported family of commuting sector products with normalization scalar one.

Proof. [✓ Proof](#) Restrict $\mathcal{K}_x$ isometrically to the range of P_x . The restricted tensor is injective and satisfies SAL, so Proposition C.8 gives its positive commuting bond product. Apply Theorem 20.13.18 in every sector. The lifted bond is supported on $P_x \otimes P_x$ and realizes $\mathcal{K}_x$ exactly. The BNT projector construction gives $P_x P_y = 0$ for $x \neq y$, which completes the sector family. $\square$

Theorem 20.13.25 (Orthogonal commuting sector products give the GSNNCH form). [Lean](#) [Lean](#) [✓ Stmt](#)
 Suppose that $\{K_x\}_{x=0}^{g-1}$ has an orthogonally supported family of commuting sector products and that, for every $N \geq 2$,

$$\rho^{(N)}(M) = \sum_{x=0}^{g-1} n_x \rho^{(N)}(K_x),$$

$$n_x \in \mathbb{N}.$$

Then M has the explicit GSNNCH sector form with outer sectors x and natural multiplicities n_x .

Proof. [✓ Proof](#) For each $N \geq 2$, the represented unnormalized sector sum is

$$\sum_{x=0}^{g-1} n_x \prod_{i=0}^{N-1} \tau_i(B^{(x)}) = \sum_{x=0}^{g-1} n_x \rho^{(N)}(K_x) = \rho^{(N)}(M).$$

Thus the positive proportionality constant in Definition 20.13.1 is one. $\square$

Theorem 20.13.26 (Nonnegative sector coefficients give the GSNNCH form). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
 Suppose that $\{K_x\}_{x=0}^{g-1}$ has an orthogonally supported family of commuting sector products, and let $n_x \geq 1$ be natural numbers. For every $N \geq 2$, let $c_x(N) \geq 0$ and suppose that

$$\rho^{(N)}(M) = \sum_{x=0}^{g-1} c_x(N) \rho^{(N)}(K_x).$$

Then M has the explicit GSNNCH sector form.

Proof. [✓ Proof](#) Fix $N \geq 2$. Replace the bond in sector x by

$$\widetilde{B}_N^{(x)} = \left(\frac{c_x(N)}{n_x} \right)^{1/N} B^{(x)}.$$

This bond is positive, has the same support, and its translates commute. Its periodic product is

$$n_x \prod_{i=0}^{N-1} \tau_i(\widetilde{B}_N^{(x)}) = c_x(N) \prod_{i=0}^{N-1} \tau_i(B^{(x)}) = c_x(N) \rho^{(N)}(K_x).$$

Taking n_x as the sector multiplicity gives the required decomposition, with positive proportionality constant one. $\square$

Theorem 20.13.27 (Proportional local orthogonal sums preserve SAL). [Lean](#) [Stmt](#) *Let $\{K_x\}_{x=0}^{g-1}$ be a nonempty finite family of MPO tensors, let n_x be positive natural numbers, and let P_x be pairwise orthogonal one-site projections. Suppose that, for every $N > 0$, there is a number $c_N > 0$ such that*

$$\rho^{(N)}(M) = c_N \sum_{x=0}^{g-1} n_x \rho^{(N)}(K_x).$$

For $N \geq 2$ and $1 \leq L \leq N$, write $\hat{\rho}_{x,N,L}$ for the normalized marginal of $\rho^{(N)}(K_x)$ on the first L sites. Assume that

$$(P_x \otimes \mathbb{1}^{\otimes(L-1)}) \hat{\rho}_{x,N,L} = \hat{\rho}_{x,N,L}.$$

If every K_x satisfies SAL, then M satisfies SAL.

Proof. [Proof](#) For fixed N , put

$$S_N = \sum_x n_x \rho^{(N)}(K_x),$$

$$p_x = \frac{c_N n_x \operatorname{tr}(\rho^{(N)}(K_x))}{\operatorname{tr}(\rho^{(N)}(M))}.$$

Trace linearity and $\rho^{(N)}(M) = c_N S_N$ give

$$\operatorname{tr}(\rho^{(N)}(M))^{-1} \rho^{(N)}(M) = \operatorname{tr}(S_N)^{-1} S_N = \sum_x p_x \hat{\rho}_{x,N,N}.$$

Thus the common positive scalar cancels from the normalized chain and from every marginal.

Put $Q_x = P_x \otimes \mathbb{1}^{\otimes(L-1)}$. The support identity gives $Q_x \hat{\rho}_{x,N,L} = \hat{\rho}_{x,N,L}$, while $P_x P_y = 0$ for $x \neq y$ gives $Q_x Q_y = 0$. The entropy of a nonempty marginal is therefore

$$S_L(M) = \sum_x (-p_x \log p_x + p_x S_L(K_x)).$$

Substitution in $I_L = S_L + S_{N-L} - S_N$ gives

$$I_L(M) = - \sum_x p_x \log p_x + \sum_x p_x I_L(K_x).$$

The probabilities depend on N but not on L . Sectorwise SAL therefore gives $I_L(M) = I_{L+1}(M)$ throughout the required range. $\square$

Theorem 20.13.28 (Exact local orthogonal sums preserve SAL). Lean ✓ Stmt Let $\{K_x\}_{x=0}^{g-1}$ be a nonempty finite family of MPO tensors, let n_x be positive natural numbers, and let P_x be pairwise orthogonal one-site projections. Suppose that, for every $N > 0$,

$$\rho^{(N)}(M) = \sum_{x=0}^{g-1} n_x \rho^{(N)}(K_x).$$

For $N \geq 2$ and $1 \leq L \leq N$, write $\hat{\rho}_{x,N,L}$ for the normalized marginal of $\rho^{(N)}(K_x)$ on the first L sites. Assume that the sector label is visible on the first retained site:

$$(P_x \otimes \mathbb{1}^{\otimes(L-1)}) \hat{\rho}_{x,N,L} = \hat{\rho}_{x,N,L}.$$

If every K_x satisfies SAL, then M satisfies SAL.

This is the conditional last inference in the proposition at [CPGSV16, Appendix C.2, lines 1801–1808]. It assumes sectorwise SAL. The printed proposition states the conclusion from the GSNNCH form alone, but its proof invokes the single-sector proposition at lines 1597–1619, whose hypotheses include ZCL. The theorem here does not supply that omitted sectorwise implication.

Proof. ✓ Proof Take $c_N = 1$ in Theorem 20.13.27. □

Theorem 20.13.29 (Proportional commuting sector products have normalized-chain support). Lean ✓ Stmt Let $\{K_x\}_{x=0}^{g-1}$ have a proportionally realized orthogonal commuting sector family with one-site projections P_x and two-site bonds $B^{(x)}$. Then, for every sector x and every $N \geq 2$,

$$(P_x \otimes \mathbb{1}^{\otimes(N-1)}) \hat{\rho}_{x,N,N} = \hat{\rho}_{x,N,N}.$$

Proof. ✓ Proof Since P_x is a projection and $B^{(x)}$ is supported on $P_x \otimes P_x$,

$$(P_x \otimes \mathbb{1}) B^{(x)} = B^{(x)}.$$

Thus the first translated bond, and hence the full product, satisfy

$$\begin{aligned} (P_x \otimes \mathbb{1}^{\otimes(N-1)}) \tau_0(B^{(x)}) &= \tau_0(B^{(x)}), \\ (P_x \otimes \mathbb{1}^{\otimes(N-1)}) \prod_{i=0}^{N-1} \tau_i(B^{(x)}) &= \prod_{i=0}^{N-1} \tau_i(B^{(x)}). \end{aligned}$$

Multiplying the second identity by $a_{x,N}$ gives

$$(P_x \otimes \mathbb{1}^{\otimes(N-1)}) \rho^{(N)}(K_x) = \rho^{(N)}(K_x).$$

Multiplying this identity by the scalar $(\text{tr } \rho^{(N)}(K_x))^{-1}$ gives

$$\begin{aligned} (P_x \otimes \mathbb{1}^{\otimes(N-1)}) \left[(\text{tr } \rho^{(N)}(K_x))^{-1} \rho^{(N)}(K_x) \right] \\ = (\text{tr } \rho^{(N)}(K_x))^{-1} \rho^{(N)}(K_x). \end{aligned}$$

The scalar multiples on both sides are the normalized chain states. This remains valid when the trace is zero, with the convention $0^{-1} = 0$. □

Theorem 20.13.30 (Proportional commuting sector products have prefix-marginal support).

[Lean](#) [✓ Stmt](#) Let $\{K_x\}_{x=0}^{g-1}$ have a proportionally realized orthogonal commuting sector family with one-site projections P_x and two-site bonds $B^{(x)}$. Then, for every sector x , every $N \geq 2$, and every $1 \leq L \leq N$,

$$(P_x \otimes \mathbb{1}^{\otimes(L-1)})\hat{\rho}_{x,N,L} = \hat{\rho}_{x,N,L}.$$

Proof. [✓ Proof](#) Write $\text{Tr}_{L+1,\dots,N}$ for the partial trace over the final $N - L$ sites. By definition,

$$\hat{\rho}_{x,N,L} = \text{Tr}_{L+1,\dots,N}(\hat{\rho}_{x,N,N}).$$

The first-site action commutes with this partial trace. The preceding theorem therefore gives

$$\begin{aligned} & (P_x \otimes \mathbb{1}^{\otimes(L-1)})\hat{\rho}_{x,N,L} \\ &= \text{Tr}_{L+1,\dots,N}((P_x \otimes \mathbb{1}^{\otimes(N-1)})\hat{\rho}_{x,N,N}) \\ &= \text{Tr}_{L+1,\dots,N}(\hat{\rho}_{x,N,N}) = \hat{\rho}_{x,N,L}. \end{aligned}$$

□

Lemma 20.13.31 (Commuting sector products have local marginal support). [Lean](#) [✓ Stmt](#) Suppose that K_x is the periodic product of translates of a positive two-site operator B_x , and that B_x is supported on $P_x \otimes P_x$. If $N \geq 2$ and $1 \leq L \leq N$, then the normalized marginal of K_x on the first L sites satisfies

$$(P_x \otimes \mathbb{1}^{\otimes(L-1)})\hat{\rho}_{x,N,L} = \hat{\rho}_{x,N,L}.$$

Proof. [✓ Proof](#) Apply the preceding theorem with $a_{x,N} = 1$. □

Theorem 20.13.32 (Proportional orthogonal commuting sector sums preserve SAL). [Lean](#) [✓ Stmt](#) Let K_x be a nonempty finite family with a proportionally realized orthogonal commuting sector family, and let n_x be positive natural numbers. Suppose that, for every $N > 0$, there is a number $c_N > 0$ such that

$$\rho^{(N)}(M) = c_N \sum_x n_x \rho^{(N)}(K_x).$$

If every K_x satisfies SAL, then M satisfies SAL.

Proof. [✓ Proof](#) The proportional sector-product support lemma supplies the support of every nonempty marginal. Apply Theorem 20.13.27; its sector weights contain the outer scalar c_N , while the inner scalars $a_{x,N}$ have already disappeared under sector normalization. □

Corollary 20.13.33 (Orthogonal commuting sector sums preserve SAL). [Lean](#) [✓ Stmt](#) Let K_x be a nonempty finite family, let an orthogonally supported commuting sector family realize the tensors K_x , and let n_x be positive natural numbers. Suppose that, for every $N > 0$,

$$\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(K_x).$$

If every K_x satisfies SAL, then M satisfies SAL.

Proof. [✓ Proof](#) Take $a_{x,N} = c_N = 1$ in Theorem 20.13.32. □

Theorem 20.13.34 (Proportional orthogonal commuting-bond sectors with ZCL satisfy SAL).

[Lean](#) [✓ Stmt](#) Let K_x be a nonempty finite family of injective tensors generating MPDOs. Suppose that there are pairwise orthogonal one-site projections P_x and fixed translation-invariant positive commuting bonds $B^{(x)}$, supported on $P_x \otimes P_x$. For every sector x and every $N \geq 2$, suppose that there is a number $a_{x,N} > 0$ such that

$$\rho^{(N)}(K_x) = a_{x,N} \prod_{i=0}^{N-1} \tau_i(B^{(x)}).$$

Suppose also that every K_x has source zero correlation length. Let n_x be positive natural numbers and suppose that, for every $N > 0$, there is a number $c_N > 0$ such that

$$\rho^{(N)}(M) = c_N \sum_x n_x \rho^{(N)}(K_x).$$

Then M satisfies SAL.

Proof. [✓ Proof](#) For each x , the fixed bond $B^{(x)}$ and the positive proportional product identities form the single-sector commuting-bond hypothesis. The single-sector theorem therefore gives SAL for every K_x . Theorem 20.13.32 then gives SAL for M . $\square$

Theorem 20.13.35 (A compatible sector decomposition with ambient ZCL satisfies SAL).

[Lean](#) [✓ Stmt](#) Let S be a nonempty sector decomposition with representatives A_x , positive copy numbers n_x , and nonzero copy weights. Suppose its total bond dimension equals that of an MPO tensor M , and a block-diagonal gauge reconstructs M from the tensor associated with S . Suppose also that this tensor has the same matrix product vectors as M at every positive length, that all copy weights over A_x have a common value μ_x , and that the physical-trace transfer of every A_x is nonnilpotent. Assume that the one-letter tuples $(A_x^i)_x$ span the full product of the representative matrix algebras, and put $K_x = \mu_x A_x$.

Suppose that the tensors K_x form a proportionally realized orthogonal commuting sector family and that each one-site operator $\rho^{(1)}(K_x)$ is positive semidefinite. If M has source zero correlation length, then M satisfies SAL.

This is [CPGSV16, Proposition C.14] with the ambient zero-correlation-length and one-site positivity hypotheses made explicit: the printed proposition omits the ambient hypothesis that its proof uses, through Proposition C.9, to obtain sectorwise zero correlation length, and its commuting-bond realization is unambiguous only from length two onward, so positivity of $\rho^{(1)}(K_x)$ is the sharp additional boundary.

Proof. [✓ Proof](#) The one-site hypothesis and Theorem 20.13.20 show that every K_x generates matrix product density operators. The simultaneous one-letter span and the nonzero absorbed weights give one-site injectivity by Theorem 20.13.10. Ambient source zero correlation length descends to every K_x by Theorem 20.15.2.31. Finally, the supplied positive-length sector-decomposition identity gives $\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(K_x)$ for every $N > 0$, so Theorem 20.13.34 applies. $\square$

Theorem 20.13.36 (Orthogonal commuting-bond sectors with ZCL satisfy SAL). [Lean](#) [✓ Stmt](#)

Let K_x be a nonempty finite family of injective tensors generating MPDOs. Suppose that there are pairwise orthogonal one-site projections P_x and fixed translation-invariant positive commuting bonds $B^{(x)}$, supported on $P_x \otimes P_x$, whose periodic products equal $\rho^{(N)}(K_x)$ for every $N \geq 2$. Suppose also that every K_x has source zero correlation length. Let n_x be positive natural numbers

and suppose that, for every $N > 0$,

$$\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(K_x).$$

Then M satisfies SAL.

Proof. [✓ Proof](#) Take $a_{x,N} = c_N = 1$ in Theorem [20.13.34](#). □

Corollary 20.13.37 (Orthogonal BNT sector sums preserve SAL). [Lean](#) [✓ Stmt](#) Let M have a nonempty basis-of-normal-tensors decomposition with positive copy numbers n_x and copy-independent weights μ_x . Put $K_x = \mu_x A_x$, and suppose that the decomposition generates the same positive-length matrix product vectors as M . If an independently supplied orthogonally supported commuting sector family realizes the tensors K_x , and every K_x satisfies SAL, then M satisfies SAL.

Proof. [✓ Proof](#) The positive-length canonical-form identity gives $\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(K_x)$. Apply Corollary [20.13.33](#). □

Corollary 20.13.38 (Orthogonal BNT sector products give the GSNNCH form). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let its BNT canonical form have normal representatives A_x , copy numbers n_x , and copy-independent weights μ_x . Put $\mathcal{K}_x = \mu_x A_x$, and assume that this canonical form generates the same matrix product vectors as M at every positive length. If the representatives $\mathcal{K}_x$ have orthogonally supported commuting sector products, then

$$\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(\mathcal{K}_x).$$

Hence the BNT copy numbers n_x are the natural multiplicities of the explicit GSNNCH form.

Proof. [✓ Proof](#) Apply Theorem [20.13.25](#) to the positive-length BNT decomposition. Its coefficients are exactly the copy numbers n_x after the common weights have been absorbed. □

Corollary 20.13.39 (Sectorwise SAL gives the BNT GSNNCH form). [Lean](#) [✓ Stmt](#) Let M be an MPO tensor and let its BNT canonical form have normal representatives A_x , copy numbers n_x , and copy-independent weights μ_x . Put $\mathcal{K}_x = \mu_x A_x$, and assume that this canonical form generates the same matrix product vectors as M at every positive length. Equivalently, for every $N > 0$,

$$\rho^{(N)}(M) = \sum_x n_x \rho^{(N)}(\mathcal{K}_x).$$

Under the BNT projector-selection hypotheses, if every $\mathcal{K}_x$ satisfies SAL, then M has the source GSNNCH form. Its outer sectors are the BNT sectors and its natural multiplicities are the copy numbers n_x .

Proof. [✓ Proof](#) Theorem [20.13.24](#) supplies the orthogonally supported commuting sector products. Apply Corollary [20.13.38](#). □

Lemma 20.13.40 (Cyclic-shift transport of an embedded local operator). [Lean](#) [✓ Stmt](#) Let B act on L consecutive spins of the periodic N -site chain, embedded at base site i , and let r denote the cyclic shift of sites. Reindexing both chain configurations by r carries the embedded operator to its translate at the shifted base site:

$$\tau_i(B)[\sigma \circ r, \eta \circ r] = \tau_{r(i)}(B)[\sigma, \eta].$$

Proof. ✓ Proof Entrywise from the definitions: the shift carries the window at i to the window at $r(i)$, preserves agreement of two configurations outside the window, and preserves the extracted window values. $\square$

Lemma 20.13.41 (Translates of a sector bond commute pairwise). Lean ✓ Stmt *For every sector x of an explicit GSNNCH decomposition and every finite chain length $N \geq 2$, all cyclic translates of the sector bond commute: $[\tau_i(B^{(x)}), \tau_j(B^{(x)})] = 0$.*

Proof. ✓ Proof Translates with disjoint windows commute because they act on disjoint sites, and two length-two windows overlap only when their starting sites coincide or are cyclic neighbors. For the neighboring pairs, applying the transport identity of Lemma 20.13.40 p times to the neighboring commutation $[\tau_0(B^{(x)}), \tau_1(B^{(x)})] = 0$ of Definition 20.13.1 gives $[\tau_p(B^{(x)}), \tau_{p+1}(B^{(x)})] = 0$ for every p , which covers all overlapping pairs. $\square$

Lemma 20.13.42 (Positivity of the sector-bond translates). Lean Lean ✓ Stmt *An operator embedded into the periodic chain from a positive semidefinite local factor is positive semidefinite; in particular every translate $\tau_i(B^{(x)})$ of a sector bond is positive semidefinite.*

Proof. ✓ Proof In the coordinates given by the window-plus-complement splitting of the chain, the embedded operator decomposes as

$$\tau_i(B) \cong B \otimes \mathbb{1}_{\text{comp}},$$

where the identity acts on the cyclic complement of the window. A tensor product of positive semidefinite operators is positive semidefinite, and $B^{(x)} \geq 0$ by Definition 20.13.1. $\square$

Lemma 20.13.43 (Positivity of the sector sum). Lean Lean Lean ✓ Stmt *Each sector product $\prod_{j=1}^N \tau_j(B^{(x)})$ of an explicit GSNNCH decomposition is positive semidefinite; hence so are the represented operator*

$$\bigoplus_x n_x \prod_{j=1}^N \tau_j(B^{(x)})$$

and every chain operator equal to a positive multiple of it.

Proof. ✓ Proof A product of pairwise commuting positive semidefinite matrices is positive semidefinite: for two commuting factors A and B , the square root of A commutes with B , so $AB = \sqrt{A}B\sqrt{A} \geq 0$, as congruence by $\sqrt{A}$ preserves positive semidefiniteness; iterate along the product. Each sector product is such a product by Lemmas 20.13.42 and 20.13.41, and the sum with natural coefficients preserves positivity. $\square$

Lemma 20.13.44 (Translation invariance of the sector sum). Lean Lean Lean ✓ Stmt *Each sector product of an explicit GSNNCH decomposition is invariant under the cyclic shift of the periodic chain, where spin $N+1$ is identified with the first; hence so are the represented operator and every chain operator equal to a positive multiple of it [CPGSV16, lines 838–842].*

Proof. ✓ Proof By the transport identity of Lemma 20.13.40, the cyclic shift carries each translate $\tau_j(B^{(x)})$ to $\tau_{j+1}(B^{(x)})$, so it permutes the factors of each sector product cyclically. All factors commute by Lemma 20.13.41, so the product is unchanged, and so is the sum over sectors. $\square$

Lemma 20.13.45 (The sector form passes to the trace normalization). Lean ✓ Stmt *If a finite-chain operator of an MPO tensor has the explicit sector form of Definition 20.13.1 and nonzero trace, then its trace normalization has the sector form as well.*

Proof. [✓ Proof](#) By Lemma 20.13.43 the operator is positive semidefinite, so its nonzero trace is positive and the normalization multiplies the sector sum by a positive constant. $\square$

Theorem 20.13.46 (Sector form with nonvanishing traces gives GSNCH). [Lean](#) [✓ Stmt](#) *If every finite-chain operator of an MPO tensor has the explicit sector form of Definition 20.13.1 and a nonzero trace, then the normalized finite-chain states are density operators with that sector form, so the tensor generates Gibbs states of a nearest-neighbor commuting Hamiltonian.*

Proof. [✓ Proof](#) Positivity of each chain operator follows from Lemma 20.13.43, and the normalized state has unit trace and keeps the sector form by Lemma 20.13.45. $\square$

20.14 Two-site and positive-length physical blocking

The two-site block is the virtual contraction of two copies of K :

$$\begin{array}{c} (j_1, j_2) \\ \downarrow \\ \boxed{M} \\ \uparrow \\ (j_1, j_2) \end{array} = \begin{array}{cc} i_1 & i_2 \\ \downarrow & \downarrow \\ \boxed{K} & \boxed{K} \\ \uparrow & \uparrow \\ j_1 & j_2 \end{array}$$

The two ket indices and the two bra indices are grouped into the blocked physical indices, as in [CPGSV16, Theorem 4.9, lines 851–856].

Definition 20.14.1 (Physical blocking of an MPO tensor). [Lean](#) [Lean](#) [✓ Stmt](#) For an integer $L \geq 0$, group L adjacent physical sites and denote the resulting local tensor by $M^{[L]}$. It has a length- L ket word $I = (i_1, \dots, i_L)$ and a length- L bra word $J = (j_1, \dots, j_L)$ as its physical indices, and

$$(M^{[L]})^{I,J} = M^{i_1 j_1} \dots M^{i_L j_L}.$$

Thus the ket and bra words are grouped separately. For $L = 0$, both words are empty and the unique local matrix is the identity. For $L > 0$, this is the MPO version of the physical blocking used for the basis of normal tensors in [CPGSV16, lines 317–345]. It is a local tensor, not the closed-chain operator $O_L(M)$ of [CPGSV16, lines 962–967].

Theorem 20.14.2 (Doubled-index form of MPO blocking). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *For every integer $L \geq 0$, pairing the ket and bra letters site by site gives the canonical identification*

$$\text{Fin}(d^L) \times \text{Fin}(d^L) \simeq (\text{Fin}(d) \times \text{Fin}(d))^{\text{Fin}(L)} \simeq \text{Fin}((d^2)^L). \quad (20.24)$$

Under this identification, the doubled-index MPS tensor of $M^{[L]}$ is the physical reindexing of the L -blocked doubled-index MPS tensor of M . In particular, one is injective if and only if the other is.

Proof. [✓ Proof](#) Decode the blocked ket and bra words, pair their letters at each site, and compare the two ordered matrix products. For $L = 0$, both words are empty and both matrix products are the identity. Physical reindexing by a bijection preserves the span of the tensor matrices. $\square$

Theorem 20.14.3 (Blocking commutes with the MPO product). [Lean](#) [✓ Stmt](#) *For every integer $L \geq 0$ and two MPO tensors M, N with the same physical dimension, let juxtaposition denote the MPO tensor product obtained by contracting the intermediate physical index. Then $(MN)^{[L]} = M^{[L]}N^{[L]}$. For $L = 0$, each blocked tensor is the unique empty-word local matrix, namely the identity, so the equality reduces to the product of two identity local tensors.*

Proof. [✓ Proof](#) Expand the product tensor and sum over its length- L intermediate physical word. Reindexing this word by one blocked physical index gives the product of the two blocked tensors. When $L = 0$, the intermediate word is empty and all three blocked tensors are identities. $\square$

Theorem 20.14.4 (Closed MPO chains under physical blocking). [Lean](#) [Lean](#) [✓ Stmt](#) Let $L \in \mathbb{N}$, and let

$$\Phi_{N,L} : \{0, \dots, d^L - 1\}^N \longrightarrow \{0, \dots, d - 1\}^{NL}$$

concatenate the N blocked words. For blocked configurations σ and τ ,

$$\rho^{(N)}(M^{[L]})_{\sigma, \tau} = \rho^{(NL)}(M)_{\Phi_{N,L}(\sigma), \Phi_{N,L}(\tau)}.$$

The algebraic identity holds for every $L \in \mathbb{N}$. For $L \geq 1$, positivity of the original nonempty chain of length NL shows that blocking L adjacent physical sites of an MPDO again gives an MPDO. This is the closed-chain identification implicit when the one-site tensor and its two-site blocking are placed in vertical canonical form in [CPGSV16, Appendix C.4, lines 1952–2017].

Proof. [✓ Proof](#) Write $\sigma_k = (\sigma_{k,0}, \dots, \sigma_{k,L-1})$ and $\tau_k = (\tau_{k,0}, \dots, \tau_{k,L-1})$. For each k , $(M^{[L]})^{\sigma_k, \tau_k} = \prod_{j=0}^{L-1} M^{\sigma_{k,j}, \tau_{k,j}}$. Taking both products in increasing index order gives

$$\prod_{k=0}^{N-1} (M^{[L]})^{\sigma_k, \tau_k} = \prod_{k=0}^{N-1} \prod_{j=0}^{L-1} M^{\sigma_{k,j}, \tau_{k,j}} = \prod_{r=0}^{NL-1} M^{(\Phi_{N,L}(\sigma))_r, (\Phi_{N,L}(\tau))_r}.$$

Taking the virtual trace gives the equality in the theorem. The map $\Phi_{N,L}$ is a bijection, and simultaneous reindexing of the rows and columns preserves positive semidefiniteness. $\square$

Definition 20.14.5 (Two-site blocking of an MPO tensor). [Lean](#) [✓ Stmt](#) For an MPO tensor K , its two-site blocking $K^{[2]}$ has physical indices (i_0, i_1) and (j_0, j_1) and matrices

$$(K^{[2]})^{(i_0, i_1), (j_0, j_1)} = K^{i_0 j_0} K^{i_1 j_1}.$$

This is the blocking used in [CPGSV16, Theorem 4.9, lines 851–856].

Theorem 20.14.6 (Concrete two-site blocking as length-two blocking). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\varphi : \{0, \dots, d^2 - 1\} \rightarrow \{0, \dots, d - 1\}^2$ be the canonical bijection. Then $(K^{[2]})^{i,j} = (K^{[L]})^{\varphi(i), \varphi(j)} \big|_{L=2}$. Consequently, for every nonempty chain length N , the closed MPO of $K^{[2]}$ is obtained from the closed MPO of the general length-two block by applying φ independently at every ket and bra site. In particular, if K generates matrix product density operators, then so does $K^{[2]}$.

Proof. [✓ Proof](#) Decode a blocked index as an ordered pair. The two local tensors are then the same product $K^{i_0 j_0} K^{i_1 j_1}$. Applying this equality at every site gives the closed-chain reindexing, and simultaneous row and column reindexing preserves positive semidefiniteness. $\square$

Theorem 20.14.7 (MPO blocking preserves literal CPSV doubled-index form). [Lean](#) [Lean](#)

[✓ Stmt](#) Let A_M be the doubled-index MPS tensor associated to an MPO tensor M . If A_M is in literal CPSV canonical form and $L \geq 1$, then the doubled-index MPS tensor associated to $M^{[L]}$ is again in literal CPSV canonical form. The same conclusion holds for the concrete two-site blocking $M^{[2]}$.

Proof. ✓ Proof The doubled-index tensor of $M^{[L]}$ is, under the canonical pairing of ket and bra words, a physical relabeling of $A_M^{[L]}$:

$$A_{M^{[L]}} = \psi_L^*(A_M^{[L]}). \quad (20.25)$$

Theorem 8.6.15 gives literal CPSV canonical form for $A_M^{[L]}$, and Theorem 8.6.8 carries it across ψ_L . For two-site blocking one takes $L = 2$ and uses the concrete identification of Theorem 20.14.6. $\square$

Theorem 20.14.8 (Literal one-site vertical decomposition). Lean ✓ Stmt *Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. Then there are normal tensors A_α , positive multiplicities r_α , positive numbers $c_{\alpha,q}$, and a coisometry U such that the A_α form a CPSV basis of normal tensors for $\widetilde{M}$ and, writing*

$$B_v = \bigoplus_{\alpha} \bigoplus_{q=0}^{r_\alpha-1} c_{\alpha,q} A_\alpha^v, \quad (20.26)$$

one has

$$\begin{aligned} UU^\dagger &= \mathbb{1}, \\ U\widetilde{M}_v U^\dagger &= B_v, \\ \widetilde{M}_v &= U^\dagger B_v U. \end{aligned} \quad (20.27)$$

Proof. ✓ Proof Group the normal vertical corners into phase classes and normalize their physical sector maps. The exact original-sector reconstruction gives positive-length matrix-product-vector equality with the corresponding weighted direct sum. Hence the spectral representatives retain the CPSV basis-of-normal-tensors property for $\widetilde{M}$. Flatten the pair (α, q) into one finite index. Orthogonality of the normalized sector maps gives $UU^\dagger = \mathbb{1}$; their intertwining identities give the middle equality in (20.27), and their exact reconstruction gives the last equality. $\square$

Theorem 20.14.9 (Literal two-site vertical decomposition). Lean ✓ Stmt *Under the hypotheses of Theorem 20.14.8, the concrete two-site block $M^{[2]}$ has a decomposition of the same form: there are normal tensors $\widehat{A}_\alpha$, $r_\alpha^{[2]} > 0$, $c_{\alpha,q}^{[2]} > 0$, and a coisometry $U^{[2]}$ such that*

$$\begin{aligned} U^{[2]}(U^{[2]})^\dagger &= \mathbb{1}, \\ U^{[2]}\widetilde{M}^{[2]}_v(U^{[2]})^\dagger &= \bigoplus_{\alpha,q} c_{\alpha,q}^{[2]} \widehat{A}_\alpha^v, \\ \widetilde{M}^{[2]}_v &= (U^{[2]})^\dagger \left(\bigoplus_{\alpha,q} c_{\alpha,q}^{[2]} \widehat{A}_\alpha^v \right) U^{[2]}. \end{aligned} \quad (20.28)$$

The tensors $\widehat{A}_\alpha$ form a CPSV basis of normal tensors for $\widetilde{M}^{[2]}$.

Proof. ✓ Proof Two-site blocking preserves both literal CPSV canonical form and matrix-product-density-operator positivity. Apply Theorem 20.14.8 to $M^{[2]}$. $\square$

Corollary 20.14.10 (Paired literal vertical decompositions). Lean ✓ Stmt *Under the hypotheses of Theorem 20.14.8, there exist simultaneously a one-site vertical decomposition of M and a two-site vertical decomposition of $M^{[2]}$ of the respective forms stated in Theorems 20.14.8 and 20.14.9.*

Proof. [✓ Proof](#) Choose the one-site and two-site decompositions independently and take their ordered pair. $\square$

Lemma 20.14.11 (Two-site blocking preserves normalized BNT-refined horizontal form). [Lean](#)

[✓ Stmt](#) *If an MPO tensor K is in normalized BNT-refined horizontal form, then its two-site blocking $K^{[2]}$ is again in that form.*

Proof. [✓ Proof](#) Block every normal representative in the horizontal sector decomposition by two sites and square each copy weight. Positive blocking preserves irreducibility, left-canonical normalization, eventual linear independence of the representative states, and inequivalence of distinct representatives. Normalized self-overlap is preserved by $O_{A_j^{[2]}A_j^{[2]}}(N) = O_{A_jA_j}(2N) \rightarrow 1$. Hence the blocked sector decomposition is again a BNT canonical form. If φ is the canonical bijection between a pair of ket–bra indices and a length-two word in the doubled alphabet, then $(K^{[2]})^{\text{MPS}} = \varphi^*(\text{block}_2(K^{\text{MPS}}))$. Thus φ only relabels the physical basis. Finally, if X is the original block-diagonal virtual gauge and S its sector tensor, word evaluation gives $((K^{[2]})^{\text{MPS}})_i = X(\varphi^*(S^{[2]}))_i X^{-1}$. Therefore this blocked decomposition and the same copy gauges witness normalized BNT-refined horizontal form for $K^{[2]}$. $\square$

Theorem 20.14.12 (One-site and two-site vertical canonical forms). [Lean](#) [✓ Stmt](#) *Let K be an MPO tensor in normalized BNT-refined horizontal form that generates matrix product density operators. Then both K and $K^{[2]}$ satisfy the vertical canonical-form conditions of Definition 17.4.6. This is the initial canonical-form step in [CPGSV16, Appendix C.4, lines 1951–1956].*

Proof. [✓ Proof](#) The preceding blocking results show that $K^{[2]}$ is in normalized BNT-refined horizontal form and generates matrix product density operators. Apply Theorem 17.4.112 first to K and then to $K^{[2]}$. $\square$

Definition 20.14.13 (Normalized vertical-sector maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) *Let the vertical sectors be labelled by α , with simple matrix algebra M_{d_α} and positive diagonal multiplicity matrix*

$$\mu_\alpha = \text{diag}(\mu_{\alpha,0}, \dots, \mu_{\alpha,r_\alpha-1}).$$

Write $m_\alpha = \text{tr}(\mu_\alpha)$. The normalized embedding and the left partial trace are

$$\begin{aligned} R_\mu \left(\bigoplus_\alpha X_\alpha \right) &= \bigoplus_\alpha \frac{\mu_\alpha}{m_\alpha} \otimes X_\alpha, \\ \widetilde{R}_\mu \left(\bigoplus_\alpha Y_\alpha \right) &= \bigoplus_\alpha \text{tr}_{\text{mult}}(Y_\alpha). \end{aligned}$$

On a weighted sector, $\widetilde{R}_\mu(\lambda_\alpha \mu_\alpha \otimes X_\alpha) = \lambda_\alpha m_\alpha X_\alpha$. Thus the partial trace is the canonical extension to the full block matrix space of the inverse map displayed in [CPGSV16, Appendix C.4, lines 1957–1971].

Theorem 20.14.14 (Complete positivity of the normalized vertical-sector maps). [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) *Let $(d_\alpha)_\alpha$ and $(r_\alpha)_\alpha$ be finite families of dimensions with $r_\alpha > 0$, and suppose that every diagonal entry of μ_α is positive. Then the normalized embedding*

$$\begin{aligned} R_\mu: \bigoplus_\alpha M_{d_\alpha}(\mathbb{C}) &\longrightarrow \bigoplus_\alpha M_{r_\alpha d_\alpha}(\mathbb{C}), \\ R_\mu((X_\alpha)_\alpha) &= \left(\frac{\mu_\alpha}{m_\alpha} \otimes X_\alpha \right)_\alpha \end{aligned}$$

is a direct-sum Kraus map. Without any condition on the multiplicities, the map

$$\begin{aligned}\widetilde{R}: \bigoplus_{\alpha} M_{r_{\alpha} d_{\alpha}}(\mathbb{C}) &\longrightarrow \bigoplus_{\alpha} M_{d_{\alpha}}(\mathbb{C}), \\ \widetilde{R}((Y_{\alpha})_{\alpha}) &= (\text{tr}_{\text{mult}}(Y_{\alpha}))_{\alpha}\end{aligned}$$

is a direct-sum Kraus map. Its restriction to the weighted sectors is the retraction in [CPGSV16, Appendix C.4, lines 1961–1970].

Proof. ✓ Proof

Exchange the two coordinates within each sector by $w(\alpha, q, i) = (\alpha, i, q)$. Let R_w be the corresponding matrix reindexing. For the normalized embedding, set $\rho_{\alpha} = m_{\alpha}^{-1} \mu_{\alpha}$. The hypotheses give $\rho_{\alpha} \succeq 0$ and $\text{tr}(\rho_{\alpha}) = 1$. Let $\mathcal{P}$ be the orthogonally controlled preparation of these density matrices. Thus

$$[\mathcal{P}(Z)]_{\langle \alpha, (i, q) \rangle, \langle \beta, (j, s) \rangle} = \begin{cases} Z_{\langle \alpha, i \rangle, \langle \alpha, j \rangle} \rho_{\alpha}(q, s), & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

The preparation Kraus operators resolve the identity in each sector, so $\mathcal{P}$ is trace-preserving and completely positive. Reindexing by w^{-1} changes $X_{\alpha} \otimes \rho_{\alpha}$ into $\rho_{\alpha} \otimes X_{\alpha}$. Consequently, the full-matrix extension satisfies

$$\begin{aligned}[\widehat{R}_{\mu}(Z)]_{\langle \alpha, (q, i) \rangle, \langle \beta, (s, j) \rangle} &= \begin{cases} \rho_{\alpha}(q, s) Z_{\langle \alpha, i \rangle, \langle \alpha, j \rangle}, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases} \\ \widehat{R}_{\mu} &= R_{w^{-1}} \circ \mathcal{P}.\end{aligned}$$

This proves complete positivity of the normalized embedding.

Now let $\mathcal{C}_{\text{tr}}$ be the controlled partial trace over the $\mathbb{C}^{r_{\alpha}}$ factors. For a matrix Y on $\bigoplus_{\alpha} (\mathbb{C}^{r_{\alpha}} \otimes \mathbb{C}^{d_{\alpha}})$, the full-matrix extension satisfies

$$[\widehat{R}(Y)]_{\langle \alpha, i \rangle, \langle \beta, j \rangle} = \begin{cases} \sum_{q=0}^{r_{\alpha}-1} Y_{\langle \alpha, (q, i) \rangle, \langle \alpha, (q, j) \rangle}, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

The diagonal case is the ordinary right partial trace after applying R_w , and the off-diagonal case vanishes under the orthogonal sector control. Hence $\widehat{R} = \mathcal{C}_{\text{tr}} \circ R_w$. Equivalence reindexing and the controlled dependent partial trace are trace-preserving completely positive maps, so their composition has a Kraus representation. □

Lemma 20.14.15 (Retraction of the normalized vertical-sector embedding). Lean Lean ✓ Stmt
Suppose that every multiplicity space is nonzero and every diagonal entry of μ_{α} is positive. Then $m_{\alpha} \neq 0$ for every α , and $\widehat{R}_{\mu} \circ R_{\mu} = \text{id}$.

Proof. ✓ Proof Positivity of the diagonal entries and nonemptiness of the multiplicity space give $m_{\alpha} = \sum_q \mu_{\alpha, q} > 0$. The Kronecker-product identity for the left partial trace then gives

$$\text{tr}_{\text{mult}}(m_{\alpha}^{-1} \mu_{\alpha} \otimes X_{\alpha}) = m_{\alpha}^{-1} \text{tr}(\mu_{\alpha}) X_{\alpha} = X_{\alpha}.$$

□

Definition 20.14.16 (Products of weighted vertical contractions). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
 For vertical BNT tensors M_α with simple matrix algebras M_{d_α} and a family of complex scalars $(c_\alpha)_\alpha$, define

$$V_c(X) = \bigoplus_{\alpha} c_\alpha M_\alpha(X),$$

$$M_\alpha(X) = \sum_{a,b} X_{ba} M_{\alpha,ab}.$$

For each $L \geq 0$, let

$$C_L(V_c) = \text{span}_{\mathbb{C}} \{V_c(X_1) \cdots V_c(X_L) : X_1, \dots, X_L \in M_D\} \subseteq \prod_{\alpha} M_{d_\alpha}.$$

Write V_m for the specialization obtained by taking $c_\alpha = m_\alpha = \text{tr}(\mu_\alpha)$.

Lemma 20.14.17 (Contraction against a matrix unit). [Lean](#) [✓ Stmt](#) *Let $M = (M_{ab})_{a,b=0}^{D-1}$ be a matrix-product tensor whose physical index is identified with the ordered pair (a, b) . Then $M(E_{ba}) = M_{ab}$.*

Lemma 20.14.18 (Contraction in the encoded product coordinate). [Lean](#) [✓ Stmt](#) *If $r \in \{0, \dots, D^2 - 1\}$ encodes the ordered pair $(\lfloor r/D \rfloor, r \bmod D)$, then $M(E_{(r \bmod D), \lfloor r/D \rfloor}) = M_r$.*

Definition 20.14.19 (Sectorwise scaling equivalence). [Lean](#) [✓ Stmt](#) For scalars $c_\alpha \neq 0$, the map

$$\varphi_c : \prod_{\alpha} M_{d_\alpha} \longrightarrow \prod_{\alpha} M_{d_\alpha},$$

$$(X_\alpha)_\alpha \mapsto (c_\alpha X_\alpha)_\alpha$$

is a linear equivalence.

Lemma 20.14.20 (Weighted contraction of matrix-unit words). [Lean](#) [✓ Stmt](#) *For a word $w = ((a_1, b_1), \dots, (a_L, b_L))$,*

$$V_c(E_{b_1 a_1}) \cdots V_c(E_{b_L a_L}) = \bigoplus_{\alpha} c_\alpha^L M_\alpha^w.$$

Lemma 20.14.21 (Transfer of simultaneous word spanning). [Lean](#) [✓ Stmt](#) *Fix $L \geq 0$ and suppose that every c_α is nonzero. If the simultaneous word tuples $(M_\alpha^w)_\alpha$ of length L span $\prod_{\alpha} M_{d_\alpha}$, then $C_L(V_c) = \prod_{\alpha} M_{d_\alpha}$.*

Theorem 20.14.22 (Nonzero-weight vertical bond contractions generate the sector algebra). [Lean](#) [✓ Stmt](#) *Suppose that $(M_\alpha)_\alpha$ is a basis of normal tensors and that $(c_\alpha)_\alpha$ is any family of nonzero complex scalars. Then there is a positive integer L such that $C_L(V_c) = \prod_{\alpha} M_{d_\alpha}$.*

Proof. [✓ Proof](#) Choose $L > 0$ for which the simultaneous word tuples $(M_\alpha^w)_\alpha$ span $\prod_{\alpha} M_{d_\alpha}$. Since every c_α is nonzero, so is c_α^L . Apply Lemma 20.14.21. $\square$

Theorem 20.14.23 (Vertical bond contractions generate the sector algebra). [Lean](#) [✓ Stmt](#) *Suppose that $(M_\alpha)_\alpha$ is a basis of normal tensors, every multiplicity space is nonzero, and every diagonal entry of μ_α is positive. Then there is a positive integer L such that $C_L(V_m) = \prod_{\alpha} M_{d_\alpha} =: \mathcal{A}_1$. This is the spanning assertion used in [CPGSV16, Appendix C.4, lines 1980–1990].*

Proof. ✓ Proof Since every multiplicity space is nonzero and every diagonal entry is positive, $m_\alpha = \sum_{q=0}^{\text{mult}(\alpha)-1} \mu_{\alpha,q} > 0$. Apply the preceding theorem to the nonzero family $c_\alpha = m_\alpha = \text{tr}(\mu_\alpha)$. □

Theorem 20.14.24 (Fixed spanning contraction products determine an endomorphism). Lean ✓ Stmt Let F be a linear endomorphism of $\prod_\alpha M_{d_\alpha}$. Suppose that $C_L(V_c)$ is the whole sector algebra and, for every choice of bond matrices $X_1, \dots, X_L$,

$$F(V_c(X_1) \cdots V_c(X_L)) = V_c(X_1) \cdots V_c(X_L).$$

Then F is the identity.

Proof. ✓ Proof Since the displayed products span the domain, every $Y \in \prod_\alpha M_{d_\alpha}$ has the form $Y = \sum_i \lambda_i V_c(X_1^{(i)}) \cdots V_c(X_L^{(i)})$. Hence

$$F(Y) = \sum_i \lambda_i F(V_c(X_1^{(i)}) \cdots V_c(X_L^{(i)})) = \sum_i \lambda_i V_c(X_1^{(i)}) \cdots V_c(X_L^{(i)}) = Y.$$

Thus F is the identity. □

Definition 20.14.25 (Products of fixed points). Lean ✓ Stmt Let F be a linear endomorphism of $\mathcal{A} = \prod_\alpha M_{d_\alpha}$, and write $\mathcal{F} = \{X \in \mathcal{A} : F(X) = X\}$. For $L \geq 0$, define

$$C_L(\mathcal{F}) = \text{span}_{\mathbb{C}}\{X_1 \cdots X_L : X_1, \dots, X_L \in \mathcal{F}\}.$$

This is the product space used in the dimension argument of [CPGSV16, Appendix C.4, lines 1980–1993].

Theorem 20.14.26 (Density blocks of a faithful fixed family bound products). Lean ✓ Stmt Let $\mathcal{A} = \prod_\alpha M_{d_\alpha}$, and let $F : \mathcal{A} \rightarrow \mathcal{A}$ be positive and trace preserving. Suppose that the trace adjoint F^* satisfies the Schwarz inequality and that F fixes a family $\rho = (\rho_\alpha)_\alpha$ for which every ρ_α is positive definite. If $\mathcal{F} = \text{Fix}(F)$, then, for every $L > 0$, $\dim C_L(\mathcal{F}) \leq \dim \mathcal{F}$.

Proof. ✓ Proof

Extend F to the full matrix algebra by diagonal compression followed by block-diagonal embedding, and choose a numbered basis of the ambient space. The extension is positive and trace preserving, its trace adjoint satisfies the Schwarz inequality, and the embedded family ρ is a positive definite fixed point. The fixed points therefore have the form

$$U \left(\bigoplus_k \sigma_k \otimes X_k \right) U^*, \quad X_k \in M_{h_k}(\mathbb{C}),$$

where $\text{tr}(\sigma_k) = 1$. The parametrization $(X_k)_k \mapsto \bigoplus_k \sigma_k \otimes X_k$ is injective, since its left partial trace is $(X_k)_k$. Every product of L fixed points lies in the range of

$$(Y_k)_k \mapsto U \left(\bigoplus_k \sigma_k^L \otimes Y_k \right) U^*.$$

Thus the span of these products has dimension at most $\sum_k h_k^2$, which is the dimension of the fixed-point space of the extension. Writing $\mathcal{E}$ for the fixed-point space of the full-matrix extension, the injective block-diagonal embedding and diagonal compression give the dimension chain

$$\dim C_L(\mathcal{F}) \leq \dim C_L(\mathcal{E}) \leq \dim \mathcal{E} \leq \dim \mathcal{F}.$$

□

Theorem 20.14.27 (Fixed contractions and the dimension of their fixed-point space). [Lean](#)

[✓ Stmt](#) Let F be a linear endomorphism of $\mathcal{A} = \prod_{\alpha} M_{d_{\alpha}}$. Suppose that the following three conditions hold, with the second required for every bond matrix X :

$$\begin{aligned} C_L(V_c) &= \mathcal{A}, \\ F(V_c(X)) &= V_c(X), \\ \dim C_L(\mathcal{F}) &\leq \dim \mathcal{F}, \quad \mathcal{F} = \{Y \in \mathcal{A} : F(Y) = Y\}. \end{aligned}$$

Then $F = \text{id}_{\mathcal{A}}$.

This is the finite-dimensional conclusion in [CPGSV16, Appendix C.4, lines 1980–1993]. It does not assume that the products $V_c(X_1) \cdots V_c(X_L)$ are fixed.

Proof. [✓ Proof](#) Since every $V_c(X)$ belongs to $\mathcal{F}$, every product occurring in $C_L(V_c)$ belongs to $C_L(\mathcal{F})$. Hence

$$\mathcal{A} = C_L(V_c) \subseteq C_L(\mathcal{F}) \subseteq \mathcal{A},$$

so $C_L(\mathcal{F}) = \mathcal{A}$. Therefore

$$\dim \mathcal{A} = \dim C_L(\mathcal{F}) \leq \dim \mathcal{F} \leq \dim \mathcal{A}.$$

Thus $\mathcal{F} = \mathcal{A}$, and F is the identity. □

Theorem 20.14.28 (Faithful fixed family in a finite product). [Lean](#) [✓ Stmt](#) Let F be a positive trace-preserving endomorphism of $\mathcal{A} = \prod_{\alpha} M_{d_{\alpha}}$. Suppose that each V_j is fixed by F and, for some $L > 0$, the identity belongs to

$$\text{span}_{\mathbb{C}}\{V_{j_1} \cdots V_{j_L} : j_1, \dots, j_L \in J\}.$$

Then there is a fixed family $\rho = (\rho_{\alpha})_{\alpha}$ such that every ρ_{α} is positive definite.

This is a local finite-product consequence used to formalize the fixed-point argument in [CPGSV16, Appendix C.4, lines 1980–1993]; CPSV16 applies Wolf’s density-block description directly rather than stating this intermediate result.

Proof. [✓ Proof](#) Extend F to the full matrix algebra by diagonal compression followed by block-diagonal embedding. This extension is positive and trace preserving. Block-diagonal embedding preserves the identity and every product $V_{j_1} \cdots V_{j_L}$, so Theorem 20.14.66 gives a positive-definite fixed point $\hat{\rho}$ of the extension. Fixed points of the extension are block diagonal. Thus

$$\begin{aligned} \hat{\rho} &= \bigoplus_{\alpha} \rho_{\alpha}, \\ F(\rho) &= \rho. \end{aligned}$$

Every diagonal block ρ_{α} of the positive-definite matrix $\hat{\rho}$ is positive definite. □

Theorem 20.14.29 (Identity criterion from fixed contractions). [Lean](#) [✓ Stmt](#) Let F be a positive trace-preserving endomorphism of $\mathcal{A} = \prod_{\alpha} M_{d_{\alpha}}$ whose trace adjoint satisfies the Schwarz inequality. Suppose that, for some $L > 0$, $C_L(V_c) = \mathcal{A}$ and $F(V_c(X)) = V_c(X)$ for every bond matrix X . Then $F = \text{id}_{\mathcal{A}}$. No product $V_c(X_1) \cdots V_c(X_L)$ is assumed to be fixed.

This is the implication used for each transported square composite in [CPGSV16, Appendix C.4, lines 1980–1995].

Proof. [✓ Proof](#) The fixed contractions and their positive-length product span give a positive-definite fixed family $\rho = (\rho_\alpha)_\alpha$. Hence Theorem 20.14.26 gives $\dim C_L(\mathcal{F}) \leq \dim \mathcal{F}$, where $\mathcal{F} = \text{Fix}(F)$. Since $C_L(V_c) = \mathcal{A}$ and $V_c(X) \in \mathcal{F}$, the preceding theorem gives $F = \text{id}_\mathcal{A}$. $\square$

Definition 20.14.30 (Retained coordinates for the vertical-sector algebra). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Retain the coordinates (α, a, i) , where α labels a simple sector, a labels a diagonal entry of μ_α , and $0 \leq i < d_\alpha$. The family $(Y_\alpha)_\alpha$ is represented on this space by $\bigoplus_\alpha Y_\alpha$. Write ι for this inclusion, π for extraction of the diagonal summands, and $P = \iota\pi$. Composing ι and π with R_μ and $\widetilde{R}_\mu$ defines their retained-coordinate forms.

Lemma 20.14.31 (Extraction after vertical-sector inclusion). [Lean](#) [✓ Stmt](#) *The diagonal-summand extraction is a left inverse of the inclusion: $\pi\iota = \text{id}$.*

Proof. [✓ Proof](#) Restricting $\bigoplus_\beta Y_\beta$ to the rows and columns with sector label α gives Y_α . $\square$

Lemma 20.14.32 (Entries retained by the vertical-sector projection). [Lean](#) [✓ Stmt](#) *For every retained-coordinate matrix Z , $P(Z)_{(\alpha,a,i),(\alpha,b,j)} = Z_{(\alpha,a,i),(\alpha,b,j)}$.*

Proof. [✓ Proof](#) Both indices lie in the same extracted diagonal summand. $\square$

Lemma 20.14.33 (Entries removed by the vertical-sector projection). [Lean](#) [✓ Stmt](#) *If $\alpha \neq \beta$, then $P(Z)_{(\alpha,a,i),(\beta,b,j)} = 0$.*

Proof. [✓ Proof](#) The two indices belong to distinct diagonal summands. $\square$

Lemma 20.14.34 (Idempotence of the vertical-sector projection). [Lean](#) [✓ Stmt](#) *The projection onto the vertical-sector diagonal summands is idempotent: $P^2 = P$.*

Proof. [✓ Proof](#) The identity $\pi\iota = \text{id}$ gives $P^2 = \iota\pi\iota\pi = \iota\pi$. $\square$

Lemma 20.14.35 (Retraction in retained vertical-sector coordinates). [Lean](#) [✓ Stmt](#) *Suppose that every multiplicity space is nonzero and every diagonal entry of μ_α is positive. Then the retained-coordinate partial trace is a left inverse of the normalized retained-coordinate embedding.*

Proof. [✓ Proof](#) First extract the diagonal summands, then apply $\widetilde{R}_\mu R_\mu = \text{id}$. $\square$

Lemma 20.14.36 (Weighted vertical contraction in retained coordinates). [Lean](#) [✓ Stmt](#) *For every horizontal bond matrix X , contraction of $\bigoplus_\alpha \mu_\alpha \otimes M_\alpha$ gives $\bigoplus_\alpha \mu_\alpha \otimes M_\alpha(X)$.*

Proof. [✓ Proof](#) Flattening the index triple (α, a, i) gives the entry formula

$$B(X)_{(\alpha,a,i),(\beta,b,j)} = \delta_{\alpha\beta} \delta_{ab} \mu_{\alpha,a} M_\alpha(X)_{ij}.$$

$\square$

Definition 20.14.37 (Bond-matrix contraction of a vertical tensor). [Lean](#) [✓ Stmt](#) Let A be a tensor whose physical label is a pair (a, b) of horizontal bond indices. For a bond matrix X , define

$$A(X) = \sum_{a,b} X_{ba} A_{ab}.$$

The order of the entries of X is chosen so that $A(X)_{ij} = \text{tr}(M^{ij} X)$ when $A_{ab,ij} = M_{ab}^{ij}$.

Lemma 20.14.38 (Contraction under fixed left and right multiplication). [Lean](#) [✓ Stmt](#) *Let L and K be fixed matrices of compatible sizes. Then $((a, b) \mapsto LA_{ab}K)(X) = LA(X)K$.*

Proof. ✓ Proof Distributivity of matrix multiplication over finite sums gives

$$\sum_{a,b} X_{ba} (L A_{ab} K) = L \left(\sum_{a,b} X_{ba} A_{ab} \right) K.$$

□

Lemma 20.14.39 (Contraction of a weighted block sum). Lean ✓ Stmt For a weighted block-diagonal tensor $A = \bigoplus_{\alpha} \mu_{\alpha} A_{\alpha}$, one has $A(X) = \bigoplus_{\alpha} \mu_{\alpha} A_{\alpha}(X)$.

Proof. ✓ Proof Since each tensor letter is block diagonal,

$$\sum_{a,b} X_{ba} \left(\bigoplus_{\alpha} \mu_{\alpha} A_{\alpha} \right)_{ab} = \bigoplus_{\alpha} \mu_{\alpha} \sum_{a,b} X_{ba} (A_{\alpha})_{ab} = \bigoplus_{\alpha} \mu_{\alpha} A_{\alpha}(X).$$

□

Lemma 20.14.40 (Vertical contraction and physical closure). Lean ✓ Stmt For the vertically viewed tensor $\widetilde{M}_{ab,ij} = M_{ab}^{ij}$, bond-matrix contraction is the one-site physical closure: $\widetilde{M}(X) = M(X)$.

Proof. ✓ Proof For every pair of physical indices,

$$\widetilde{M}(X)_{ij} = \sum_{a,b} X_{ba} M_{ab}^{ij} = \text{tr}(M^{ij} X) = M(X)_{ij}.$$

□

Lemma 20.14.41 (Contraction of the assembled vertical tensor). Lean ✓ Stmt The contraction of the assembled vertical tensor is the weighted block diagonal of the sector contractions:

$$\left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right) (X) = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X).$$

Proof. ✓ Proof Applying the preceding lemma to the block form $\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}$ of the assembled tensor gives

$$\left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right) (X) = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X).$$

□

Lemma 20.14.42 (Contraction of a letterwise forward identity). Lean ✓ Stmt Let B_{ab} be an arbitrary vertical tensor. If $U \widetilde{M}_{ab} U^{\dagger} = B_{ab}$ for every pair (a, b) , then, for every bond matrix X , $UM(X)U^{\dagger} = B(X)$.

Proof. ✓ Proof By the definition of bond-matrix contraction,

$$UM(X)U^{\dagger} = U \left(\sum_{a,b} X_{ba} \widetilde{M}_{ab} \right) U^{\dagger} = \sum_{a,b} X_{ba} (U \widetilde{M}_{ab} U^{\dagger}) = \sum_{a,b} X_{ba} B_{ab} = B(X).$$

□

Lemma 20.14.43 (Contraction of a letterwise reconstruction identity). [Lean](#) [Stmt](#) Let B_{ab} be an arbitrary vertical tensor. If $\widetilde{M}_{ab} = U^\dagger B_{ab} U$ for every pair (a, b) , then, for every bond matrix X , $M(X) = U^\dagger B(X) U$.

Proof. [Proof](#) Contracting the letterwise identity gives

$$M(X) = \sum_{a,b} X_{ba} \widetilde{M}_{ab} = \sum_{a,b} X_{ba} (U^\dagger B_{ab} U) = U^\dagger \left(\sum_{a,b} X_{ba} B_{ab} \right) U = U^\dagger B(X) U.$$

□

Lemma 20.14.44 (Contraction of the one-site vertical canonical form). [Lean](#) [Lean](#) [Stmt](#) Suppose that

$$U \widetilde{M}_{ab} U^\dagger = \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right)_{ab}$$

is a one-site vertical canonical-form identity. Contracting against an arbitrary bond matrix X gives

$$UM(X)U^\dagger = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X).$$

If the reverse letterwise identity is also given, then

$$M(X) = U^\dagger \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X) \right) U.$$

These are the contracted canonical-form identities used in [CPGSV16, Appendix C.4, lines 1955–1979].

Proof. [Proof](#) Since $A(X)_{ij} = \sum_{a,b} X_{ba} A_{ab,ij}$ and $\widetilde{M}_{ab,ij} = M_{ab}^{ij}$, one has $M(X) = \sum_{a,b} X_{ba} \widetilde{M}_{ab}$. Distributivity of fixed left and right multiplication over finite sums gives

$$U \left(\sum_{a,b} X_{ba} \widetilde{M}_{ab} \right) U^\dagger = \sum_{a,b} X_{ba} (U \widetilde{M}_{ab} U^\dagger).$$

Substituting the forward hypothesis into this equality gives

$$UM(X)U^\dagger = \sum_{a,b} X_{ba} \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right)_{ab} = \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right) (X).$$

Applied blockwise, the same calculation gives

$$\left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right) (X) = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X).$$

If the reverse letterwise identity is given, the same calculation yields

$$\begin{aligned} M(X) &= \sum_{a,b} X_{ba} \widetilde{M}_{ab} \\ &= U^\dagger \left(\sum_{a,b} X_{ba} \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha} \right)_{ab} \right) U \\ &= U^\dagger \left(\bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X) \right) U. \end{aligned}$$

□

Definition 20.14.45 (Four-site physical closure). [Lean](#) [Lean](#) [✓ Stmt](#) For a virtual matrix X , the four-site physical closure of K is the operator $K_4(X)$ whose matrix coefficients are

$$K_4(X)_{(i_0, (i_1, (i_2, i_3))), (j_0, (j_1, (j_2, j_3)))} = \text{tr}(K^{i_0 j_0} K^{i_1 j_1} K^{i_2 j_2} K^{i_3 j_3} X).$$

Lemma 20.14.46 (Coordinates of the four-site regrouping). [Lean](#) [Lean](#) [✓ Stmt](#) The canonical four-coordinate regrouping and its inverse satisfy

$$\begin{aligned} R_4(\sigma) &= (\sigma_0, (\sigma_1, (\sigma_2, \sigma_3))), \\ R_4^{-1}(i_0, (i_1, (i_2, i_3))) &= (i_0, i_1, i_2, i_3). \end{aligned}$$

Proof. [✓ Proof](#) Both identities follow from the definition of the canonical right-associated regrouping. □

Theorem 20.14.47 (General and right-associated four-site closures). [Lean](#) [✓ Stmt](#) Under the canonical right-associated identification of four-site configurations with quadruples of physical indices,

$$K_4(X)_{(i_0, (i_1, (i_2, i_3))), (j_0, (j_1, (j_2, j_3)))} = \text{tr}(K^{i_0 j_0} K^{i_1 j_1} K^{i_2 j_2} K^{i_3 j_3} X).$$

Proof. [✓ Proof](#) Expand the general four-site closure in the right-associated coordinates. □

Theorem 20.14.48 (One-site closure of the two-site blocking). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) After identifying one blocked physical index with two original indices, $(K^{[2]})_1(X) = K_2(X)$.

Proof. [✓ Proof](#) Expand one blocked tensor matrix and decode its two physical indices. □

Definition 20.14.49 (Physical maps in vertical canonical coordinates). [Lean](#) [Lean](#) [✓ Stmt](#) Let U_1 and U_2 be the physical coordinate changes in vertical canonical forms of M and its two-site blocking, respectively. Let J be the canonical relabelling from matrices indexed by pairs of one-site physical indices to matrices indexed by one blocked physical index. For the refinement and coarse-graining maps T and S , define

$$\begin{aligned} T_v(Z) &= U_2 J(T(U_1^\dagger Z U_1)) U_2^\dagger, \\ S_v(W) &= U_1 S(J^{-1}(U_2^\dagger W U_2)) U_1^\dagger. \end{aligned}$$

These are the physical maps after absorbing the two vertical coordinate changes, as in [CPGSV16, Appendix C.4, lines 1955–1979]. They are intermediate maps on the retained physical coordinate spaces; the normalized sector maps $R_1, R_2, \widetilde{R}_1, \widetilde{R}_2$ have not yet been composed with them.

Theorem 20.14.50 (Action of the transported physical maps). [Lean](#) [Lean](#) [✓ Stmt](#) For a bond matrix X , write $\mathcal{M}^{(1)}(X)$ and $\mathcal{M}^{(2)}(X)$ for the one-site and two-site physical closures, respectively, and write

$$\begin{aligned} B_1(X) &= \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}(X), \\ B_2(X) &= \bigoplus_{\beta} \nu_{\beta} \otimes A_{\beta}(X) \end{aligned}$$

for the contracted one-site and two-site vertical canonical forms. If

$$\begin{aligned} T(\mathcal{M}^{(1)}(X)) &= \mathcal{M}^{(2)}(X), \\ S(\mathcal{M}^{(2)}(X)) &= \mathcal{M}^{(1)}(X), \end{aligned}$$

then

$$\begin{aligned} T_v(B_1(X)) &= B_2(X), \\ S_v(B_2(X)) &= B_1(X). \end{aligned}$$

These are the physical-coordinate identities which, after composition with $R_1, R_2, \widetilde{R}_1, \widetilde{R}_2$, give the two displayed identities for $\widetilde{T}$ and $\widetilde{S}$ in [CPGSV16, Appendix C.4, lines 1955–1979].

Proof. ✓ Proof The reverse one-site canonical-form identity gives $U_1^\dagger B_1(X) U_1 = \mathcal{M}^{(1)}(X)$. Applying T , relabelling its two-site output by J , and using the forward blocked canonical-form identity gives

$$T_v(B_1(X)) = U_2 J(T(\mathcal{M}^{(1)}(X))) U_2^\dagger = U_2 J(\mathcal{M}^{(2)}(X)) U_2^\dagger = B_2(X).$$

The reverse blocked canonical-form identity and the equation for S give, in the other direction,

$$S_v(B_2(X)) = U_1 S(\mathcal{M}^{(2)}(X)) U_1^\dagger = U_1 \mathcal{M}^{(1)}(X) U_1^\dagger = B_1(X).$$

□

Definition 20.14.51 (Refinement and coarse-graining on vertical sectors). Lean Lean ✓ Stmt Define maps between the one-site and two-site simple-sector algebras by

$$\begin{aligned} \widetilde{T} &= \widetilde{R}_2 T_v R_1, \\ \widetilde{S} &= \widetilde{R}_1 S_v R_2. \end{aligned}$$

The partial trace and diagonal-summand extraction define these maps on the full matrix algebras. Their agreement with the normalized maps in [CPGSV16, Appendix C.4, lines 1972–1979] is asserted on the weighted block-diagonal image. At this stage there is no assertion that the image is preserved, that the two maps are inverse, completely positive, or trace preserving, or that the sector labels have been matched. The final renormalization fixed-point conclusion is separate.

Definition 20.14.52 (Positive vertical-sector families and total trace). Lean Lean ✓ Stmt A vertical-sector family $X = (X_\alpha)_\alpha$ is positive when every X_α is positive semidefinite. Its total sector trace is

$$\text{Tr}_{\text{sec}}(X) = \sum_{\alpha} \text{tr}(X_\alpha).$$

Lemma 20.14.53 (Positivity and trace in normalized sector coordinates). Lean Lean Lean Lean Lean Lean ✓ Stmt Suppose that every multiplicity space is nonzero and every diagonal entry of μ_α is positive. The normalized embedding R_μ sends a positive sector family to a positive retained matrix and satisfies $\text{tr}(R_\mu(X)) = \text{Tr}_{\text{sec}}(X)$. Conversely, the sectorwise partial trace sends a positive retained matrix to a positive sector family and satisfies $\text{Tr}_{\text{sec}}(\widetilde{R}_\mu(Z)) = \text{tr}(Z)$. These trace identities follow by summing the traces of the diagonal sectors; deleting the off-diagonal sector entries does not change the trace.

Proof. [✓ Proof](#) A positive diagonal weight matrix has positive trace, and its Kronecker product with each positive X_α is positive. Partial trace preserves positivity. The two trace formulas follow from $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$ and the normalization by $\text{tr}(\mu_\alpha)$. $\square$

Lemma 20.14.54 (Normalized embedding of multiplicity-trace-scaled sectors). [Lean](#) [✓ Stmt](#) Suppose that every multiplicity space is nonzero and every diagonal entry of μ_α is positive. For $m_\alpha = \text{tr}(\mu_\alpha)$,

$$R_\mu \left(\bigoplus_\alpha m_\alpha X_\alpha \right) = \bigoplus_\alpha \mu_\alpha \otimes X_\alpha.$$

Proof. [✓ Proof](#) In each sector,

$$R_\mu(m_\alpha X_\alpha) = m_\alpha^{-1} m_\alpha (\mu_\alpha \otimes X_\alpha) = \mu_\alpha \otimes X_\alpha.$$

$\square$

Lemma 20.14.55 (Partial trace of weighted vertical sectors). [Lean](#) [✓ Stmt](#) For $m_\alpha = \text{tr}(\mu_\alpha)$,

$$\widetilde{R}_\mu \left(\bigoplus_\alpha \mu_\alpha \otimes X_\alpha \right) = \bigoplus_\alpha m_\alpha X_\alpha.$$

Proof. [✓ Proof](#) Taking the partial trace over the multiplicity factor gives

$$\widetilde{R}_\mu(\mu_\alpha \otimes X_\alpha) = \text{tr}(\mu_\alpha) X_\alpha = m_\alpha X_\alpha.$$

$\square$

Lemma 20.14.56 (Trace under coisometric compression). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $U : \mathbb{C}^n \rightarrow \mathbb{C}^r$ satisfy $UU^\dagger = 1$. If $Z \geq 0$, then

$$\begin{aligned} \text{tr}(UZU^\dagger) &\leq \text{tr}(Z), \\ \text{tr}(UZU^\dagger) &= \text{tr}(Z) \iff U^\dagger UZ = Z. \end{aligned}$$

For every $W \in \mathbb{C}^{r \times r}$, one also has $\text{tr}(U^\dagger WU) = \text{tr}(W)$.

Proof. [✓ Proof](#) Put $P = U^\dagger U$. The matrices P and $1 - P$ are orthogonal projections, and

$$\begin{aligned} \text{tr}(UZU^\dagger) &= \text{tr}(PZ), \\ \text{tr}(Z) - \text{tr}(PZ) &= \text{tr}((1 - P)Z) \geq 0. \end{aligned}$$

The last trace vanishes precisely when $(1 - P)Z = 0$. The formula for W follows by cyclicity of trace and $UU^\dagger = 1$. $\square$

Definition 20.14.57 (Matrices before active-sector compression). [Lean](#) [Lean](#) [✓ Stmt](#) Let $Z_T(X)$ be the matrix obtained by applying the normalized one-site embedding, expanding by $U_1^\dagger$, applying T , and relabelling the two-site indices, before compression by U_2 . Define $Z_S(Y)$ in the reverse direction, before compression by U_1 .

[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the physical maps are completely positive and trace

$$\mathrm{tr}(Z_S(Y)) = \mathrm{Tr}_{\mathrm{sec}}(Y).$$

After the final compression and sectorwise partial trace,

$$\begin{aligned}\mathrm{Tr}_{\mathrm{sec}}(\tilde{T}(X)) &= \mathrm{tr}(U_2 Z_T(X) U_2^\dagger), \\ \mathrm{Tr}_{\mathrm{sec}}(\tilde{S}(Y)) &= \mathrm{tr}(U_1 Z_S(Y) U_1^\dagger).\end{aligned}$$

Proof. ✓ Proof Positivity follows successively from the normalized embedding, expansion by an adjoint, the physical map, relabelling, and partial trace. The trace identities follow from the preceding two lemmas and trace preservation of the physical maps. $\square$

Theorem 20.14.59 (Positivity and trace loss of the transported sector maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✔ Stmt](#) Suppose that the multiplicity spaces are nonzero, their diagonal weights are positive, the physical maps T and S are completely positive and trace preserving, and the vertical coordinate maps satisfy $U_i U_i^\dagger = 1$. Then $\tilde{T}$ and $\tilde{S}$ preserve pointwise positive semidefiniteness. For pointwise positive families $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$, their total sector traces obey

$$\begin{aligned}\mathrm{Tr}_{\mathrm{sec}}(\widetilde{T}(X)) &\leq \mathrm{Tr}_{\mathrm{sec}}(X), \\ \mathrm{Tr}_{\mathrm{sec}}(\widetilde{S}(Y)) &\leq \mathrm{Tr}_{\mathrm{sec}}(Y).\end{aligned}$$

The trace loss vanishes precisely when

$$\begin{aligned} U_2^\dagger U_2 Z_T(X) &= Z_T(X), \\ U_1^\dagger U_1 Z_S(Y) &= Z_S(Y), \end{aligned}$$

respectively.

This result concerns positivity and trace on positive elements. It does not assert a complete-positivity structure for the finite products, a trace-preserving extension to their full ambient matrix spaces, or an inverse relation between the two transported maps.

Proof. ✓ **Proof** The normalized diagonal embedding and the sectorwise partial trace preserve positivity. Before the final compression, trace preservation of the physical map and $U_i U_i^\dagger = 1$ give the original sector trace. For a positive matrix Z and a coisometry U , put $P = U^\dagger U$. Then

$$\text{tr}(UZU^\dagger) = \text{tr}(PZ) \leq \text{tr}(Z).$$

Since $1 - P$ is an orthogonal projection, equality is equivalent to $\text{tr}((1 - P)Z) = 0$, hence to $(1 - P)Z = 0$. $\square$

Lemma 20.14.60 (Two-site support after refinement). [Lean] [✓ Stmt] For a bond matrix X , set $V_1(X) = \bigoplus_{\alpha} m_{\alpha} M_{\alpha}(X)$. Suppose that both vertical canonical forms satisfy their exact reconstruction identities, the one-site multiplicity spaces are nonzero, their diagonal weights are positive, $T(\mathcal{M}^{(1)}(X)) = \mathcal{M}^{(2)}(X)$, and $U_2 U_2^{\dagger} = 1$. Then $U_2^{\dagger} U_2 Z_T(V_1(X)) = Z_T(V_1(X))$. This is the refinement support identity used in [CPGSV16, Appendix C.4, lines 1955–1980].

Proof. ✓ Proof Let $\mathcal{R}^{-1}$ relabel a matrix indexed by two one-site indices as a matrix indexed by the blocked two-site index. Then

$$Z_T(V_1(X)) = \mathcal{R}^{-1} [T(U_1^\dagger B_1(X) U_1)] = \mathcal{R}^{-1} [T(\mathcal{M}^{(1)}(X))] = \mathcal{R}^{-1} [\mathcal{M}^{(2)}(X)] = U_2^\dagger B_2(X) U_2.$$

Here the first equality is the normalized embedding, the second and last are the two reconstruction identities, and the middle equality is the refinement identity. The equality before the last is precisely the blocked-index relabelling theorem. Therefore

$$U_2^\dagger U_2 Z_T(V_1(X)) = U_2^\dagger (U_2 U_2^\dagger) B_2(X) U_2 = Z_T(V_1(X)).$$

□

Lemma 20.14.61 (One-site support after coarse-graining). Lean ✓ Stmt For a bond matrix X , set $V_2(X) = \bigoplus_\beta n_\beta A_\beta(X)$. Suppose that both vertical canonical forms satisfy their exact reconstruction identities, the two-site multiplicity spaces are nonzero, their diagonal weights are positive, $S(\mathcal{M}^{(2)}(X)) = \mathcal{M}^{(1)}(X)$, and $U_1 U_1^\dagger = 1$. Then $U_1^\dagger U_1 Z_S(V_2(X)) = Z_S(V_2(X))$. This is the coarse-graining support identity used in [CPGSV16, Appendix C.4, lines 1955–1980].

Proof. ✓ Proof Let $\mathcal{R}$ decode a matrix indexed by the blocked two-site index as a matrix indexed by two one-site indices. Then

$$Z_S(V_2(X)) = S(\mathcal{R}[U_2^\dagger B_2(X) U_2]) = S(\mathcal{M}^{(2)}(X)) = \mathcal{M}^{(1)}(X) = U_1^\dagger B_1(X) U_1.$$

Here the first equality is the normalized embedding, the next equality is the blocked-index relabelling theorem, the middle equality is the coarse-graining identity, and the last is the one-site reconstruction. Therefore

$$U_1^\dagger U_1 Z_S(V_2(X)) = U_1^\dagger (U_1 U_1^\dagger) B_1(X) U_1 = Z_S(V_2(X)).$$

□

Lemma 20.14.62 (Trace preservation on one-site canonical-form contractions). Lean ✓ Stmt Under the hypotheses of Lemma 20.14.60, suppose in addition that $U_1 U_1^\dagger = 1$ and that T is completely positive and trace preserving. Then $\text{Tr}_{\text{sec}}(\tilde{T}(V_1(X))) = \text{Tr}_{\text{sec}}(V_1(X))$.

Proof. ✓ Proof Cyclicity of trace and the preceding support identity give

$$\text{Tr}_{\text{sec}}(\tilde{T}(V_1(X))) = \text{tr}(U_2^\dagger U_2 Z_T(V_1(X))) = \text{tr}(Z_T(V_1(X))) = \text{Tr}_{\text{sec}}(V_1(X)).$$

□

Lemma 20.14.63 (Trace preservation on two-site canonical-form contractions). Lean ✓ Stmt Under the hypotheses of Lemma 20.14.61, suppose in addition that $U_2 U_2^\dagger = 1$ and that S is completely positive and trace preserving. Then $\text{Tr}_{\text{sec}}(\tilde{S}(V_2(X))) = \text{Tr}_{\text{sec}}(V_2(X))$.

Proof. ✓ Proof Cyclicity of trace and the preceding support identity give

$$\text{Tr}_{\text{sec}}(\tilde{S}(V_2(X))) = \text{tr}(U_1^\dagger U_1 Z_S(V_2(X))) = \text{tr}(Z_S(V_2(X))) = \text{Tr}_{\text{sec}}(V_2(X)).$$

□

Theorem 20.14.64 (Action of the normalized vertical-sector maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
 $\checkmark$ **Stmt** Suppose that every multiplicity space for μ_α and ν_β is nonzero, every diagonal entry of these two matrices is positive, and

$$\begin{aligned} T_v \left(\bigoplus_\alpha \mu_\alpha \otimes X_\alpha \right) &= \bigoplus_\beta \nu_\beta \otimes Y_\beta, \\ S_v \left(\bigoplus_\beta \nu_\beta \otimes Y_\beta \right) &= \bigoplus_\alpha \mu_\alpha \otimes X_\alpha. \end{aligned}$$

Writing $m_\alpha = \text{tr}(\mu_\alpha)$ and $n_\beta = \text{tr}(\nu_\beta)$, one has

$$\begin{aligned} \tilde{T} \left(\bigoplus_\alpha m_\alpha X_\alpha \right) &= \bigoplus_\beta n_\beta Y_\beta, \\ \tilde{S} \left(\bigoplus_\beta n_\beta Y_\beta \right) &= \bigoplus_\alpha m_\alpha X_\alpha. \end{aligned}$$

In particular, take $X_\alpha = M_\alpha(X)$ and $Y_\beta = A_\beta(X)$ from the contractions of the one-site and two-site vertical canonical forms. The physical closure identities for T and S then imply the two hypotheses above, so the same conclusions hold for these source-generated sector operators.

Proof. $\checkmark$ **Proof** Since $\tilde{T} = \tilde{R}_2 T_v R_1$, one has

$$\tilde{T} \left(\bigoplus_\alpha m_\alpha X_\alpha \right) = \tilde{R}_2 \left(T_v \left(\bigoplus_\alpha \mu_\alpha \otimes X_\alpha \right) \right) = \tilde{R}_2 \left(\bigoplus_\beta \nu_\beta \otimes Y_\beta \right) = \bigoplus_\beta n_\beta Y_\beta.$$

Likewise, $\tilde{S} = \tilde{R}_1 S_v R_2$ gives

$$\tilde{S} \left(\bigoplus_\beta n_\beta Y_\beta \right) = \tilde{R}_1 \left(S_v \left(\bigoplus_\beta \nu_\beta \otimes Y_\beta \right) \right) = \tilde{R}_1 \left(\bigoplus_\alpha \mu_\alpha \otimes X_\alpha \right) = \bigoplus_\alpha m_\alpha X_\alpha.$$

□

Lemma 20.14.65 (Fixed contraction families of the transported composites). [Lean](#) [Lean](#)
 $\checkmark$ **Stmt** For every bond matrix X , set

$$\begin{aligned} V_1(X) &= \bigoplus_\alpha m_\alpha M_\alpha(X), \\ V_2(X) &= \bigoplus_\beta n_\beta A_\beta(X). \end{aligned}$$

Under the hypotheses of the preceding theorem, the transported composites satisfy

$$\begin{aligned} (\tilde{S} \circ \tilde{T})(V_1(X)) &= V_1(X), \\ (\tilde{T} \circ \tilde{S})(V_2(X)) &= V_2(X). \end{aligned}$$

Thus the one-site and two-site contraction families lie in the respective fixed-point spaces, as in Appendix C.4, lines 1974–1980. These equations do not assert that either composite is the identity on the whole sector algebra; that conclusion requires the fixed-point structure argument of Appendix C.4, lines 1980–1993.

Proof. ✓ Proof The preceding theorem gives $\tilde{T}(V_1(X)) = V_2(X)$ and $\tilde{S}(V_2(X)) = V_1(X)$. Hence

$$\begin{aligned} (\tilde{S} \circ \tilde{T})(V_1(X)) &= \tilde{S}(V_2(X)) = V_1(X), \\ (\tilde{T} \circ \tilde{S})(V_2(X)) &= \tilde{T}(V_1(X)) = V_2(X). \end{aligned}$$

□

Theorem 20.14.66 (Faithful fixed point from fixed product generators). Lean ✓ Stmt Let $\mathcal{M}$ be a finite full matrix algebra over $\mathbb{C}$, and let $F : \mathcal{M} \rightarrow \mathcal{M}$ be positive and trace nonincreasing. Suppose that a family $(V_j)_{j \in J}$ consists of fixed points of F and that, for some $L > 0$, the identity belongs to the linear span of the products $V_{j_1} \cdots V_{j_L}$ in $\mathcal{M}$. Then F has a positive-definite fixed point. This is the local full-matrix intermediate implication used in [CPGSV16, Appendix C.4, lines 1980–1995]; it is not stated there as a separate theorem.

Proof. ✓ Proof Boundedness of the forward orbits gives the mean-ergodic projection P . Set $\rho = P(\mathbf{1})$ and let Q be the support projection of ρ . For every j , the maximal-support property of ρ gives $QV_jQ = V_j$. Since Q is a projection, this implies $QV_j = V_j = V_jQ$. By induction, for every $k \geq 1$,

$$Q \cdot V_{j_1} \cdots V_{j_k} \cdot Q = V_{j_1} \cdots V_{j_k}.$$

The product-span hypothesis and linearity then give $Q\mathbf{1}Q = \mathbf{1}$. Since $Q^2 = Q$, one has $Q = Q^2 = Q\mathbf{1}Q = \mathbf{1}$. Thus ρ is positive definite, and the construction of P gives $F(\rho) = \rho$. □

Theorem 20.14.67 (Trace preservation from fixed product generators). Lean ✓ Stmt Let $\mathcal{M}$ be a finite full matrix algebra over $\mathbb{C}$, and let $F : \mathcal{M} \rightarrow \mathcal{M}$ be positive and trace nonincreasing. Suppose that $F(V_j) = V_j$ for every $j \in J$. Put $\mathcal{V} = \text{span}_{\mathbb{C}}\{V_j : j \in J\}$ and, in the notation of [CPGSV16, Appendix C.4], define

$$C_L(\mathcal{V}) = \text{span}_{\mathbb{C}}\{v_1 \cdots v_L : v_1, \dots, v_L \in \mathcal{V}\}.$$

If $\mathbf{1} \in C_L(\mathcal{V})$ for some $L > 0$, then F is trace preserving.

Proof. ✓ Proof Multilinearity gives

$$C_L(\mathcal{V}) = \text{span}_{\mathbb{C}}\{V_{j_1} \cdots V_{j_L} : j_1, \dots, j_L \in J\}.$$

By Theorem 20.14.66, there is a positive-definite matrix ρ such that $F(\rho) = \rho$. Let F^* be the trace-pairing adjoint and set $\Delta = \mathbf{1} - F^*(\mathbf{1})$. For every positive semidefinite X , trace nonincrease gives $\text{tr}(\Delta X) = \text{tr}(X) - \text{tr}(F(X)) \geq 0$. Positivity of F^* and self-duality of the positive semidefinite cone therefore give $\Delta \geq 0$. The fixed-point equation gives $\text{tr}(\rho \Delta) = 0$, so positive definiteness of ρ forces $\Delta = 0$. Thus $F^*(\mathbf{1}) = \mathbf{1}$, which is equivalent to trace preservation. □

Theorem 20.14.68 (Trace transfer through the canonical full-matrix extension). Lean Lean ✓ Stmt

Let $\mathcal{A} = \bigoplus_{\alpha \in I} \mathcal{M}_{d_\alpha \times d_\alpha}$, let $\iota : \mathcal{A} \rightarrow \text{End}(\bigoplus_{\alpha} \mathbb{C}^{d_\alpha})$ be the block-diagonal embedding, and let π be diagonal compression. For a linear map $F : \mathcal{A} \rightarrow \mathcal{A}$, set $\widehat{F} = \iota \circ F \circ \pi$. Then:

1. if F is trace nonincreasing on positive elements of $\mathcal{A}$, then $\widehat{F}$ is trace nonincreasing;
2. if $\widehat{F}$ is trace preserving, then F preserves the total trace on $\mathcal{A}$.

Proof. ✓ Proof If $Y \geq 0$, then $\pi(Y) \geq 0$, and the trace identities for diagonal compression and block-diagonal embedding give

$$\mathrm{tr}(\widehat{F}(Y)) = \mathrm{tr}_{\mathcal{A}}(F(\pi(Y))) \leq \mathrm{tr}_{\mathcal{A}}(\pi(Y)) = \mathrm{tr}(Y).$$

This proves the first assertion. For $X \in \mathcal{A}$, the identity $\pi \circ \iota = \mathrm{id}_{\mathcal{A}}$ gives, whenever $\widehat{F}$ is trace preserving,

$$\mathrm{tr}_{\mathcal{A}}(F(X)) = \mathrm{tr}(\widehat{F}(\iota(X))) = \mathrm{tr}(\iota(X)) = \mathrm{tr}_{\mathcal{A}}(X).$$

□

Theorem 20.14.69 (Trace preservation on a finite sum from fixed products). Lean ✓ Stmt Let $\mathcal{A} = \bigoplus_{\alpha \in I} \mathcal{M}_{d_{\alpha} \times d_{\alpha}}$ be a finite sum of full matrix algebras, with total trace $\mathrm{tr}_{\mathcal{A}}(X) = \sum_{\alpha \in I} \mathrm{tr}(X_{\alpha})$. Let $F : \mathcal{A} \rightarrow \mathcal{A}$ be positive and trace nonincreasing. Suppose that $F(V_j) = V_j$ for every $j \in J$. With $\mathcal{V} = \mathrm{span}_{\mathbb{C}}\{V_j : j \in J\}$, put

$$C_L(\mathcal{V}) = \mathrm{span}_{\mathbb{C}}\{v_1 \cdots v_L : v_1, \dots, v_L \in \mathcal{V}\}.$$

If $\mathbf{1}_{\mathcal{A}} \in C_L(\mathcal{V})$ for some $L > 0$, then F preserves the total trace on $\mathcal{A}$.

Proof. ✓ Proof Multilinearity first gives

$$C_L(\mathcal{V}) = \mathrm{span}_{\mathbb{C}}\{V_{j_1} \cdots V_{j_L} : j_1, \dots, j_L \in J\}.$$

Let $\iota : \mathcal{A} \rightarrow \mathrm{End}(\bigoplus_{\alpha} \mathbb{C}^{d_{\alpha}})$ be the block-diagonal embedding and let $\widehat{F}$ be the canonical full-matrix extension. These maps satisfy

$$\begin{aligned} \iota(XY) &= \iota(X)\iota(Y), \\ \iota(\mathbf{1}_{\mathcal{A}}) &= \mathbf{1}, \\ \widehat{F}(\iota(X)) &= \iota(F(X)). \end{aligned}$$

The extension is positive and trace nonincreasing. The matrices $\iota(V_j)$ are fixed by $\widehat{F}$, and the displayed identities carry the product-span hypothesis to the full matrix algebra. Theorem 20.14.67 makes $\widehat{F}$ trace preserving. Restricting to block-diagonal matrices proves preservation of $\mathrm{tr}_{\mathcal{A}}$ by F . □

Theorem 20.14.70 (Trace sandwich on finite sums of matrix algebras). Lean ✓ Stmt Let $\mathcal{A}$ and $\mathcal{B}$ be finite sums of full matrix algebras, and let $T : \mathcal{A} \rightarrow \mathcal{B}$ and $S : \mathcal{B} \rightarrow \mathcal{A}$ be linear maps. Suppose that T is positive, both maps are trace nonincreasing, and $S \circ T$ is trace preserving. Then T is trace preserving.

Proof. ✓ Proof For every positive $X \in \mathcal{A}$, positivity of T and the two trace-nonincreasing inequalities give

$$\mathrm{tr}_{\mathcal{A}}(X) = \mathrm{tr}_{\mathcal{A}}(S(T(X))) \leq \mathrm{tr}_{\mathcal{B}}(T(X)) \leq \mathrm{tr}_{\mathcal{A}}(X).$$

Hence the middle term equals the endpoints. Equality extends from the positive cone to Hermitian matrices by positive and negative parts. For arbitrary X , put

$$\begin{aligned} H_1 &= X + X^{\dagger}, \\ H_2 &= \mathrm{i}(X - X^{\dagger}), \\ X &= \frac{1}{2}(H_1 - \mathrm{i}H_2). \end{aligned}$$

Both H_1 and H_2 are Hermitian, so linearity proves trace preservation for every X . □

Theorem 20.14.71 (Trace preservation of the transported vertical-sector maps). [Lean](#) [✓ Stmt](#)
 Suppose that every one-site and two-site multiplicity space is nonzero and every corresponding diagonal weight is positive. Let $\widetilde{M}^{(1)}$ and $\widetilde{M}^{(2)}$ be the vertical tensors of M and its two-site blocking, and let B_1 and B_2 be their assembled one-site and two-site canonical forms. Assume that both canonical forms are bases of normal tensors and that, for every vertical letter ℓ ,

$$\begin{aligned} U_i \widetilde{M}_\ell^{(i)} U_i^\dagger &= (B_i)_\ell, \\ \widetilde{M}_\ell^{(i)} &= U_i^\dagger (B_i)_\ell U_i, \\ U_i U_i^\dagger &= \mathbf{1} \quad (i = 1, 2). \end{aligned}$$

Assume in addition that the physical maps T and S are completely positive and trace preserving and that, for every horizontal bond matrix Z , the two physical-coordinate identities hold:

$$\begin{aligned} T(\mathcal{M}^{(1)}(Z)) &= \mathcal{M}^{(2)}(Z), \\ S(\mathcal{M}^{(2)}(Z)) &= \mathcal{M}^{(1)}(Z). \end{aligned}$$

Then each transported map and both square composites preserve the appropriate total sector trace. For the individual maps,

$$\begin{aligned} \sum_\beta \text{tr}(\widetilde{T}(X)_\beta) &= \sum_\alpha \text{tr}(X_\alpha), \\ \sum_\alpha \text{tr}(\widetilde{S}(Y)_\alpha) &= \sum_\beta \text{tr}(Y_\beta). \end{aligned}$$

For the square composites,

$$\begin{aligned} \sum_\alpha \text{tr}((\widetilde{S}\widetilde{T})(X)_\alpha) &= \sum_\alpha \text{tr}(X_\alpha), \\ \sum_\beta \text{tr}((\widetilde{T}\widetilde{S})(Y)_\beta) &= \sum_\beta \text{tr}(Y_\beta). \end{aligned}$$

Neither of the stronger range inclusions is assumed:

$$\begin{aligned} T(R_1(\mathcal{A}_1)) &\subseteq R_2(\mathcal{A}_2), \\ S(R_2(\mathcal{A}_2)) &\subseteq R_1(\mathcal{A}_1). \end{aligned}$$

Proof. [✓ Proof](#) Each transported map is positive and trace nonincreasing, and hence so are the square composites $\widetilde{S}\widetilde{T}$ and $\widetilde{T}\widetilde{S}$. The fixed-generator lemma and the product-spanning theorem give the hypotheses of Theorem 20.14.69 for both composites. Thus both preserve the corresponding total trace.

For positive $X \in \mathcal{A}_1$, one has

$$\text{tr}_{\mathcal{A}_1}(X) = \text{tr}_{\mathcal{A}_1}((\widetilde{S}\widetilde{T})(X)) \leq \text{tr}_{\mathcal{A}_2}(\widetilde{T}(X)) \leq \text{tr}_{\mathcal{A}_1}(X).$$

Theorem 20.14.70 therefore makes $\widetilde{T}$ trace preserving. Applying the same theorem to $\widetilde{S}$ and $\widetilde{T}\widetilde{S}$ proves the assertion for $\widetilde{S}$. $\square$

Lemma 20.14.72 (Complete positivity of refinement in vertical coordinates). [Lean](#) [✓ Stmt](#) *If the physical refinement map T is completely positive, then the transported map T_v in vertical canonical coordinates is completely positive.*

Proof. ✓ Proof By Definition 20.14.49,

$$T_v(Z) = U_2 J(T(U_1^\dagger Z U_1)) U_2^\dagger.$$

The two single-operator conjugations and the reindexing J are completely positive, so the assertion follows by composition. $\square$

Lemma 20.14.73 (Complete positivity of coarse-graining in vertical coordinates). Lean ✓ Stmt *If the physical coarse-graining map S is completely positive, then the transported map S_v in vertical canonical coordinates is completely positive.*

Proof. ✓ Proof By Definition 20.14.49,

$$S_v(W) = U_1 S(J^{-1}(U_2^\dagger W U_2)) U_1^\dagger.$$

Again, single-operator conjugation, reindexing, and composition preserve complete positivity. $\square$

Lemma 20.14.74 (Retained-coordinate block-diagonal inclusion). Lean ✓ Stmt *Let ι be block-diagonal inclusion of the weighted vertical-sector block family into its full matrix algebra, and let W be the canonical reindexing to retained physical coordinates. Then the retained-coordinate inclusion is $\iota_v = W \circ \iota$.*

Proof. ✓ Proof This is the defining relation between the direct-sum and retained coordinates. $\square$

Lemma 20.14.75 (Retained-coordinate diagonal-block extraction). Lean ✓ Stmt *Let π be diagonal-block extraction to the weighted vertical-sector block family, and let W be the canonical reindexing to retained physical coordinates. Then the retained-coordinate extraction is $\pi_v = \pi \circ W^{-1}$.*

Proof. ✓ Proof This is the inverse defining relation between the retained and direct-sum coordinates. $\square$

Lemma 20.14.76 (Full-matrix factorization of transported refinement). Lean ✓ Stmt *Write $\widehat{R}_1$ for normalized state preparation, $\widehat{R}_2$ for controlled partial trace, and W_i for retained-coordinate reindexing. The full-matrix extension of the transported refinement map is*

$$\widehat{T} = \widehat{R}_2 \circ W_2^{-1} \circ T_v \circ W_1 \circ \widehat{R}_1. \quad (20.29)$$

Proof. ✓ Proof Substitute the preceding inclusion and extraction identities into $\widetilde{T} = \widetilde{R}_2 T_v R_1$. $\square$

Lemma 20.14.77 (Full-matrix factorization of transported coarse-graining). Lean ✓ Stmt *Write $\widehat{R}_2$ for normalized state preparation, $\widehat{R}_1$ for controlled partial trace, and W_i for retained-coordinate reindexing. The full-matrix extension of the transported coarse-graining map is*

$$\widehat{S} = \widehat{R}_1 \circ W_1^{-1} \circ S_v \circ W_2 \circ \widehat{R}_2. \quad (20.30)$$

Proof. ✓ Proof Substitute the preceding inclusion and extraction identities into $\widetilde{S} = \widetilde{R}_1 S_v R_2$. $\square$

Theorem 20.14.78 (Complete positivity and adjoint Schwarz for the transported maps). Lean Lean ✓ Stmt *Under the hypotheses of Theorem 20.14.71, the maps*

$$\begin{aligned} \widetilde{T} &: \mathcal{A}_1 \longrightarrow \mathcal{A}_2, \\ \widetilde{S} &: \mathcal{A}_2 \longrightarrow \mathcal{A}_1 \end{aligned}$$

are completely positive. The trace adjoints of the two square composites satisfy the Schwarz inequality in every simple summand.

Proof. ✓ Proof Let ι_i and π_i be block-diagonal inclusion and diagonal-block extraction, respectively, and let W_i denote the canonical reindexing from the weighted vertical-sector coordinates to the retained physical coordinates. The full-matrix extension of the transported refinement map has the factorization (20.29). Here $\widehat{R}_1$ is normalized state preparation, $\widetilde{\widehat{R}}_2$ is controlled partial trace, and

$$T_v(Z) = U_2 J \left(T(U_1^\dagger Z U_1) \right) U_2^\dagger.$$

Every factor is completely positive. The analogous identities

$$S_v(W) = U_1 S \left(J^{-1}(U_2^\dagger W U_2) \right) U_1^\dagger$$

and (20.30) prove complete positivity of $\widetilde{S}$. These factorizations are exact on the full matrix algebras: $\pi_i W_i^{-1}$ is precisely extraction of the retained diagonal sector blocks. Thus no range inclusion is used.

Composition gives complete positivity of $\widetilde{S}\widetilde{T}$ and $\widetilde{T}\widetilde{S}$. The preceding theorem gives trace preservation of these two composites. Their trace adjoints are therefore unital completely positive maps, and the Kadison–Schwarz inequality applies in each simple summand. $\square$

Theorem 20.14.79 (Transported vertical-sector composites are identities). Lean ✓ Stmt Under precisely the hypotheses of Theorem 20.14.71, the transported maps satisfy

$$\widetilde{S} \circ \widetilde{T} = \text{id}_{\mathcal{A}_1}, \tag{20.31}$$

$$\widetilde{T} \circ \widetilde{S} = \text{id}_{\mathcal{A}_2}. \tag{20.32}$$

Thus $\widetilde{T}$ and $\widetilde{S}$ are mutually inverse. This is the identity-composition conclusion of [CPGSV16, Appendix C.4, lines 1974–1995].

Proof. ✓ Proof For each square composite F , positivity follows from positivity of the two transported maps. Theorem 20.14.71 and Theorem 20.14.78 give trace preservation and the Schwarz inequality for F^* , respectively. Lemma 20.14.65 gives $F(V_i(X)) = V_i(X)$ for every bond matrix X . Theorem 20.14.23 gives a positive integer L_i such that $C_{L_i}(V_i) = \mathcal{A}_i$. Applying Theorem 20.14.29 first to $F = \widetilde{S} \circ \widetilde{T}$ and then to $F = \widetilde{T} \circ \widetilde{S}$ proves (20.31) and (20.32). $\square$

Theorem 20.14.80 (Unitary relabelling of transported vertical sectors). Lean ✓ Stmt Under the hypotheses of Theorem 20.14.79, write

$$\mathcal{A}_1 = \prod_{\alpha} M_{D_{\alpha}^{(1)}}(\mathbb{C}), \quad \mathcal{A}_2 = \prod_{\beta} M_{D_{\beta}^{(2)}}(\mathbb{C}).$$

There are an equivalence σ of the two sector index sets, equalities $D_{\alpha}^{(1)} = D_{\sigma(\alpha)}^{(2)}$, and unitaries $V_{\alpha} \in M_{D_{\sigma(\alpha)}^{(2)}}(\mathbb{C})$ such that

$$\widetilde{T}(0, \dots, 0, X, 0, \dots, 0) = (0, \dots, 0, V_{\alpha} \iota_{\alpha}(X) V_{\alpha}^*, 0, \dots, 0),$$

where the nonzero entry on the right is in position $\sigma(\alpha)$, and

$$\widetilde{S}(0, \dots, 0, Y, 0, \dots, 0) = (0, \dots, 0, \iota_{\alpha}^{-1}(V_{\alpha}^* Y V_{\alpha}), 0, \dots, 0),$$

where the entries on the left and right are in positions $\sigma(\alpha)$ and α , respectively. These identities hold for every $X \in M_{D_{\alpha}^{(1)}}(\mathbb{C})$ and $Y \in M_{D_{\sigma(\alpha)}^{(2)}}(\mathbb{C})$. Here ι_{α} is the reindexing induced by the

displayed dimension equality. These are the mutually inverse unitary formulas of [CPGSV16, Appendix C.4, line 1997]. They make no assertion about the multiplicities, weights, or coefficient identity at lines 2001–2008.

Proof. ✓ Proof The two basis-of-normal-tensors decompositions have positive matrix dimensions. The transported maps are completely positive and preserve the total block trace, and the preceding theorem gives both inverse laws. Apply Theorem [LTC26, “Unitary action of an invertible CPTP map”] to this pair. Its equivalence of summands, dimension equalities, and unitary matrices give the two displayed formulas without changing the chosen relabelling. $\square$

Theorem 20.14.81 (Coefficient comparison for transported vertical sectors). Lean ✓ Stmt Under the hypotheses and with the notation of Theorem 20.14.80, let μ_α and ν_β be the positive diagonal multiplicity matrices in the one-site and two-site vertical canonical forms, and set $m_\alpha = \text{tr}(\mu_\alpha)$ and $n_\beta = \text{tr}(\nu_\beta)$. For every horizontal bond matrix X and every sector α ,

$$A_{\sigma(\alpha)}^{(2)}(X) = \frac{m_\alpha}{n_{\sigma(\alpha)}} V_\alpha \iota_\alpha(A_\alpha^{(1)}(X)) V_\alpha^*.$$

Consequently, for every tensor letter a ,

$$A_{\sigma(\alpha),a}^{(2)} = \frac{m_\alpha}{n_{\sigma(\alpha)}} V_\alpha \iota_\alpha(A_{\alpha,a}^{(1)}) V_\alpha^*.$$

This is the coefficient identity in [CPGSV16, Appendix C.4, lines 2001–2008]. It compares the traces of the multiplicity matrices; it does not assert equality of their dimensions or of their individual diagonal entries.

Proof. ✓ Proof Decompose the weighted contraction family into its individual matrix summands. At the summand paired with α , the unitary formula gives

$$m_\alpha V_\alpha \iota_\alpha(A_\alpha^{(1)}(X)) V_\alpha^* = n_{\sigma(\alpha)} A_{\sigma(\alpha)}^{(2)}(X).$$

Positivity of $\nu_{\sigma(\alpha)}$ makes $n_{\sigma(\alpha)}$ nonzero, so division gives the first identity. Taking X to be a matrix unit gives the tensor-letter identity. $\square$

Theorem 20.14.82 (Simultaneous representations of the blocked vertical operator). Lean Lean Lean Lean Lean ✓ Stmt Under the hypotheses and with the sector correspondence of Theorem 20.14.81, let B be the vertical reading of the two-site blocking, and write $O_L(A) = \text{Tr}_{\text{bond}}(A_{i_1} \cdots A_{i_L})$ for the length- L closed-chain operator. If $L > 0$, then the same equivalence σ , dimension identifications, and unitaries appearing in the tensor-letter identity satisfy

$$\begin{aligned} O_L(B) &= \sum_\alpha \left(\frac{m_\alpha}{n_{\sigma(\alpha)}} \right)^L \text{tr}(\nu_{\sigma(\alpha)}^L) O_L(A_\alpha^{(1)}), \\ O_L(B) &= \sum_{\alpha, \beta} \text{tr}(\mu_\alpha^L) \text{tr}(\mu_\beta^L) O_L(A_\alpha^{(1)}) O_L(A_\beta^{(1)}). \end{aligned}$$

These are the two representations in [CPGSV16, Appendix C.4, lines 2011–2018].

Local fix (blocked coefficient exponent): CPSV16 Appendix C.4, line 2013 prints m_γ^L/n_γ . The line-2008 tensor scaling and line 2040 give $(m_\gamma/n_\gamma)^L$. This is documented in [con26x].

Proof. ✓ Proof The two-site vertical canonical form first gives $O_L(B) = \sum_{\delta} \text{tr}(\nu_{\delta}^L) O_L(A_{\delta}^{(2)})$. Reindex this sum by $\delta = \sigma(\alpha)$. The tensor-letter identity from the preceding theorem multiplies every letter of $A_{\alpha}^{(1)}$ by $m_{\alpha}/n_{\sigma(\alpha)}$. Along a chain of length L this scalar therefore occurs to the power L ; unitary conjugation and the bond-index identification leave the closing trace unchanged. This proves the first formula.

For the second formula, the vertical reading of a two-site block is the product tensor of two copies of the one-site vertical reading. Closed chains turn this tensor product into multiplication of operators. Expand each copy by the one-site vertical canonical form, $O_L(\tilde{M}) = \sum_{\alpha} \text{tr}(\mu_{\alpha}^L) O_L(A_{\alpha}^{(1)})$, and distribute the product of the two finite sums. $\square$

Theorem 20.14.83 (Canonical hypotheses for the retained vertical product). Lean Lean Lean Lean ✓ Stmt *Suppose that the vertical reading of M is reconstructed from a retained tensor A by a coisometry U . The two-site blocking is then reconstructed from the product of two copies of A by the Kronecker square of U . Moreover, if M is in normalized BNT-refined horizontal form and generates positive operators, this retained product tensor has invariant-projector closure and has no nontrivial periodic vectors. No full-support assumption is imposed on the retained bond space.*

Proof. ✓ Proof The Kronecker square of a coisometry is again a coisometry. Expanding a blocked vertical letter gives the product of the two one-site reconstructions and hence the asserted exact compression formula. Normalized BNT-refined horizontal form and positivity are preserved by two-site blocking. Invariant-projector closure descends to an exact reducing compression, while a periodic vector in the compression would embed isometrically as a periodic vector of the blocked vertical tensor. $\square$

Theorem 20.14.84 (Canonical hypotheses for every retained copy pair). Lean Lean Lean Lean ✓ Stmt *In the retained coordinates, the product tensor is the orthogonal direct sum over ordered pairs of BNT copies. Its summand indexed by $((\alpha, q), (\beta, r))$ is $\mu_{\alpha, q} \mu_{\beta, r} (M_{\alpha} M_{\beta})$. Every such summand has invariant-projector closure and has no nontrivial periodic vectors. This includes zero-dimensional summands and requires no nonvanishing assumption on the displayed scalar.*

Proof. ✓ Proof The canonical inclusion of a dependent direct-sum block is an isometry. After transporting this inclusion through the retained-coordinate equivalence, it intertwines the chosen summand with the full retained product tensor on both sides. Its range projection therefore commutes with every tensor letter. Invariant-projector closure descends by exact compression, and any periodic vector of the summand would embed as a periodic vector of the full retained product. $\square$

Theorem 20.14.85 (Normal-corner decomposition of a retained copy pair). Lean ✓ Stmt *For every ordered pair of retained BNT copies, the weighted product tensor has an exact orthogonal decomposition into positive multiples of normal tensors. The corresponding corner isometries are retained, intertwine in both directions, and reconstruct every tensor letter. The active family is allowed to be empty; in particular, no normal block is introduced for a zero-dimensional or identically zero copy-pair tensor.*

Proof. ✓ Proof Apply the canonical-form sufficient condition to the invariant-projector closure and periodic-exclusion properties of the preceding theorem. Its spectral construction discards precisely the corners on which every letter vanishes and retains the isometry of every nonzero corner. $\square$

resulting coefficients are positive, the blocks are normal, and the inclusion ranges are pairwise orthogonal. For every tensor letter,

$$\begin{aligned} C^i &= \sum_j W_j(\lambda_j A_j^i) W_j^\dagger, \\ W_j^\dagger W_l &= \delta_{jl}, \\ C^i W_j &= W_j(\lambda_j A_j^i), \\ W_j^\dagger C^i &= (\lambda_j A_j^i) W_j^\dagger. \end{aligned}$$

Each compression $W_j^\dagger C^i W_j$ is $\lambda_j A_j^i$, and C has the same positive-length closed chains as the finite direct sum $\bigoplus_j \lambda_j A_j$.

Proof. ✓ Proof First reconstruct the retained product as the sum of its canonical copy-pair blocks. Substitute the exact local spectral reconstruction in every pair, distribute the two inclusion maps across the finite sums, and identify the iterated sum with the dependent active-label sum. Finally transport this sum through the active-label enumeration. Isometry, orthogonality, intertwining, and compression are inherited by composition; the closed-chain equality follows from the exact isometric reconstruction. This is the active-corner decomposition in [CPGSV16, Appendix C.4, lines 2020–2029]. □

Theorem 20.14.90 (Coverage of active product corners by the blocked BNT). Lean Lean Lean

✓ Stmt Suppose that the one-site vertical tensor has the stated exact coisometric reconstruction, and let $(A_\gamma^{(2)})_\gamma$ be a basis of normal tensors for the blocked vertical tensor. For every active product corner A_j there are a label $\gamma(j)$, an equality of bond dimensions, an invertible matrix X_j , and a scalar ζ_j of modulus one such that, after transporting the blocked tensor along the dimension equality,

$$A_j^i = \zeta_j X_j A_{\gamma(j)}^{(2),i} X_j^{-1}.$$

The choices may be made simultaneously. No converse coverage or surjectivity of $j \mapsto \gamma(j)$ is asserted.

Scope restriction (active product BNT): The conclusion gives only one-way coverage of active product corners. It does not assert that every blocked BNT label occurs. This is documented in [con26c].

Local fix (per-pair support): Appendix C.4, lines 2011–2029 of [CPGSV16] compares a sum over all retained copy pairs and does not isolate a fixed pair. Here fixed-pair support follows from the exact squared reconstruction and the canonical direct-sum corner; a zero pair contributes an empty active family. See [con26q].

Proof. ✓ Proof The squared coisometric reconstruction gives equality of all positive-length closed chains between the blocked vertical tensor and the retained product. Transport the blocked basis of normal tensors along this equality. Proposition 2.7, applied to the flattened positive normal decomposition, then covers each active corner by a blocked BNT tensor up to an invertible gauge and a unit-modulus phase. Classical choice makes these comparisons simultaneous. □

Theorem 20.14.91 (Distinguished blocked reference corners). Lean Lean Lean Lean Lean

Lean Lean Lean ✓ Stmt Assume that every blocked BNT label has positive multiplicity and that the blocked vertical tensor has the exact coisometric reconstruction $T = U_2^\dagger C_2 U_2$. Let $E_{\gamma,0}$ be the canonical inclusion of the first copy of label γ , and put $F_\gamma = U_2^\dagger E_{\gamma,0}$. Then F_γ is

an isometry, intertwines T with the distinguished weighted BNT copy in both directions, and compresses T to that copy exactly. For an active product corner covered by γ , transport F_γ along the stated equality of bond dimensions. The transported map remains an isometry and obeys the corresponding transported compression identity.

Proof. ✓ Proof The copy inclusion is isometric and selects one diagonal block of the assembled tensor. Inserting $T = U_2^\dagger C_2 U_2$ and $U_2 U_2^\dagger = 1$ gives both intertwining identities and compression. Transport along equality of bond dimensions preserves these equations. $\square$

Theorem 20.14.92 (Active product positivity from sector separation). Lean ✓ Stmt Assume that M generates matrix product density operators and that the one-site and blocked vertical tensors admit exact coisometric reconstructions. Assume positive multiplicities and weights for the blocked BNT and normality of its representatives. Suppose moreover that every projection with a nonzero two-sided corner of the blocked vertical tensor has a nonzero finite-chain sector compression. Then every active product corner j satisfies $0 < \lambda_j \zeta_j$.

Proof. ✓ Proof Compare the active corner with the distinguished positive-weight copy of its blocked BNT representative. The active coefficient is nonzero because $\lambda_j > 0$ and $|\zeta_j| = 1$. The assumed separation property supplies a nonzero finite-chain compression, so the sector-weight comparison gives $\lambda_j \zeta_j > 0$. $\square$

Theorem 20.14.93 (Positivity of active product coefficients). Lean ✓ Stmt Suppose that M is in normalized BNT-refined horizontal form and generates positive operators. Assume exact coisometric reconstructions of its one-site and blocked vertical tensors, positive multiplicities and weights for the blocked BNT, and normality of its BNT representatives. For every active product corner j , one has $0 < \lambda_j \zeta_j$.

Proof. ✓ Proof Normalized BNT-refined horizontal form separates every nonzero range-projection corner at some finite chain length. Apply the preceding theorem with this separation property. $\square$

Theorem 20.14.94 (Literal positivity of active product coefficients). Lean ✓ Stmt Suppose that the doubled-index MPS tensor of M is in literal CPSV canonical form and that M generates matrix product density operators. Assume exact coisometric reconstructions of the one-site and blocked vertical tensors, positive multiplicities and weights for the blocked BNT, and normality of its representatives. For every active product corner j , one has $0 < \lambda_j \zeta_j$. This is the positivity step of [CPGSV16, Appendix C.4, lines 2025–2029].

Proof. ✓ Proof The literal sector-compression separation theorem supplies the finite chain on which the active corner is nonzero. Apply Theorem 20.14.92. $\square$

Theorem 20.14.95 (Unitary normalization of active product gauges). Lean ✓ Stmt Under the hypotheses of the preceding positivity theorem, for every active product corner j there is a real number $\omega_j > 0$ such that

$$\begin{aligned} X_j^\dagger X_j &= \omega_j \mathbb{1}, \\ \omega_j^{-1/2} X_j &\in \mathbf{U}(D_j). \end{aligned}$$

This is the gauge normalization used in [CPGSV16, Appendix C.4, lines 2025–2029].

Proof. ✓ **Proof** Apply the Figure 8 comparison to the active corner and the distinguished blocked reference corner. Both coefficients are positive, so horizontal canonicity and positivity identify their Gram-dressed normal tensors. Normal-tensor rigidity then makes $X_j^\dagger X_j$ a positive real multiple of the identity. Division by the positive square root gives a unitary matrix. $\square$

Theorem 20.14.96 (Literal unitary normalization of active product gauges). Lean ✓ **Stmt** Under the hypotheses of the preceding literal positivity theorem, for every active product corner j there is a real number $\omega_j > 0$ such that

$$\begin{aligned} X_j^\dagger X_j &= \omega_j \mathbb{1}, \\ \omega_j^{-1/2} X_j &\in \mathbf{U}(D_j). \end{aligned}$$

Proof. ✓ **Proof** Compress the blocked vertical tensor to the active corner and to the distinguished blocked reference corner. Their coefficients are positive by the preceding theorem and by positivity of the blocked multiplicity weight. The literal Figure 8 comparison gives $X_j^\dagger X_j = \omega_j \mathbb{1}$ with $\omega_j > 0$; division by $\sqrt{\omega_j}$ gives a unitary matrix. $\square$

Definition 20.14.97 (Original-label corner family). Lean ✓ **Stmt** An original-label corner family for the retained product records, for every active corner of each copy pair, an original one-site BNT label, a positive coefficient, and an isometric inclusion into the raw product bond space. Distinct corners belonging to one copy pair have orthogonal ranges. Each inclusion intertwines the raw product tensor with the stated positive multiple of its original BNT representative, and the resulting corners reconstruct every tensor letter exactly.

Local fix (Figure-11 fixed-pair support): A copy pair with no active corner is represented by an empty family; no unsupported corner is inserted. See [con26q].

Theorem 20.14.98 (Positive original-label corner decomposition). Lean ✓ **Stmt** Let the active product corners be covered by blocked BNT representatives, let the blocked representatives be identified with the original BNT by the trace-ratio unitary conjugacies, and suppose that the active gauge-phase coefficients are positive. If $X_j^\dagger X_j = \omega_j \mathbb{1}$ with $\omega_j > 0$, then every retained copy pair has an original-label corner family. Its coefficient at an active corner is

$$\frac{\lambda_j \zeta_j}{w_{\alpha, q} w_{\beta, r}} \frac{\mathrm{tr} D_{\gamma(j)}}{\mathrm{tr} D_{\sigma(\gamma(j))}^{(2)}},$$

which is positive. The normalized inclusions are isometries with pairwise orthogonal ranges, intertwine the raw product tensor with these positive multiples of the original BNT representatives, and reconstruct every raw product letter exactly.

Local fix (Figure-11 fixed-pair support): Empty active families remain empty. Exact weighted reconstruction and positivity of the outer copy weight imply that the corresponding raw product vanishes. See [con26q].

Proof. ✓ **Proof** Transport each local inclusion first by the normalized active comparison gauge and then by the unitary identifying the blocked representative with its original one-site label. The first normalization preserves isometry and orthogonality; the second unitary transport does likewise. The two conjugacy equations give the stated coefficient and intertwining identity. Positivity follows from the active gauge-phase coefficient, the two positive multiplicity traces, and the positive outer copy weight. Conjugating each corner term through the two transports recovers its local spectral term. Summing these identities gives the weighted reconstruction, and cancellation of the nonzero outer weight gives the raw reconstruction. $\square$

Theorem 20.14.99 (Fusion coisometries from original-label corners). Lean ✓ Stmt Given an original-label corner family, choose the distinguished retained copy of every BNT label. For fixed labels α, β , let the multiplicity of γ be the number of active corners of the distinguished copy pair whose original label is γ , and put the corresponding positive corner coefficients on the diagonal of $\chi_{\alpha, \beta, \gamma}$. The block row of the adjoints of the corner inclusions, regrouped by these label fibers, defines a fusion coisometry. It satisfies

$$\begin{aligned}\chi_{\alpha, \beta, \gamma; r} &> 0, \\ U_{\alpha, \beta} U_{\alpha, \beta}^\dagger &= 1, \\ U_{\alpha, \beta} (M_\alpha M_\beta)^{ij} U_{\alpha, \beta}^\dagger &= \bigoplus_\gamma \chi_{\alpha, \beta, \gamma} \otimes M_\gamma^{ij},\end{aligned}$$

together with the exact reconstruction

$$(M_\alpha M_\beta)^{ij} = U_{\alpha, \beta}^\dagger \left(\bigoplus_\gamma \chi_{\alpha, \beta, \gamma} \otimes M_\gamma^{ij} \right) U_{\alpha, \beta}.$$

Empty label fibers contribute zero-dimensional diagonal blocks. This is the decomposition of CPSV16, Appendix C.4, lines 2020–2029.

Scope restriction (active product BNT): Only active product corners are retained. A label absent from the distinguished copy pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): The distinguished copy pair may have an empty active family; no unsupported corner is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): The retained-row map is a coisometry onto the active direct sum. Exact reconstruction includes any common zero corner discarded by the forward map. See [con26j].

Proof. ✓ Proof For each distinguished copy pair, form the block row from the adjoints of its pairwise orthogonal isometric corner inclusions. The block-row identity makes this map a coisometry. Its forward conjugation is block diagonal over the active corners, while the corner reconstruction gives the reverse identity. Regrouping the active-corner sum by the original label turns each label fiber into the diagonal matrix whose entries are precisely the positive corner coefficients. These identities are valid without separate cases when a fiber or the whole active family is empty. $\square$

Theorem 20.14.100 (Fusion coisometries from normalized product corners). Lean ✓ Stmt Suppose that every active product corner has positive gauge-phase coefficient and that its gauge Gram matrix is a positive scalar multiple of the identity. Given the trace-ratio unitary correspondence between the one-site and two-site vertical sectors, there are positive diagonal matrices $\chi_{\alpha, \beta, \gamma}$ and coisometries $U_{\alpha, \beta}$ satisfying the forward fusion identity and exact reconstruction for every product tensor $M_\alpha M_\beta$. This is the active-corner construction in [CPGSV16, Appendix C.4, lines 2020–2029].

Scope restriction (active product BNT): Only active product corners occur. A label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): A fixed product pair may have no active corner, and no unsupported sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): The retained-row map is a coisometry onto the active direct sum, and its adjoint gives the exact reconstruction. See [con26j].

Proof. ✓ Proof The positive coefficients and normalized gauges give an original-label corner family. Apply the preceding block-row construction to its orthogonal isometric inclusions. $\square$

Theorem 20.14.101 (Positive fusion from transported vertical sectors). [Lean](#) [✓ Stmt](#) Under the one-site and two-site vertical basis-of-normal-tensors decompositions and the two completely positive, trace-preserving physical-closure transport identities, assume that the one-site tensor is in normalized BNT-refined horizontal form and defines a matrix product density operator. Then there are positive diagonal multiplicity matrices $\chi_{\alpha,\beta,\gamma}$ and matrices $U_{\alpha,\beta}$ such that every diagonal entry is positive and

$$U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1, \\ U_{\alpha,\beta}(M_\alpha M_\beta)^{ij}U_{\alpha,\beta}^\dagger = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij},$$

while the exact reconstruction is

$$(M_\alpha M_\beta)^{ij} = U_{\alpha,\beta}^\dagger \left(\bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij} \right) U_{\alpha,\beta}.$$

This is the positive fusion decomposition of CPSV16, Appendix C.4, lines 2020–2029.

Scope restriction (active product BNT): Only active product corners occur. A label absent from a fixed product pair has zero fusion multiplicity. See [\[con26c\]](#).

Local fix (Figure-11 fixed-pair support): A fixed product pair may have no active corner, and no unsupported sector is inserted. See [\[con26q\]](#).

Local fix (Figure-11 fusion coisometry): The retained-row map is a coisometry onto the active direct sum, and its adjoint gives the exact reconstruction. See [\[con26j\]](#).

Proof. [✓ Proof](#) First decompose every retained copy-pair tensor into its active normal corners and compare these corners with the two-site vertical basis. Positivity of matrix product density operators makes every resulting scalar positive, while the Gram comparison normalizes each gauge to a unitary. The inverse vertical-sector transport then returns the corners to the original one-site labels. Their orthogonal isometric inclusions form the rows of $U_{\alpha,\beta}$. Grouping these rows by the original label gives the positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$; corner intertwining and corner reconstruction give the two displayed identities. $\square$

Theorem 20.14.102 (Positive fusion from literal CPSV canonical form). [Lean](#) [✓ Stmt](#) Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. Given one-site and two-site vertical canonical decompositions and their trace-ratio unitary sector correspondence, there are positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$ and matrices $U_{\alpha,\beta}$ such that

$$U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1, \\ U_{\alpha,\beta}(M_\alpha M_\beta)^{ij}U_{\alpha,\beta}^\dagger = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij}, \\ (M_\alpha M_\beta)^{ij} = U_{\alpha,\beta}^\dagger \left(\bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij} \right) U_{\alpha,\beta}.$$

Every diagonal entry of $\chi_{\alpha,\beta,\gamma}$ is positive. This is the positive fusion decomposition in [\[CPGSV16, Appendix C.4, lines 2020–2029\]](#).

Scope restriction (active product BNT): Only active product corners occur. A label absent from a fixed product pair has zero fusion multiplicity. See [\[con26c\]](#).

Local fix (Figure-11 fixed-pair support): A fixed product pair may have no active corner, and no unsupported sector is inserted. See [\[con26q\]](#).

Local fix (Figure-11 fusion coisometry): The retained-row map is a coisometry onto the active direct sum, and its adjoint gives the exact reconstruction. See [con26j].

Proof. ✓ **Proof** Literal canonical form gives simultaneous spectra for all retained product corners. Compare each active corner with the blocked vertical BNT. Literal sector separation makes its coefficient positive, and the literal Figure 8 comparison normalizes its gauge to a unitary. Transporting these corners through the one-site–two-site sector correspondence gives positive original-label corners; their adjoints form the rows of the coisometries $U_{\alpha,\beta}$ and yield both displayed identities. □

Theorem 20.14.103 (Idempotent trace scalars from blocked representations). Lean ✓ **Stmt** Fix one-site and two-site vertical canonical decompositions whose sectors are identified by the transported unitary relabelling. Suppose that the product tensors have positive diagonal fusion matrices $\chi_{\alpha,\beta,\gamma}$ and exact forward and reverse coisometric fusion identities. If the blocked operator has the two simultaneous representations of Theorem 20.14.82, then

$$m_\gamma = \sum_{\alpha,\beta} \text{tr}(\chi_{\alpha,\beta,\gamma}) m_\alpha m_\beta.$$

This is the idempotent coefficient identity in [CPGSV16, Appendix C.4, lines 2030–2042].

Proof. ✓ **Proof** For every sufficiently large L , expand the product of one-site sector operators with the fusion identity. Eventual linear independence of the BNT operators and the two blocked representations then identify, for every γ , the two positive power sums

$$\text{tr} \left[\left(\frac{m_\gamma}{n_{\sigma(\gamma)}} \nu_{\sigma(\gamma)} \right)^L \right] = \sum_{\alpha,\beta} \text{tr} \left[(\mu_\alpha \otimes \mu_\beta \otimes \chi_{\alpha,\beta,\gamma})^L \right].$$

The eventual power-sum comparison identifies the two multisets of positive entries. Summing this multiset equality and cancelling $n_{\sigma(\gamma)}$ gives the displayed length-one identity. □

Theorem 20.14.104 (Positive fusion decomposition of vertical products). Lean ✓ **Stmt** Let M be in normalized BNT-refined horizontal form, generate matrix product density operators, and satisfy the renormalization fixed-point condition of Definition 20.2.1. Then one may choose a vertical basis of normal tensors $\{M_\alpha\}_\alpha$ for which there are positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$ and matrices $U_{\alpha,\beta}$. Every diagonal entry of $\chi_{\alpha,\beta,\gamma}$ is positive, and

$$\begin{aligned} U_{\alpha,\beta} U_{\alpha,\beta}^\dagger &= 1, \\ U_{\alpha,\beta} (M_\alpha M_\beta)^{ij} U_{\alpha,\beta}^\dagger &= \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij}, \\ (M_\alpha M_\beta)^{ij} &= U_{\alpha,\beta}^\dagger \left(\bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij} \right) U_{\alpha,\beta}. \end{aligned}$$

Thus the renormalization fixed-point and matrix-product-density-operator assumptions, together with normalized BNT-refined horizontal form, imply the positive fusion decomposition of CPSV16, Appendix C.4, lines 2020–2029.

Scope restriction (active product BNT): Only active product corners occur. A label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): A fixed product pair may have no active corner, and no unsupported sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): The retained-row map is a coisometry onto the active direct sum, and its adjoint gives the exact reconstruction. See [con26j].

Proof. [✓ Proof](#) Apply the proved normalized BNT-refined specialization of Proposition 4.13 to M and to its two-site blocking. In each case, retain the CPSV basis-of-normal-tensors assertion, the positive multiplicity matrices, and both coisometric decomposition identities. Unpack the two completely positive, trace-preserving maps in the renormalization fixed-point condition. The preceding transported-sector theorem then supplies the positive diagonal family, the row coisometries, the forward fusion identity, and exact reconstruction for the chosen one-site vertical basis. $\square$

Theorem 20.14.105 (Two-site closure of the two-site blocking). [Lean](#) [✓ Stmt](#) *After identifying two blocked physical indices with four original indices, one has $(K^{[2]})_2(X) = K_4(X)$.*

Proof. [✓ Proof](#) Expand the two blocked tensor matrices and reassociate their product. $\square$

20.15 Simple local structure from SAL and ZCL

20.15.1 Simple canonical-form weights

Lemma 20.15.1.1 (Blockwise zero-correlation-length equation). [Lean](#) [✓ Stmt](#) *Suppose a tensor satisfying the literal zero-correlation-length equation is gauge-equivalent, through block-diagonal copy gauges, to an assembled sector decomposition. Every copy satisfies*

$$(\mu_{j,q} \mathcal{T}_{A_j})^2 = \mu_{j,q} \mathcal{T}_{A_j}. \quad (20.33)$$

This is the block equation in [CPGSV16, Appendix C.2, lines 1652–1657].

Proof. [✓ Proof](#) Write $\mathcal{T}_M = G \mathcal{T}_S G^{-1}$, where G is the induced block-diagonal gauge. The equation $\mathcal{T}_M^2 = \mathcal{T}_M$ therefore becomes $\mathcal{T}_S^2 = \mathcal{T}_S$ after conjugation by G^{-1} . Restriction to the (j, q) -th diagonal block gives (20.33). $\square$

Theorem 20.15.1.2 (Copy independence for a non-nilpotent sector decomposition). [Lean](#) [✓ Stmt](#) *Let a tensor satisfying the literal zero-correlation-length equation be gauge-equivalent, through block-diagonal copy gauges, to an assembled sector decomposition whose basis elements all have non-nilpotent physical-trace transfer. Then the weights do not depend on the copy index: $\mu_{j,q} = \mu_j$ [CPGSV16, Appendix C.2, lines 1646–1661].*

Proof. [✓ Proof](#) Literal zero correlation length states $\mathcal{T}_M^2 = \mathcal{T}_M$. If $\mathcal{T}_M = X \mathcal{T}_S X^{-1}$ is the block-diagonal gauge to the transfer of the assembled sector decomposition, then cancellation of X and X^{-1} gives $\mathcal{T}_S^2 = \mathcal{T}_S$. By Lemmas 20.11.2 and 20.11.3, the identity descends to every copy:

$$\begin{aligned} \mu_{j,q}^2 \mathcal{T}_{A_j}^2 &= \mu_{j,q} \mathcal{T}_{A_j}, \\ \mathcal{T}_{A_j}^2 &= \frac{1}{\mu_{j,q}} \mathcal{T}_{A_j}. \end{aligned}$$

The left side does not depend on q , and $\mathcal{T}_{A_j} \neq 0$ because it is not nilpotent, so $1/\mu_{j,q}$ is independent of q , which gives $\mu_{j,q} = \mu_{j,q'}$. $\square$

Corollary 20.15.1.3 (Simple canonical-form tensors with zero correlation length). [Lean](#) [✓ Stmt](#) *A simple canonical-form tensor (Definition 20.11.20) with zero correlation length has non-nilpotent basis transfers and weights independent of the copy index.*

Proof. [✓ Proof](#) Apply Theorem 20.15.1.2 to the witness supplied by the simple canonical-form predicate. $\square$

Lemma 20.15.1.4 (Scale-invariant blockwise zero-correlation-length equation). [Lean](#) [✓ Stmt](#) Suppose a tensor with scale-invariant source zero correlation length is gauge-equivalent, through block-diagonal copy gauges, to an assembled sector decomposition. There is a scalar $\lambda > 0$ such that every copy satisfies

$$(\mu_{j,q} \mathcal{T}_{A_j})^2 = \lambda \mu_{j,q} \mathcal{T}_{A_j}.$$

Proof. [✓ Proof](#) Conjugate the scale-invariant physical-trace equation through the induced block-diagonal gauge and restrict it to the (j, q) -th diagonal block. $\square$

Theorem 20.15.1.5 (Scale-invariant copy-independence generalization). [Lean](#) [✓ Stmt](#) More generally, suppose that, for some $\lambda > 0$, the nonzero physical-trace transfer satisfies

$$\mathcal{T}_M^2 = \lambda \mathcal{T}_M.$$

Under the same canonical-form and non-nilpotency conditions, the weights remain independent of the copy index. This scale-invariant statement is distinct from the literal fixed-representative assertion of Theorem 20.15.1.2.

Proof. [✓ Proof](#) The weighted block equation gives $\mathcal{T}_{A_j}^2 = (\lambda/\mu_{j,q}) \mathcal{T}_{A_j}$. Comparing two copy indices and using $\lambda \neq 0$ and $\mathcal{T}_{A_j} \neq 0$ gives $\mu_{j,q} = \mu_{j,q'}$. $\square$

Corollary 20.15.1.6 (Simple canonical-form tensors with scale-invariant zero correlation length). [Lean](#) [✓ Stmt](#) A simple canonical-form tensor with scale-invariant source zero correlation length has non-nilpotent basis transfers and weights independent of the copy index.

Proof. [✓ Proof](#) Apply Theorem 20.15.1.5 to the witness supplied by the simple canonical-form predicate. $\square$

Definition 20.15.1.7 (Common sector weight and absorbed representative). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose the weights over each representative do not depend on the copy index. Choose the common weight μ_j from any copy over j and absorb it into the representative:

$$\mathcal{K}_j = \mu_j A_j.$$

This is the convention of [CPGSV16, Appendix C.2, equation CFK, lines 1660–1665].

Lemma 20.15.1.8 (Every copy has the common weight). [Lean](#) [✓ Stmt](#) For every copy q over j , one has $\mu_{j,q} = \mu_j$.

Proof. [✓ Proof](#) Compare the q -th copy with the distinguished copy used to define μ_j . $\square$

Theorem 20.15.1.9 (Coefficient after common-weight absorption). [Lean](#) [✓ Stmt](#) Let r_j be the number of horizontal copies of representative j . Then its coefficient on a chain of length N is

$$\sum_q \mu_{j,q}^N = r_j \mu_j^N.$$

Thus the natural multiplicity is the horizontal copy count r_j .

Proof. [✓ Proof](#) Every term in the copy sum equals μ_j^N , and there are r_j terms. $\square$

Theorem 20.15.1.10 (Nonvanishing of the common-weight coefficient). [Lean](#) [Lean](#) [✓ Stmt](#) The common weight is nonzero, and hence for every chain length N ,

$$q_j = \sum_q \mu_{j,q}^N = r_j \mu_j^N \neq 0.$$

This is the coefficient nonvanishing used in the choice $m_j = q_j \tilde{m}_j$ in [CPGSV16, Appendix C.2, lines 1714–1718]. It does not assert nonvanishing of the independent virtual tail contraction $\tilde{m}_j$.

Proof. [✓ Proof](#) The distinguished copy weight is nonzero by the sector data. Since $r_j > 0$, both factors in $r_j \mu_j^N$ are nonzero. $\square$

Theorem 20.15.1.11 (Common nonzero BNT tail selection). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose the weights over each representative are independent of the copy index, and none of the physical-trace transfers $\mathcal{B}_j = \sum_i A_j^{ii}$ is nilpotent. There is one positive tail length L such that, for every representative A_j , some virtual indices α, β satisfy $\tilde{m}_j = [\mathcal{B}_j^L]_{\alpha,\beta} \neq 0$. At total chain length $L + 3$, the corresponding copy coefficient is $q_j = \sum_q \mu_{j,q}^{L+3} \neq 0$. Thus the three-site closing matrix satisfies

$$\begin{aligned} R_j &= q_j \mathcal{B}_j^L, \\ (R_j)_{\alpha,\beta} &= q_j \tilde{m}_j \neq 0. \end{aligned}$$

This is the traced-tail selection made in [CPGSV16, Appendix C.2, lines 1714–1718].

Proof. [✓ Proof](#) Since $\mathcal{B}_j$ is not nilpotent, every positive power of $\mathcal{B}_j$ is nonzero; in particular, one common positive power has a nonzero entry for every j . The preceding theorem supplies the nonzero coefficient at length $L + 3$. Therefore its product with the selected entry of $\mathcal{B}_j^L$ is nonzero. $\square$

Theorem 20.15.1.12 (Canonical form with natural multiplicities). [Lean](#) [✓ Stmt](#) After absorbing the common weight, the matrix product vector of the assembled tensor is

$$\mathcal{V}_N(S) = \sum_j r_j \mathcal{V}_N(\mathcal{K}_j).$$

This is equation CFK of [CPGSV16, Appendix C.2, lines 1660–1665]; the same multiplicities occur in equation sigmaNKj of [CPGSV16, Appendix C.2, lines 1756–1759].

Proof. [✓ Proof](#) Expand the assembled tensor by representatives, use Theorem 20.15.1.9, and absorb μ_j^N into the length- N matrix product vector of A_j . $\square$

Definition 20.15.1.13 (Three-site family closure and outer inverse contraction). [Lean](#) [Lean](#) [✓ Stmt](#) Let $(\mathcal{K}_s)_s$ be a finite family of MPO tensors, with $\mathcal{K}_s^{ij} \in M_{D_s}(\mathbb{C})$, and let $R_s \in M_{D_s}(\mathbb{C})$. A three-site operator is a family closure when

$$\rho_{i_1 i_2 i_3, j_1 j_2 j_3} = \sum_s \text{tr}(\mathcal{K}_s^{i_1 j_1} \mathcal{K}_s^{i_2 j_2} \mathcal{K}_s^{i_3 j_3} R_s).$$

Given a simultaneous left inverse C as in Theorem 21.3.83, contracting C against the first and third physical sites of ρ defines the outer inverse contraction

$$(\Gamma_{x,y})_{i_2 j_2} = \sum_{i_1, j_1, i_3, j_3} C_{x, (i_1, j_1)} C_{y, (i_3, j_3)} \rho_{(i_1, i_2, i_3), (j_1, j_2, j_3)}.$$

The closure is the sector-sum form of the displayed equation for $\sigma_3^{(N)}$ in [CPGSV16, Appendix C.2, lines 1343–1348], used with the block-canonical tensor in the Case II calculation at [CPGSV16, Appendix C.2, lines 1726–1732]. The outer contraction is the sector-family form of the Case I contraction in [CPGSV16, Appendix C.2, lines 1415–1438]; the same construction is used with sector projectors at lines 1719–1725.

Theorem 20.15.1.14 (Normalized three-site BNT family closure). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Suppose that the doubled-index tensor has the horizontal BNT representation S , the copy weights over each representative agree, the physical-trace transfers $\mathcal{B}_j = \sum_i A_j^{ii}$ are not nilpotent, and the original tensor satisfies SAL. After identifying length-three configurations with ordered triples, the normalized three-site marginal of the four-site state is the family closure

$$(\sigma_3^{(4)})_{i_1 i_2 i_3, j_1 j_2 j_3} = \sum_j \text{tr}(A_j^{i_1 j_1} A_j^{i_2 j_2} A_j^{i_3 j_3} \widehat{R}_j),$$

where

$$\begin{aligned} \widehat{R}_j &= \text{tr}(\rho^{(4)})^{-1} q_j \mathcal{B}_j, \\ q_j &= \sum_q \mu_{j,q}^4. \end{aligned}$$

Every $\widehat{R}_j$ is nonzero. More generally, a tail of length L gives $\widehat{R}_j = \text{tr}(\rho^{(L+3)})^{-1} q_j \mathcal{B}_j^L$. This is the normalized traced-tail expansion used in [CPGSV16, Appendix C.2, lines 1714–1732].

Proof. [✓ Proof](#) Equality of the positive-length matrix-product-vector families expands every matrix entry of the original operator as the sum of the BNT entries with coefficients q_j . Summing the diagonal physical configurations of the tail gives $\mathcal{B}_j^L$. The global normalization multiplies every closing matrix by $\text{tr}(\rho^{(L+3)})^{-1}$. At $L = 1$, SAL makes this scalar nonzero, copy independence makes q_j nonzero, and nonnilpotence makes $\mathcal{B}_j$ nonzero. $\square$

Theorem 20.15.1.15 (Collapse of the outer inverse contraction). [Lean](#) [✓ Stmt](#) Let $x = (s, \alpha_1, \beta_1)$ and $y = (t, \alpha_3, \beta_3)$. For a three-site family closure and a simultaneous left inverse, the outer inverse contraction vanishes when $s \neq t$. When $s = t$, it is $(\mathcal{K}_s^{i_2 j_2})_{\beta_1, \alpha_3} (R_s)_{\beta_3, \alpha_1}$. In particular, the contraction retains precisely one entry of the middle physical slice and one entry of the closing matrix, with the orientation shown above. This is the sector-family form of the Case I contraction in [CPGSV16, Appendix C.2, lines 1415–1438]. The basis-of-normal-tensors inverse is constructed at lines 1666–1676 and the resulting contraction is used in the Case II calculation at lines 1719–1732 of the same appendix. The closing entry $R_s(\beta_3, \alpha_1)$ is the corrected tail-index orientation of the display at lines 1422–1438, recorded in [con26t].

Proof. [✓ Proof](#) Expand the family closure and interchange the finite sector and physical sums. For $x = (x_s, x_\alpha, x_\beta)$, the inverse identity is

$$\sum_{i,j} C_{x,(i,j)} \mathcal{K}_s^{ij} = \begin{cases} E_{x_\alpha, x_\beta}, & x_s = s, \\ 0, & x_s \neq s. \end{cases}$$

Hence distinct labels give zero. For equal labels, the remaining trace is

$$\text{tr}(E_{\alpha_1, \beta_1} \mathcal{K}_s^{i_2 j_2} E_{\alpha_3, \beta_3} R_s) = (\mathcal{K}_s^{i_2 j_2})_{\beta_1, \alpha_3} (R_s)_{\beta_3, \alpha_1}.$$

$\square$

Corollary 20.15.1.16 (Three-site collapse from the one-letter span). [Lean](#) [✓ Stmt](#) *Under the one-letter product-algebra spanning hypothesis, there is a simultaneous left inverse for which every three-site family closure has the preceding diagonal-sector contraction and vanishing off-diagonal contractions.*

Proof. [✓ Proof](#) Choose the simultaneous left inverse from Theorem 21.3.83 and apply Theorem 20.15.1.15. $\square$

20.15.2 Markov decomposition and inverse-map sector factorization

For the simple MPDO case in Appendix C.2 of [CPGSV16], the strong-area-law hypothesis is used locally through the equality case of strong subadditivity for a normalized three-site reduced state.

Definition 20.15.2.1 (Local η -structure for a three-site reduced state). [Lean](#) [✓ Stmt](#) For a three-site density operator ρ_{ABC} , the local η -structure is the quantum Markov decomposition on the middle subsystem B produced by equality in strong subadditivity. Concretely, after a unitary change of basis on B , one has a finite direct-sum decomposition

$$B = \bigoplus_k (b_1^{(k)} \otimes b_2^{(k)}),$$

$$\rho_{ABC} = \bigoplus_k \rho_{Ab_1}^{(k)} \otimes \rho_{b_2C}^{(k)}.$$

Definition 20.15.2.2 (Canonical projections of a Hayashi decomposition). [Lean](#) [Lean](#) [✓ Stmt](#) Let $B = \bigoplus_k (b_1^{(k)} \otimes b_2^{(k)})$ be the middle-site decomposition, and let E_k be the coordinate projection onto its k -th summand. In the original physical basis, define $Q_k = U_B^* E_k U_B$. These are the projectors introduced in [CPGSV16, Appendix C.2, lines 1364–1368]; the unitary implements the basis change described in [CPGSV16, Appendix C.2, lines 1437–1440].

Theorem 20.15.2.3 (Resolution of the identity by Hayashi-sector projections). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Each Q_k is an orthogonal projection, distinct summands are mutually orthogonal, and the family resolves the identity:*

$$Q_k Q_\ell = 0 \quad \text{if } k \neq \ell,$$

$$\sum_k Q_k = \mathbb{1}.$$

The first two assertions follow from the orthogonal direct-sum decomposition in [CPGSV16, Appendix C.2, lines 1364–1368]; the resolution of the identity is stated in [CPGSV16, Appendix C.2, lines 1693–1697].

Proof. [✓ Proof](#) The coordinate projections satisfy

$$E_k^* = E_k,$$

$$E_k^2 = E_k,$$

$$\sum_k E_k = \mathbb{1}.$$

Reindexing the direct sum and conjugating by U_B preserve these identities. Hence each $Q_k = U_B^* E_k U_B$ is a star projection and therefore an orthogonal projection, while $\sum_k Q_k = U_B^* (\sum_k E_k) U_B =$

1. Finally, multiplying the resolution on both sides by Q_k gives

$$\sum_{\ell} Q_k Q_{\ell} Q_k = Q_k,$$

$$Q_k Q_{\ell} Q_k = (Q_{\ell} Q_k)^* (Q_{\ell} Q_k) \geq 0.$$

The $\ell = k$ term already equals Q_k , so every term with $\ell \neq k$ vanishes. Thus $(Q_{\ell} Q_k)^* (Q_{\ell} Q_k) = 0$, hence $Q_{\ell} Q_k = 0$. Taking adjoints gives $Q_k Q_{\ell} = 0$ for $\ell \neq k$. $\square$

Definition 20.15.2.4 (Markov factors of a simultaneous inverse). [Lean](#) [Lean](#) [✓ Stmt](#) Let C_x and C_y be two rows of a simultaneous left inverse, and choose a Hayashi decomposition of the three-site operator. In its k -th Markov sector, set

$$A_x^{(k)}(l, l') = \sum_{i_1, j_1} C_{x, (i_1, j_1)} \rho_{Ab_1; (i_1, l), (j_1, l')}^{(k)},$$

$$B_y^{(k)}(r, r') = \sum_{i_3, j_3} C_{y, (i_3, j_3)} \rho_{b_2 C; (r, i_3), (r', j_3)}^{(k)}.$$

These are the two factors obtained in the projected contraction of [\[CPGSV16, Appendix C.2, lines 1719–1725\]](#).

Theorem 20.15.2.5 (Outer inverse contraction in Markov coordinates). [Lean](#) [✓ Stmt](#) Let $\Gamma_{x,y}$ denote the outer inverse contraction. In the basis of the chosen Hayashi decomposition,

$$[U_B \Gamma_{x,y} U_B^{\dagger}]_{(k,l,r), (k',l',r')} = \begin{cases} p_k A_x^{(k)}(l, l') B_y^{(k)}(r, r'), & k = k', \\ 0, & k \neq k'. \end{cases}$$

In particular, the probability p_k remains explicit. This identity requires neither a nonvanishing condition on p_k nor assumptions of saturated area law or zero correlation length.

Proof. [✓ Proof](#) Expand the two matrix products and the outer inverse contraction. The Hayashi block decomposition makes the middle-site contraction zero for $k \neq k'$. For $k = k'$, the remaining four outer sums separate into the product defining $A_x^{(k)}$ and $B_y^{(k)}$, with the common factor p_k . $\square$

Theorem 20.15.2.6 (Entrywise BNT–Markov identity). [Lean](#) [✓ Stmt](#) Let $x = (s, \alpha_1, \beta_1)$ and $y = (t, \alpha_3, \beta_3)$. Comparing the two expressions for the same outer inverse contraction gives

$$\begin{aligned} & \begin{cases} p_k A_x^{(k)}(l, l') B_y^{(k)}(r, r'), & k = k', \\ 0, & k \neq k' \end{cases} \\ &= \begin{cases} (R_s)_{\beta_3, \alpha_1} [U_B \kappa_{\beta_1, \alpha_3}^{(s)} U_B^{\dagger}]_{(k,l,r), (k',l',r')}, & s = t, \\ 0, & s \neq t. \end{cases} \end{aligned}$$

Here $\kappa_{\beta_1, \alpha_3}^{(s)}(i, j) = (\mathcal{K}_s^{ij})_{\beta_1, \alpha_3}$. Thus both the normal-sector and Markov-sector off-diagonal entries are retained in the statement, rather than being suppressed by a prior choice of equal labels. This is the full entrywise identity used in [\[CPGSV16, Appendix C.2, lines 1719–1732\]](#).

The closing entry $(R_s)_{\beta_3, \alpha_1}$ is the corrected tail-index orientation of the display at lines 1422–1438, recorded in [\[con26t\]](#).

Proof. [✓ Proof](#) The preceding theorem identifies the left side with the transformed outer inverse contraction. The collapse theorem identifies the contraction with zero when $s \neq t$ and with $(R_s)_{\beta_3, \alpha_1} \kappa_{\beta_1, \alpha_3}^{(s)}$ when $s = t$. Conjugating this matrix by U_B gives the right side. $\square$

Theorem 20.15.2.7 (Off-diagonal Markov blocks of a BNT sector). [Lean](#) [✓ Stmt](#) Fix a normal sector s and virtual indices α_1, β_3 for which $(R_s)_{\beta_3, \alpha_1} \neq 0$. Then, for every β_1, α_3 and every pair of distinct Markov sectors $k \neq k'$, one has

$$[U_B \kappa_{\beta_1, \alpha_3}^{(s)} U_B^\dagger]_{(k, l, r), (k', l', r')} = 0.$$

Thus each physical slice of the s -th normal tensor is block diagonal in the Markov decomposition. This is equation Qks in [\[CPGSV16, Appendix C.2, lines 1682–1688 and 1730–1735\]](#).

The nonzero closing entry uses the corrected orientation $(R_s)_{\beta_3, \alpha_1}$ recorded in [\[con26t\]](#).

Proof. [✓ Proof](#) In the entrywise BNT–Markov identity, take equal normal-sector labels and distinct Markov-sector labels. The left side vanishes, while the right side is the selected closing entry times the displayed block. Cancel the nonzero closing entry. $\square$

Definition 20.15.2.8 (Diagonal BNT block in a Markov sector). [Lean](#) [Lean](#) [✓ Stmt](#) For a normal sector s and a Markov sector k , let

$$O_s^{(k)}(\beta, \alpha) = [U_B \kappa_{\beta, \alpha}^{(s)} U_B^\dagger]_{k, k}.$$

The assertion $O_s^{(k)} \neq 0$ means that this block is nonzero for at least one pair of virtual indices β, α .

Theorem 20.15.2.9 (Existence of a normal-sector label). [Lean](#) [✓ Stmt](#) If $p_k \neq 0$, then there is a normal sector s for which $O_s^{(k)} \neq 0$.

This is the existence part of equation QkKjs in [\[CPGSV16, Appendix C.2, lines 1733–1737\]](#). Restricting to $p_k \neq 0$ is the literal version of the source’s preceding removal of summands for which $Q_k \sigma_3^{(N)} Q_k = 0$.

Proof. [✓ Proof](#) Choose nonzero diagonal entries of the trace-one left and right states in sector k . The corresponding diagonal entry of the Hayashi decomposition is nonzero because $p_k \neq 0$. Expanding the same entry by the three-site normal-sector family therefore leaves at least one nonzero summand, whose middle tensor has $O_s^{(k)} \neq 0$. $\square$

Theorem 20.15.2.10 (Uniqueness from nonzero closing matrices). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose, in addition, that every closing matrix R_s is nonzero. If $p_k \neq 0$, there is a unique normal sector j_k for which $O_{j_k}^{(k)} \neq 0$.

This is the cancellation argument in equation QkKjs. The stated nonzero-closing-matrix condition isolates the step obtained in the source from simplicity and the common-weight choice at [\[CPGSV16, Appendix C.2, lines 1714–1718\]](#). Its derivation from those preceding hypotheses is [Theorem 20.15.1.14](#).

Proof. [✓ Proof](#) Assume that sectors $s \neq t$ both have nonzero k -blocks. A diagonal instance of the BNT–Markov identity for s gives a nonzero left factor. A mixed instance with normal-sector labels s, t forces the right factor selected from t to vanish. A diagonal instance for t then has zero left-hand side and a nonzero right-hand side, a contradiction. $\square$

Definition 20.15.2.11 (Normal-sector projections on the active support). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Under the hypotheses of [Theorem 20.15.2.10](#), define

$$P_s = \sum_{\substack{k: p_k \neq 0 \\ j_k = s}} Q_k,$$

$$P_{\text{act}} = \sum_{k: p_k \neq 0} Q_k.$$

This is equation Pis, restricted to the positive-weight Markov support retained here.

Theorem 20.15.2.12 (Orthogonality and resolution on the active support). [Lean](#) [Lean](#)
[Lean](#) [✓ Stmt](#) Every P_s is an orthogonal projection, distinct P_s are mutually orthogonal, and $\sum_s P_s = P_{\text{act}}$.

Proof. [✓ Proof](#) Each P_s is a sum of mutually orthogonal Markov-sector projections. Distinct labels have disjoint sums. Finally, summing over s counts each positive-weight Markov sector exactly once. $\square$

Theorem 20.15.2.13 (SAL and arbitrary-cut equality in strong subadditivity). [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $1 \leq m \leq \lfloor N/2 \rfloor$, and divide the chain into four consecutive regions A, B, C, D of lengths $m-1, 1, N-m-1, 1$. If the tensor saturates the area law, then its marginal on ABC satisfies $S(ABC) + S(B) = S(AB) + S(BC)$. Indeed, this is the entropy form of $I_1 = I_m$. Taking $m = 2$ gives the regions of lengths $1, 1, N-3$, and then taking $N = 4$ shows that the three-site reduced state of the four-site chain satisfies equality in strong subadditivity.

Conversely, suppose the tensor generates positive semidefinite periodic operators and their traces are nonzero at every positive chain length. If the displayed equality holds for every N and every $1 \leq m \leq \lfloor N/2 \rfloor$, then the tensor saturates the area law. This is the choice of regions used in [CPGSV16, Appendix C.2, lines 1760–1780]. The endpoint corrects the strict inequality printed at source line 1771: Definition 4.6 and the concluding comparison both require $m = \lfloor N/2 \rfloor$. This local correction is recorded in [con26t].

Proof. [✓ Proof](#) By saturation, $I_1 = I_m$. Expanding the two mutual informations and cancelling the entropy of the full chain gives $S_{N-1} + S_1 = S_m + S_{N-m}$, which is the stated equality for the regions A, B, C . Conversely, this equality gives $I_1 = I_m$. If it holds for every admissible m , comparing the cases $m = L$ and $m = L + 1$ gives $I_L = I_{L+1}$ and hence saturation of the area law. $\square$

Theorem 20.15.2.14 (All-cut quantum Markov structure implies SAL). [Lean](#) [✓ Stmt](#) Suppose that the periodic operators are positive semidefinite and have nonzero trace at every positive chain length. For every N and every $1 \leq m \leq \lfloor N/2 \rfloor$, divide the chain into four consecutive regions of lengths $m-1, 1, N-m-1, 1$. If the marginal on the first three regions admits a quantum Markov decomposition on its one-site middle system, then the tensor saturates the area law.

Proof. [✓ Proof](#) The quantum Markov decomposition gives equality in strong subadditivity at every admissible cut: $S(\rho_{ABC}) + S(\rho_B) = S(\rho_{AB}) + S(\rho_{BC})$. These equalities at every admissible cut imply saturation of the area law by Theorem 20.15.2.13. $\square$

Theorem 20.15.2.15 (SAL implies local η -structure). [Lean](#) [✓ Stmt](#) If a normalized three-site reduced state satisfies the equality case of strong subadditivity — the local entropy form of the simple-MPDO strong area law — then it admits the local η -structure of Definition 20.15.2.1.

Proof. [✓ Proof](#) This is the forward implication of the Hayashi equality characterization. The product-reference raw Petz formula, including its singular-support compression, is given by the corresponding theorem in the companion quantum-channel volume [LTC26, “Raw Petz factorization for a product reference”]. The singular-support equality-to-recovery theorem is still separate. Beyond it, the remaining structural ingredient is the family-level Koashi–Imoto decomposition and the induced action of the middle-system channel on its common direct-sum factors. $\square$

Lemma 20.15.2.16 (Partial trace along a right tensor factor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#)
 Suppose that $C \simeq C' \otimes E$. The partial trace over E is given entrywise by

$$(\mathrm{tr}_E X)_{(a,c'),(a',d')} = \sum_e X_{(a,c',e),(a',d',e)}.$$

It preserves positivity and trace.

Proof. [✓ Proof](#) Reindex C as $C' \otimes E$ and apply the ordinary right partial trace. Its entry formula gives the displayed sum. Positivity follows by tracing a positive matrix, and invariance of the full trace under reindexing gives trace preservation. $\square$

Theorem 20.15.2.17 (Right marginals preserve a quantum Markov decomposition). [Lean](#) [Lean](#) [Stmnt](#)
 Suppose that $C \simeq C' \otimes E$ and

$$\rho_{ABC} = \bigoplus_k p_k \rho_{Ab_1}^{(k)} \otimes \rho_{b_2C}^{(k)}$$

is a quantum Markov decomposition on B . Then the marginal $\rho_{ABC'} = \mathrm{tr}_E(\rho_{ABC})$ has the decomposition

$$\rho_{ABC'} = \bigoplus_k p_k \rho_{Ab_1}^{(k)} \otimes \mathrm{tr}_E(\rho_{b_2C}^{(k)}).$$

Proof. [✓ Proof](#) The middle-system decomposition, its unitary change of basis, and the probabilities p_k are unchanged. Partial trace preserves positivity and trace, so each $\mathrm{tr}_E(\rho_{b_2C}^{(k)})$ is a density operator. Applying tr_E to the block-diagonal identity gives the displayed formula. $\square$

Lemma 20.15.2.18 (Tracing a terminal block). [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let $a, b, c, N \in \mathbb{N}$ satisfy $a + b + c \leq N$. For configurations u, v on the first $a + b$ sites,

$$\sum_y \sigma_{a+b+c}^{(N)}(K)_{u\|y, v\|y} = \sigma_{a+b}^{(N)}(K)_{u,v},$$

where the sum ranges over configurations on the last c sites. The same identity holds after replacing an arithmetically equal block length by its canonical identification.

Proof. [✓ Proof](#) Expand both reduced states as partial traces of the normalized periodic state. The left-hand side first traces the complement of the $(a + b + c)$ -site block and then its last c sites; combining the two sums gives the partial trace onto the first $a + b$ sites. $\square$

Theorem 20.15.2.19 (SAL three-site marginals have Hayashi decompositions). [Lean](#) [Stmnt](#)
 Let K satisfy the strong area law and let $N \geq 4$. If $\sigma_3^{(N)}(K)$ is obtained from the normalized N -site periodic state by tracing out all but its first three sites, then $\sigma_3^{(N)}(K)$ admits a quantum Markov decomposition on its middle site. This is the three-site marginal result proved in [CPGSV16, Appendix C.2, lines 1351–1371]. The decomposition may depend on N .

Proof. [✓ Proof](#) Divide the first $N - 1$ sites into consecutive regions of lengths 1, 1, and $N - 3$. Let $\rho_{ABC}^{\mathrm{flat}}$ denote this marginal in the corresponding index set $\{0, \dots, d^1 - 1\} \times \{0, \dots, d^1 - 1\} \times \{0, \dots, d^{N-3} - 1\}$. By Theorem 20.15.2.13,

$$S(\rho_{ABC}^{\mathrm{flat}}) + S(\rho_B^{\mathrm{flat}}) = S(\rho_{AB}^{\mathrm{flat}}) + S(\rho_{BC}^{\mathrm{flat}}).$$

The first two singleton blocks have the canonical identification $e_{\text{site}} : \{0, \dots, d-1\} \simeq \{0, \dots, d^1-1\}$, and reindexing these factors preserves the displayed equality. Positivity and unit trace follow from the normalized reduced state, so the Hayashi characterization gives a quantum Markov decomposition of the $(N-1)$ -site marginal. Finally split the third region as $C \simeq c \otimes E$, where c is its first site and E contains the remaining $N-4$ sites. Theorem 20.15.2.17 gives the same middle-site direct sum after tracing out E . The contiguous-block reduction gives

$$\sigma_3^{(N)}(K) = \text{tr}_{4, \dots, N-1} \left[\sigma_{N-1}^{(N)}(K) \right],$$

which identifies the resulting operator with $\sigma_3^{(N)}(K)$. $\square$

Theorem 20.15.2.20 (Four-site SAL marginals have Hayashi decompositions). [Lean](#) [✓ Stmt](#)
Let K satisfy the strong area law. If $\sigma_3^{(4)}(K)$ is obtained from its normalized four-site periodic state by tracing out the fourth site, then $\sigma_3^{(4)}(K)$ admits a quantum Markov decomposition on its middle site.

Proof. [✓ Proof](#) This is the case $N = 4$ of Theorem 20.15.2.19. $\square$

Theorem 20.15.2.21 (BNT sector projectors for the four-site SAL marginal). [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) *Suppose that the common blocking has been performed, so that the BNT representatives have a simultaneous one-letter span. For the normalized three-site marginal $\sigma_3^{(4)}$, there are a simultaneous inverse C and a Hayashi decomposition such that every Markov sector k with $p_k \neq 0$ has a unique BNT label j_k . Consequently,*

$$P_s = \sum_{\substack{k: p_k \neq 0 \\ j_k = s}} Q_k.$$

Each P_s is an orthogonal projection, $P_s P_t = 0$ for $s \neq t$, and $\sum_s P_s = P_{\text{act}}$. No independent nonzero-closing-matrix hypothesis is required. This is the four-site specialization of equations QkKjs and Pis in [CPGVS16, Appendix C.2, lines 1714–1737]. The Hayashi decomposition uses the forward equality characterization of strong subadditivity recorded in the corresponding theorem in the companion quantum-channel volume [LTC26, “Hayashi SSA-equality characterization”]; no additional analytic assumption is introduced here.

Proof. [✓ Proof](#) The canonical ordered-triple reindexing is the submatrix convention of the SAL-to-Hayashi theorem. Choose the simultaneous inverse from the one-letter span and the Hayashi decomposition from SAL. By Theorem 20.15.1.14, the same marginal is a BNT family closure whose closing matrices are all nonzero. The generic uniqueness and projector theorems now give the unique labels, orthogonality, mutual disjointness, and resolution of the active support. $\square$

Definition 20.15.2.22 (Sitewise physical action and absorbed BNT representatives). [Lean](#)

[Lean](#) [✓ Stmt](#) For a one-site matrix V , write $V^{\otimes N}$ in the configuration basis as

$$(V^{\otimes N})_{s,t} = \prod_{n=1}^N V_{s_n, t_n}.$$

If the copies of the j -th normal tensor have common coefficient μ_j , let $\mathcal{K}_j = \mu_j A_j$ denote the representative with this coefficient absorbed into its local tensor.

Theorem 20.15.2.23 (Local selection by the BNT projectors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
Under the hypotheses of Theorem 20.15.2.21, the resulting projectors satisfy $P_s A_i P_t = 0$ whenever $s \neq t$ or $i \neq s$. For the matching label, they satisfy $P_i A_i P_i = A_i$. Consequently, $\sum_s P_s A_i P_s = A_i$, which is the local diagonal decomposition required for the sector entropy argument. Equivalently, compressing the physical legs of A_i by P_s retains A_i for $s = i$ and gives the zero local tensor otherwise. This is the local projection relation in [CPGSV16, Appendix C.2, lines 1680–1755].

The Markov summands of zero Hayashi weight are retained in the ambient decomposition. They annihilate every normal-sector physical slice, so the displayed local identities do not require the stronger assertion $\sum_s P_s = \mathbb{1}$ outside the active support.

Proof. [✓ Proof](#) Distinct Hayashi sectors give zero off-diagonal corners. On an active diagonal sector, uniqueness of the BNT label kills every normal tensor except the matching one. A zero-weight diagonal sector also kills every physical slice: otherwise the nonzero closing matrix would make the corresponding three-site Markov block nonzero. Summing these statements over the Hayashi sectors gives the asserted $P_s A_i P_t$ identities. $\square$

Definition 20.15.2.24 (Completion on the zero-weight Markov sectors). [Lean](#) [Lean](#) [✓ Stmt](#)
Fix a BNT label s_0 . Each positive-weight Markov sector retains its unique BNT label, while every zero-weight Markov sector is assigned the label s_0 . If $\widehat{S}_s$ is the set of Markov sectors with completed label s , define

$$\widehat{P}_s = \sum_{k \in \widehat{S}_s} Q_k.$$

Theorem 20.15.2.25 (Separating projectors with copy-independent weights). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
Let K be simple, in biCF, and satisfy SAL. Assume in addition that the common blocking has been performed, the copy weights over each BNT representative agree, and the representatives have a simultaneous one-site span. There is a distinguished BNT label s_0 such that the completed projectors are orthogonal and, for distinct labels s and t ,

$$\begin{aligned} \widehat{P}_s \widehat{P}_t &= 0, \\ \sum_s \widehat{P}_s &= \mathbb{1}. \end{aligned}$$

For every physical slice of the i -th BNT representative, whenever $s \neq t$ or $i \neq s$,

$$\widehat{P}_s K_i^{\beta\alpha} \widehat{P}_t = 0.$$

This is the copy-independent-weight specialization of the separating-projector lemma in [CPGSV16, Appendix C.2, lines 1680–1737].

Proof. [✓ Proof](#) A unit-modulus BNT weight supplies the label s_0 . The Markov-sector projections Q_k are mutually orthogonal and sum to the identity, so partitioning all labels by $\widehat{S}_s$ proves orthogonality and the identity resolution. If $p_k = 0$, then $Q_k K_i^{\beta\alpha} Q_k = 0$ for every i, β, α . If $p_k \neq 0$, then $Q_k K_i^{\beta\alpha} Q_k = 0$ whenever the BNT label of sector k is not i . For distinct Markov sectors, $Q_k K_i^{\beta\alpha} Q_l = 0$ whenever $k \neq l$. Hence, whenever $s \neq t$ or $i \neq s$,

$$\widehat{P}_s K_i^{\beta\alpha} \widehat{P}_t = \sum_{k \in \widehat{S}_s} \sum_{l \in \widehat{S}_t} Q_k K_i^{\beta\alpha} Q_l = 0,$$

since every summand vanishes by one of the three preceding cases. $\square$

Theorem 20.15.2.26 (Separating projectors for a simple tensor satisfying ZCL and SAL).

[Lean](#) [✓ Stmt](#) Let K be a simple tensor in block-injective canonical form which satisfies ZCL and SAL. For the representatives $\mathcal{K}_i$ of its BNT, there are orthogonal projectors P_s such that

$$\sum_s P_s = \mathbb{1},$$

$$P_s K_i^{\beta\alpha} P_t = 0 \quad \text{whenever } s \neq t \text{ or } i \neq s.$$

This is the separating-projector lemma in [\[CPGSV16, Appendix C.2, lines 1626–1691\]](#).

Proof. [✓ Proof](#) Write the horizontal canonical form as $K = G(\bigoplus_{j,q} \mu_{j,q} \mathcal{K}_j) G^{-1}$. Restricting the ZCL equation to the (j, q) -th block and using the non-nilpotence supplied by simplicity gives $\mu_{j,q} = \mu_{j,q'}$ for all j, q, q' . The block-diagonal gauge preserves every positive-length periodic state, so the canonical-form tensor and K generate the same such states. The preceding copy-independent-weight theorem now supplies the projectors. $\square$

Theorem 20.15.2.27 (Positive-length BNT-sector compression). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every nonempty periodic chain and every normal-sector label s ,

$$P_s^{\otimes N} \sigma^{(N)}(K) P_s^{\otimes N} = n_s \sigma^{(N)}(\mathcal{K}_s), \quad N \geq 1. \quad (20.34)$$

This is equation sigmaNKj in [\[CPGSV16, Appendix C.2, lines 1756–1759\]](#).

Here and throughout the MPDO discussion, a periodic chain has positive length.

Proof. [✓ Proof](#) Sitewise multiplication by P_s is the periodic MPO obtained by changing every local physical matrix by P_s . The preceding theorem replaces each normal representative by itself when its label is s and by the zero tensor otherwise. For $N \geq 1$, the latter has zero periodic MPO. The surviving coefficient is $n_s \mu_s^N$, which is precisely n_s times the MPO of the representative with μ_s absorbed locally. $\square$

Corollary 20.15.2.28 (Each absorbed BNT element generates MPDOs). [Lean](#) [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem [20.15.2.27](#), every absorbed BNT element $\mathcal{K}_s$ generates positive semidefinite operators on all nonempty periodic chains. Indeed, $P_s^{\otimes N} \sigma^{(N)}(K) P_s^{\otimes N} \geq 0$ and $n_s > 0$, so [\(20.34\)](#) gives $\sigma^{(N)}(\mathcal{K}_s) \geq 0$. This is the first positivity conclusion in [\[CPGSV16, Appendix C.2, lines 1753–1759\]](#).

Proof. [✓ Proof](#) Congruence by the Hermitian projection $P_s^{\otimes N}$ preserves positive semidefiniteness. Multiplication by the inverse of the positive integer n_s then gives positivity of the sector operator. $\square$

Theorem 20.15.2.29 (Physical-trace transfer after common-weight absorption). [Lean](#) [Lean](#)

[✓ Stmt](#) Suppose the weights over each BNT representative are copy-independent. For $\mathcal{K}_s = \mu_s A_s$ and $\mathcal{B}_s = \mathcal{T}_{A_s}$, one has

$$\mathcal{T}_{\mathcal{K}_s} = \mu_s \mathcal{B}_s.$$

Proof. [✓ Proof](#) Substitute $\mathcal{K}_s = \mu_s A_s$ into the defining diagonal sum for the physical-trace transfer and factor out the scalar μ_s . $\square$

Theorem 20.15.2.30 (Literal ZCL descends to absorbed BNT representatives). [Lean](#) [Lean](#)

[✓ Stmt](#) Let K be gauge equivalent to the weighted direct sum of BNT representatives A_s , with

copy weights $\mu_{s,q}$. If its physical-trace transfer is literally idempotent, then every weighted copy satisfies

$$\mathcal{T}_{\mu_{s,q}A_s}^2 = \mathcal{T}_{\mu_{s,q}A_s}.$$

If the copy weights over A_s agree and $\mathcal{K}_s = \mu_s A_s$, then

$$\mathcal{T}_{\mathcal{K}_s}^2 = \mathcal{T}_{\mathcal{K}_s}.$$

Proof. ✓ Proof Conjugating $\mathcal{T}_K^2 = \mathcal{T}_K$ by the block-diagonal gauge gives the same identity for the weighted direct sum. Its diagonal block indexed by (s, q) is $\mathcal{T}_{\mu_{s,q}A_s}^2 = \mathcal{T}_{\mu_{s,q}A_s}$. When the weights are copy-independent, $\mathcal{T}_{\mathcal{K}_s} = \mu_s \mathcal{T}_{A_s}$ is that block for any copy over s . $\square$

Theorem 20.15.2.31 (Scale-invariant source ZCL descends to absorbed BNT representatives). Lean Lean ✓ Stmt Let K be a simple tensor in its chosen canonical form and suppose that K has scale-invariant source zero correlation length. Then the copy weights over each normal representative A_s agree. For every A_s with non-nilpotent physical-trace transfer $\mathcal{B}_s$, absorb the common weight and write $\mathcal{K}_s = \mu_s A_s$. Each absorbed BNT representative $\mathcal{K}_s$ has scale-invariant source zero correlation length. More precisely, $\mathcal{T}_{\mathcal{K}_s} = \mu_s \mathcal{B}_s \neq 0$ and, for some $\lambda > 0$,

$$\mathcal{T}_{\mathcal{K}_s}^2 = \lambda \mathcal{T}_{\mathcal{K}_s}.$$

This is the final zero-correlation-length conclusion in [CPGSV16, Appendix C.2, line 1781].

Proof. ✓ Proof Corollary 20.15.1.6 gives the common weight on every copy. Lemma 20.15.1.4 then gives the displayed quadratic identity. The common weight is nonzero, and nonnilpotence of $\mathcal{B}_s$ implies $\mathcal{B}_s \neq 0$, so the absorbed transfer is nonzero. $\square$

Remark 20.15.2.32 (Sectorwise hypotheses in the canonical BNT application). In the canonical absorbed BNT presentation, ambient source zero correlation length supplies source zero correlation length for every absorbed representative by Theorem 20.15.2.31. The standing biCF one-letter span and the nonzero common weights give one-site injectivity by Theorem 20.13.10.

The positive commuting-bond realization begins at length two. It therefore proves positivity of $\rho^{(N)}(K_x)$ for all $N \geq 2$, while positivity of $\rho^{(1)}(K_x)$ remains the only separate sector-positivity hypothesis. Theorem 20.13.35 combines these facts and proves SAL at every original cut, without a normality hypothesis on the absorbed sectors. The printed GSNNCH-only proposition remains incomplete because it omits ambient source zero correlation length; an arbitrary GSNNCH presentation need not inherit this tensor-level property.

Theorem 20.15.2.33 (Strict normalization at positive length). Lean Lean Lean ✓ Stmt Let M generate matrix product density operators and have source zero correlation length. Then, for every $N \geq 1$, $\text{tr}(\sigma^{(N)}(M)) > 0$. In particular, after a common blocking has made the simultaneous BNT word span full at one letter, the absorbed BNT elements selected in Theorem 20.15.2.31 have a well-defined normalized state at every physical chain length. This is the biCF hypothesis imposed at the start of Case II in the source; its relation with finite physical blocking is recorded in [con26u]. This supplies the strict form of the sector normalization used in [CPGSV16, Appendix C.2, lines 1760–1780].

Proof. ✓ Proof Write the physical-trace transfer as $\mathcal{T}_M = \lambda E$, where $\lambda > 0$ and E is a nonzero idempotent. A nonzero idempotent on a finite-dimensional complex vector space has nonzero trace, while $E^N = E$ for $N \geq 1$. Hence

$$\text{tr}(\sigma^{(N)}(M)) = \text{tr}(\mathcal{T}_M^N) = \lambda^N \text{tr}(E) \neq 0.$$

Positivity of the density operator then makes this trace strictly positive. $\square$

Theorem 20.15.2.34 (Saturated area law for the absorbed BNT elements). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K saturate the area law and have zero correlation length, and let its horizontal BNT canonical form, up to block-diagonal gauge, have normal representatives A_s , common copy weights μ_s , and copy numbers n_s . Assume that the physical-trace transfer of every A_s is nonnilpotent. After a common blocking, assume explicitly that the simultaneous one-letter word tuples of the A_s span the direct sum of their full matrix algebras. Put $\mathcal{K}_s = \mu_s A_s$.

For every nonempty chain, define

$$p_s^{(N)} = \frac{n_s \operatorname{tr}(\sigma^{(N)}(\mathcal{K}_s))}{\operatorname{tr}(\sigma^{(N)}(K))}.$$

Then $p_s^{(N)} > 0$, and every nonempty prefix marginal has the same sector decomposition with these probabilities. Translation invariance identifies the other contiguous blocks of the same length. Consequently, every absorbed representative $\mathcal{K}_s$ saturates the area law.

The one-letter simultaneous span is the block-injective canonical-form hypothesis imposed at the start of Case II in [CPGSV16, line 1628]. Its relation with finite physical blocking is recorded in [con26u]. The entropy conclusion is the argument of [CPGSV16, Appendix C.2, lines 1748–1781].

Proof. [✓ Proof](#) The BNT projections carry every normalized marginal on mutually annihilating supports, so the preceding support form of entropy additivity gives

$$S_L(K) = H(p^{(N)}) + \sum_s p_s^{(N)} S_L(\mathcal{K}_s), \quad 1 \leq L \leq N.$$

The four marginals entering strong subadditivity have the same probabilities. Substitution in the equality for K cancels all four Shannon terms. Strong subadditivity gives the corresponding inequality in each sector, and strict positivity of every $p_s^{(N)}$ forces every such inequality to be an equality. Comparing the equalities for block lengths L and $L + 1$ proves saturation for $\mathcal{K}_s$. $\square$

Theorem 20.15.2.35 (Analytic properties of absorbed BNT representatives). [Lean](#) [✓ Stmt](#) Let K be an MPDO tensor in biCF, written up to a block-diagonal gauge as a weighted BNT direct sum whose BNT representatives have nonnilpotent physical-trace transfers. Suppose that the weights over each BNT label are copy-independent, $\mu_{s,q} = \mu_{s,q'}$, and that K saturates the area law and satisfies literal zero correlation length, $\mathcal{T}_K^2 = \mathcal{T}_K$. For every BNT label s , let $\mathcal{K}_s = \mu_s A_s$ be the absorbed BNT representative. Then $\mathcal{K}_s$ is one-site injective, generates positive semidefinite operators on every nonempty periodic chain, saturates the area law, and satisfies

$$\mathcal{T}_{\mathcal{K}_s}^2 = \mathcal{T}_{\mathcal{K}_s}.$$

Proof. [✓ Proof](#) The block-diagonal gauge identifies the MPV families of K and its weighted BNT direct sum. Saturation gives $\operatorname{tr}(\sigma^{(1)}(K)) > 0$. Since $\operatorname{tr}(\sigma^{(1)}(K)) = \operatorname{tr}(\mathcal{T}_K)$, one has $\mathcal{T}_K \neq 0$. Together with $\mathcal{T}_K^2 = \mathcal{T}_K$, this gives scale-invariant source zero correlation length with scalar 1. The one-letter BNT span gives injectivity, the sector compression gives positivity, and the sector entropy decomposition gives saturation of the area law. Theorem 20.15.2.30 supplies the final literal idempotence identity. $\square$

Definition 20.15.2.36 (Injective inverse map for a simple MPO tensor). [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) A simple MPO tensor K is called injective if its doubled-index MPS tensor is injective. For such a tensor, the chosen decomposition map of the doubled-index MPS furnishes a concrete inverse tensor K^{-1} together with right- and left-physical realization maps that turn virtual bond insertions back into local physical operators.

Theorem 20.15.2.37 (Inverse-map identities for an injective simple MPO tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For an injective simple MPO tensor, contracting K^{-1} with K recovers the matrix units on the virtual bond space. Likewise, every right virtual insertion admits a physical realization, this realization is multiplicative, and the analogous left-insertion statement also holds.

For an injective simple tensor, contracting K^{-1} with K through the physical index identifies the corresponding virtual legs. In components, this is the matrix-unit identity

$$\sum_p (K^{-1})_p^{\alpha,\beta} (K^p)_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \quad (20.35)$$

Its two Kronecker factors are the two bare cups:

This is the identity in [CPGSV16, Appendix C.2, lines 1376–1380].

Applying this inverse map to the doubled physical index and closing its virtual legs against a further copy of K recovers K :

This is the single-block identity in [CPGSV16, Appendix C.2, lines 1671–1676].

Definition 20.15.2.38 (Normalized fourth-site virtual tail). [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be an MPO tensor and set $Z_4 = \text{tr}[\rho^{(4)}(\mathcal{K})]$. Its normalized fourth-site virtual tail is

$$R_4 := Z_4^{-1} \mathcal{J}_{\mathcal{K}} = \text{tr}[\rho^{(4)}(\mathcal{K})]^{-1} \sum_{i_4=0}^{d-1} \mathcal{K}^{i_4 i_4}. \quad (20.36)$$

Thus the contraction of the fourth ket and bra indices is incorporated into the virtual closing matrix. This is the $N = 4$ specialization of the normalization convention and three-site marginal in [CPGSV16, lines 792–793 and 1343–1348].

Theorem 20.15.2.39 (Three-site marginal closed by the normalized fourth-site tail). [Lean](#) [✓ Stmt](#) Let $u = (i_1, i_2, i_3)$ and $v = (j_1, j_2, j_3)$ be physical words of length three. The corresponding entry of the first-three-site marginal of the normalized four-site periodic operator is

$$\sigma_3^{(4)}(\mathcal{K})_{u,v} = \text{tr}(\mathcal{K}^{u,v} R_4) = \text{tr}(\mathcal{K}^{i_1 j_1} \mathcal{K}^{i_2 j_2} \mathcal{K}^{i_3 j_3} R_4). \quad (20.37)$$

Proof. [✓ Proof](#) Expanding the partial trace over the fourth site and factoring each length-four word at its last letter gives

$$\sigma_3^{(4)}(\mathcal{K})_{u,v} = Z_4^{-1} \sum_{i_4=0}^{d-1} \text{tr}(\mathcal{K}^{u,v} \mathcal{K}^{i_4 i_4}).$$

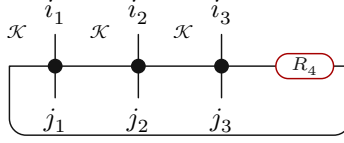


Figure 20.1: Tracing the fourth site of the normalized four-site periodic MPO leaves the physical legs of sites 1, 2, 3 open and replaces the fourth site by the virtual closing operator $R_4 = \text{tr}[\rho^{(4)}(\mathcal{K})]^{-1} \sum_{i_4} \mathcal{K}^{i_4 i_4}$; compare the normalization convention and three-site marginal in [CPGSV16, lines 792–793 and 1343–1348].

Linearity of the trace and (20.36) now yield

$$Z_4^{-1} \sum_{i_4=0}^{d-1} \text{tr}(\mathcal{K}^{u,v} \mathcal{K}^{i_4 i_4}) = \text{tr} \left(\mathcal{K}^{u,v} \left(Z_4^{-1} \sum_{i_4=0}^{d-1} \mathcal{K}^{i_4 i_4} \right) \right) = \text{tr}(\mathcal{K}^{u,v} R_4).$$

□

Theorem 20.15.2.40 (Non-vanishing of the normalized fourth-site tail). [Lean](#) [✓ Stmt](#) If $Z_4 = \text{tr}[\rho^{(4)}(\mathcal{K})] \neq 0$, then $R_4 \neq 0$.

Proof. [✓ Proof](#) If $R_4 = 0$, then the tail formula makes every entry of $\sigma_3^{(4)}(\mathcal{K})$ vanish, so $\text{tr}[\sigma_3^{(4)}(\mathcal{K})] = 0$. On the other hand, $Z_4 \neq 0$ implies $\text{tr}[\sigma_3^{(4)}(\mathcal{K})] = 1$, a contradiction. □

Theorem 20.15.2.41 (A nonzero entry of the normalized fourth-site tail). [Lean](#) [✓ Stmt](#) If $Z_4 \neq 0$, there exist virtual indices β and α such that $(R_4)_{\beta,\alpha} \neq 0$. These are the indices selected before the sector factorization in [CPGSV16, Appendix C.2, lines 1431–1437].

Proof. [✓ Proof](#) If every matrix entry of R_4 vanished, then R_4 would be the zero matrix, contrary to the preceding theorem. □

Definition 20.15.2.42 (Three-site inverse-map contraction). [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be an injective simple MPO tensor, and let $R \in M_D(\mathbb{C})$ be the virtual contraction of the sites following a three-site word. For virtual indices $\alpha_1, \beta_1, \alpha_3, \beta_3$, define

$$C_{\alpha_1, \beta_1, \alpha_3, \beta_3}(p_2; R) := \sum_{p_1, p_3} (\mathcal{K}^{-1})_{p_1}^{\alpha_1, \beta_1} (\mathcal{K}^{-1})_{p_3}^{\alpha_3, \beta_3} \text{tr}(\mathcal{K}^{p_1} \mathcal{K}^{p_2} \mathcal{K}^{p_3} R). \quad (20.38)$$

This is the contraction used in the proof of [CPGSV16, Appendix C.2, lines 1415–1438].

Theorem 20.15.2.43 (Evaluation of the three-site inverse-map contraction). [Lean](#) [✓ Stmt](#) For every middle physical index p_2 ,

$$C_{\alpha_1, \beta_1, \alpha_3, \beta_3}(p_2; R) = \mathcal{K}_{\beta_1, \alpha_3}^{p_2} R_{\beta_3, \alpha_1}. \quad (20.39)$$

Proof. [✓ Proof](#) Move the two finite sums through the trace. By the inverse-map identity

$$\sum_p (\mathcal{K}^{-1})_p^{\alpha, \beta} \mathcal{K}^p = E_{\alpha, \beta},$$

which is the matrix form of (20.35), the double sum collapses to

$$\text{tr}(E_{\alpha_1, \beta_1} \mathcal{K}^{p_2} E_{\alpha_3, \beta_3} R) = \mathcal{K}_{\beta_1, \alpha_3}^{p_2} R_{\beta_3, \alpha_1}.$$

□

$$\left(\alpha_1 \text{---} \overset{\mathcal{K}^{-1}}{\bullet} \text{---} \beta_1 \otimes \mathbb{1} \otimes \alpha_3 \text{---} \overset{\mathcal{K}^{-1}}{\bullet} \text{---} \beta_3 \right) \underbrace{\left(\beta_1 \text{---} \overset{p_1}{\bullet} \text{---} \overset{p_2}{\bullet} \text{---} \overset{p_3}{\bullet} \text{---} \overset{R}{\bullet} \right)}_{\substack{\mathcal{K} \quad \mathcal{K} \quad \mathcal{K}}} = \beta_1 \text{---} \overset{p_2}{\bullet} \text{---} \alpha_3 \otimes \beta_3 \text{---} \overset{R}{\bullet} \text{---} \alpha_1$$

Figure 20.2: Applying $\mathcal{K}^{-1}$ at the first and third sites collapses the three-site contraction to $\mathcal{K}_{\beta_1, \alpha_3}^{p_2} R_{\beta_3, \alpha_1}$, where $p_2 = (i_2, j_2)$ is the doubled physical index; compare [CPGSV16, Appendix C.2, lines 1415–1438].

Definition 20.15.2.44 (Three-site closure against a virtual tail). [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be an MPO tensor and let $R \in M_D(\mathbb{C})$. A tripartite operator ρ is the three-site closure of $\mathcal{K}$ against R if

$$\rho_{i_1 i_2 i_3, j_1 j_2 j_3} = \text{tr}(\mathcal{K}^{i_1 j_1} \mathcal{K}^{i_2 j_2} \mathcal{K}^{i_3 j_3} R). \quad (20.40)$$

This is the algebraic form of the three-site reduced operator in [CPGSV16, Appendix C.2, lines 1422–1433].

Definition 20.15.2.45 (Physical slice and Hayashi inverse factors). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Fix virtual indices β, α and set $\kappa_{\beta, \alpha}(i, j) = \mathcal{K}_{\beta, \alpha}^{ij}$. For a Hayashi sector k , define the two inverse factors

$$A_{\alpha_1, \beta_1}^{(k)}(l, l') := \sum_{i_1, j_1} (\mathcal{K}^{-1})_{(i_1, j_1)}^{\alpha_1, \beta_1} (\rho_{Ab_1}^{(k)})_{(i_1, l), (j_1, l')},$$

$$B_{\alpha_3, \beta_3}^{(k)}(r, r') := \sum_{i_3, j_3} (\mathcal{K}^{-1})_{(i_3, j_3)}^{\alpha_3, \beta_3} (\rho_{b_2 C}^{(k)})_{(r, i_3), (r', j_3)}.$$

These are the two factors in the sectorwise inverse-map calculation of [CPGSV16, Appendix C.2, lines 1407–1428].

Theorem 20.15.2.46 (Outer inverse-map contraction of a closure). [Lean](#) [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be injective and let ρ be its three-site closure against R . Applying the inverse tensor at the two outer sites collapses the closure to one entry of the middle physical slice and one entry of the tail:

$$\sum_{i_1, j_1, i_3, j_3} (\mathcal{K}^{-1})_{(i_1, j_1)}^{\alpha_1, \beta_1} (\mathcal{K}^{-1})_{(i_3, j_3)}^{\alpha_3, \beta_3} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} = \mathcal{K}_{\beta_1, \alpha_3}^{i_2 j_2} R_{\beta_3, \alpha_1}. \quad (20.41)$$

Consequently, for any matrix U on the middle site,

$$R_{\beta_3, \alpha_1} (U \kappa_{\beta_1, \alpha_3} U^\dagger)_{b, b'} = \sum_{i_1, j_1, i_3, j_3} (\mathcal{K}^{-1})_{(i_1, j_1)}^{\alpha_1, \beta_1} (\mathcal{K}^{-1})_{(i_3, j_3)}^{\alpha_3, \beta_3} \sum_{i_2, j_2} U_{b, i_2} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} \overline{U_{b', j_2}}. \quad (20.42)$$

Proof. [✓ Proof](#) Substituting the three-site closure relation (20.40) converts the left-hand side of (20.41) to $C_{\alpha_1, \beta_1, \alpha_3, \beta_3}((i_2, j_2); R)$. Theorem 20.15.2.43 then gives

$$C_{\alpha_1, \beta_1, \alpha_3, \beta_3}((i_2, j_2); R) = \mathcal{K}_{\beta_1, \alpha_3}^{i_2 j_2} R_{\beta_3, \alpha_1},$$

which is (20.41). The second identity follows by expanding the matrix product entrywise,

$$(U\kappa_{\beta_1,\alpha_3}U^\dagger)_{b,b'} = \sum_{i_2,j_2} U_{b,i_2}\kappa_{\beta_1,\alpha_3}(i_2,j_2)\overline{U_{b',j_2}},$$

multiplying by R_{β_3,α_1} , substituting

$$R_{\beta_3,\alpha_1}\kappa_{\beta_1,\alpha_3}(i_2,j_2) = \sum_{i_1,j_1,i_3,j_3} (\mathcal{K}^{-1})_{(i_1,j_1)}^{\alpha_1,\beta_1} (\mathcal{K}^{-1})_{(i_3,j_3)}^{\alpha_3,\beta_3} \rho_{i_1 i_2 i_3, j_1 j_2 j_3},$$

and reordering the finite sums. $\square$

Theorem 20.15.2.47 (Sectorwise inverse-map comparison). Lean ✓ Stmt Suppose that $\mathcal{K}$ is injective, ρ is its three-site closure against R , and a Hayashi decomposition of ρ has been chosen. Write

$$\tilde{\kappa}_{\beta_1,\alpha_3}^{(k)}((l,r),(l',r')) := [U_B \kappa_{\beta_1,\alpha_3} U_B^\dagger]_{(k,l,r),(k,l',r')}.$$

Then every diagonal sector satisfies

$$R_{\beta_3,\alpha_1} \tilde{\kappa}_{\beta_1,\alpha_3}^{(k)}((l,r),(l',r')) = p_k A_{\alpha_1,\beta_1}^{(k)}(l,l') B_{\alpha_3,\beta_3}^{(k)}(r,r'). \quad (20.43)$$

The probability p_k remains explicit because both sector density operators are normalized. No zero-correlation-length or non-vanishing hypothesis is used in this comparison.

For fixed k, α_1, β_3 , the identity (20.43) reads coefficientwise as follows.

$$R_{\beta_3,\alpha_1} \begin{array}{c} (l,r) \\ \beta_1 \text{---} \boxed{\tilde{\kappa}^{(k)}} \text{---} \alpha_3 \\ (l',r') \end{array} = p_k A_{\alpha_1,\beta_1}^{(k)}(l,l') B_{\alpha_3,\beta_3}^{(k)}(r,r').$$

Proof. ✓ Proof At fixed indices (k, l, r) and (k, l', r') , the Hayashi block equality is

$$\begin{aligned} & \sum_{i_2,j_2} [U_B]_{(k,l,r),i_2} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} \overline{[U_B]_{(k,l',r'),j_2}} \\ &= p_k (\rho_{Ab_1}^{(k)})_{(i_1,l),(j_1,l')} (\rho_{b_2C}^{(k)})_{(r,i_3),(r',j_3)}. \end{aligned}$$

For each i_2, j_2 , the preceding theorem gives

$$\begin{aligned} & \sum_{i_1,j_1,i_3,j_3} (\mathcal{K}^{-1})_{(i_1,j_1)}^{\alpha_1,\beta_1} (\mathcal{K}^{-1})_{(i_3,j_3)}^{\alpha_3,\beta_3} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} \\ &= \mathcal{K}_{\beta_1,\alpha_3}^{i_2 j_2} R_{\beta_3,\alpha_1}. \end{aligned}$$

Apply

$$\sum_{i_1,j_1,i_3,j_3} (\mathcal{K}^{-1})_{(i_1,j_1)}^{\alpha_1,\beta_1} (\mathcal{K}^{-1})_{(i_3,j_3)}^{\alpha_3,\beta_3} (-)$$

to the Hayashi block equality. Substituting the preceding display and interchanging the finite sums gives

$$\begin{aligned} & \sum_{i_1, j_1, i_3, j_3} (\mathcal{K}^{-1})_{(i_1, j_1)}^{\alpha_1, \beta_1} (\mathcal{K}^{-1})_{(i_3, j_3)}^{\alpha_3, \beta_3} \\ & \times \left(\sum_{i_2, j_2} [U_B]_{(k, l, r), i_2} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} \overline{[U_B]_{(k, l', r'), j_2}} \right) \\ & = \sum_{i_2, j_2} [U_B]_{(k, l, r), i_2} \mathcal{K}_{\beta_1, \alpha_3}^{i_2 j_2} R_{\beta_3, \alpha_1} \overline{[U_B]_{(k, l', r'), j_2}}. \end{aligned}$$

The right-hand side is

$$\begin{aligned} & R_{\beta_3, \alpha_1} \sum_{i_2, j_2} [U_B]_{(k, l, r), i_2} \kappa_{\beta_1, \alpha_3}(i_2, j_2) \overline{[U_B]_{(k, l', r'), j_2}} \\ & = R_{\beta_3, \alpha_1} \tilde{\kappa}_{\beta_1, \alpha_3}^{(k)}((l, r), (l', r')). \end{aligned}$$

The contraction of the right-hand side of the Hayashi block equality is $p_k A_{\alpha_1, \beta_1}^{(k)}(l, l') B_{\alpha_3, \beta_3}^{(k)}(r, r')$. Since the Hayashi block equality holds, equating the two contractions gives (20.43). $\square$

Theorem 20.15.2.48 (The four-site marginal is a three-site closure). [Lean](#) [✓ Stmt](#) *Carried onto the tripartite site index, the three-site marginal of the normalized four-site chain is the three-site closure of $\mathcal{K}$ against the normalized fourth-site tail R_4 .*

Proof. [✓ Proof](#) The entry formula (20.37) for $\sigma_3^{(4)}(\mathcal{K})$, carried onto the tripartite site index (i_1, i_2, i_3) , gives

$$\rho_{i_1 i_2 i_3, j_1 j_2 j_3} = \text{tr}(\mathcal{K}^{i_1 j_1} \mathcal{K}^{i_2 j_2} \mathcal{K}^{i_3 j_3} R_4),$$

which is the three-site closure condition (20.40) against R_4 . $\square$

Theorem 20.15.2.49 (Vanishing of the off-diagonal Hayashi sectors). [Lean](#) [✓ Stmt](#) *In the setting of Theorem 20.15.2.47, for all $k \neq k'$, every off-diagonal sector satisfies*

$$R_{\beta_3, \alpha_1} [U_B \kappa_{\beta_1, \alpha_3} U_B^\dagger]_{(k, l, r), (k', l', r')} = 0. \quad (20.44)$$

Together with the diagonal comparison this gives the direct-sum structure of the factorized tensor: the direct sum over sectors is inherited from the splitting of the middle site.

Proof. [✓ Proof](#) For b in sector k and b' in sector k' with $k \neq k'$, the Hayashi block equality at off-diagonal indices gives

$$\sum_{i_2, j_2} (U_B)_{b, i_2} \rho_{i_1 i_2 i_3, j_1 j_2 j_3} \overline{(U_B)_{b', j_2}} = 0.$$

Substituting this vanishing into the expansion of Theorem 20.15.2.46 gives (20.44). $\square$

Definition 20.15.2.50 (Sector tensors). [Lean](#) [Lean](#) [✓ Stmt](#) Fix outer virtual indices α_1, β_3 and a Hayashi sector k . The sector tensors are

$$\begin{aligned} (l_k)_{\beta_1} &:= R_{\beta_3, \alpha_1}^{-1} p_k A_{\alpha_1, \beta_1}^{(k)}, \\ (r_k)_{\alpha_3} &:= B_{\alpha_3, \beta_3}^{(k)}. \end{aligned}$$

These are the tensors of the factorization displayed in [CPGSV16, Appendix C.2, lines 1435–1448].

Lemma 20.15.2.51 (Zero-weight left sector tensor). [Lean](#) [Lean](#) [✓ Stmt](#) If $p_k = 0$, then $(l_k)_\beta = 0$ for every virtual index β . Consequently, $\eta_{h,k} = 0$ for every sector h .

Proof. [✓ Proof](#) The scalar weight p_k is a factor of every matrix entry of $(l_k)_\beta$. Each neighboring operator entering sector k contains $(l_k)_\beta$ as its target factor. $\square$

Theorem 20.15.2.52 (Sector factorization of an injective simple tensor). [Lean](#) [✓ Stmt](#) Suppose that $\mathcal{K}$ is injective, ρ is its three-site closure against R , a Hayashi decomposition of ρ has been chosen, and the outer indices satisfy $R_{\beta_3, \alpha_1} \neq 0$. Then for all virtual indices β_1, α_3 ,

$$U_B \kappa_{\beta_1, \alpha_3} U_B^\dagger = \bigoplus_k (l_k)_{\beta_1} \otimes (r_k)_{\alpha_3}. \quad (20.45)$$

The direct sum refers to the physical indices and is inherited from the splitting of the middle site. This is the factorization displayed at [CPGSV16, Appendix C.2, lines 1435–1448].

In the middle-site basis selected by U_B , the same factorization (20.45) is displayed as

Proof. [✓ Proof](#) On a diagonal sector, the sectorwise comparison gives $R_{\beta_3, \alpha_1} \tilde{\kappa}_1^{(k)} = p_k A^{(k)} B^{(k)}$, and dividing by the nonzero entry R_{β_3, α_1} produces $(l_k)_{\beta_1} \otimes (r_k)_{\alpha_3}$. On an off-diagonal sector, the product with R_{β_3, α_1} vanishes, and since $R_{\beta_3, \alpha_1} \neq 0$ the sector itself vanishes. $\square$

Theorem 20.15.2.53 (Existence of the sector tensors). [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be injective with nonzero four-site trace, let ρ be a three-site closure of $\mathcal{K}$ against the normalized fourth-site tail R_4 , and choose a Hayashi decomposition of ρ . Then there exist sector tensors l_k and r_k such that, for all virtual indices β_1, α_3 ,

$$U_B \kappa_{\beta_1, \alpha_3} U_B^\dagger = \bigoplus_k (l_k)_{\beta_1} \otimes (r_k)_{\alpha_3}.$$

The neighboring operators $\eta_{k,h}$ are assembled from these tensors in Definition 20.15.3.1; the remaining steps of [CPGSV16, Appendix C.2, Lemma C.4] prove the all-length identity and primitivity.

Proof. [✓ Proof](#) A nonzero four-site trace supplies outer indices with $(R_4)_{\beta_3, \alpha_1} \neq 0$, and the sector factorization applies. $\square$

20.15.3 Neighboring operators and commuting bond products

Definition 20.15.3.1 (Neighboring operators). [Lean](#) [✓ Stmt](#) Fix outer virtual indices α_1, β_3 with the chosen Hayashi decomposition, and let l_k, r_k be the sector tensors of Definition 20.15.2.50. For each pair of sectors (k, h) , the neighboring operator contracts the right sector tensor of one site with the left sector tensor of the following site over the virtual bond they share:

$$\eta_{k,h} = \sum_\gamma (r_k)_\gamma \otimes (l_h)_\gamma.$$

Definition 20.15.3.6 (Virtual-matrix transport of a physical-sector factorization). [Lean](#) [✓ Stmt](#) Let K and L have virtual dimensions D and E , respectively. Suppose that matrices $X \in M_{E \times D}(\mathbb{C})$ and $Y \in M_{D \times E}(\mathbb{C})$ satisfy the following identity for every physical index i :

$$L^i = XK^iY.$$

Every physical-sector factorization of K induces one of L with the same physical decomposition. Its virtual tensor families are

$$(l_k^L)_\beta = \sum_\gamma X_{\beta,\gamma} (l_k^K)_\gamma, \quad (r_k^L)_\alpha = \sum_\delta Y_{\delta,\alpha} (r_k^K)_\delta.$$

This construction abstracts the virtual corner in [CPGSV16, Section 2.3, lines 195–219] and the physical-sector factorization in [CPGSV16, Appendix C.2, lines 1381–1439].

Theorem 20.15.3.7 (Neighboring operators under virtual-matrix transport). [Lean](#) [✓ Stmt](#) *The neighboring operators of the transported factorization satisfy*

$$\eta_{k,h}^L = \eta_{k,h}^K(YX).$$

This identity combines the virtual corner in [CPGSV16, Section 2.3, lines 195–219] with the neighboring contraction in [CPGSV16, Appendix C.2, lines 1441–1450].

Proof. [✓ Proof](#) Replace each left factor by $\sum_\gamma X_{\beta,\gamma} (l_k^K)_\gamma$ and each right factor by $\sum_\delta Y_{\delta,\alpha} (r_k^K)_\delta$. Contracting the common virtual index gives

$$\sum_\beta Y_{\delta,\beta} X_{\beta,\gamma} = (YX)_{\delta,\gamma},$$

which proves the neighboring-operator identity. $\square$

Definition 20.15.3.8 (Virtual compression of a physical-sector factorization). [Lean](#) [✓ Stmt](#) Let K and L have virtual dimensions D and E , respectively, and suppose that a rectangular matrix $V \in M_{D \times E}(\mathbb{C})$ satisfies the following identity for every physical index i :

$$L^i = V^\dagger K^i V.$$

Every physical-sector factorization of K induces one of L with the same physical decomposition. This construction combines the virtual corner in [CPGSV16, Section 2.3, lines 195–219] with the physical-sector factorization in [CPGSV16, Appendix C.2, lines 1381–1439].

Theorem 20.15.3.9 (Range projection of a matrix-block inclusion). [Lean](#) [✓ Stmt](#) *For the canonical inclusion V_s of a matrix block, $V_s V_s^\dagger$ is the projection onto that block.*

Proof. [✓ Proof](#) Compute the entries of the inclusion and its conjugate transpose. $\square$

Theorem 20.15.3.10 (Flattened sector as an isometric virtual corner). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $A = \bigoplus_t \mu_t A_t$ be a weighted block-diagonal tensor in flattened virtual coordinates. For every block s , the canonical inclusion V_s of that block satisfies*

$$\begin{aligned} V_s^\dagger V_s &= 1, \\ V_s V_s^\dagger &= P_s, \\ V_s^\dagger A^i V_s &= \mu_s A_s^i. \end{aligned}$$

Here P_s is the corresponding block projection, transported to the flattened coordinates. Moreover,

$$\begin{aligned} A^i V_s &= V_s (\mu_s A_s^i), \\ V_s^\dagger A^i &= (\mu_s A_s^i) V_s^\dagger. \end{aligned}$$

This is the exact blockwise virtual-corner identity from [CPGSV16, Section 2.3, equation II_Aiplusk1, lines 195–219]. Appendix C.2, Case I, lines 1374–1450 and 1571–1619 is the downstream application context; no positivity after insertion of P_s is asserted here.

Proof. ✓ Proof Let E_s be the canonical inclusion of the s -th summand in the dependent direct sum, let Q_s be the corresponding unflattened block projection, and let e flatten the direct-sum coordinates. Thus V_s is the reindexing of E_s , while P_s is the reindexing of Q_s . Multiplicativity of reindexing transports

$$\begin{aligned} E_s^\dagger E_s &= 1, \\ E_s E_s^\dagger &= Q_s \end{aligned}$$

to the two corresponding identities for V_s . If $B_t^i = \mu_t A_t^i$, block diagonality gives

$$\begin{aligned} A^i V_s &= V_s B_s^i, \\ V_s^\dagger A^i &= B_s^i V_s^\dagger. \end{aligned}$$

Therefore

$$V_s^\dagger A^i V_s = V_s^\dagger V_s B_s^i = B_s^i = \mu_s A_s^i.$$

□

Theorem 20.15.3.11 (Neighboring operators under virtual compression). Lean ✓ Stmt The neighboring operators of the factorization obtained by virtual compression satisfy

$$\eta_{k,h}^L = \eta_{k,h}^K (VV^\dagger).$$

No positivity of the right-hand side is asserted. This identity combines the virtual corner in [CPGSV16, Section 2.3, lines 195–219] with the neighboring contraction in [CPGSV16, Appendix C.2, lines 1441–1450].

Proof. ✓ Proof Apply the virtual-matrix transport identity with $X = V^\dagger$ and $Y = V$. □

Theorem 20.15.3.12 (Virtual-gauge invariance of physical-sector factorizations). Lean Lean ✓ Stmt Suppose that two MPO tensors of the same virtual dimension differ by an invertible virtual gauge X . The induced factorization is precisely the virtual-matrix transport by X and X^{-1} . Thus every physical-sector factorization of the first tensor induces one of the second tensor with the same physical decomposition and the same neighboring operators. In particular, positive semidefiniteness of all neighboring operators is invariant under the gauge.

Proof. ✓ Proof Specialize the virtual-matrix neighboring identity to a gauge and its inverse. Their product is the identity matrix, and contraction through the identity matrix is the ordinary neighboring operator. □

Definition 20.15.3.13 (One-site matrix spaces of sector sets). [Lean](#) [Lean](#) [✓ Stmt](#) For a sector k and physical matrix indices x, y in its summand, let $A_{k;x,y} \in M_D(\mathbb{C})$ be the corresponding entry of the sector-coordinate tensor. For a set S of sectors, define

$$\mathcal{A}_S = \text{span}\{A_{k;x,y} : k \in S\}.$$

These are the matrix spaces associated with the physical projections in [CPGSV16, Appendix C.2, lines 1463–1467].

Theorem 20.15.3.14 (Directed-cut annihilation of one-site matrix spaces). [Lean](#) [Lean](#) [✓ Stmt](#) Let S and C be sets of sectors. If $\eta_{k,h} = 0$ for every $k \in S$ and $h \in C$, then $XY = 0$ for every $X \in \mathcal{A}_S$ and $Y \in \mathcal{A}_C$. No conclusion in the reverse product order is asserted. This is the directed local-orthogonality implication used in [CPGSV16, Appendix C.2, lines 1463–1470].

Proof. [✓ Proof](#) Expand two generating matrices in sector coordinates. Their product has entries

$$\sum_{\gamma} (l_k)_{\beta} (r_k)_{\gamma} (l_h)_{\gamma} (r_h)_{\alpha}.$$

The middle contraction is a matrix entry of $\eta_{k,h}$ and therefore vanishes. Bilinearity extends the conclusion from the generators to the two linear spans. $\square$

Definition 20.15.3.15 (Vertex rephasing of the sector tensors). [Lean](#) [✓ Stmt](#) Let $z_k \in \mathbb{C}$ satisfy $|z_k| = 1$ for every sector. Replacing

$$\begin{aligned} (l_k)_a &\mapsto z_k (l_k)_a, \\ (r_k)_a &\mapsto z_k^{-1} (r_k)_a \end{aligned}$$

leaves every physical-slice factorization unchanged.

Theorem 20.15.3.16 (Rephasing identities for the sector data). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Vertex rephasing transforms the neighboring operators by $\eta_{k,h} \mapsto z_k^{-1} z_h \eta_{k,h}$. It leaves every sector virtual matrix unchanged, preserves the nonzero neighboring support, and leaves every complete cyclic product of neighboring operators unchanged. These are the rephasing identities used in [CPGSV16, Appendix C.2, lines 1434–1450].

Definition 20.15.3.17 (Vanishing right tensors on inactive sectors). [Lean](#) [Lean](#) [✓ Stmt](#) Let S be a set of active sectors in a physical-sector factorization, and suppose that $(l_k)_{\beta} = 0$ for every $k \notin S$ and every β . Retain every physical summand and define

$$\begin{aligned} \hat{l}_k &= l_k, \\ \hat{r}_k &= \begin{cases} r_k, & k \in S, \\ 0, & k \notin S. \end{cases} \end{aligned}$$

Then $\hat{l}_k \otimes \hat{r}_k = l_k \otimes r_k$ for every k , so the physical-slice factorization is unchanged. The neighboring operators become

$$\hat{\eta}_{k,h} = \begin{cases} \eta_{k,h}, & k \in S, \\ 0, & k \notin S. \end{cases}$$

This is the reparameterization freedom in the factorization and neighboring contraction of [CPGSV16, Appendix C.2, lines 1434–1445].

Definition 20.15.3.18 (Inverse-map physical-sector factorization). [Lean](#) [✓ Stmt](#) Let $\mathcal{K}$ be injective, let ρ be its three-site closure against a virtual tail R , and choose a Hayashi decomposition of ρ . Fix outer indices such that $R_{\beta_3, \alpha_1} \neq 0$. The inverse-map tensors l_k, r_k , together with the Hayashi physical decomposition and unitary, determine a physical-sector factorization of $\mathcal{K}$.

The left and right factor spaces are nonzero because the corresponding Hayashi density matrices have trace one; their nonzero dimensions are conclusions rather than hypotheses. No positivity of the neighboring operators is asserted. This gives the factorization in [CPGSV16, Appendix C.2, equations `AppUkU=r1` and `formK`, lines 1383–1387 and 1434–1439].

Theorem 20.15.3.19 (Identification of the neighboring operators). [Lean](#) [✓ Stmt](#) For all sectors k, h , the neighboring contraction of the resulting physical-sector factorization is the inverse-map operator

$$\eta_{k,h} = \sum_{\gamma} (r_k)_{\gamma} \otimes (l_h)_{\gamma}.$$

This is equation `etar1` in [CPGSV16, Appendix C.2, lines 1441–1445].

Definition 20.15.3.20 (Zero-weight inverse-map reparameterization). [Lean](#) [✓ Stmt](#) In the inverse-map factorization, retain every physical sector and its factor spaces, and set

$$\begin{aligned} \hat{l}_k &= l_k, \\ \hat{r}_k &= \begin{cases} r_k, & p_k \neq 0, \\ 0, & p_k = 0. \end{cases} \end{aligned}$$

The physical-slice factorization is unchanged. This treats the zero-weight freedom left by the Hayashi decomposition used in [CPGSV16, Appendix C.2, lines 1415–1445].

Theorem 20.15.3.21 (Neighboring operators after zero-weight reparameterization). [Lean](#) [Lean](#) [✓ Stmt](#) The neighboring operators of the reparameterized factorization are

$$\hat{\eta}_{k,h} = \begin{cases} \eta_{k,h}, & p_k \neq 0, \\ 0, & p_k = 0. \end{cases}$$

Thus a zero-weight sector has no outgoing support edge. Since $p_h = 0$ also gives $l_h = 0$, it has no incoming support edge either.

Proof. [✓ Proof](#) The neighboring contraction uses r_k at its source and l_h at its target. The first assertion follows from the definition of $\hat{r}_k$, and the last assertion follows from Lemma 20.15.2.51. $\square$

Corollary 20.15.3.22 (SAL gives a raw physical-sector factorization). [Lean](#) [✓ Stmt](#) Every injective MPO tensor satisfying SAL admits a physical-sector factorization. This is only the raw algebraic factorization from Lemma C.4 and equations `formK` and `etar1`; coherent positivity of the neighboring operators, recurrence, and primitivity are not part of this conclusion.

Definition 20.15.3.23 (Finite-chain coordinates of a physical-sector factorization). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $N \geq 1$. For a cyclic sector configuration $k = (k_0, \dots, k_{N-1})$, write

$$I_k = \prod_{n=0}^{N-1} (B_{k_n}^L \times B_{k_n}^R).$$

Applying the one-site sector decomposition at every site gives a bijection

$$\Phi_N : \{0, \dots, d-1\}^N \longrightarrow \prod_{k_0, \dots, k_{N-1}} I_k.$$

On I_k , define the cyclic neighboring product by

$$\Omega_k(x, y) = \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}((x_n^R, x_{n+1}^L), (y_n^R, y_{n+1}^L)),$$

where site labels are read modulo N .

Theorem 20.15.3.24 (Finite-chain decomposition of a physical-sector factorization). [Lean](#) [Lean](#)

[✓ Stmt](#) Suppose that the physical-sector factorization is given. For every $N \geq 1$, the physically transformed MPO is block diagonal in the cyclic sector configurations, and

$$\text{reindex}_{\Phi_N}(M_N(\tilde{\mathcal{K}})) = \bigoplus_{k_0, \dots, k_{N-1}} \Omega_k.$$

This is the all-length algebraic decomposition in [\[CPGSV16, Appendix C.2, lines 1435–1450\]](#), conditional here on the displayed physical-sector factorization. It does not assert positivity of the individual neighboring operators.

Proof. [✓ Proof](#) Fix a sector configuration k and entries $x, y \in I_k$. Expanding the closed horizontal contraction gives

$$\sum_g \prod_n (l_{k_n})_{g_n} (x_n^L, y_n^L) (r_{k_n})_{g_{n+1}} (x_n^R, y_n^R).$$

Cyclically shifting the summation variables and collecting the factors with the same horizontal index turns this expression into $\Omega_k(x, y)$. If the row and column sector configurations differ, one transformed physical slice lies between distinct summands and vanishes. Thus all off-diagonal sector blocks vanish. $\square$

Corollary 20.15.3.25 (SAL gives positive cyclic sector products). [Lean](#) [✓ Stmt](#) Every injective MPO tensor satisfying SAL admits a physical-sector factorization such that, for every $N \geq 1$ and every cyclic sector configuration $k = (k_0, \dots, k_{N-1})$ with $k_N = k_0$,

$$\bigotimes_{n=0}^{N-1} \eta_{k_n, k_{n+1}} \geq 0.$$

This is the projected-chain inequality in [\[CPGSV16, Appendix C.2, Lemma C.4, lines 1446–1450\]](#). It does not assert positivity of the individual neighboring operators.

Proof. [✓ Proof](#) Choose the raw physical-sector factorization supplied by SAL. The physical basis change sends the positive N -site MPO to a positive congruence. Each cyclic neighboring product is a diagonal sector compression of this congruence and is therefore positive semidefinite. $\square$

Definition 20.15.3.26 (Boundary contraction against a virtual matrix). [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) For an arbitrary virtual matrix X , define the outer boundary operator on $B_k^L \otimes B_h^R$ by

$$B_{k,h}(X) = \sum_{a,b} X_{b,a} (l_k)_a \otimes (r_h)_b.$$

The map $X \mapsto B_{k,h}(X)$ is complex-linear. No positivity of X or $B_{k,h}(X)$ is assumed.

Definition 20.15.3.27 (Regrouped two-site sector closure). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) First pass to the physical basis selected by U , so that each slice is written in the sector coordinates of Definition 20.15.3.3. Form the two-site closure of this sector-coordinate tensor, restrict it to the physical sectors (k, h) , and reindex its physical factors by

$$((L_k, R_k), (L_h, R_h)) \mapsto ((L_k, R_h), (R_k, L_h)).$$

Denote the resulting matrix by $\mathcal{K}_{2;k,h}(X)$.

Theorem 20.15.3.28 (Two-site closure factorization). [Lean](#) [✓ Stmt](#) For every virtual matrix X and every sector pair (k, h) ,

$$\mathcal{K}_{2;k,h}(X) = B_{k,h}(X) \otimes \eta_{k,h}.$$

Proof. [✓ Proof](#) At the regrouped matrix entry indexed by $((\lambda_k, \rho_h), (\rho_k, \lambda_h))$ and $((\lambda'_k, \rho'_h), (\rho'_k, \lambda'_h))$, the virtual trace expands as the scalar

$$\sum_{a,b,c} X_{b,a} [(l_k)_a]_{\lambda_k, \lambda'_k} [(r_k)_c]_{\rho_k, \rho'_k} [(l_h)_c]_{\lambda_h, \lambda'_h} [(r_h)_b]_{\rho_h, \rho'_h}.$$

Commuting scalar factors and collecting the sums over (a, b) and c gives respectively $B_{k,h}(X)$ and $\eta_{k,h}$. $\square$

Definition 20.15.3.29 (Regrouped three-site sector closure). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) First pass to the physical basis selected by U and form the three-site closure of the resulting sector-coordinate tensor. Restrict this closure to the sectors (k, l, h) and reindex its physical factors by

$$((L_k, R_k), ((L_l, R_l), (L_h, R_h))) \mapsto ((L_k, R_h), ((R_k, L_l), (R_l, L_h))).$$

Denote the resulting matrix by $\mathcal{K}_{3;k,l,h}(X)$.

Theorem 20.15.3.30 (Three-site closure factorization). [Lean](#) [✓ Stmt](#) For every virtual matrix X and fixed sectors (k, l, h) ,

$$\mathcal{K}_{3;k,l,h}(X) = B_{k,h}(X) \otimes (\eta_{k,l} \otimes \eta_{l,h}).$$

Proof. [✓ Proof](#) At the regrouped matrix entry indexed by $((\lambda_k, \rho_h), ((\rho_k, \lambda_l), (\rho_l, \lambda_h)))$ and its primed counterpart, the left-hand side is the scalar

$$\sum_{a,b,c,e} X_{b,a} [(l_k)_a]_{\lambda_k, \lambda'_k} [(r_k)_c]_{\rho_k, \rho'_k} [(l_l)_c]_{\lambda_l, \lambda'_l} [(r_l)_e]_{\rho_l, \rho'_l} [(l_h)_e]_{\lambda_h, \lambda'_h} [(r_h)_b]_{\rho_h, \rho'_h}.$$

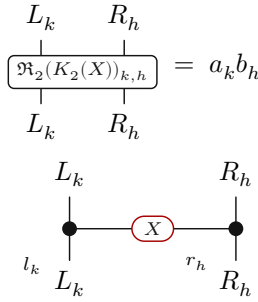
Reordering the four finite sums separates the (a, b) boundary contraction from the neighboring contractions indexed by c and e , which gives the three factors on the right-hand side. $\square$

Definition 20.15.3.31 (Three-site neighboring operator). [Lean](#) [✓ Stmt](#) For every outer-sector pair (k, h) , define

$$\Omega_{k,h} = \bigoplus_l (\eta_{k,l} \otimes \eta_{l,h}).$$

This is the direct sum of the two neighboring contractions appearing in the three-site closure factorization [CPGSV16, Appendix C.2, lines 1510–1516].

$$\mathfrak{R}_2(\mathcal{K}_2(X))_{k,h} = a_k b_h B_{k,h}(X).$$



$$\mathfrak{R}_3(\mathcal{K}_3(X))_{k,l,h} = B_{k,h}(X) \otimes \eta_{k,l} \otimes \eta_{l,h}.$$

Figure 20.3: First each slice is conjugated by U and expressed in the corresponding sector coordinates. After restriction to fixed sectors and the displayed regroupings, the arbitrary- X two- and three-site closures separate into the boundary factor and respectively one or two neighboring contractions. The displayed identities do not assert positivity, preparation, recovery, or the full conclusion of Proposition C.7. They follow the contractions in [CPGSV16, lines 638–655 and Appendix C.2, lines 1435–1448].

Definition 20.15.3.32 (Neighboring-operator trace factorization). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that every neighboring operator is positive semidefinite and that there are real numbers a_k, b_h such that

$$\begin{aligned} \text{tr}(\eta_{k,h}) &= a_k b_h, \\ \sum_k a_k b_k &= 1. \end{aligned}$$

These are precisely the remaining hypotheses used in the construction of Proposition C.7 after the physical-sector factorization has been fixed [CPGSV16, Appendix C.2, lines 1389–1403].

Theorem 20.15.3.33 (Unitary-between physical-coordinate invariance of neighboring trace factorizations). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $V : \mathbb{C}^d \rightarrow \mathbb{C}^e$ satisfy $V^\dagger V = \mathbb{1}_d$ and $VV^\dagger = \mathbb{1}_e$, and define $K_{\beta,\alpha}^V = VK_{\beta,\alpha}V^\dagger$. Then K admits a physical-sector factorization with

$$\begin{aligned} \eta_{k,h} &\geq 0, \\ \text{tr}(\eta_{k,h}) &= a_k b_h, \\ \sum_k a_k b_k &= 1. \end{aligned}$$

if and only if K^V admits one. The same sector spaces, tensors l_k, r_k , neighboring operators $\eta_{k,h}$, and numbers a_k, b_h may be used on both sides. This invariance is project-derived; the data being transported are those of [CPGSV16, Theorem 4.9(iv) and Appendix C.2, lines 1381–1450], but the equivalence itself is not stated there.

Proof. ✓ Proof The two one-sided unitary identities give $d \leq e$ and $e \leq d$, respectively, so $d = e$. If the physical isometry for K is U , use $UV^\dagger$ for K^V ; then $(UV^\dagger)K_{\beta,\alpha}^V(UV^\dagger)^\dagger = UK_{\beta,\alpha}U^\dagger$. Conversely, if the isometry for K^V is U' , use $U'V$ for K . In both directions the tensors l_k, r_k are unchanged, so $\eta_{k,h} = \sum_\gamma (r_k)_\gamma \otimes (l_h)_\gamma$ is unchanged. The positivity, trace, and normalization identities therefore remain the same identities. $\square$

Theorem 20.15.3.34 (Gauge and support-coordinate independence of neighboring trace factorizations). Lean Lean Lean Lean Lean Lean ✓ Stmt Suppose that two MPO tensors of the same physical and virtual dimensions differ by an invertible virtual gauge. One admits a physical-sector factorization with neighboring data $\eta_{k,h}$, a_k , and b_h if and only if the other does. The same neighboring operators and the same numbers a_k, b_h may be used on both sides.

Let $V_F : \mathbb{C}^{d_F} \rightarrow \mathbb{C}^d$ and $V_G : \mathbb{C}^{d_G} \rightarrow \mathbb{C}^d$ be two isometric inclusions with the same range projection P :

$$\begin{aligned} V_F^\dagger V_F &= \mathbb{1}_{d_F}, & V_G^\dagger V_G &= \mathbb{1}_{d_G}, \\ V_F V_F^\dagger &= P, & V_G V_G^\dagger &= P. \end{aligned} \tag{20.46}$$

Set $Q = V_F^\dagger V_G$. Then

$$\begin{aligned} Q^\dagger Q &= \mathbb{1}_{d_G}, & Q Q^\dagger &= \mathbb{1}_{d_F}, \\ Q(V_G^\dagger K V_G)Q^\dagger &= V_F^\dagger K V_F. \end{aligned} \tag{20.47}$$

Consequently, the restrictions of K along V_F and V_G admit neighboring trace factorizations simultaneously.

Both equivalences are project-derived and are not stated in [CPGSV16]. The gauge transformation is the one discussed in Section 2.3, lines 187–195; the support-restriction setting is that of Appendix C.2, lines 1680–1691 and 1733–1770.

Proof. ✓ Proof For the virtual gauge $L^i = X K^i X^{-1}$, write $\eta_{k,h}^K(M)$ for the neighboring contraction with M inserted between the transported virtual factors. Their composition is $X^{-1}X$, and hence

$$\eta_{k,h}^L = \eta_{k,h}^K(X^{-1}X) = \eta_{k,h}^K(\mathbb{1}) = \eta_{k,h}^K.$$

The families a_k and b_h are retained literally, together with their positivity, trace, and normalization identities. The reverse implication uses the inverse gauge.

For the support coordinates, (20.46) gives

$$\begin{aligned} Q^\dagger Q &= V_G^\dagger (V_F V_F^\dagger) V_G = V_G^\dagger P V_G = \mathbb{1}_{d_G}, \\ Q Q^\dagger &= V_F^\dagger (V_G V_G^\dagger) V_F = V_F^\dagger P V_F = \mathbb{1}_{d_F}, \\ Q V_G^\dagger &= V_F^\dagger (V_G V_G^\dagger) = V_F^\dagger P = V_F^\dagger. \end{aligned}$$

The last identity gives (20.47) by composition of physical coordinate changes. The unitary-between invariance above, with the resulting equivalence reversed, proves the claim. $\square$

Theorem 20.15.3.35 (Trace of the three-site neighboring operator). [Lean](#) [Lean](#) [✓ Stmt](#) Assume the neighboring-operator trace factorization of Definition 20.15.3.32. Then, for every outer-sector pair (k, h) ,

$$\mathrm{tr}(\Omega_{k,h}) = \sum_l \mathrm{tr}(\eta_{k,l}) \mathrm{tr}(\eta_{l,h}) = a_k b_h.$$

This is the normalization used in the coarse-graining channel of [CPGSV16, Appendix C.2], lines 1547–1555.

Proof. [✓ Proof](#) The trace of a direct sum is the sum of the traces of its blocks, and the trace of a tensor product is the product of the traces. Hence

$$\mathrm{tr}(\Omega_{k,h}) = \sum_l (a_k b_l)(a_l b_h) = a_k b_h \sum_l a_l b_l = a_k b_h.$$

□

Lemma 20.15.3.36 (Restricting finite sums to nonzero weights). [Lean](#) [✓ Stmt](#) Let I be finite, let p_i be weights in a type with zero, and let f_i take values in a commutative additive monoid. If $f_i = 0$ whenever $p_i = 0$, then

$$\sum_{i \in I} f_i = \sum_{\substack{i \in I \\ p_i \neq 0}} f_i.$$

Proof. [✓ Proof](#) Split the full sum according to whether p_i vanishes. Every term in the zero-weight part vanishes by hypothesis. □

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “Cyclic trace identity for rectangular powers”].

Theorem 20.15.3.37 (Unit closure traces from a neighboring trace factorization). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M be an MPO tensor admitting a physical-sector factorization whose neighboring operators satisfy the trace factorization of Definition 20.15.3.32. Closing the physical legs of the factorized slices writes the physical-trace transfer as a rectangular product

$$\mathcal{T}_M = LQ, \quad L_{\beta,k} = \mathrm{tr}((l_k)_\beta), \quad Q_{k,\alpha} = \mathrm{tr}((r_k)_\alpha),$$

whose opposite product is the rank-one coefficient matrix

$$(QL)_{k,h} = \mathrm{tr}(\eta_{k,h}) = a_k b_h.$$

Consequently every positive power of the physical-trace transfer and every nonempty closed chain has unit trace:

$$\mathrm{tr}(\mathcal{T}_M^N) = 1, \quad \mathrm{tr} \rho^{(N)}(M) = 1, \quad N \geq 1.$$

In particular, a tensor whose closed chains do not all have unit trace admits no physical-sector factorization with positive neighboring operators and normalized rank-one trace coefficients. The normalization $\mathrm{tr}(T) = 1$ invoked in [CPGSV16, Appendix C.2, line 1498] is therefore a constraint on the tensor rather than a convention, and the ambient normalization at [CPGSV16, Appendix C.2, lines 1755–1759] constrains only the weighted sum over the representatives of a basis of normal tensors, not each representative separately.

Proof. ✓ Proof Closing the physical legs of each factorized slice and using unitary invariance of the trace expresses each entry of the physical-trace transfer as $\sum_k \text{tr}((l_k)_\beta) \text{tr}((r_k)_\alpha)$, which is the rectangular product LQ . The entries of the opposite product are the contractions

$$(QL)_{k,h} = \sum_\alpha \text{tr}((r_k)_\alpha) \text{tr}((l_h)_\alpha) = \text{tr}(\eta_{k,h}) = a_k b_h.$$

Cyclic invariance of the trace gives $\text{tr}((LQ)^N) = \text{tr}((QL)^N)$ for every $N \geq 1$, and every positive power of the rank-one matrix $(a_k b_h)_{k,h}$ with $\sum_k a_k b_k = 1$ equals the matrix itself, of trace one. The closed chain of length N has trace $\text{tr}(\mathcal{T}_M^N)$. □

Theorem 20.15.3.38 (Partial traces of the sector closures). Lean Lean Lean Lean ✓ Stmt
For every virtual matrix X ,

$$\text{tr}_{R_k \otimes L_h}(\mathcal{K}_{2;k,h}(X)) = \text{tr}(\eta_{k,h}) B_{k,h}(X) = a_k b_h B_{k,h}(X).$$

Similarly,

$$\text{tr}_{(R_k \otimes L_l) \otimes (R_l \otimes L_h)}(\mathcal{K}_{3;k,l,h}(X)) = \text{tr}(\eta_{k,l}) \text{tr}(\eta_{l,h}) B_{k,h}(X),$$

and summing the scalar coefficient over l gives $a_k b_h$.

Proof. ✓ Proof The partial trace of $A \otimes B$ is $\text{tr}(B)A$. The two-site identity follows immediately. For three sites, the coefficient is

$$\sum_l (a_k b_l)(a_l b_h) = a_k \left(\sum_l a_l b_l \right) b_h = a_k b_h.$$

□

Definition 20.15.3.39 (Global two- and three-site sector coordinates). Lean Lean Lean Lean ✓ Stmt The two- and three-site physical spaces are reindexed by their canonical direct sums over sector pairs and triples, with each summand further regrouped into its boundary and neighboring factors. Write $\mathfrak{G}_2$ and $\mathfrak{G}_3$ for these reindexings of matrices.

Theorem 20.15.3.40 (Global direct-sum form of the sector closures). Lean Lean ✓ Stmt After the canonical regrouping of all physical-sector indices, the full closures are

$$\begin{aligned} \mathfrak{G}_2(\mathcal{K}_2(X)) &= \bigoplus_{k,h} B_{k,h}(X) \otimes \eta_{k,h}, \\ \mathfrak{G}_3(\mathcal{K}_3(X)) &= \bigoplus_{k,l,h} B_{k,h}(X) \otimes (\eta_{k,l} \otimes \eta_{l,h}). \end{aligned}$$

Every matrix entry between distinct sector tuples vanishes.

Proof. ✓ Proof On a diagonal sector tuple these are the fixed-sector factorizations. For distinct sector labels $k \neq p$, the block structure of the sector-coordinate tensor gives $[U\kappa_{\beta,\alpha}U^\dagger]_{k,p} = 0$. Consequently, if $(k,h) \neq (p,q)$ or $(k,l,h) \neq (p,m,q)$, respectively, then

$$\begin{aligned} [\mathfrak{G}_2(\mathcal{K}_2(X))]_{(k,h),(p,q)} &= 0, \\ [\mathfrak{G}_3(\mathcal{K}_3(X))]_{(k,l,h),(p,m,q)} &= 0. \end{aligned}$$

□

Definition 20.15.3.41 (Two-site sector bond). [Lean](#) [✓ Stmt](#) In the two-site sector coordinates, define

$$B_{\text{sector}} = \bigoplus_{k,h} \left(\mathbb{1}_{B_k^L \otimes B_h^R} \otimes \eta_{k,h} \right).$$

Thus the outer factors carry the identity, while the neighboring factors carry the operator $\eta_{k,h}$. This is the local bond in [CPGSV16, Appendix C.2, Proposition C.8, lines 1581–1593].

Theorem 20.15.3.42 (Positivity of the two-site sector bond). [Lean](#) [✓ Stmt](#) Suppose that $\eta_{k,h}$ is positive semidefinite for every pair of sectors. Then B_{sector} is positive semidefinite. The positivity premise is the conclusion at [CPGSV16, Appendix C.2, lines 1446–1450].

Proof. [✓ Proof](#) Each summand $\mathbb{1}_{B_k^L \otimes B_h^R} \otimes \eta_{k,h}$ is positive semidefinite. Positivity is preserved by finite block diagonals and by the reindexing from the regrouped direct sum to the two-site sector coordinates. $\square$

Definition 20.15.3.43 (Two-site bond in physical coordinates). [Lean](#) [Lean](#) [✓ Stmt](#) Let V_2 be the two-site tensor product of the project’s coordinate map, so that $V_2 X V_2^\dagger$ expresses a physical operator X in sector coordinates. Thus $V_2 = U_{\text{paper}}^\dagger$ in the convention of [CPGSV16, Appendix C.2, lines 1581–1583]. Define

$$B_{\text{physical}} = V_2^\dagger B_{\text{sector}} V_2,$$

and identify pairs of physical indices with functions $\{0, 1\} \rightarrow \{0, \dots, d-1\}$. This is the physical two-site bond in [CPGSV16, Appendix C.2, Proposition C.8, lines 1581–1593].

Theorem 20.15.3.44 (Two-site translation identities). [Lean](#) [Lean](#) [✓ Stmt](#) Under the canonical identification of two-site configurations with ordered pairs, the translates beginning at sites zero and one are respectively $B_{01} = B_{\text{physical}}$ and $B_{10} = S B_{\text{physical}} S^{-1}$, where $S(x \otimes y) = y \otimes x$ exchanges the two physical factors.

Proof. [✓ Proof](#) Apply Theorem 20.12.3 to B_{physical} . Its pair-coordinate matrix is the operator in Definition 20.15.3.43, so the two generic identities give the stated formulas. $\square$

Theorem 20.15.3.45 (Positivity of the physical two-site bond). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that $\eta_{k,h}$ is positive semidefinite for every pair of sectors. Then B_{physical} is positive semidefinite, both with pair indices and with two-site configuration indices.

Proof. [✓ Proof](#) Apply positivity of B_{sector} to the congruence $V_2^\dagger B_{\text{sector}} V_2$. Reindexing the resulting matrix by two-site configurations preserves positivity. $\square$

Definition 20.15.3.46 (Three-site lifts of a two-site operator). [Lean](#) [Lean](#) [✓ Stmt](#) Let B be an operator on two copies of a finite-dimensional space H . On the right-associated space $H \otimes (H \otimes H)$, define

$$B_{01} = \alpha(B \otimes \mathbb{1}_H) \alpha^{-1}, \quad B_{12} = \mathbb{1}_H \otimes B,$$

where $\alpha : (H \otimes H) \otimes H \rightarrow H \otimes (H \otimes H)$ is the canonical associator.

Theorem 20.15.3.47 (Three-site translation identities). [Lean](#) [Lean](#) [✓ Stmt](#) Under the canonical identification of three-site configurations with $H \otimes (H \otimes H)$, the translates of the physical bond beginning at sites 0 and 1 are respectively $(B_{\text{physical}})_{01} = B_{01}$ and $(B_{\text{physical}})_{12} = B_{12}$.

Proof. ✓ Proof An entry of the first translate vanishes unless the third indices agree; its remaining entry is that of B_{physical} . Similarly, an entry of the second translate vanishes unless the first indices agree. These are precisely the matrix entries of B_{01} and B_{12} . $\square$

Theorem 20.15.3.48 (Commutativity on a fixed sector triple). Lean ✓ Stmt Fix sectors (k, l, h) . On the regrouped sector space, set

$$\begin{aligned} C_{01}^{k,l,h} &= \mathbb{1}_{B_k^L \otimes B_h^R} \otimes (\eta_{k,l} \otimes \mathbb{1}_{B_l^R \otimes B_h^L}), \\ C_{12}^{k,l,h} &= \mathbb{1}_{B_k^L \otimes B_h^R} \otimes (\mathbb{1}_{B_k^R \otimes B_l^L} \otimes \eta_{l,h}). \end{aligned}$$

Then $C_{01}^{k,l,h} C_{12}^{k,l,h} = C_{12}^{k,l,h} C_{01}^{k,l,h}$. This is the fixed-sector calculation in [CPGSV16, Appendix C.2, Proposition C.8, lines 1589–1593].

On the sector triple (k, l, h) , the two operators act on distinct neighboring factors, while the outer factors B_k^L and B_h^R remain unchanged. Their commutativity is therefore the local identity $\eta_{k,l}^{(12)} \eta_{l,h}^{(23)} = \eta_{l,h}^{(23)} \eta_{k,l}^{(12)}$.

Proof. ✓ Proof The operators $\eta_{k,l}$ and $\eta_{l,h}$ act on the distinct factors $B_k^R \otimes B_l^L$ and $B_l^R \otimes B_h^L$, respectively. The claim follows from the mixed-product identity for tensor products. $\square$

Theorem 20.15.3.49 (Commutativity of adjacent physical bonds). Lean ✓ Stmt The two translates of the physical bond on three sites commute:

$$(B_{\text{physical}})_{01} (B_{\text{physical}})_{12} = (B_{\text{physical}})_{12} (B_{\text{physical}})_{01}.$$

This is the adjacent-bond calculation in [CPGSV16, Appendix C.2, Proposition C.8, lines 1589–1593].

Proof. ✓ Proof Regroup the three-site sector coordinates into the direct sum over triples (k, l, h) . The two bonds become block diagonal, with blocks $C_{01}^{k,l,h}$ and $C_{12}^{k,l,h}$, which commute by Theorem 20.15.3.48. Conjugating by the three-fold physical coordinate unitary preserves the equality. The three-site translation identities then give the displayed physical commutator. $\square$

Theorem 20.15.3.50 (Agreement outside a cyclic window). Lean ✓ Stmt Let $W_i^L = \{i, i+1, \dots, i+L-1\}$ be a cyclic window of length $L \leq N$. Two configurations agree outside W_i^L precisely when their values agree at every site not contained in W_i^L .

Proof. ✓ Proof Replacing the entries in W_i^L does not affect any complementary site. Conversely, if the two configurations agree on the complement, replacing the window of the first by the window of the second reconstructs the second configuration. $\square$

Theorem 20.15.3.51 (Multiplicativity of cyclic local embeddings). Lean ✓ Stmt Let E_i^L embed an operator on L consecutive sites into an N -site periodic chain at the cyclic window beginning at i . Then $E_i^L(BC) = E_i^L(B)E_i^L(C)$.

Proof. ✓ Proof Identify a periodic configuration with its restriction to W_i^L and its restriction to the cyclic complement. In these coordinates, $E_i^L(B) = B \otimes \mathbb{1}$, so the assertion is the mixed-product identity. $\square$

Theorem 20.15.3.52 (Commutativity on disjoint cyclic windows). Lean ✓ Stmt Let $L \leq N$, and let W_i^L and W_j^L be disjoint cyclic windows in an N -site periodic chain. For arbitrary operators B and C on L sites, $E_i^L(B)E_j^L(C) = E_j^L(C)E_i^L(B)$.

Proof. ✓ Proof Apply both sides to a chain amplitude. The first order of application is a double sum over replacements on W_i^L and W_j^L . Since the windows are disjoint, each replacement leaves the other restricted configuration unchanged, and the two replacements commute. Interchanging the two finite sums gives the reverse order. $\square$

Theorem 20.15.3.53 (Adjacent physical bonds on periodic chains). Lean ✓ Stmt Let $N \geq 3$. For every site $i \in \mathbb{Z}/N\mathbb{Z}$, including $i = N - 1$, the adjacent translates of the physical bond commute:

$$(B_{\text{physical}})_{i,i+1}(B_{\text{physical}})_{i+1,i+2} = (B_{\text{physical}})_{i+1,i+2}(B_{\text{physical}})_{i,i+1}.$$

This is the periodic-chain form of the adjacent-bond calculation in [CPGSV16, Appendix C.2, Proposition C.8, lines 1571–1593]. The case $N = 2$ is treated separately below, because its two translated bonds have the same support with opposite cyclic order.

Proof. ✓ Proof Enclose the two bonds in the cyclic three-site window beginning at i . In these coordinates they are the bonds beginning at sites 0 and 1, respectively. Embedding an operator into a fixed cyclic window preserves products, so the assertion follows from Theorem 20.15.3.49. $\square$

Theorem 20.15.3.54 (Commutativity of the crossed two-site translates). Lean ✓ Stmt On the periodic chain of length two, the translates beginning at sites zero and one read the physical sites in the opposite cyclic orders $(0, 1)$ and $(1, 0)$. Nevertheless they commute: $B_{01}B_{10} = B_{10}B_{01}$. This is the length-two instance of the translated-bond commutativity in [CPGSV16, Appendix C.2, Proposition C.8]. The source does not discuss the crossed finite-size ordering separately.

Proof. ✓ Proof Fix sectors (k, h) and regroup the two physical sites as $(L_k \otimes R_h) \otimes (R_k \otimes L_h)$. The translate in the order $(0, 1)$ is $\mathbb{1}_{L_k \otimes R_h} \otimes \eta_{k,h}$, while the translate in the order $(1, 0)$ is $\eta_{h,k}^{\text{op}} \otimes \mathbb{1}_{R_k \otimes L_h}$, where $\eta_{h,k}^{\text{op}}$ denotes the same matrix after exchanging the factors R_h and L_k . These operators act on complementary factors and commute. Taking the direct sum over (k, h) and conjugating by the two-fold physical coordinate unitary proves the claim. $\square$

Theorem 20.15.3.55 (Disjoint physical bonds on periodic chains). Lean ✓ Stmt Let $N \geq 2$, and suppose that the cyclic bonds beginning at i and j have disjoint two-site supports. Then their physical bond operators commute: $(B_{\text{physical}})_{i,i+1}(B_{\text{physical}})_{j,j+1} = (B_{\text{physical}})_{j,j+1}(B_{\text{physical}})_{i,i+1}$. This is the locality part of the pairwise commutation assertion in [CPGSV16, Appendix C.2, Proposition C.8, lines 1571–1593].

Proof. ✓ Proof This is Theorem 20.15.3.52 applied to the same physical bond operator in both windows. No positivity or additional property of the sector factorization is needed for this locality step. $\square$

Theorem 20.15.3.56 (Pairwise commutativity of the physical bonds). Lean ✓ Stmt Suppose that the physical-sector factorization is given. For every periodic chain of length $N \geq 2$ and every pair of sites $i, j \in \mathbb{Z}/N\mathbb{Z}$, the corresponding translates of the physical bond commute:

$$(B_{\text{physical}})_{i,i+1}(B_{\text{physical}})_{j,j+1} = (B_{\text{physical}})_{j,j+1}(B_{\text{physical}})_{i,i+1}.$$

This is the pairwise commutation assertion in [CPGSV16, Appendix C.2, Proposition C.8, lines 1571–1593], conditional here on the physical-sector factorization constructed in the preceding results.

Proof. ✓ Proof For $N = 2$, equal translates commute trivially, while the two distinct translates commute by Theorem 20.15.3.54. Let $N \geq 3$. If the two cyclic supports are disjoint, apply Theorem 20.15.3.55. Otherwise, Lemma 13.1.12 shows that the two starting sites either coincide or are cyclic neighbors. The coincident case is immediate, and the two neighboring orientations follow from Theorem 20.15.3.53 and its symmetric equality. $\square$

Theorem 20.15.3.57 (Exact product of physical-sector bonds). Lean ✓ Stmt Suppose that K is equipped with a physical-sector factorization F . Let B_i denote the cyclic translate beginning at site i of the physical two-site bond determined by F . For every $N \geq 2$, the N -site periodic operator is

$$\rho_N = B_0 B_1 \cdots B_{N-1}.$$

No positivity, zero-correlation-length, injectivity, or normalization hypothesis is required. This is the product identity in [CPGSV16, Appendix C.2, Proposition C.8, lines 1581–1593], conditional here on the given physical-sector factorization.

Proof. ✓ Proof For a fixed cyclic sector sequence, contraction of the virtual index between the right tensor at site i and the left tensor at site $i + 1$ produces $\eta_{k_i, k_{i+1}}$. Taking the virtual trace therefore gives the cyclic product of these neighboring operators. The translated sector bonds give the same ordered product. Conjugating by the tensor power of the one-site coordinate unitary gives the stated physical identity. For $N = 2$, the second translate has the opposite cyclic order; the two factors are the ordinary and crossed bonds. $\square$

Theorem 20.15.3.58 (Unit scalar in the physical-sector product). Lean ✓ Stmt Under the physical-sector factorization F of the preceding theorem, for every $N \geq 2$ there is a real number $c > 0$ such that $\rho_N = c B_0 B_1 \cdots B_{N-1}$. Here the bonds are the canonical bonds determined by F , and one may take $c = 1$. No positivity hypothesis on the neighboring operators $\eta_{k,h}$ is required.

Proof. ✓ Proof Take $c = 1$ in Theorem 20.15.3.57. $\square$

Theorem 20.15.3.59 (Positive translation-invariant bond datum). Lean ✓ Stmt Suppose that every neighboring operator $\eta_{k,h}$ is positive semidefinite. Then the physical bond is positive semidefinite and all of its cyclic translates commute on every periodic chain of length at least two. Thus it determines a single translation-invariant positive bond datum. This assertion does not include the product formula for the finite-chain density operator.

Proof. ✓ Proof Positivity is Theorem 20.15.3.45; pairwise commutativity is Theorem 20.15.3.56. $\square$

Theorem 20.15.3.60 (η -local structure from a positive physical-sector factorization). Lean ✓ Stmt Suppose that a physical-sector factorization of K is given and that every neighboring operator $\eta_{k,h}$ is positive semidefinite. Then the physical two-site bond determines an η -local structure for K . Its translated copies commute pairwise, and for every $N \geq 2$, $\rho_N = B_0 B_1 \cdots B_{N-1}$. If the virtual dimension of K is positive, then K itself is a fixed tensor representation of this product family, with the same virtual dimension. Thus the positive normalization scalar may be chosen to be 1. This is Proposition C.8 conditional on the coherently positive physical-sector factorization that the source derives from SAL.

Proof. ✓ Proof Theorem 20.15.3.59 gives the positive translation-invariant bond and its pairwise commuting translates. Theorem 20.15.3.57 realizes every finite-chain operator with normalization scalar 1. $\square$

Corollary 20.15.3.61 (η -local structure from SAL). [Lean](#) [✓ Stmt](#) *Every injective MPO tensor satisfying SAL admits an η -local structure. Thus there is a single positive two-site bond whose cyclic translates commute pairwise and whose ordered product is the finite-chain density operator at every length $N \geq 2$, with normalization scalar one. This is the conclusion of Proposition C.8.*

Proof. [✓ Proof](#) Theorem 20.15.4.41 supplies the positive physical-sector factorization. Apply Theorem 20.15.3.60. $\square$

Corollary 20.15.3.62 (Normal representative for the SAL-selected bond). [Lean](#) [✓ Stmt](#) *Let K be injective and satisfy SAL, and suppose that its doubled-index matrix-product tensor is normal. There is a positive physical-sector factorization whose selected two-site bond B has K itself as a normal fixed tensor representation. Its virtual dimension is the virtual dimension of K , and for every $N \geq 2$, $\rho_N(K) = B_0 B_1 \cdots B_{N-1}$. In particular, the scalar is one. This is an existential statement about the bond selected in the proof of Proposition C.8; it does not assert uniqueness or normal rescalability for an arbitrary proportional commuting-bond presentation.*

Proof. [✓ Proof](#) SAL implies that the virtual dimension is positive. The positive physical-sector factorization is supplied by Theorem 20.15.4.41. For its selected bond, use K as the fixed tensor. The exact product identity is Theorem 20.15.3.57, and normality is the given hypothesis on the doubled-index tensor. $\square$

20.15.4 Closed-sector contractions and coherent rephasing

Definition 20.15.4.1 (Closed sector tensors). [Lean](#) [Lean](#) [✓ Stmt](#) Tracing the two physical legs of a sector tensor leaves a vector over the virtual bond:

$$\begin{aligned} |l_k)_\gamma &:= \text{tr} [(l_k)_\gamma], \\ (r_k|_\gamma &:= \text{tr} [(r_k)_\gamma]. \end{aligned}$$

These are the closed tensors of [CPGSV16, Appendix C.2, lines 1473–1477].

Definition 20.15.4.2 (Closed-sector pairing operator). [Lean](#) [✓ Stmt](#) For sector tensors $(l_k)_\beta$ and $(r_k)_\alpha$, close their physical legs and set

$$S := \sum_k |l_k)(r_k|. \quad (20.48)$$

This is the pairing operator displayed in [CPGSV16, Appendix C.2, lines 1473–1493].

Theorem 20.15.4.3 (Sector trace pairing). [Lean](#) [✓ Stmt](#) *For all sectors k, h , the trace of the neighboring operator is the pairing of the closed sector tensors: $T_{k,h} = \text{tr}(\eta_{k,h}) = (r_k|l_h)$. This is [CPGSV16, Appendix C.2, lines 1452–1455 and 1478–1481].*

Proof. [✓ Proof](#) Expanding the definition of the neighboring operator and applying $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$ gives

$$\text{tr}(\eta_{k,h}) = \sum_\gamma \text{tr}((r_k)_\gamma \otimes (l_h)_\gamma) = \sum_\gamma \text{tr}[(r_k)_\gamma] \text{tr}[(l_h)_\gamma] = (r_k|l_h).$$

$\square$

Definition 20.15.4.4 (Concrete closed-sector trace matrix). [Lean](#) [Lean](#) [✓ Stmt](#) For the inverse-map sector tensors, define the complex matrix $T_{k,h} := (r_k|l_h)$. Equivalently, $T_{k,h} = \text{tr}(\eta_{k,h})$. This is the trace matrix in [CPGSV16, Appendix C.2, lines 1478–1481].

Theorem 20.15.4.5 (Closed-sector form of the physical-trace transfer). [Lean](#) [✓ Stmt](#) *More generally, any sector factorization $U_B \kappa_{\beta, \alpha} U_B^\dagger = \bigoplus_k (l_k)_\beta \otimes (r_k)_\alpha$ gives the closed-sector pairing operator*

$$\mathcal{T}_K = \sum_k |l_k)(r_k|. \quad (20.49)$$

This identifies the transfer matrix of Definition 20.5.1 with the operator on the left-hand side of the zero-correlation-length identity in [CPGSV16, Appendix C.2, lines 1489–1493].

Proof. [✓ Proof](#) Trace the sector factorization over the physical index. Unitary invariance gives $\text{tr}(U_B \kappa_{\beta, \alpha} U_B^\dagger) = \text{tr}(\kappa_{\beta, \alpha})$. The trace of the block diagonal is the sum of the block traces. In each sector,

$$\text{tr}((l_k)_\beta \otimes (r_k)_\alpha) = \text{tr}[(l_k)_\beta] \text{tr}[(r_k)_\alpha] = [|l_k)(r_k|]_{\beta, \alpha}.$$

Summing over k gives (20.49). □

Corollary 20.15.4.6 (Physical-trace transfer from the inverse-map sectors). [Lean](#) [✓ Stmt](#) *Let K be injective. Let R and ρ give a three-site closure of K , choose a Hayashi decomposition of ρ , and fix virtual indices α_1, β_3 such that $R_{\beta_3, \alpha_1} \neq 0$. For the corresponding closed sector tensors, $\mathcal{T}_K = \sum_k |l_k)(r_k|$.*

Proof. [✓ Proof](#) The inverse-map sector tensors satisfy the sector factorization of Theorem 20.15.2.52. Apply Theorem 20.15.4.5. □

Theorem 20.15.4.7 (Normalized closed-sector pairing operator). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let K be injective. Let R and ρ give a three-site closure of K , choose a Hayashi decomposition of ρ , and fix virtual indices α_1, β_3 such that $R_{\beta_3, \alpha_1} \neq 0$. Form the corresponding closed sector tensors $|l_k)$ and $(r_k|$. If K satisfies the positive-real up-to-scalar physical-trace relation, set $S := \sum_k |l_k)(r_k|$. Then there is a real number $\lambda > 0$ such that*

$$\begin{aligned} S^2 &= \lambda S, \\ (\lambda^{-1} S)^2 &= \lambda^{-1} S. \end{aligned} \quad (20.50)$$

If K satisfies the paper’s literal source zero-correlation-length identity $\mathcal{T}_K^2 = \mathcal{T}_K$, then the raw pairing operator itself satisfies $S^2 = S$. This is the display preceding [CPGSV16, Appendix C.2, Lemma C.5]. The first identity records the broader relation used in this development, which is invariant under positive real rescaling; the paper’s identity is its $\lambda = 1$ specialization.

Proof. [✓ Proof](#) Substitute $S = \mathcal{T}_K$ from Corollary 20.15.4.6 into the defining relation $\mathcal{T}_K^2 = \lambda \mathcal{T}_K$, and then apply Lemma 20.5.7 to obtain (20.50) and its normalized form. Under literal idempotence, the same substitution gives $S^2 = S$ directly. □

Theorem 20.15.4.8 (Normalized closed-sector trace relations). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 20.15.4.7, suppose that K satisfies the positive-real up-to-scalar physical-trace relation. There is a real number $\lambda > 0$ such that, for $\widehat{T} := \lambda^{-1} T$, one has*

$$\widehat{T}^2 = \widehat{T}^3. \quad (20.51)$$

For every positive integer N , one also has

$$\text{tr}(\widehat{T}^N) = \text{tr}(\widehat{T}). \quad (20.52)$$

For the literal $\lambda = 1$ source zero-correlation-length identity, this specializes to the trace-power display in [CPGSV16, Appendix C.2, lines 1490–1497]. No idempotence or rank-one conclusion for $\widehat{T}$ is asserted.

Proof. ✓ Proof Let L be the matrix whose h th column is $|l_h\rangle$, and let Q be the matrix whose k th row is $\langle r_k|$. Then $S = LQ$ and $T = QL$. Place the normalization scalar on L . Since $\lambda^{-1}LQ$ is idempotent, associativity gives

$$(Q\lambda^{-1}L)^2 = Q(\lambda^{-1}LQ)\lambda^{-1}L = Q(\lambda^{-1}LQ)^2\lambda^{-1}L = (Q\lambda^{-1}L)^3.$$

Cyclicity of trace and the idempotence of $\lambda^{-1}LQ$ give

$$\text{tr}(Q\lambda^{-1}L) = \text{tr}(\lambda^{-1}LQ) = \text{tr}((\lambda^{-1}LQ)^2) = \text{tr}((Q\lambda^{-1}L)^2).$$

By (20.51), $\widehat{T}^N = \widehat{T}^2$ for every $N \geq 2$, which proves (20.52). □

Definition 20.15.4.9 (Explicit neighboring η -operator data). Lean Lean ✓ Stmt Fix a Hayashi decomposition witness h_η . An explicit neighboring η -family is a dependent family $(\eta_{k,h})$ indexed by sector pairs, where $\eta_{k,h}$ acts on the neighboring bond space $B_k^R \otimes B_h^L$. Requiring positivity for each pair gives the corresponding positive η -data.

Definition 20.15.4.10 (Neighboring operators from sector tensors). Lean ✓ Stmt Let $(l_k)_a$ and $(r_k)_a$ be sector tensors indexed by the horizontal virtual index a . For neighboring sectors k, h , define an operator on $B_k^R \otimes B_h^L$ by

$$\eta_{k,h}((x_R, x_L), (y_R, y_L)) = \sum_a (r_k)_a(x_R, y_R)(l_h)_a(x_L, y_L). \quad (20.53)$$

Thus $\eta_{k,h} = r_k l_h$ is the contraction over the shared horizontal index in [CPGSV16, Appendix C.2, equation (etarl)]. Positivity is not part of this definition.

Lemma 20.15.4.11 (Entries of the neighboring operators). Lean ✓ Stmt For every pair of sectors and every pair of matrix indices, the entry of $\eta_{k,h}$ is the contraction displayed in (20.53).

Proof. ✓ Proof This is immediate from the definition of $\eta_{k,h}$. □

Neighboring right and left sector tensors meet in the bond operator $\eta_{k,h} = r_k l_h$:

$$\begin{array}{c} x_R \quad x_L \\ | \quad | \\ \boxed{\eta_{k,h}} \\ | \quad | \\ y_R \quad y_L \end{array} = r_k \begin{array}{c} x_R \quad a \quad x_L \\ | \quad \bullet \quad | \\ | \quad \text{---} \quad | \\ | \quad \bullet \quad | \\ y_R \quad y_L \end{array} l_h$$

This is the contraction in [CPGSV16, Appendix C.2, lines 1441–1445].

Theorem 20.15.4.12 (Cyclic contraction of the sector tensors). Lean ✓ Stmt Let $N \geq 1$ and let $k_0, \dots, k_{N-1}$ be a cyclic sequence of sectors. Fix matrix indices in the left and right factor of every sector, and abbreviate

$$\begin{aligned} L_n(a) &= (l_{k_n})_a(x_n^L, y_n^L), \\ R_n(a) &= (r_{k_n})_a(x_n^R, y_n^R). \end{aligned}$$

With all site labels read modulo N ,

$$\sum_{g_0, \dots, g_{N-1}} \prod_{n=0}^{N-1} L_n(g_n) R_n(g_{n+1}) = \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}((x_n^R, x_{n+1}^L), (y_n^R, y_{n+1}^L)). \quad (20.54)$$

This is the entrywise cyclic contraction at [CPGSV16, Appendix C.2, lines 1446–1450].

Proof. [✓ Proof](#) Expand the product of the sums defining the neighboring operators and set $g'_n = g_{n+1}$. Cyclic invariance of the finite product and commutativity of scalar multiplication give

$$\begin{aligned} \sum_{g_0, \dots, g_{N-1}} \prod_{n=0}^{N-1} L_n(g_n) R_n(g_{n+1}) &= \sum_{g'_0, \dots, g'_{N-1}} \prod_{n=0}^{N-1} R_n(g'_n) L_{n+1}(g'_n) \\ &= \prod_{n=0}^{N-1} \sum_a R_n(a) L_{n+1}(a). \end{aligned}$$

The last expression is the product of the corresponding entries of $\eta_{k_n, k_{n+1}}$, as in (20.54). $\square$

Theorem 20.15.4.13 (Closed sector word as neighboring contractions). [Lean](#) [✓ Stmt](#) Under the hypotheses of the preceding theorem, define $M_n(a, b) = L_n(a) R_n(b)$. Then

$$\text{tr}(M_0 M_1 \cdots M_{N-1}) = \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}((x_n^R, x_{n+1}^L), (y_n^R, y_{n+1}^L)). \quad (20.55)$$

$$\text{tr}(M_{k_0} \cdots M_{k_{N-1}}) = \text{tr}(\eta_{k_0, k_1} \otimes \cdots \otimes \eta_{k_{N-1}, k_0}).$$

Figure 20.4: Closing the horizontal indices of the factorized sector tensors pairs each right factor with the left factor at the next site. The cycle therefore becomes the product of the neighboring operators $\eta_{k_n, k_{n+1}}$; compare [CPGSV16, Appendix C.2, lines 1446–1450].

Proof. [✓ Proof](#) Expanding the trace of the matrix word gives

$$\text{tr}(M_0 M_1 \cdots M_{N-1}) = \sum_{g_0, \dots, g_{N-1}} \prod_{n=0}^{N-1} M_n(g_n, g_{n+1}).$$

Substituting $M_n(a, b) = L_n(a) R_n(b)$ gives the cyclic sum in (20.54); applying that identity yields (20.55). $\square$

Definition 20.15.4.14 (Sector-adapted chain coordinates). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a sector configuration $k = (k_0, \dots, k_{N-1})$, let

$$I_k = \prod_{n=0}^{N-1} (\{0, \dots, d_L(k_n) - 1\} \times \{0, \dots, d_R(k_n) - 1\}).$$

Applying the one-site Hayashi decomposition at every site gives a bijection

$$\Phi_N : \{0, \dots, d-1\}^N \longrightarrow \prod_{k_0, \dots, k_{N-1}} I_k.$$

We write $\text{reindex}_{\Phi_N}(M)$ for the matrix obtained by transporting both indices of M along this bijection.

Definition 20.15.4.15 (Physical conjugation of an MPO tensor). [Lean](#) [✓ Stmt](#) Let U be a matrix on the physical space. The physical conjugation of K by U is the tensor $\tilde{K}$ defined by

$$\tilde{\kappa}_{\beta,\alpha}^{ij} = (U\kappa_{\beta,\alpha}U^\dagger)_{ij}.$$

Its horizontal bond indices are unchanged. When U is unitary, this is the corresponding physical basis change.

Definition 20.15.4.16 (Cyclic tensor product of neighboring operators). [Lean](#) [✓ Stmt](#) Let $N \geq 1$ and let k be a sector configuration. Define the matrix Ω_k on I_k by

$$\Omega_k(x, y) = \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}((x_n^R, x_{n+1}^L), (y_n^R, y_{n+1}^L)). \quad (20.56)$$

Thus $\Omega_k = \eta_{k_0, k_1} \otimes \cdots \otimes \eta_{k_{N-1}, k_0}$ after the neighboring tensor factors are ordered cyclically.

Theorem 20.15.4.17 (A fixed sector block is the cyclic neighboring product). [Lean](#) [✓ Stmt](#) Suppose that the physical slices in the Hayashi basis satisfy

$$U_B \kappa_{\beta,\alpha} U_B^\dagger = \bigoplus_q (l_q)_\beta \otimes (r_q)_\alpha.$$

Let $\eta_{q,h} = r_q l_h$. For every $N \geq 1$ and every sector configuration k , the corresponding diagonal block of $\text{reindex}_{\Phi_N}(M_N(\tilde{K}))$ is Ω_k .

Proof. [✓ Proof](#) Fix $x, y \in I_k$ and expand the closed horizontal trace of the MPO word. The sector factorization writes each local entry as a product of entries of l_{k_n} and r_{k_n} . Set

$$M_n(\beta, \alpha) = (l_{k_n})_\beta(x_n^L, y_n^L)(r_{k_n})_\alpha(x_n^R, y_n^R).$$

Then

$$\begin{aligned} [\text{reindex}_{\Phi_N}(M_N(\tilde{K}))]_{(k,x),(k,y)} &= \text{tr}(M_0 M_1 \cdots M_{N-1}) \\ &= \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}((x_n^R, x_{n+1}^L), (y_n^R, y_{n+1}^L)) = \Omega_k(x, y). \end{aligned}$$

The second equality is Theorem 20.15.4.13, and the last is (20.56). $\square$

Theorem 20.15.4.18 (Sector-adapted MPO decomposition). [Lean](#) [✓ Stmt](#) Under the same sector factorization hypothesis, for every $N \geq 1$,

$$\text{reindex}_{\Phi_N}(M_N(\tilde{K})) = \bigoplus_{k_0, \dots, k_{N-1}} \eta_{k_0, k_1} \otimes \eta_{k_1, k_2} \otimes \cdots \otimes \eta_{k_{N-1}, k_0}. \quad (20.57)$$

This is the algebraic sector decomposition in [CPGSV16, Appendix C.2, lines 1446–1450]. It does not assert that the individual neighboring operators are positive.

Proof. [✓ Proof](#) If the row and column sector configurations coincide, the corresponding block is Theorem 20.15.4.17. If they differ, choose a site n at which their labels k_n and h_n are unequal. Block diagonality gives $[U_B \kappa_{\beta,\alpha} U_B^\dagger]_{(k_n, \cdot), (h_n, \cdot)} = 0$ for every pair of horizontal indices. This local factor occurs in every cyclic horizontal-index product, so every summand in the closed trace is zero, proving (20.57). $\square$

Theorem 20.15.4.19 (Positive rescaling of Kronecker factors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let A and B be nonzero complex square matrices whose Kronecker product $A \otimes B$ is positive semidefinite. Then there is a scalar $c \neq 0$ with $cA \geq 0$ and $c^{-1}B \geq 0$. Since $(cA) \otimes (c^{-1}B) = A \otimes B$, the rescaled factors represent the same product.

Proof. [Stmnt](#) [Proof](#) On a product vector $x \otimes y$ the quadratic form of $A \otimes B$ factors, so $(x^\dagger A x)(y^\dagger B y) \geq 0$ for all vectors x and y . By polarization a complex matrix is determined by its quadratic form; since A and B are nonzero there are vectors with $a = v_0^\dagger A v_0 \neq 0$ and $c = w_0^\dagger B w_0 \neq 0$. Every value $x^\dagger (cA) x = (x^\dagger A x) c$ is non-negative, and over the complex numbers a non-negative quadratic form already forces the matrix to be Hermitian, hence positive semidefinite. For the second factor, $ac > 0$ and $a(y^\dagger B y) \geq 0$ give $c^{-1}(y^\dagger B y) = (a(y^\dagger B y))(ac)^{-1} \geq 0$. $\square$

Lemma 20.15.4.20 (A matrix and its negative cannot both be positive). [Lean](#) [Stmnt](#) If X and $-X$ are positive semidefinite complex matrices, then $X = 0$.

Proof. [Stmnt](#) [Proof](#) The quadratic form of X is both non-negative and nonpositive, hence vanishes identically. Polarization gives $X = 0$. $\square$

Definition 20.15.4.21 (Finite Kronecker product). [Lean](#) [Stmnt](#) For square matrices A_n indexed by $n = 0, \dots, N-1$, possibly with different index types, their finite Kronecker product is the matrix with entries

$$\left(\bigotimes_{n=0}^{N-1} A_n \right)_{x,y} = \prod_{n=0}^{N-1} (A_n)_{x_n, y_n}.$$

Lemma 20.15.4.22 (Elementary finite Kronecker identities). [Lean](#) [Lean](#) [Stmnt](#) A finite Kronecker product of nonzero matrices is nonzero. Replacing one factor A_i by cA_i multiplies the whole finite product by c .

Proof. [Stmnt](#) [Proof](#) Choose a nonzero entry in each factor and evaluate the product at the resulting row and column. The rescaling identity follows immediately from the entry formula in Definition 20.15.4.21. $\square$

Theorem 20.15.4.23 (Uniqueness of the positive rescaling phase). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmnt](#) Let $M \neq 0$. If $c_1, c_2 \neq 0$ and both $c_1 M$ and $c_2 M$ are positive semidefinite, then $c_1 = tc_2$ for some positive real number t . Every complex scalar of unit modulus is nonzero. In particular, every nonzero scalar c has a unit-modulus part $u = c/|c|$, positivity of cM implies positivity of uM , and there is at most one such unit scalar when $M \neq 0$.

Proof. [Stmnt](#) [Proof](#) Hermiticity of $c_1 M$ and $c_2 M$ gives $c_2 \bar{c}_1 = c_1 \bar{c}_2$, so c_1/c_2 is real. It cannot be negative: otherwise $c_1 M$ and its negative would both be positive semidefinite, forcing $c_1 M = 0$. Thus the quotient is positive. Multiplication by $|c|^{-1} > 0$ replaces c by its unit-modulus part without changing positivity. Two admissible unit scalars differ by a positive real number of modulus one, and are therefore equal. $\square$

Theorem 20.15.4.24 (Positive rescaling of a finite Kronecker product). [Lean](#) [Stmnt](#) Let $N \geq 1$, and let $A_0, \dots, A_{N-1}$ be nonzero complex square matrices, with possibly different dimensions. If their finite Kronecker product is positive semidefinite, then there are nonzero scalars $c_0, \dots, c_{N-1}$ such that

$$\prod_{n=0}^{N-1} c_n = 1, \\ c_n A_n \geq 0 \quad (0 \leq n < N).$$

Proof. [✓ Proof](#) The assertion is proved by induction on N . Separate the first factor from the remaining Kronecker product and apply Theorem 20.15.4.19. Absorb the inverse of the first rescaling coefficient into one factor of the tail, then apply the induction hypothesis. The resulting coefficients are nonzero, their product is one, and every rescaled factor is positive semidefinite. $\square$

Lemma 20.15.4.25 (One-sided product obstruction in a full matrix algebra). [Lean](#) [✓ Stmt](#) Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq M_D(\mathbb{C})$ be nonzero linear subspaces. If $XY = 0$ for every $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$, then $\mathcal{A}_1 + \mathcal{A}_2 \neq M_D(\mathbb{C})$. No vanishing condition on products in the reverse order is required. This is the matrix-algebra obstruction used in [CPGSV16, Appendix C.2, lines 1465–1470].

Proof. [✓ Proof](#) Choose nonzero matrices $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$, together with nonzero entries X_{ip} and Y_{qj} . If the two subspaces spanned the full matrix algebra, write the matrix unit $E_{pq} = E_1 + E_2$ with $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. The one-sided product hypothesis gives

$$XE_{pq}Y = X(E_1Y) + (XE_2)Y = 0.$$

Its (i, j) entry is $X_{ip}Y_{qj} \neq 0$, a contradiction. $\square$

Definition 20.15.4.26 (Nonzero sector support and recurrence). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The nonzero support graph of a neighboring-operator family has a directed edge $k \rightarrow h$ precisely when $\eta_{k,h} \neq 0$. Its support is recurrent when every such edge belongs to a directed cycle, equivalently when h can reach k whenever $k \rightarrow h$ is an edge.

Definition 20.15.4.27 (Sector-coordinate virtual matrix families). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) In sector coordinates, let $A_{k;x,y}$ be the virtual matrix obtained from the (x, y) entry within sector k . The sector-coordinate family contains all transformed physical matrices, while the virtual-matrix family consists of all matrices $A_{k;x,y}$.

Theorem 20.15.4.28 (Sector spanning from injectivity). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be injective and let

$$U\kappa_{\beta,\alpha}U^\dagger = \bigoplus_k (l_k)_\beta \otimes (r_k)_\alpha$$

be a physical-sector factorization. Then the virtual matrices $A_{k;x,y}$ obtained from all within-sector physical entries span the full virtual matrix algebra $M_D(\mathbb{C})$.

Proof. [✓ Proof](#) Since U is unitary, every original physical matrix of K is a linear combination of the matrices in the transformed physical coordinates. The off-diagonal sector entries in those coordinates vanish, while each diagonal sector entry is one of the matrices $A_{k;x,y}$. Injectivity says that the original physical matrices span $M_D(\mathbb{C})$, and the conclusion follows. $\square$

Theorem 20.15.4.29 (Triangle closure from sector spanning). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a physical-sector factorization, let $A_{k;x,y} \in M_D(\mathbb{C})$ be the virtual matrix obtained from the physical matrix entry (x, y) within sector k . Suppose that all such matrices span $M_D(\mathbb{C})$ and that both endpoint sectors of every nonzero neighboring operator contain a nonzero matrix of this form. Then every nonzero $\eta_{k,h}$ admits a two-edge return: there is a sector j for which

$$\begin{aligned} \eta_{h,j} &\neq 0, \\ \eta_{j,k} &\neq 0. \end{aligned}$$

In particular, the nonzero neighboring support is recurrent.

Proof. ✓ Proof A nonzero entry of $\eta_{k,h}$, together with nonzero sector matrices at k and h , gives matrices A from sector k and B from sector h with $AB \neq 0$. By nondegeneracy of the trace pairing, choose Y with $\text{tr}(ABY) \neq 0$. Expanding Y in the spanning family, some sector matrix C from a sector j satisfies $\text{tr}(ABC) \neq 0$. Cyclicity of the trace gives $BC \neq 0$ and $CA \neq 0$. The sector multiplication identity then forces $\eta_{h,j} \neq 0$ and $\eta_{j,k} \neq 0$. $\square$

Lemma 20.15.4.30 (Equivalent forms of sector reachability). Lean Lean Lean ✓ Stmt *A vertex h is reachable from k in the nonzero sector support if and only if there is a finite directed walk from k to h . Consequently, the support is recurrent if and only if every directed edge admits a directed return walk.*

Proof. ✓ Proof A walk gives an element of the reflexive transitive closure by induction on its edges. Conversely, append the final edge at each induction step in the reflexive transitive closure. $\square$

Lemma 20.15.4.31 (Finite cyclic form of a closed directed walk). Lean Lean Lean ✓ Stmt *Every nonempty closed directed walk of length N determines a cyclic family of vertices $v_0, \dots, v_{N-1}$. Each $v_n \rightarrow v_{n+1}$ is an edge, where $v_N = v_0$, and for any edge weights in a commutative monoid, $\text{wt}(w) = \prod_{n=0}^{N-1} \kappa_{v_n, v_{n+1}}$.*

Proof. ✓ Proof Enumerate the sources of the successive edges. The last edge ends at the initial vertex because the walk is closed. The product identity follows by induction on the number of edges. $\square$

Theorem 20.15.4.32 (Closed-walk criterion for vertex phases). Lean ✓ Stmt *Let E be a directed relation on a set V , and suppose that every edge $a \rightarrow b$ admits a directed return walk from b to a . Let G be a group and assign a weight $\kappa_{a,b} \in G$ to each ordered pair of vertices. If the product of the edge weights along every closed directed walk is one, then there are vertex weights $z_a \in G$ such that $\kappa_{a,b} = z_a^{-1} z_b$ whenever $a \rightarrow b$ is an edge. This is the graph-theoretic coboundary step needed to make the phase choices in [CPGSV16, Appendix C.2, lines 1446–1450] coherent across a recurrent support component.*

Proof. ✓ Proof Reachability is an equivalence relation because every edge has a return walk. Choose one vertex in each reachability class and, for every vertex a , choose a walk from the class representative to a . Define z_a as the product of the edge weights along this walk. If two such walks have the same endpoints, append a common return walk; the two resulting closed walks have weight one, so the original walk weights agree. Comparing the chosen walk to b with the chosen walk to a followed by the edge $a \rightarrow b$ gives $z_b = z_a \kappa_{a,b}$. $\square$

Definition 20.15.4.33 (Horizontal bond fiber of a sector cycle). Lean Lean ✓ Stmt For a cyclic sector assignment $(k_n)_{n \in \mathbb{Z}/N\mathbb{Z}}$, the horizontal bond fiber at n is

$$H_n = B_{k_n}^R \times B_{k_{n+1}}^L.$$

Shifting the left component by one site identifies $\prod_n H_n$ with the vertex-indexed sector fiber used in the cyclic neighboring-operator product.

Lemma 20.15.4.34 (A cyclic block is a finite Kronecker product). Lean ✓ Stmt *Reindexing a cyclic neighboring-operator block along its horizontal bond fiber gives the finite Kronecker product $\bigotimes_{n=0}^{N-1} \eta_{k_n, k_{n+1}}$.*

Proof. ✓ Proof Expand both sides entrywise. The shifted left component of the bond fiber at site $n+1$ is precisely the left index paired with the right component at site n . $\square$

Theorem 20.15.4.35 (Positive rescaling around one sector cycle). [Lean](#) [✓ Stmt](#) Let $N \geq 1$, and fix a cyclic sector assignment $k_0, \dots, k_{N-1}$ such that every operator $\eta_{k_n, k_{n+1}}$ is nonzero. If the corresponding cyclic tensor product is positive semidefinite, then there are nonzero scalars c_n satisfying

$$\prod_{n=0}^{N-1} c_n = 1, \\ c_n \eta_{k_n, k_{n+1}} \geq 0 \quad (0 \leq n < N).$$

This is a choice on one fixed cycle. It does not assert that choices made on two different cycles agree on a common edge.

Proof. [✓ Proof](#) Reindex the cyclic sector block by the shared horizontal-bond indices. Its matrix entries then become the finite Kronecker product of the consecutive neighboring operators. Apply Theorem 20.15.4.24. $\square$

Corollary 20.15.4.36 (Positive rescaling of the concrete sector operators around one cycle). [Lean](#) [✓ Stmt](#) Let K be an injective MPDO, let R give a three-site closure, and fix an η -structure and a nonzero tail entry. Let $N \geq 1$. For a cyclic sector assignment $k_0, \dots, k_{N-1}$, suppose that every concrete operator $\eta_{k_n, k_{n+1}}$ obtained from these data is nonzero. Then there are nonzero scalars c_n such that

$$\prod_{n=0}^{N-1} c_n = 1, \\ c_n \eta_{k_n, k_{n+1}} \geq 0 \quad (0 \leq n < N).$$

This is the specialization of Theorem 20.15.4.35 to the inverse-map sector operators; positivity of the cyclic product follows from the MPDO property.

Proof. [✓ Proof](#) The fixed-sector cyclic product is positive semidefinite by Theorem 20.15.4.38. Apply Theorem 20.15.4.35. $\square$

Theorem 20.15.4.37 (The conjugated chain is a congruence). [Lean](#) [Lean](#) [✓ Stmt](#) For every matrix U on the physical space and every $N \geq 1$,

$$\rho^{(N)}(\tilde{K}) = U^{\otimes N} \rho^{(N)}(K) (U^{\otimes N})^\dagger.$$

In particular $\rho^{(N)}(\tilde{K}) \geq 0$ whenever $\rho^{(N)}(K) \geq 0$.

Proof. [✓ Proof](#) Expand the closed trace over bond configurations g (with $g_{N+1} = g_1$), and each conjugated local entry over its two physical indices:

$$\begin{aligned} [\rho^{(N)}(\tilde{K})]_{\sigma, \tau} &= \sum_g \prod_{n=1}^N \tilde{\kappa}_{g_n, g_{n+1}}^{\sigma_n \tau_n} \\ &= \sum_g \prod_{n=1}^N \sum_{p_n, q_n} U_{\sigma_n p_n} \kappa_{g_n, g_{n+1}}^{p_n q_n} \overline{U_{\tau_n q_n}}. \end{aligned}$$

Distributing the site product over the physical sums factors the two U -products out of the bond sum, which leaves the matrix entry of $U^{\otimes N} \rho^{(N)}(K) (U^{\otimes N})^\dagger$. Positivity is preserved by any congruence. $\square$

Theorem 20.15.4.38 (Positivity of the projected chain blocks). [Lean](#) [✓ Stmt](#) Let K be an MPDO whose transformed physical slices satisfy the sector factorization hypothesis. Then for every $N \geq 1$ and every cyclic sector assignment $k_1, \dots, k_N$ with $k_{N+1} = k_1$,

$$\begin{aligned} 0 &\leq [Q_{k_1} \otimes \dots \otimes Q_{k_N}] \tilde{\sigma} [Q_{k_1} \otimes \dots \otimes Q_{k_N}] \\ &= \bigotimes_{n=1}^N \eta_{k_n, k_{n+1}}. \end{aligned}$$

This is the positivity half of the display at [CPGSV16, Appendix C.2, lines 1446–1450].

Proof. [✓ Proof](#) The cyclic tensor product Ω_k of Definition 20.15.4.16 is a diagonal sector block of the basis-conjugated chain by Theorem 20.15.4.17. The conjugated chain is positive semidefinite by Theorem 20.15.4.37 and the MPDO property, and a principal submatrix of a positive semidefinite matrix is positive semidefinite. $\square$

Theorem 20.15.4.39 (Positivity of the diagonal and paired neighboring operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be an injective MPDO with the chosen Hayashi decomposition and a nonzero tail entry. Then for all sectors k, h :

1. $\eta_{k,k} \geq 0$;
2. $\eta_{k,h} \otimes \eta_{h,k} \geq 0$;
3. if $\eta_{k,h} \neq 0$ and $\eta_{h,k} \neq 0$, there is a scalar $c \neq 0$ with $c\eta_{k,h} \geq 0$, $c^{-1}\eta_{h,k} \geq 0$, and $(c\eta_{k,h}) \otimes (c^{-1}\eta_{h,k}) = \eta_{k,h} \otimes \eta_{h,k}$.

Proof. [✓ Proof](#) The chain of length one projected to the sector k is $\eta_{k,k}$ up to the exchange of its two tensor factors, and the chain of length two projected to (k, h) is $\eta_{k,h} \otimes \eta_{h,k}$, both positive by Theorem 20.15.4.38. The third item, including the Kronecker-product-preserving equality, is Theorem 20.15.4.19. $\square$

Theorem 20.15.4.40 (Coherent positive rephasing under recurrent support). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $(\eta_{k,h})$ be a neighboring-operator family for which every nonempty cyclic tensor product is positive semidefinite. Suppose, in addition, that every nonzero edge $k \rightarrow h$ admits a directed return path from h to k . Then there are unit complex numbers z_k such that $z_k^{-1} z_h \eta_{k,h} \geq 0$ for all k and h . Consequently, if the neighboring operators arise from the inverse-map physical-sector factorization of an injective MPDO and its nonzero support is recurrent, the sector tensors admit a reciprocal rephasing for which all neighboring operators are positive semidefinite. This factorization determines the positive commuting nearest-neighbor product structure of Proposition C.8.

This is a scope-restricted result. Recurrence is not a hypothesis of Lemma C.4 in [CPGSV16, Appendix C.2, lines 1406–1450]. Its derivation for the zero-weight reparameterized inverse-map factorization is treated in Theorem 20.15.4.41 and recorded in [con26t].

Proof. [✓ Proof](#) For a nonzero edge, choose a directed return path and close it with the edge. Positive rescaling of the corresponding cyclic tensor product supplies a scalar that makes the edge operator positive; retain only its unit-modulus part. On any closed directed walk, positive rescaling of the associated cyclic tensor product gives phases whose product is one. Uniqueness of the unit phase identifies them edge by edge with the chosen phases. Hence the chosen edge phases have product one around every closed walk. The closed-walk criterion gives $\kappa_{k,h} = z_k^{-1} z_h$ on every nonzero edge. Zero edge operators are already positive semidefinite. Finally, reciprocal rephasing of the left and right sector tensors multiplies each neighboring operator by this phase and preserves the physical-slice factorization. The conditional eta-local construction then gives the positive commuting product data. $\square$

Theorem 20.15.4.41 (Coherent positive choice of the neighboring operators). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be an injective MPO tensor satisfying SAL. Then it admits a physical-sector factorization for which every neighboring operator $\eta_{k,h} = \sum_a (r_k)_a \otimes (l_h)_a$ is positive semidefinite. Zero-weight Hayashi sectors remain in the physical direct sum, but their right tensors may be set to zero without changing the physical slices.

Proof. [✓ Proof](#) First set the right tensor to zero in every zero-weight sector. The left tensor already vanishes there, so this does not alter the factorization and removes every support edge incident to such a sector. Every remaining endpoint has positive weight. If all virtual matrices in a positive-weight sector vanished, then the corresponding middle-site block of the three-site closure would vanish; this contradicts its Hayashi expression as a nonzero weight times two trace-one sector states. Thus every endpoint of a nonzero edge contains a nonzero virtual matrix. Injectivity supplies spanning by all sector virtual matrices, and triangle closure gives recurrence. The projected chain identity makes every cyclic Kronecker product positive semidefinite. Apply Theorem 20.15.4.40 to obtain the coherent vertex rephasing. $\square$

Theorem 20.15.4.42 (Positivity choice with symmetric support). [Lean](#) [✓ Stmt](#) In the setting of Theorem 20.15.4.39, assume in addition that the nonzero neighboring operators occur in symmetric pairs: $\eta_{k,h} = 0$ forces $\eta_{h,k} = 0$. Then there is a positive semidefinite family $\eta'_{k,h}$ with $\eta'_{k,k} = \eta_{k,k}$ and, for every pair of sectors, scalars $c_{k,h} \neq 0$ with $\eta'_{k,h} = c_{k,h} \eta_{k,h}$ and $\eta'_{h,k} = c_{k,h}^{-1} \eta_{h,k}$. In particular every two-site block $\eta'_{k,h} \otimes \eta'_{h,k} = \eta_{k,h} \otimes \eta_{h,k}$ of the sector-adapted decomposition is unchanged.

The source asserts at [CPGSV16, Appendix C.2, lines 1446–1450] that the η 's can be chosen positive semidefinite, without the symmetric-support hypothesis. That hypothesis excludes a scalar obstruction: a nonzero neighboring operator whose reverse pair vanishes is constrained only by cycles through at least three sectors, and a coherent choice for the whole family can then fail.

Proof. [✓ Proof](#) The diagonal operators are positive by Theorem 20.15.4.39, and their scalar is taken to be one. For a pair of distinct sectors, either both neighboring operators vanish and any nonzero scalar works, or both are nonzero by symmetric support and Theorem 20.15.4.39 supplies the reciprocal pair of scalars. Choosing the scalar once per unordered pair makes the two assignments reciprocal, and the block-preserving equality of Theorem 20.15.4.19 applied to that scalar gives $\eta'_{k,h} \otimes \eta'_{h,k} = \eta_{k,h} \otimes \eta_{h,k}$. $\square$

20.15.5 Normalized preparations and controlled partial traces

Definition 20.15.5.1 (Trace matrices of explicit η -data). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Every explicit neighboring η -family determines a sector-by-sector trace matrix. We give both its complex-valued form and the real-part version, which is the direct data for the real Perron–Frobenius matrix T used in the rank-one step.

Definition 20.15.5.2 (Rank-one factorization of neighboring-operator traces). [Lean](#) [✓ Stmt](#) Let $(\eta_{k,h})$ be a positive neighboring-operator family. A rank-one trace factorization consists of real families (a_k) and (b_h) such that

$$\begin{aligned} \text{tr}(\eta_{k,h}) &= a_k b_h, \\ \sum_l a_l b_l &= 1. \end{aligned}$$

These identities occur in [CPGSV16, Appendix C.2, lines 1395–1402].

Definition 20.15.5.3 (Direct-sum neighboring operator). [Lean](#) [Lean](#) [✓ Stmt](#) For sectors k, h , define the positive-operator candidate

$$\Omega_{k,h} = \bigoplus_l (\eta_{k,l} \otimes \eta_{l,h}).$$

The summand indexed by l acts on $(B_k^R \otimes B_l^L) \otimes (B_l^R \otimes B_h^L)$. This is the operator adjointed by $\mathcal{T}_{k,h}$ in [CPGSV16, Appendix C.2, lines 1527–1535].

Theorem 20.15.5.4 (Positivity of finite dependent block diagonals). [Lean](#) [✓ Stmt](#) Let $(A_i)_{i \in I}$ be a finite family of positive semidefinite matrices, where the size of A_i may depend on i . Then the block-diagonal matrix $\bigoplus_{i \in I} A_i$ is positive semidefinite.

Proof. [✓ Proof](#) Write each $A_i = B_i^* B_i$ using its positive square root. The dependent block diagonal then satisfies $\bigoplus_i A_i = (\bigoplus_i B_i)^* (\bigoplus_i B_i)$. $\square$

Theorem 20.15.5.5 (Positivity of the direct-sum neighboring operator). [Lean](#) [✓ Stmt](#) Every operator $\Omega_{k,h}$ is positive semidefinite.

Proof. [✓ Proof](#) For every l , positivity is preserved by the Kronecker product: $\eta_{k,l} \succeq 0$ and $\eta_{l,h} \succeq 0$ imply $\eta_{k,l} \otimes \eta_{l,h} \succeq 0$. Apply Theorem 20.15.5.4 to the family $(\eta_{k,l} \otimes \eta_{l,h})_l$. $\square$

Theorem 20.15.5.6 (Trace of the direct-sum neighboring operator). [Lean](#) [Lean](#) [✓ Stmt](#) The trace is

$$\text{tr}(\Omega_{k,h}) = \sum_l \text{tr}(\eta_{k,l}) \text{tr}(\eta_{l,h}).$$

Under the rank-one trace factorization, this becomes

$$\text{tr}(\Omega_{k,h}) = a_k b_h \sum_l a_l b_l = a_k b_h.$$

This is the trace calculation in [CPGSV16, Appendix C.2, lines 1531–1535].

Proof. [✓ Proof](#) The direct-sum and Kronecker trace identities give

$$\text{tr}(\Omega_{k,h}) = \sum_l \text{tr}(\eta_{k,l} \otimes \eta_{l,h}) = \sum_l \text{tr}(\eta_{k,l}) \text{tr}(\eta_{l,h}).$$

Substitute $\text{tr}(\eta_{k,h}) = a_k b_h$ and $\sum_l a_l b_l = 1$. $\square$

Theorem 20.15.5.7 (Vanishing of zero-weight neighboring operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Every product $a_k b_h$ is non-negative. If $a_k b_h = 0$, then $\eta_{k,h} = 0$ and $\Omega_{k,h} = 0$.

Proof. [✓ Proof](#) Since $\eta_{k,h} \succeq 0$, its trace is non-negative, and hence $a_k b_h = \text{tr}(\eta_{k,h}) \geq 0$. If $a_k b_h = 0$, then $\eta_{k,h} \succeq 0$ and $\text{tr}(\eta_{k,h}) = a_k b_h = 0$, which imply $\eta_{k,h} = 0$. The same implication gives $\Omega_{k,h} = 0$ because $\Omega_{k,h} \succeq 0$ and $\text{tr}(\Omega_{k,h}) = a_k b_h = 0$. $\square$

Definition 20.15.5.8 (Normalized active neighboring operators). [Lean](#) [Lean](#) [✓ Stmt](#) With the convention $0^{-1} = 0$, define for every pair (k, h)

$$\begin{aligned} \hat{\eta}_{k,h} &= (a_k b_h)^{-1} \eta_{k,h}, \\ \hat{\Omega}_{k,h} &= (a_k b_h)^{-1} \Omega_{k,h}. \end{aligned}$$

These operators are normalized when the pair is active, namely when $a_k b_h \neq 0$.

Theorem 20.15.5.9 (Positivity of normalized neighboring operators). [Lean](#) [Lean](#) [✓ Stmt](#) For every pair (k, h) , the operators $\hat{\eta}_{k,h}$ and $\hat{\Omega}_{k,h}$ are positive semidefinite.

Proof. [✓ Proof](#) By Theorem 20.15.5.7, $a_k b_h \geq 0$, and hence $(a_k b_h)^{-1} \geq 0$. Since $\eta_{k,h} \succeq 0$ and $\Omega_{k,h} \succeq 0$, multiplication by $(a_k b_h)^{-1}$ gives $\hat{\eta}_{k,h} \succeq 0$ and $\hat{\Omega}_{k,h} \succeq 0$. $\square$

Theorem 20.15.5.10 (Normalized active preparations). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $a_k b_h \neq 0$, then $\hat{\eta}_{k,h}$ and $\hat{\Omega}_{k,h}$ have trace one. Consequently the maps

$$\begin{aligned} X &\mapsto X \otimes \hat{\eta}_{k,h}, \\ X &\mapsto X \otimes \hat{\Omega}_{k,h} \end{aligned}$$

are trace-preserving and completely positive. These are the individual preparations $\mathcal{S}_{k,h}$ and $\mathcal{T}_{k,h}$ in [CPGSV16, Appendix C.2, lines 1527–1535 and 1551–1555].

Proof. [✓ Proof](#) On an active pair, $\text{tr}(\hat{\eta}_{k,h}) = (a_k b_h)^{-1} \text{tr}(\eta_{k,h}) = (a_k b_h)^{-1} a_k b_h = 1$. Likewise, $\text{tr}(\hat{\Omega}_{k,h}) = 1$ follows from $\text{tr}(\Omega_{k,h}) = a_k b_h$. Apply Lemma [LTC26, “State preparation is trace-preserving completely positive”] to each operator. $\square$

Lemma 20.15.5.11 (Trace-one matrices have nonempty index spaces). [Lean](#) [✓ Stmt](#) If a matrix indexed by a finite type has trace one, then its index type is nonempty.

Proof. [✓ Proof](#) A matrix on an empty index space has trace zero, contrary to the hypothesis. $\square$

Lemma 20.15.5.12 (Nonemptiness of the Hayashi sector factors). [Lean](#) [Lean](#) [✓ Stmt](#) Every left and right factor in a Hayashi sector is nonzero-dimensional.

Proof. [✓ Proof](#) A matrix on an empty index space has trace zero. Hence $\text{tr}(\rho_k^R) = 1$ implies $\dim(B_k^R \otimes C) > 0$, which implies $\dim B_k^R > 0$, and the same argument applied to ρ_k^L gives $\dim B_k^L > 0$. $\square$

Lemma 20.15.5.13 (Existence of a Hayashi sector). [Lean](#) [✓ Stmt](#) The set of Hayashi sectors is nonempty.

Proof. [✓ Proof](#) The sector-weight identity $\sum_k p_k = 1$ implies $m > 0$. $\square$

Lemma 20.15.5.14 (Nonemptiness of the neighboring auxiliary spaces). [Lean](#) [Lean](#) [✓ Stmt](#) For every pair of sectors (k, h) , the spaces carrying $\eta_{k,h}$ and $\Omega_{k,h}$ are nonzero-dimensional.

Proof. [✓ Proof](#) The tensor product carrying $\eta_{k,h}$ is nonzero-dimensional by Lemma 20.15.5.12. Choosing an intermediate sector l by Lemma 20.15.5.13 then supplies an element of the direct-sum space carrying $\Omega_{k,h}$. $\square$

Definition 20.15.5.15 (Faithful uniform density matrix). [Lean](#) [✓ Stmt](#) On a nonzero finite-dimensional space H , define

$$\tau_H = \frac{1}{\dim H} I_H.$$

Theorem 20.15.5.16 (The uniform density matrix is faithful and normalized). [Lean](#) [Lean](#) [✓ Stmt](#) The matrix τ_H is positive definite and has trace one.

Proof. [✓ Proof](#) Since $\dim H > 0$, the scalar $(\dim H)^{-1}$ is positive. Thus τ_H is positive definite, and $\text{tr}(\tau_H) = (\dim H)^{-1} \dim H = 1$. $\square$

Definition 20.15.5.17 (Completed neighboring-state preparation). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) For every sector pair define

$$\bar{\eta}_{k,h} = \begin{cases} \hat{\eta}_{k,h}, & a_k b_h \neq 0, \\ \tau_{B_k^R \otimes B_h^L}, & a_k b_h = 0, \end{cases}$$

$$\bar{\mathcal{S}}_{k,h}(X) = X \otimes \bar{\eta}_{k,h}.$$

On an active pair this is precisely the preparation $\mathcal{S}_{k,h}$ of [CPGSV16, Appendix C.2, lines 1551–1555]; the second branch gives a normalized extension on a pair where the displayed source quotient is undefined.

Theorem 20.15.5.18 (Completed neighboring-state preparations are trace-preserving completely positive). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmt](#) For every pair (k, h) , the operator $\bar{\eta}_{k,h}$ is positive semidefinite with trace one. Hence $\bar{\mathcal{S}}_{k,h}$ is trace-preserving and completely positive. If $a_k b_h \neq 0$, then both the density matrix and the preparation equal their normalized active forms.

Proof. [Stmt](#) The two branches give

$$\begin{aligned} a_k b_h \neq 0 &\implies \bar{\eta}_{k,h} = \hat{\eta}_{k,h} \succeq 0, \quad \text{tr}(\bar{\eta}_{k,h}) = 1, \\ a_k b_h = 0 &\implies \bar{\eta}_{k,h} = \tau_{B_k^R \otimes B_h^L} \succ 0, \quad \text{tr}(\bar{\eta}_{k,h}) = 1. \end{aligned}$$

The state-preparation result applies in either case. □

Definition 20.15.5.19 (Completed direct-sum preparation). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) For every sector pair define

$$\mathcal{H}_{k,h}^\Omega = \bigoplus_l (B_k^R \otimes B_l^L) \otimes (B_l^R \otimes B_h^L),$$

$$\bar{\Omega}_{k,h} = \begin{cases} \hat{\Omega}_{k,h}, & a_k b_h \neq 0, \\ \tau_{\mathcal{H}_{k,h}^\Omega}, & a_k b_h = 0, \end{cases}$$

$$\bar{\mathcal{T}}_{k,h}(X) = X \otimes \bar{\Omega}_{k,h}.$$

Thus $\mathcal{H}_{k,h}^\Omega$ is the carrier space, whereas $\Omega_{k,h}$, $\hat{\Omega}_{k,h}$, and $\bar{\Omega}_{k,h}$ are operators on it: respectively the unnormalized neighboring operator, its normalization on an active pair, and its completed density on an arbitrary pair. On an active pair this is precisely the preparation $\mathcal{T}_{k,h}$ of [CPGSV16, Appendix C.2, lines 1527–1535]; the inactive branch is a normalized extension.

Theorem 20.15.5.20 (Completed direct-sum preparations are trace-preserving completely positive). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Stmt](#) For every pair (k, h) , the operator $\bar{\Omega}_{k,h}$ is positive semidefinite with trace one. Hence $\bar{\mathcal{T}}_{k,h}$ is trace-preserving and completely positive. On every active pair it equals the normalized active preparation.

Proof. [Stmt](#) The two branches give

$$\begin{aligned} a_k b_h \neq 0 &\implies \bar{\Omega}_{k,h} = \hat{\Omega}_{k,h} \succeq 0, \quad \text{tr}(\bar{\Omega}_{k,h}) = 1, \\ a_k b_h = 0 &\implies \bar{\Omega}_{k,h} = \tau_{\mathcal{H}_{k,h}^\Omega} \succ 0, \quad \text{tr}(\bar{\Omega}_{k,h}) = 1. \end{aligned}$$

The state-preparation result therefore applies to $\bar{\mathcal{T}}_{k,h}$. □

Definition 20.15.5.21 (Globally controlled neighboring-state preparation). [Lean](#) [✓ Stmt](#) Let $H = \bigoplus_{k,h} H_{k,h}$. Orthogonally controlling the completed preparations over the sector pairs gives a map

$$\bar{\mathcal{S}}_1(X) = \sum_{k,h,a} \tilde{A}_{k,h,a} X \tilde{A}_{k,h,a}^\dagger,$$

where $(A_{k,h,a})_a$ is a Kraus family for $\bar{\mathcal{S}}_{k,h}$ and the tilde denotes its extension by zero outside $H_{k,h}$. Thus the map discards inter-sector coherences and applies the completed preparation in each diagonal sector. On active sectors this is the sector-control formula for $\mathcal{S}_1$ in [CPGSV16, Appendix C.2, lines 1548–1555].

Theorem 20.15.5.22 (Diagonal-sector action of the completed neighboring preparation). [Lean](#) [✓ Stmt](#) If $X_{k,h}$ is the (k,h) diagonal block of X , then

$$[\bar{\mathcal{S}}_1(X)]_{k,h} = \bar{\mathcal{S}}_{k,h}(X_{k,h}) = X_{k,h} \otimes \bar{\eta}_{k,h}.$$

Proof. [✓ Proof](#)

The diagonal-block formula and invariance under Kraus-index relabeling give

$$\begin{aligned} [\bar{\mathcal{S}}_1(X)]_{k,h} &= \Phi_{\tilde{A}_{k,h}}(X_{k,h}) = \Phi_{A_{k,h}}(X_{k,h}), \\ \Phi_{A_{k,h}}(X_{k,h}) &= X_{k,h} \otimes \bar{\eta}_{k,h}. \end{aligned}$$

□

Theorem 20.15.5.23 (The globally controlled neighboring-state preparation is trace-preserving completely positive). [Lean](#) [✓ Stmt](#) The map $\bar{\mathcal{S}}_1$ is trace-preserving and completely positive.

Proof. [✓ Proof](#) If $(A_{k,h,a})_a$ is the sectorwise Kraus family, then

$$\sum_{k,h,a} \tilde{A}_{k,h,a}^\dagger \tilde{A}_{k,h,a} = \bigoplus_{k,h} \left(\sum_a A_{k,h,a}^\dagger A_{k,h,a} \right) = \bigoplus_{k,h} I_{H_{k,h}} = I_H.$$

Thus the controlled Kraus family is trace-preserving. □

Definition 20.15.5.24 (Globally controlled direct-sum preparation). [Lean](#) [✓ Stmt](#) Orthogonally controlling the completed direct-sum preparations gives

$$\bar{\mathcal{T}}_1(X) = \sum_{k,h,a} \tilde{B}_{k,h,a} X \tilde{B}_{k,h,a}^\dagger,$$

where $(B_{k,h,a})_a$ is a Kraus family for $\bar{\mathcal{T}}_{k,h}$. This map discards inter-sector coherences and, on active sectors, is the sector-control formula for $\mathcal{T}_1$ in [CPGSV16, Appendix C.2, lines 1523–1535].

Theorem 20.15.5.25 (Diagonal-sector action of the completed direct-sum preparation). [Lean](#)

[✓ Stmt](#) If $X_{k,h}$ is the (k,h) diagonal block of X , then

$$[\bar{\mathcal{T}}_1(X)]_{k,h} = \bar{\mathcal{T}}_{k,h}(X_{k,h}) = X_{k,h} \otimes \bar{\Omega}_{k,h}.$$

Proof. [✓ Proof](#)

The diagonal-block formula and invariance under Kraus-index relabeling give

$$\begin{aligned} [\bar{\mathcal{T}}_1(X)]_{k,h} &= \Phi_{\tilde{B}_{k,h}}(X_{k,h}) = \Phi_{B_{k,h}}(X_{k,h}), \\ \Phi_{B_{k,h}}(X_{k,h}) &= X_{k,h} \otimes \bar{\Omega}_{k,h}. \end{aligned}$$

□

Proof. ✓ Proof For the sectorwise Kraus family $(B_{k,h,a})_a$,

$$\sum_{k,h,a} \widetilde{B}_{k,h,a}^\dagger \widetilde{B}_{k,h,a} = \bigoplus_{k,h} \left(\sum_a B_{k,h,a}^\dagger B_{k,h,a} \right) = \bigoplus_{k,h} I_{H_{k,h}} = I_H.$$

☐

The corresponding generic result is proved in the companion quantum-channel volume [LTC26, “The controlled dependent partial trace is trace-preserving completely positive”].

$$((l_k, r_k), (l_h, r_h)) \mapsto ((l_k, r_h), (r_k, l_h)).$$
$$\left(\bigoplus_k L_k \otimes R_k\right)^{\otimes 2} \simeq \bigoplus_{k,h} (L_k \otimes R_h) \otimes (R_k \otimes L_h).$$

Definition 20.15.5.29 (The two-site partial-trace map $\mathcal{T}_0$). [Lean](#) [Lean](#) [✓ Stmt](#) Define $\mathcal{T}_0$ by the two-site regrouping followed, in each pair (k, h) , by the partial trace over $R_k \otimes L_h$. Its output is indexed by the retained outer factors $L_k \otimes R_h$. Thus, writing R_{Φ_2} for the regrouping reindexing,

$$\mathcal{T}_0 = \text{CTr}_{(R_k \otimes L_h)_{k,h}} \circ R_{\Phi_2}. \quad (20.58)$$

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Proof. ✓ Proof

By (20.58), $\mathcal{T}_0$ is the composition of the controlled dependent partial trace with reindexing along the two-site equivalence. The first map is trace-preserving and completely positive by Theorem [LTC26, “The controlled dependent partial trace is trace-preserving completely positive”], and the second by Theorem [LTC26, “Equivalence reindexing is trace-preserving completely positive”]. Their composition is therefore trace-preserving and completely positive. $\square$

Definition 20.15.5.31 (The two-site shift $\mathcal{S}_2$). Lean Lean Lean Lean ✓ Stmt The map $\mathcal{S}_2$ is matrix reindexing along the inverse two-site equivalence. Thus

$$((l_k, r_h), (r_k, l_h)) \mapsto ((l_k, r_k), (l_h, r_h)).$$

Theorem 20.15.5.32 ($\mathcal{S}_2$ is trace-preserving completely positive). Lean Lean ✓ Stmt The shift $\mathcal{S}_2$ is trace-preserving and completely positive.

Proof. ✓ Proof

If Φ_2 is the two-site regrouping, then $\mathcal{S}_2 = R_{\Phi_2^{-1}}$. It is therefore trace-preserving and completely positive by Theorem [LTC26, “Equivalence reindexing is trace-preserving completely positive”]. $\square$

Definition 20.15.5.33 (Three-site subspin regrouping). Lean Lean Lean Lean Lean Lean ✓ Stmt Regroup three sites by

$$((l_k, r_k), ((l_l, r_l), (l_h, r_h))) \mapsto ((l_k, r_k), ((r_k, l_l), (r_l, l_h))).$$

Equivalently,

$$\left(\bigoplus_k L_k \otimes R_k \right)^{\otimes 3} \simeq \bigoplus_{k,h} (L_k \otimes R_h) \otimes \left[\bigoplus_l (R_k \otimes L_l) \otimes (R_l \otimes L_h) \right].$$

The forward equivalence prepares the four middle subspins for $\mathcal{S}_0$; its inverse is the shift $\mathcal{T}_2$ in [CPGSV16, Appendix C.2, lines 1535–1540 and 1547].

Definition 20.15.5.34 (The three-site partial-trace map $\mathcal{S}_0$). Lean Lean Lean ✓ Stmt Define $\mathcal{S}_0$ by the three-site regrouping followed, for every outer pair (k, h) , by the partial trace over

$$\bigoplus_l (R_k \otimes L_l) \otimes (R_l \otimes L_h).$$

Thus, writing R_{Φ_3} for the regrouping reindexing,

$$\mathcal{S}_0 = \text{CTr}_{(\bigoplus_l (R_k \otimes L_l) \otimes (R_l \otimes L_h))_{k,h}} \circ R_{\Phi_3}. \quad (20.59)$$

Theorem 20.15.5.35 ($\mathcal{S}_0$ is trace-preserving completely positive). Lean Lean ✓ Stmt The map $\mathcal{S}_0$ is trace-preserving and completely positive.

Proof. ✓ Proof

By (20.59), $\mathcal{S}_0$ is the composition of the controlled dependent partial trace with reindexing along the three-site equivalence. The controlled trace and equivalence reindexing are trace-preserving and completely positive by Theorems [LTC26, “The controlled dependent partial trace is trace-preserving completely positive”] and [LTC26, “Equivalence reindexing is trace-preserving completely positive”], respectively. Their composition has the same property. $\square$

Definition 20.15.5.36 (The three-site shift $\mathcal{T}_2$). [Lean](#) [Lean](#) [✓ Stmt](#) The map $\mathcal{T}_2$ is matrix reindexing along the inverse three-site equivalence. On each summand it sends

$$((l_k, r_h), ((r_k, l_l), (r_l, l_h))) \mapsto ((l_k, r_k), ((l_l, r_l), (l_h, r_h))).$$

Theorem 20.15.5.37 ($\mathcal{T}_2$ is trace-preserving completely positive). [Lean](#) [Lean](#) [✓ Stmt](#) The shift $\mathcal{T}_2$ is trace-preserving and completely positive.

Proof. [✓ Proof](#)

If Φ_3 is the three-site regrouping, then $\mathcal{T}_2 = R_{\Phi_3^{-1}}$. It is therefore trace-preserving and completely positive by Theorem [\[LTC26\]](#), “Equivalence reindexing is trace-preserving completely positive”]. $\square$

Definition 20.15.5.38 (Fiberwise partial-trace maps). [Lean](#) [Lean](#) [✓ Stmt](#) For fixed sectors (k, h) , define the fiberwise form of $\mathcal{T}_0$ by regrouping two sites and tracing $R_k \otimes L_h$. For fixed (k, l, h) , define the fiberwise form of $\mathcal{S}_0$ by regrouping three sites and tracing $(R_k \otimes L_l) \otimes (R_l \otimes L_h)$.

Theorem 20.15.5.39 (The fiberwise partial traces are trace-preserving completely positive). [Lean](#) [Lean](#) [✓ Stmt](#) Every fiberwise form of $\mathcal{T}_0$ and $\mathcal{S}_0$ is trace-preserving and completely positive.

Proof. [✓ Proof](#)

The fiberwise forms factor as

$$\begin{aligned} \mathcal{T}_0^{(k,h)} &= \text{tr}_{R_k \otimes L_h} \circ R_{\phi_{k,h}}, \\ \mathcal{S}_0^{(k,l,h)} &= \text{tr}_{(R_k \otimes L_l) \otimes (R_l \otimes L_h)} \circ R_{\phi_{k,l,h}}, \end{aligned}$$

where $\phi_{k,h}$ is the two-site equivalence of Definition [20.15.5.28](#) on the (k, h) sector, $\phi_{k,l,h}$ is the three-site equivalence of Definition [20.15.5.33](#) on the (k, l, h) sector, and R_ϕ denotes equivalence reindexing. Both factors in each composition are trace-preserving and completely positive, so the composition is as well. $\square$

Definition 20.15.5.40 (Fiberwise shifts). [Lean](#) [Lean](#) [✓ Stmt](#) On fixed sector pairs, define $\mathcal{S}_2$ and $\mathcal{T}_2$ by the same inverse regroupings as the corresponding global shifts.

Theorem 20.15.5.41 (The fiberwise shifts are trace-preserving completely positive). [Lean](#) [Lean](#) [✓ Stmt](#) Every fiberwise shift is trace-preserving and completely positive.

Proof. [✓ Proof](#)

The shifts $\mathcal{S}_2^{(k,h)}$ and $\mathcal{T}_2^{(k,h)}$ are reindexings along the inverse two-site and three-site equivalences of Definitions [20.15.5.28](#) and [20.15.5.33](#), respectively. They are therefore trace-preserving and completely positive by Theorem [\[LTC26\]](#), “Equivalence reindexing is trace-preserving completely positive”]. $\square$

To record the domains and codomains, set

$$\mathcal{H}_{k,h}^\Omega = \bigoplus_l (B_k^R \otimes B_l^L) \otimes (B_l^R \otimes B_h^L),$$

and introduce the matrix algebras

$$\begin{aligned}\mathcal{A}_2 &= \text{End}\left(\left[\bigoplus_j (B_j^L \otimes B_j^R)\right]^{\otimes 2}\right), & \mathcal{A}_\partial &= \text{End}\left(\bigoplus_{k,h} B_k^L \otimes B_h^R\right), \\ \mathcal{A}_\eta &= \text{End}\left(\bigoplus_{k,h} [(B_k^L \otimes B_h^R) \otimes (B_k^R \otimes B_h^L)]\right), & \mathcal{A}_\Omega &= \text{End}\left(\bigoplus_{k,h} [(B_k^L \otimes B_h^R) \otimes \mathcal{H}_{k,h}^\Omega]\right), \\ \mathcal{A}_3 &= \text{End}\left(\left[\bigoplus_j (B_j^L \otimes B_j^R)\right]^{\otimes 3}\right).\end{aligned}$$

Thus the global maps have types

$$\begin{aligned}\mathcal{T}_0 &: \mathcal{A}_2 \rightarrow \mathcal{A}_\partial, \\ \mathcal{S}_2 &: \mathcal{A}_\eta \rightarrow \mathcal{A}_2, \\ \mathcal{S}_0 &: \mathcal{A}_3 \rightarrow \mathcal{A}_\partial, \\ \mathcal{T}_2 &: \mathcal{A}_\Omega \rightarrow \mathcal{A}_3.\end{aligned}$$

The controlled maps $\mathcal{T}_0$ and $\mathcal{S}_0$ discard coherences between distinct outer-sector pairs. For fixed (k, h) , the global map $\mathcal{S}_0$ traces the entire carrier $\mathcal{H}_{k,h}^\Omega$, including its direct sum over l . The shifts $\mathcal{S}_2$ and $\mathcal{T}_2$ are global reindexings, with all displayed tensor factors present at their inputs [CPGSV16, Appendix C.2, lines 1521–1559].

For the fiberwise maps, write

$$\begin{aligned}\mathcal{A}_2^{k,h} &= \text{End}((B_k^L \otimes B_k^R) \otimes (B_h^L \otimes B_h^R)), & \mathcal{A}_\partial^{k,h} &= \text{End}(B_k^L \otimes B_h^R), \\ \mathcal{A}_\eta^{k,h} &= \text{End}((B_k^L \otimes B_h^R) \otimes (B_k^R \otimes B_h^L)), & \mathcal{A}_\Omega^{k,h} &= \text{End}((B_k^L \otimes B_h^R) \otimes \mathcal{H}_{k,h}^\Omega), \\ \mathcal{A}_3^{k,l,h} &= \text{End}((B_k^L \otimes B_k^R) \otimes (B_l^L \otimes B_l^R) \otimes (B_h^L \otimes B_h^R)), \\ \mathcal{A}_3^{k,\bullet,h} &= \text{End}\left(\bigoplus_l [(B_k^L \otimes B_k^R) \otimes (B_l^L \otimes B_l^R) \otimes (B_h^L \otimes B_h^R)]\right).\end{aligned}$$

The map $\mathcal{S}_0^{(k,l,h)}$ fixes the middle sector l before taking the partial trace; it is therefore distinct from the fixed- (k, h) restriction of the global controlled map $\mathcal{S}_0$. The four fiberwise types are

$$\mathcal{A}_2^{k,h} \xrightarrow{\mathcal{T}_0^{(k,h)}} \mathcal{A}_\partial^{k,h}$$

$$\mathcal{A}_\eta^{k,h} \xrightarrow{\mathcal{S}_2^{(k,h)}} \mathcal{A}_2^{k,h}$$

$$\mathcal{A}_3^{k,l,h} \xrightarrow{\mathcal{S}_0^{(k,l,h)}} \mathcal{A}_\partial^{k,h}$$

$$\mathcal{A}_\Omega^{k,h} \xrightarrow{\mathcal{T}_2^{(k,h)}} \mathcal{A}_3^{k,\bullet,h}$$

20.15.6 Primitivity and rank-one trace matrices

Definition 20.15.6.1 (Explicit η -data from the sector tensors). [Lean](#) [✓ Stmt](#) If every neighboring operator $\eta_{k,h}$ of Definition 20.15.3.1 is positive semidefinite, the family constitutes explicit

neighboring η -data. The source obtains the positivity from the projected chain identity: writing $\tilde{\sigma} = U^{\dagger \otimes N} \sigma^{(N)}(K) U^{\otimes N}$ for the N -site operator in the sector basis and Q_k for the projector onto the k -th sector of the physical splitting, every cyclic sector assignment $k_1, \dots, k_N$ (with $k_{N+1} = k_1$) satisfies

$$0 \leq [Q_{k_1} \otimes \dots \otimes Q_{k_N}] \tilde{\sigma} [Q_{k_1} \otimes \dots \otimes Q_{k_N}] = \bigotimes_{n=1}^N \eta_{k_n, k_{n+1}},$$

and the sector tensors can be chosen so that each factor is positive semidefinite [CPGSV16, Appendix C.2, lines 1446–1450]. The all-length sector identity is Theorem 20.15.4.18. That algebraic identity does not by itself choose coherent positive representatives for all neighboring pairs, so positive semidefiniteness remains a hypothesis on the chosen representatives here. Theorem 20.15.4.39 provides the diagonal and paired positivity, and Theorem 20.15.4.42 a rescaled positive family under symmetric support.

Theorem 20.15.6.2 (Trace matrix of the sector-tensor η -data). Lean ✓ Stmt *Under the hypotheses of Definition 20.15.6.1, the trace matrix of the explicit η -data is the pairing of the closed sector tensors:*

$$T_{k,h} = \text{tr}(\eta_{k,h}) = (r_k | l_h).$$

Proof. ✓ Proof Immediate from Theorem 20.15.4.3 and the definition of the trace matrix. □

Theorem 20.15.6.3 (Primitivity from returns of lengths two and three). Lean ✓ Stmt *Let T be a non-negative irreducible matrix on a finite index set. If $(T^2)_{k,k} > 0$ and $(T^3)_{k,k} > 0$ for every index k , then T is primitive.*

Proof. ✓ Proof Choose a positive path between every ordered pair of indices, and let L be the sum of the chosen path lengths. Every chosen length is at most L . For a chosen path of length m , write the difference $2L + 2 - m$ as $2a + 3b$ with $a, b \geq 0$. Appending these returns at the terminal index gives a positive path of common length $2L + 2$ between every ordered pair. □

Definition 20.15.6.4 (Nonzero-weight sector trace matrix). Lean Lean Lean Lean Lean Lean ✓ Stmt For sector weights p_k , let $I_* = \{k : p_k \neq 0\}$. The one-site matrices and their spaces on I_* are

$$A_{k;x,y}, \\ A_S = \text{span}\{A_{k;x,y} : k \in S\}, \quad S \subseteq I_*.$$

The real trace matrix on this index set is

$$T_{k,h}^* = \text{Re tr}(\eta_{k,h}), \quad k, h \in I_*.$$

Only the trace matrix is restricted: the physical direct sum still contains every zero-weight summand.

Theorem 20.15.6.5 (Identification and spanning of active one-site matrices). Lean Lean ✓ Stmt *For every sector entry, the one-site matrix used in the directed-cut argument equals the virtual matrix used in the spanning argument. If all sector virtual matrices span $M_D(\mathbb{C})$ and the matrices in every zero-weight sector vanish, then $\text{span}\{A_{k;x,y} : k \in I_*\} = M_D(\mathbb{C})$.*

Theorem 20.15.6.6 (Complementary active-sector spaces). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the active one-site matrices span $M_D(\mathbb{C})$ and each active sector contains a nonzero such matrix. For every $S \subseteq I_*$, $A_S + A_{I_* \setminus S} = M_D(\mathbb{C})$, and both spaces are nonzero when the corresponding sector sets are nonempty. If $\eta_{k,h} = 0$ for $k \in S$ and $h \in C$, then $A_S A_C = 0$.

Theorem 20.15.6.7 (Strong connectivity of the active neighboring support). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the active one-site matrices span $M_D(\mathbb{C})$ and each active sector contains a nonzero such matrix. Then for every $k, h \in I_*$ there is a directed walk

$$k = k_0 \longrightarrow k_1 \longrightarrow \dots \longrightarrow k_m = h,$$

such that $\eta_{k_j, k_{j+1}} \neq 0$. In particular, every active sector has an outgoing nonzero neighboring operator.

$$P_j \mathcal{K}_i P_{j'} = 0 \quad \text{unless } i = j = j'.$$

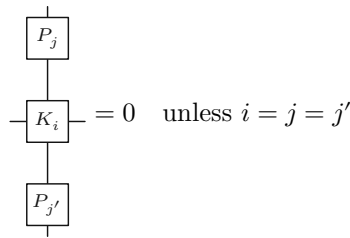


Figure 20.5: The local-orthogonality contraction of [CPGSV16, Appendix C.2, lines 1457–1470]. After decomposing the sector trace matrix into primitive components, the source assumes that two components occur, obtains two locally orthogonal summands, and chooses orthogonal projectors P_1 and P_2 . The displayed identity gives the cross contractions $P_1 K P_2 = P_2 K P_1 = 0$, and the corresponding two-site contraction also vanishes. For $A_i = \text{span}\{\text{tr}(X P_i K P_i) : X\}$, these vanishings and injectivity imply $A_1 + A_2 = M_D(\mathbb{C})$ and $A_1 A_2 = 0$, a contradiction. Theorem 20.15.6.7 does not construct P_1 or P_2 : an arbitrary reachability cut $I_* = S \sqcup C$ supplies only $A_S A_C = 0$. The common final step is the same one-sided obstruction: two nonzero matrix spaces cannot sum to $M_D(\mathbb{C})$ while their ordered products vanish.

Theorem 20.15.6.8 (Positive entries of the active trace matrix). [Lean](#) [Lean](#) [✓ Stmt](#) If every neighboring operator is positive semidefinite, then T^* is entrywise non-negative and $T_{k,h}^* > 0 \iff \eta_{k,h} \neq 0$.

Theorem 20.15.6.9 (Two-step return through every active sector). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the active one-site matrices span $M_D(\mathbb{C})$ and each active sector contains a nonzero such matrix. For every $k \in I_*$ there is an $h \in I_*$ such that

$$\begin{aligned} \eta_{k,h} &\neq 0, \\ \eta_{h,k} &\neq 0. \end{aligned}$$

If the neighboring operators are positive semidefinite, then $(T^{*2})_{k,k} > 0$.

Theorem 20.15.6.10 (Irreducibility of the active trace matrix). [Lean](#) [✓ Stmt](#) Suppose that the neighboring operators are positive semidefinite, the active one-site matrices span $M_D(\mathbb{C})$, and each active sector contains a nonzero such matrix. Then the non-negative matrix T^* is irreducible.

Theorem 20.15.6.11 (Three-step returns and a common positive power). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the neighboring operators are positive semidefinite, the active one-site matrices span $M_D(\mathbb{C})$, each active sector contains a nonzero such matrix, and every nonzero edge closes through a third active sector. Then $(T^{*3})_{k,k} > 0$ for every $k \in I_*$. Together with the length-two returns and irreducibility, this implies that T^* is primitive.

Proof. [✓ Proof](#) Triangle closure supplies a positive closed walk of length three through each active sector. Combine these walks with the length-two returns and apply Theorem 20.15.6.3. $\square$

Theorem 20.15.6.12 (Retained witnesses for the active inverse-map factorization). [Lean](#) [Lean](#) [✓ Stmt](#) Let K be an injective tensor generating MPDOs, let R close a three-site state ρ , and choose a Hayashi decomposition with weights p_k . Fix outer indices for which the corresponding entry of R is nonzero, and let F be the zero-weight reparameterized inverse-map factorization. There are coherent sector phases for which all neighboring operators are positive semidefinite and the active one-site matrices span $M_D(\mathbb{C})$. Every active sector contains a nonzero virtual matrix, every virtual matrix in a zero-weight sector vanishes, every nonzero neighboring edge closes through a third active sector, every inactive left tensor vanishes, and I_* is nonempty.

Proof. [✓ Proof](#) Choose the coherent positive phases. Injectivity gives the full virtual matrix span, while zero-weight reparameterization removes the inactive summands from the active span. A nonzero virtual matrix exists in every active sector, and the neighboring-support recurrence gives the three-edge return. Rephasing preserves the nonzero neighboring edges, so this return persists after the coherent rephasing. Inactive left tensors vanish by construction. Finally, $\sum_k p_k = 1$ gives a nonempty active sector. $\square$

Theorem 20.15.6.13 (Primitivity of the active sector trace matrix). [Lean](#) [Lean](#) [✓ Stmt](#) Let K be an injective tensor that generates MPDOs and satisfies SAL, and choose the positive semidefinite neighboring operators of Theorem 20.15.6.12. Restrict the sector index to the positive-weight summands of the Hayashi decomposition. Then positive semidefiniteness makes every neighboring trace real and non-negative, and the matrix

$$T_{k,h}^* = \text{Re tr}(\eta_{k,h})$$

is primitive. Zero-weight summands remain in the physical direct sum but are not indices of T^* .

Proof. [✓ Proof](#) Theorem 20.15.2.20 supplies the four-site Hayashi decomposition, and Theorem 20.15.2.41 supplies nonzero outer indices for the normalized four-site tail. Its reduced block state gives the required three-site closure; form the zero-weight reparameterized inverse-map factorization. Then Theorem 20.15.6.12 supplies one rephased factorization with positive neighboring operators, full active span, nonzero active virtual matrices, and the three-edge return condition. Apply Theorem 20.15.6.11 to conclude that the active trace matrix is primitive. $\square$

Theorem 20.15.6.14 (Zero-correlation-length relations on the primitive active trace matrix). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be injective and suppose that it satisfies SAL and source zero correlation length. Then there are a physical-sector factorization, weights $p_k \geq 0$ with $\sum_k p_k = 1$, and positive semidefinite neighboring operators $\eta_{k,h}$ such that the active trace matrix

$$T_{k,h}^* = \text{Re tr}(\eta_{k,h}), \quad k, h \in I_* := \{j : p_j \neq 0\},$$

is primitive. There is also a real number $\lambda > 0$ for which, upon setting $\widehat{T}^* := \lambda^{-1}T^*$, one has $(\widehat{T}^*)^2 = (\widehat{T}^*)^3$. For every positive integer N , $\text{tr}((\widehat{T}^*)^N) = \text{tr}(\widehat{T}^*)$. Thus the primitivity and the two normalized identities concern the same active, positively rephased trace matrix. No idempotence, rank-one, or semisimplicity conclusion is asserted.

Proof. [✓ Proof](#) Theorem 20.15.6.12 gives one rephased factorization with positive neighboring operators and vanishing left tensors on the zero-weight sectors. Apply Theorem 20.15.6.11 to obtain primitivity of its active trace matrix. Lemma 20.5.7 supplies normalized idempotence of the physical-trace transfer, from which the same active trace matrix satisfies the square-cube identity and the positive-power trace identities. $\square$

Lemma 20.15.6.15 (SAL makes the physical-trace transfer nonzero). [Lean](#) [✓ Stmt](#) If an MPO tensor K satisfies SAL, then its physical-trace transfer is nonzero:

$$\mathcal{T}_K \neq 0.$$

Proof. [✓ Proof](#) The SAL normalization at one site gives a nonzero trace for the one-site periodic operator. Lemma 17.4.58 identifies this trace with that of $\mathcal{T}_K$, which therefore cannot vanish. $\square$

Theorem 20.15.6.16 (SAL and literal idempotence imply source zero correlation length). [Lean](#) [✓ Stmt](#) If K satisfies SAL and

$$\mathcal{T}_K^2 = \mathcal{T}_K,$$

then K has source zero correlation length.

Proof. [✓ Proof](#) The preceding lemma gives $\mathcal{T}_K \neq 0$. Apply Theorem 20.5.6. $\square$

Theorem 20.15.6.17 (Literal-ZCL specialization of the active trace relations). [Lean](#) [✓ Stmt](#) Let K be injective and satisfy SAL and the literal identity

$$\mathcal{T}_K^2 = \mathcal{T}_K.$$

Then K admits a physical-sector factorization, nonnegative weights of sum one, and positive semidefinite neighboring operators for which the active trace matrix T^* is primitive. For some $\lambda > 0$, the normalized matrix $\widehat{T}^* = \lambda^{-1}T^*$ satisfies $(\widehat{T}^*)^2 = (\widehat{T}^*)^3$ and, for every positive integer N ,

$$\text{tr}((\widehat{T}^*)^N) = \text{tr}(\widehat{T}^*).$$

No idempotence or rank-one conclusion is asserted.

Proof. [✓ Proof](#) SAL makes the one-site periodic trace nonzero, and hence $\mathcal{T}_K \neq 0$. Theorem 20.15.6.16 therefore gives source zero correlation length. Apply Theorem 20.15.6.14. $\square$

Theorem 20.15.6.18 (Virtual spanning does not force active-sector idempotence). [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [✓ Stmt](#) There is a tensor of bond dimension two with four one-dimensional physical sectors and weights p_k such that the following statements hold simultaneously: the tensor is injective and has source zero correlation length; its physical-trace transfer is idempotent; for every $k \in \{0, 1, 2, 3\}$, the weights satisfy

$$\begin{aligned} p_k &= \frac{1}{4}, \\ 0 &\leq p_k, \\ \sum_{k=0}^3 p_k &= 1; \end{aligned}$$

its sector virtual matrices span $M_2(\mathbb{C})$; every neighboring operator is positive semidefinite; and its active trace matrix is primitive and has trace one. Nevertheless, if $S = LQ$ is the physical-trace transfer and $T = QL$ is the active trace matrix, then $Q(1 - S)L = T - T^2 \neq 0$. In particular, full virtual-matrix spanning does not by itself imply that the active trace matrix is idempotent.

Proof. [✓ Proof](#) Take

$$L = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & \frac{1}{8} \\ 1 & \frac{1}{8} \\ 1 & -\frac{1}{8} \\ 1 & -\frac{1}{8} \end{pmatrix}.$$

For each k , the k th sector virtual matrix is the outer product of the k th column of L with the k th row of Q . Their four coordinate functions are proportional to the Walsh functions $1, x, y, xy$ on four points. More explicitly, the matrix whose columns are their coordinates in the order $(00, 01, 10, 11)$ has determinant $-1/64$. Hence they form a basis of $M_2(\mathbb{C})$. Placing them on the diagonal physical entries gives an injective tensor and a physical-sector factorization with scalar sectors.

Direct multiplication gives

$$LQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$QL = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

Thus LQ is idempotent, whereas every entry of QL is strictly positive, so QL is primitive. Its trace is one. Since each sector is one-dimensional, the neighboring operators are the scalar matrices $\eta_{k,h} = ((QL)_{k,h}) \succeq 0$.

The $(0, 1)$ entry of $(QL)^2$ is $1/4$, while the corresponding entry of QL is $1/8$. Hence QL is not idempotent, and associativity gives $Q(1 - LQ)L = QL - (QL)^2 \neq 0$. $\square$

Theorem 20.15.6.19 (A one-sector factorization of the four-sector witness). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The tensor in Theorem 20.15.6.18 also admits a physical-sector factorization with one sector. Write each physical label as a pair $(x, y) \in \{0, 1\}^2$ and put $Z = \text{diag}(1, -1)$. The left and right factors are

$$l_0 = \frac{1}{4}I, \quad l_1 = Z,$$

$$r_0 = I, \quad r_1 = \frac{1}{8}Z.$$

Its sole neighboring operator is

$$\eta = I \otimes \frac{1}{4}I + \frac{1}{8}Z \otimes Z = \text{diag}\left(\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right).$$

In particular, η is positive semidefinite and $\text{tr}(\eta) = 1$. Taking $a = b = 1$ gives the normalized rank-one trace factorization. Thus the four sectors selected by the inverse-map construction are nonminimal; their non-rank-one trace matrix is not an obstruction to the existence of another physical-sector factorization for this tensor.

Proof. [✓ Proof](#) Reindex the four physical basis vectors by parity and quotient. The diagonal entries of the tensor then factor as $(l_\beta)_{x,x}(r_\alpha)_{y,y}$, and all off-diagonal entries vanish. Contracting the common virtual label gives the displayed expression for η . Its four diagonal entries prove positivity and trace one. $\square$

Theorem 20.15.6.20 (Pair-preserving trace obstruction). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [✓ Stmt](#) Let $\eta_{k,h}$ be the neighboring operators of a physical-sector factorization, and define

$$C_{k,h} = \text{tr}(\eta_{k,h}),$$

$$\Theta_{k,h} = \bigoplus_{l,m} \eta_{k,l} \otimes \eta_{l,m} \otimes \eta_{m,h}.$$

Then

$$\text{tr}(\Theta_{k,h}) = (C^3)_{k,h}.$$

Suppose that, for every fixed pair (k,h) , there is either a trace-preserving linear map from the four-site middle algebra to the two-site middle algebra which sends $\Theta_{k,h}$ to $\eta_{k,h}$, or a trace-preserving linear map in the reverse direction which sends $\eta_{k,h}$ to $\Theta_{k,h}$. If the same direction is chosen for all pairs, then

$$C = C^3.$$

Proof. [✓ Proof](#) Multiplicativity of the trace on tensor products and additivity on direct sums give

$$\text{tr}(\Theta_{k,h}) = \sum_{l,m} C_{k,l} C_{l,m} C_{m,h} = (C^3)_{k,h}.$$

Trace preservation identifies this quantity with $\text{tr}(\eta_{k,h}) = C_{k,h}$ in either direction. The entrywise identities give $C = C^3$. $\square$

Theorem 20.15.6.21 (Doubled-index self-overlap is the Hilbert–Schmidt trace). [Lean](#) [Lean](#)
[✓ Stmt](#) Let K be an MPO tensor. For every chain length N , the self-overlap of the doubled-index MPS family equals the Hilbert–Schmidt trace of the closed operator:

$$O_{K^{\text{MPS}} K^{\text{MPS}}}(N) = \text{tr}(\rho^{(N)}(K) \rho^{(N)}(K)^\dagger).$$

This is a project derivation preparing the Lemma C.5 route of [CPGSV16, Appendix C.2, lines 1473–1499]; the pure-state analogue is the normalized-MPV self-overlap lemma of [CPGSV16, lines 1080–1100]. It is not a statement of that source. Until a trace normalization is established, the right-hand side is the raw squared Hilbert–Schmidt norm of the MPO family, not a normalized-state purity.

Proof. [✓ Proof](#) Both sides are sums over physical configurations. Expanding the trace of the product gives the Frobenius inner product

$$\text{tr}(\rho^{(N)}(K) \rho^{(N)}(K)^\dagger) = \sum_{\sigma,\tau} \rho^{(N)}(K)_{\sigma,\tau} \overline{\rho^{(N)}(K)_{\sigma,\tau}}$$

over pairs (σ, τ) of ket and bra configurations, while the overlap sums $V^{(N)}(K^{\text{MPS}})_\rho \overline{V^{(N)}(K^{\text{MPS}})_\rho}$ over doubled-index words ρ . The product encoding of Definition 17.1.2 reindexes each configuration pair (σ, τ) as the doubled-index word $n \mapsto (\sigma_n, \tau_n)$, and decoding each doubled index as a ket–bra pair identifies the summands: $V^{(N)}(K^{\text{MPS}})_{(\sigma,\tau)} = \rho^{(N)}(K)_{\sigma,\tau}$. $\square$

Corollary 20.15.6.22 (Self-overlap as the trace of the square for an MPDO). [Lean](#) [✓ Stmt](#) If K generates MPDOs and $N \geq 1$, then

$$O_{K^{\text{MPS}} K^{\text{MPS}}}(N) = \text{tr}(\rho^{(N)}(K) \rho^{(N)}(K)).$$

Proof. [✓ Proof](#) Positivity on the nonempty chain makes $\rho^{(N)}(K)$ positive semidefinite, hence Hermitian. Substituting $\rho^{(N)}(K)^\dagger = \rho^{(N)}(K)$ into Theorem 20.15.6.21 gives the identity. $\square$

Corollary 20.15.6.23 (Real part of the self-overlap is the squared Frobenius norm). [Lean](#) [✓ Stmt](#) For every MPO tensor K and every chain length N ,

$$\text{Re } O_{K^{\text{MPS}} K^{\text{MPS}}}(N) = \|\rho^{(N)}(K)\|_F^2 = \sum_{\sigma, \tau} |\rho^{(N)}(K)_{\sigma, \tau}|^2.$$

Proof. [✓ Proof](#) Cyclicity of the trace rewrites the Hilbert–Schmidt trace as $\text{tr}(\rho^{(N)}(K)^\dagger \rho^{(N)}(K))$, whose real part is the sum of the squared absolute entries. $\square$

Theorem 20.15.6.24 (Trace factorization of the cyclic neighboring product). [Lean](#) [Lean](#) [✓ Stmt](#) Let $N \geq 1$ and let $k = (k_0, \dots, k_{N-1})$ be a cyclic sector configuration of a physical-sector factorization. The trace of the cyclic neighboring product factors into the product of the neighboring traces, and so does the trace of its square:

$$\begin{aligned} \text{tr}(\Omega_k) &= \prod_{n=0}^{N-1} \text{tr}(\eta_{k_n, k_{n+1}}), \\ \text{tr}(\Omega_k^2) &= \prod_{n=0}^{N-1} \text{tr}(\eta_{k_n, k_{n+1}}^2), \end{aligned}$$

with site labels read modulo N . These are the coefficient-extraction identities for the cyclic direct-sum form of [CPGSV16, Appendix C.2, lines 1580–1593].

Proof. [✓ Proof](#) The fiber bijection of Definition 20.15.3.23 identifies the diagonal entries of Ω_k with tuples of diagonal entries of the neighboring factors. Hence

$$\text{tr}(\Omega_k) = \sum_x \prod_{n=0}^{N-1} \eta_{k_n, k_{n+1}}(x_n, x_n) = \prod_{n=0}^{N-1} \sum_{x_n} \eta_{k_n, k_{n+1}}(x_n, x_n),$$

expanding the product of sums. For the square, write $\text{tr}(\Omega_k^2) = \sum_{x, y} \Omega_k(x, y) \Omega_k(y, x)$; the same reindexing and expansion turn the inner sums into the diagonal entries of $\eta_{k_n, k_{n+1}}^2$. $\square$

Lemma 20.15.6.25 (Trace-squared matrix of the neighboring operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be an MPO tensor with a physical-sector factorization and sector weights p_k , and suppose that every neighboring operator $\eta_{k, h}$ is positive semidefinite. On the active sector set I_* , define the trace-squared matrix

$$S_{k, h}^* := \text{Re } \text{tr}(\eta_{k, h}^2), \quad k, h \in I_*.$$

Then, entrywise,

$$0 \leq S_{k, h}^* \leq (T_{k, h}^*)^2.$$

This matrix is a project derivation preparing the Case-I route toward [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]; it is not a matrix defined in that source.

Proof. [✓ Proof](#) Hermiticity of $\eta_{k,h}$ gives $\eta_{k,h}^2 = \eta_{k,h} \eta_{k,h}^\dagger$, which is positive semidefinite, so its trace is a non-negative real number. The upper bound is the corresponding theorem in the companion quantum-channel volume [LTC26, “Forward Lorentz-cone trace inequality”] applied to $\eta_{k,h}$. $\square$

Theorem 20.15.6.26 (Literal zero correlation length stabilizes the active trace matrix). [Lean](#) [✓ Stmt](#) Let K be an MPO tensor with a physical-sector factorization and sector weights p_k . Suppose that every neighboring operator is positive semidefinite, that the physical-trace transfer is literally idempotent,

$$\mathcal{T}_K^2 = \mathcal{T}_K,$$

as in [CPGSV16, Definition 4.2, lines 735–741], and that the left tensor of every zero-weight sector vanishes. Then the active trace matrix stabilizes after the second power:

$$(T^*)^2 = (T^*)^3.$$

If, in addition, the active one-site matrices span $M_D(\mathbb{C})$, each active sector contains a nonzero such matrix, every nonzero neighboring operator closes through a third active sector, and there is at least one active sector, then $(T^*)^2$ is strictly positive entrywise.

These are matrix-algebra components of the Case-I argument under the assumptions of [CPGSV16, Appendix C.2, lines 1374–1381], for the factorization supplied by [CPGSV16, Appendix C.2, Lemma C.4, lines 1406–1471] and the zero-correlation-length step of [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]. They are project derivations, not statements of that source.

Proof. [✓ Proof](#) On the active sectors, collect the left and right sector-tensor traces into rectangular matrices L and Q with

$$L_{\beta,h} = \text{tr}((l_h)_\beta), \quad Q_{k,\alpha} = \text{tr}((r_k)_\alpha).$$

The left tensor vanishes on every zero-weight sector, so the rectangular product LQ is the full physical-trace transfer, and literal zero correlation length makes LQ idempotent. Associativity then gives

$$(QL)^2 = Q(LQ)L = Q(LQ)(LQ)L = (QL)^3.$$

Positivity of the neighboring operators makes their traces real, and expanding the products identifies QL with the complexification of T^* ; injectivity of the real-to-complex inclusion gives $(T^*)^2 = (T^*)^3$.

Under the additional hypotheses, Theorem 20.15.6.11 makes T^* primitive: some power $m \geq 1$ has strictly positive entries. Stabilization gives $(T^*)^{2m} = (T^*)^2$, so

$$((T^*)^2)_{k,h} = \sum_j ((T^*)^m)_{k,j} ((T^*)^m)_{j,h} > 0,$$

which is the positivity argument of Theorem 20.15.6.55. $\square$

Lemma 20.15.6.27 (Indicator-path expansion of a matrix power). [Lean](#) [✓ Stmt](#) Let S be a square matrix with entries in a commutative semiring, indexed by a finite set I , and let $N \geq 0$. Then, for $a, b \in I$,

$$(S^N)_{a,b} = \sum_{\substack{v:\{0,\dots,N\}\rightarrow I \\ v_0=a, v_N=b}} \prod_{i=0}^{N-1} S_{v_i, v_{i+1}},$$

the sum, over length- N walks v in I from a to b , of the product of matrix entries along the walk. This is an elementary combinatorial identity used to derive the cyclic-product form of $\text{tr}(S^N)$ in the Case-I route.

Proof. ✓ Proof Induction on N : the base case is the identity-matrix entry $\delta_{a,b}$, indexed by the unique length-0 walk; the inductive step expands $(S^{N+1})_{a,b} = \sum_c (S^N)_{a,c} S_{c,b}$ and reindexes the walk-and-endpoint pair (v, c) as the length- $(N+1)$ walk obtained by appending c to v . $\square$

Theorem 20.15.6.28 (Trace of a matrix power as a sum over cyclic products). Lean ✓ Stmt Let S be a square matrix with entries in a commutative semiring, indexed by a finite set I , and let $N \geq 1$. Then

$$\text{tr}(S^N) = \sum_{k: \{0, \dots, N-1\} \rightarrow I} \prod_{i=0}^{N-1} S_{k_i, k_{i+1}},$$

with indices read modulo N , the sum over labelings k of the N cyclic positions of the product of matrix entries along the cycle. Applied to the trace-squared matrix S^* of Lemma 20.15.6.25, this is the combinatorial identity used to express the self-overlap $O_{K^{\text{MPS}} K^{\text{MPS}}}(N)$ in terms of $\text{tr}((S^*)^N)$.

Proof. ✓ Proof Expand $\text{tr}(S^N) = \sum_a (S^N)_{a,a}$ by Lemma 20.15.6.27, collapse the sum over the shared endpoint $a = v_0 = v_N$, and reindex the resulting closed length- N walks v (with $v_0 = v_N$) as cyclic labelings $k: \{0, \dots, N-1\} \rightarrow I$ by dropping the repeated final coordinate. $\square$

Lemma 20.15.6.29 (Trace invariance under the physical basis change). Lean Lean ✓ Stmt Let K be an MPO tensor and U an isometry on the physical space, $U^\dagger U = 1$. Then

$$\text{tr}(\rho^{(N)}(\tilde{K}) \rho^{(N)}(\tilde{K})) = \text{tr}(\rho^{(N)}(K) \rho^{(N)}(K)),$$

where $\tilde{K}$ is K conjugated by U at every physical site. This is a project derivation preparing the Case-I route of [CPGSV16, Appendix C.2, lines 1434–1448] (the basis change defining σ_{tilde}); it is not a claim from that source.

Proof. ✓ Proof The sitewise product matrix $W_{\sigma, \sigma'} = \prod_n U(\sigma_n, \sigma'_n)$ satisfies $W^\dagger W = 1$ because U does: expanding $W^\dagger W$ as a sum over intermediate configurations and using $\sum_x \overline{U(x, a)} U(x, b) = \delta_{a,b}$ site by site gives the identity. Since $\rho^{(N)}(\tilde{K}) = W \rho^{(N)}(K) W^\dagger$, cyclicity of the trace and $W^\dagger W = 1$ cancel the W 's from $\text{tr}(W \rho^{(N)}(K) \rho^{(N)}(K) W^\dagger)$. $\square$

Lemma 20.15.6.30 (Reduction to the active sector). Lean Lean Lean ✓ Stmt Let K be an MPO tensor with a physical-sector factorization and sector weights p_k , such that the left tensor of every zero-weight sector vanishes. Then, for $N \geq 1$, the sum over all sector labelings of the cyclic trace-square product reduces to the sum over active-sector labelings:

$$\sum_{k: \{0, \dots, N-1\} \rightarrow I} \prod_{i=0}^{N-1} \text{tr}(\eta_{k_i, k_{i+1}}^2) = \sum_{k: \{0, \dots, N-1\} \rightarrow I_*} \prod_{i=0}^{N-1} \text{tr}(\eta_{k_i, k_{i+1}}^2).$$

This is the reconciliation, flagged as an open reconciliation gap when Lemma 20.15.6.27 and Theorem 20.15.6.28 were added, between the full sector index set I and the active sector set I_* .

Proof. ✓ Proof A neighboring operator with an inactive target sector vanishes, because the left tensor of that sector vanishes identically. Hence any cyclic labeling touching an inactive sector has a vanishing factor (that sector is the target of some edge in the cycle), so its product is zero. The active-sector sum is therefore exactly the full-sector sum, restricted to a labeling-by-labeling bijection with the inactive terms discarded. $\square$

Theorem 20.15.6.31 (The overlap formula). Lean ✓ Stmt *Let K be an MPO tensor with a physical-sector factorization and sector weights p_k , such that every neighboring operator is positive semidefinite and the left tensor of every zero-weight sector vanishes. Then, for $N \geq 1$,*

$$O_{K^{\text{MPS}} K^{\text{MPS}}}(N) = \text{tr}((S^*)^N).$$

This is a project derivation completing the first step of the Case-I route toward [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]; it is not a statement of that source.

Proof. ✓ Proof Write $\rho = \rho^{(N)}(K)$, $\tilde{\rho} = \rho^{(N)}(\tilde{K})$ for the physically conjugated MPO, and Ω_k for the cyclic neighboring product of Theorem 20.15.6.24. The overlap unwinds into a chain of trace identities:

$$\begin{aligned} O_{K^{\text{MPS}} K^{\text{MPS}}}(N) &= \text{tr}(\rho \rho) && \text{Theorem 20.15.6.21} \\ &= \text{tr}(\tilde{\rho} \tilde{\rho}) && \text{Lemma 20.15.6.29} \\ &= \text{tr}\left(\left(\bigoplus_k \Omega_k\right)^2\right) && \text{block-diagonal product realization} \\ &= \sum_{k:\{0, \dots, N-1\} \rightarrow I} \text{tr}(\Omega_k^2) && \text{trace of a block-diagonal matrix} \\ &= \sum_{k:\{0, \dots, N-1\} \rightarrow I} \prod_{i=0}^{N-1} \text{tr}(\eta_{k_i, k_{i+1}}^2) && \text{Theorem 20.15.6.24} \\ &= \sum_{k:\{0, \dots, N-1\} \rightarrow I_*} \prod_{i=0}^{N-1} \text{tr}(\eta_{k_i, k_{i+1}}^2) && \text{Lemma 20.15.6.30} \\ &= \sum_{k:\{0, \dots, N-1\} \rightarrow I_*} \prod_{i=0}^{N-1} S_{k_i, k_{i+1}}^* && \text{Hermiticity, } \text{tr}(A) = \text{Re tr}(A) \text{ for } A = A^\dagger \\ &= \text{tr}((S^*)^N) && \text{Theorem 20.15.6.28} \end{aligned}$$

The third step combines the block-diagonal product realization of the sector-coordinate MPO with the reindexing invariance of trace-of-square. The sixth step uses Hermiticity of $\eta_{k,h}$ to identify its complex trace-square with the real entry $S_{k,h}^* = \text{Re tr}(\eta_{k,h}^2)$. $\square$

Theorem 20.15.6.32 (Trace invariance under squared diagonal similarity). Lean ✓ Stmt *Let I be a finite index set, let $S \in \mathbb{R}^{I \times I}$, and let $v \in \mathbb{R}^I$ satisfy $v_i > 0$ for every $i \in I$. Put*

$$D = \text{diag}(v_i^2).$$

Then, for every $N \geq 0$,

$$\text{tr}((D^{-1}SD)^N) = \text{tr}(S^N).$$

This project lemma isolates the diagonal-conjugation calculation used in the Case-I argument. It is not a separately stated result of [CPGSV16].

Proof. ✓ Proof Every diagonal entry of D is nonzero, so D is invertible. Induction on N gives

$$(D^{-1}SD)^N = D^{-1}S^N D.$$

Cyclicity of trace and $DD^{-1} = 1$ then give the claimed equality. □

Theorem 20.15.6.33 (Trace-power decay below a stochastic Hadamard square). Lean ✓ Stmt *Let I be a finite set with at least two elements, and let $P, S \in \mathbb{R}^{I \times I}$. Suppose that P is primitive and row-stochastic, and that, for all $i, j \in I$,*

$$0 \leq S_{i,j} \leq (P \odot P)_{i,j} = P_{i,j}^2.$$

Then, as $N \rightarrow \infty$,

$$\text{tr}(S^N) \rightarrow 0.$$

This project matrix-theoretic lemma is used in the Case-I argument. It is not a separately stated result of [CPGSV16].

Proof. ✓ Proof Entrywise non-negativity gives $\text{tr}(S^N) \geq 0$. Entrywise domination is preserved by matrix powers, and hence

$$\text{tr}(S^N) \leq \text{tr}((P \odot P)^N).$$

Every power of $P \odot P$ has L^∞ operator norm at most one. Primitivity gives a positive power whose L^∞ operator norm is strictly below one; the division algorithm then shows that $(P \odot P)^N \rightarrow 0$. Thus its trace tends to zero, and the squeeze theorem proves the result. □

Theorem 20.15.6.34 (The active sector is unique). Lean ✓ Stmt *Let K be an MPO tensor of positive bond dimension D with a physical-sector factorization and sector weights p_k , and let $I_* = \{k : p_k \neq 0\}$. Suppose that every neighboring operator $\eta_{k,h}$ is positive semidefinite,*

$$\text{span}_{\mathbb{C}}\{A_{k;x,y} : k \in I_*\} = M_D(\mathbb{C}),$$

and, for every $k \in I_*$, some virtual matrix $A_{k;x,y}$ is nonzero. Suppose also that every nonzero $\eta_{k,h}$ with $k, h \in I_*$ admits $j \in I_*$ such that $\eta_{h,j} \neq 0$ and $\eta_{j,k} \neq 0$, that the physical-trace transfer is literally idempotent,

$$\mathcal{T}_K^2 = \mathcal{T}_K,$$

that $p_k = 0$ implies that the left tensor of sector k vanishes, that the underlying tensor is normal in the sense of Definition 8.6.1, and that I_* is nonempty. Then

$$|I_*| = 1.$$

This is a project derivation completing the Case-I route toward [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]; it is not a statement of that source.

Proof. ✓ Proof By contradiction, suppose $|I_*| \geq 2$. The overlap formula and the normal self-overlap limit ([CPGSV16, lines 1080–1100]) give $\text{tr}((S^*)^N) \rightarrow 1$. The relation $(T^*)^2 = (T^*)^3$ and primitivity supply a positive Perron eigenvector with eigenvalue 1. Conjugating by its diagonal gives a row-stochastic primitive matrix P . The matching squared diagonal conjugate $\hat{S}$ of S^* is entrywise non-negative and bounded by $P \odot P$. On the nontrivial active-sector set, Theorem 20.15.6.33 gives $\text{tr}(\hat{S}^N) \rightarrow 0$. Theorem 20.15.6.32 then gives $\text{tr}((S^*)^N) \rightarrow 0$, contradicting the limit 1. □

Theorem 20.15.6.35 (The Case-I relation $(T^*)^2 = T^*$). [Lean](#) [✓ Stmt](#) Let K be an MPO tensor of positive bond dimension D with a physical-sector factorization and sector weights p_k , and let $I_* = \{k : p_k \neq 0\}$. Suppose that every neighboring operator $\eta_{k,h}$ is positive semidefinite,

$$\text{span}_{\mathbb{C}}\{A_{k;x,y} : k \in I_*\} = M_D(\mathbb{C}),$$

and, for every $k \in I_*$, some virtual matrix $A_{k;x,y}$ is nonzero. Suppose also that every nonzero $\eta_{k,h}$ with $k, h \in I_*$ admits $j \in I_*$ such that $\eta_{h,j} \neq 0$ and $\eta_{j,k} \neq 0$, that the physical-trace transfer is literally idempotent,

$$\mathcal{T}_K^2 = \mathcal{T}_K,$$

that $p_k = 0$ implies that the left tensor of sector k vanishes, that the underlying tensor is normal in the sense of Definition 8.6.1, and that I_* is nonempty. Then

$$(T^*)^2 = T^*.$$

This is the Case-I input to the proof of Lemma C.5 ([CPGSV16, Appendix C.2, lines 1473–1499]), not the lemma’s headline statement.

Proof. [✓ Proof](#) On the singleton active sector set, T^* is a scalar t . Stabilization gives $t^2 = t^3$, and positivity of $(T^*)^2$ gives $t^2 > 0$, hence $t = 1$ and $t^2 = t$. $\square$

Theorem 20.15.6.36 (Normalized Case-I rank-one coefficients). [Lean](#) [✓ Stmt](#) Under the complete normal Case-I hypotheses of Theorem 20.15.6.35, there are real functions $a, b : I_* \rightarrow \mathbb{R}$ such that, for all $k, h \in I_*$,

$$T_{k,h}^* = a_k b_h,$$

and

$$\sum_{k \in I_*} a_k b_k = 1.$$

This is the normal Case-I active-sector packaging of the displayed rank-one and normalization equations in [CPGSV16, Appendix C.2, lines 1394–1401 and 1484–1499]. It neither transfers a conclusion to Case II nor uses the paper’s constant-trace-powers rank-one argument.

Proof. [✓ Proof](#) Write $I_* = \{k_0\}$. Strict positivity of $(T^*)^2$ and $(T^*)^2 = T^*$ give

$$0 < (T_{k_0,k_0}^*)^2 = T_{k_0,k_0}^*.$$

Thus the sole entry is $T_{k_0,k_0}^* = 1$. Taking $a_k = b_k = 1$ gives $T_{k,h}^* = a_k b_h$ and $\sum_{k \in I_*} a_k b_k = 1$. $\square$

Theorem 20.15.6.37 (Strong-area-law form of the normalized Case-I coefficients). [Lean](#) [✓ Stmt](#) Let K be an injective MPO tensor. Suppose that K satisfies the strong area law, that its physical-trace transfer is literally idempotent,

$$\mathcal{T}_K^2 = \mathcal{T}_K,$$

and that the underlying tensor is normal in the sense of Definition 8.6.1. Then there is a physical-sector factorization with sector weights $p_k \geq 0$ satisfying $\sum_k p_k = 1$, for which every

neighboring operator is positive semidefinite. There are real functions a, b on $I_* = \{k : p_k \neq 0\}$. For all $k, h \in I_*$,

$$T_{k,h}^* = a_k b_h,$$

and

$$\sum_{k \in I_*} a_k b_k = 1.$$

This is the active-sector part of the normal Case-I conclusion of [CPGSV16, Appendix C.2, Case I, lines 1374–1381]. The active Hayashi factorization comes from [CPGSV16, Appendix C.2, Lemma C.4, lines 1406–1471]; Theorem 20.15.6.40 extends these coefficients to every sector label, as in Lemma C.5.

Proof. ✓ Proof Apply Theorem 20.15.6.39 and retain its active coefficient functions and equations. □

Theorem 20.15.6.38 (Vanishing at a zero-weight incident sector). Lean ✓ Stmt Let $(\eta_{k,h})_{k,h}$ be the neighboring operators of the zero-weight-reparameterized inverse-map factorization, after a coherent sector rephasing, and let $(p_k)_k$ be its Hayashi weights. If $p_k = 0$ or $p_h = 0$, then

$$\eta_{k,h} = 0.$$

The reparameterization makes the right tensor of an inactive source sector zero, while its left tensor is already zero. Thus this is a statement about the retained reparameterized and rephased witness, not about the raw inverse-map factorization.

Proof. ✓ Proof Rephasing changes a neighboring operator by

$$(\eta_z)_{k,h} = z_k^{-1} z_h \eta_{k,h}.$$

Hence $\eta_{k,h} = 0$ implies $(\eta_z)_{k,h} = 0$. If $p_k = 0$, the reparameterized right tensor vanishes, so $\eta_{k,h} = 0$. If $p_h = 0$, the left tensor vanishes and again $\eta_{k,h} = 0$. □

Theorem 20.15.6.39 (Rephased inverse-map Case-I coefficient witness). Lean ✓ Stmt Let K be an injective MPO tensor satisfying the strong area law. Suppose that its physical-trace transfer is literally idempotent and that its underlying tensor is normal. Then one can choose a single coherently rephased, zero-weight-reparameterized inverse-map factorization with nonnegative weights $(p_k)_k$ satisfying $\sum_k p_k = 1$, positive semidefinite neighboring operators $\eta_{k,h}$, and real functions a, b on $I_* = \{k : p_k \neq 0\}$. For all $k, h \in I_*$,

$$T_{k,h}^* = a_k b_h,$$

and

$$\sum_{k \in I_*} a_k b_k = 1.$$

For all ambient sector labels k, h ,

$$p_k = 0 \text{ or } p_h = 0 \implies \eta_{k,h} = 0.$$

The incident-vanishing clause belongs to this chosen reparameterized and rephased factorization, not to the raw inverse-map factors. This is the normal Case-I construction from [CPGSV16, Appendix C.2, Case I, Lemmas C.4–C.5, lines 1374–1505]; the incident-vanishing clause is the chosen zero-weight convention for inactive summands.

Proof. ✓ **Proof** Normality makes the bond dimension positive. The strong area law supplies the four-site Hayashi decomposition, while the normalized four-site tail gives nonzero outer indices and the required three-site closure. Choose the coherently rephased zero-weight-reparameterized inverse-map factorization. Its active-sector witnesses give the normalized rank-one equations, and Theorem 20.15.6.38 gives the final incident-vanishing clause for the same factorization. $\square$

Theorem 20.15.6.40 (All-sector normalized Case-I rank-one coefficients). Lean ✓ **Stmt** *Let K be an injective normal MPO tensor satisfying the strong area law and literal physical-trace idempotence,*

$$\mathcal{T}_K^2 = \mathcal{T}_K.$$

Then there is a physical-sector factorization with nonnegative weights $(p_k)_k$ satisfying $\sum_k p_k = 1$, positive semidefinite neighboring operators $\eta_{k,h}$, and real families $(a_k)_k, (b_k)_k$ indexed by all sector labels. For all k, h ,

$$\begin{aligned} \text{Re tr}(\eta_{k,h}) &= a_k b_h, \\ \sum_k a_k b_k &= 1. \end{aligned}$$

On the nonzero-weight sectors these are the coefficient families of Theorem 20.15.6.39; if $p_k = 0$, then $a_k = b_k = 0$. This is a project-derived realization of the all-index coefficient statement in [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]. The source does not introduce the active–inactive split or ambient filler sectors used here. This theorem concerns literal zero correlation length only; it makes no assertion for the scale-invariant source-ZCL condition or in Case II.

Proof. ✓ **Proof** Extend the coupled witness’s active coefficient families by zero. In the chosen zero-weight-reparameterized and rephased factorization, $p_k = 0$ makes every neighboring operator incident to k vanish. Therefore the trace factorization is zero whenever either index is inactive, while it is the active factorization otherwise. Filtering the full diagonal sum to the nonzero-weight sectors gives

$$\sum_k a_k b_k = \sum_{k: p_k \neq 0} a_k b_k = 1.$$

$\square$

Theorem 20.15.6.41 (Rectangular form of the Case-I relation). Lean ✓ **Stmt** *Under the hypotheses of Theorem 20.15.6.35, define the rectangular trace-pairing matrices*

$$L_{\beta,h} = \text{tr}((l_h)_\beta), \quad Q_{k,\alpha} = \text{tr}((r_k)_\alpha).$$

Then

$$Q(1 - LQ)L = 0.$$

This is a project-derived rectangular form, in project notation, of the l_h, r_k factors from [CPGSV16, Appendix C.2, Lemma C.5, lines 1473–1499]; the source does not state it as a separate displayed theorem.

Proof. ✓ **Proof** The trace pairing gives $QL = T^*$. Therefore

$$Q(1 - LQ)L = QL - (QL)^2 = T^* - (T^*)^2 = 0,$$

where the last equality is Theorem 20.15.6.35. $\square$

Theorem 20.15.6.42 (Positivity of bipartite partial traces). [Lean](#) [Lean](#) [✓ Stmt](#) *If ρ is positive semidefinite on a bipartite space $A \otimes B$, then both reduced operators $\text{tr}_A \rho$ and $\text{tr}_B \rho$ are positive semidefinite.*

Proof. [✓ Proof](#) Write each partial trace as the finite sum, over the traced index, of a principal submatrix of ρ . Principal submatrices of a positive semidefinite matrix are positive semidefinite, and finite sums of positive semidefinite matrices remain positive semidefinite. $\square$

Definition 20.15.6.43 (Hayashi–Markov extraction of explicit η -operators). [Lean](#) [✓ Stmt](#) Every Hayashi decomposition witness h_η canonically produces an explicit neighboring η -family by Kronecker-multiplying the sector-reduced states on the neighboring bond spaces:

$$\eta_{k,h} := \text{tr}_C(\rho_{b_2C}^{(k)}) \otimes \text{tr}_A(\rho_{Ab_1}^{(h)}).$$

Positive semidefiniteness of each $\eta_{k,h}$ follows from the fact that partial traces preserve positivity and that positive semidefiniteness is closed under Kronecker products. This is the sector-reduced extraction; the remaining connection to [CPGSV16, Appendix C.2] is the identification of this family with the inverse-map construction in the original simple-MPDO tensor coordinates.

Theorem 20.15.6.44 (Trace matrix of the Hayashi–Markov η -operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The trace matrix induced by the extracted family is entrywise $T_{k,h} = 1$ in both its complex-valued form and its real-valued form. If there are m sectors, then $T = \mathbf{1}\mathbf{1}^\top$ and $\text{tr}(T) = m$.*

Proof. [✓ Proof](#) Each $\rho_{b_2C}^{(k)}$ is a density matrix, hence has unit trace; the partial trace tr_C preserves the trace. The analogous statement holds for $\text{tr}_A \rho_{Ab_1}^{(h)}$. Multiplicativity of the trace under Kronecker products gives $T_{k,h} = 1 \cdot 1 = 1$, and taking real parts gives the real-valued statement. Summing the m diagonal entries gives $\text{tr}(T) = m$. $\square$

Theorem 20.15.6.45 (Positive semidefiniteness of the Hayashi–Markov trace matrix). [Lean](#) [✓ Stmt](#) *The real trace matrix of the sector-reduced family is positive semidefinite.*

Proof. [✓ Proof](#) This trace matrix is the all-ones matrix $\mathbf{1}\mathbf{1}^\top$, hence is positive semidefinite. $\square$

Theorem 20.15.6.46 (Entrywise nonnegativity of the real trace matrix). [Lean](#) [✓ Stmt](#) *Positivity of the neighboring operators gives entrywise nonnegativity of the real trace matrix. Matrix-level positive semidefiniteness of T is a separate structural condition.*

Proof. [✓ Proof](#) The trace of a positive semidefinite neighboring operator is non-negative in the complex order. Taking real parts gives the stated entrywise nonnegativity. $\square$

Definition 20.15.6.47 (Auxiliary matrix predicates for the rank-one step). [Lean](#) [Lean](#) [✓ Stmt](#) The auxiliary predicates state two matrix-theoretic ingredients used in the rank-one step of [CPGSV16, Appendix C.2, Lemma C.5]: constant traces of all positive powers and existence of a rank-one factorization $T = ab^\top$.

The proposed implication from primitivity and constant trace powers to a rank-one factorization is not a valid general theorem. A concrete 3×3 counterexample appears in the archive. The corrected criterion proved here adds positive semidefiniteness and trace normalization, which provide the diagonalizability needed to turn the trace-power condition into a rank-one factorization.

Theorem 20.15.6.48 (Trace square of a Hermitian matrix). [Lean](#) [✓ Stmt](#) *Let T be a finite Hermitian matrix over a real or complex scalar field, with Hermitian eigenvalues λ_i . Then $\text{tr}(T^2) = \sum_i \lambda_i^2$.*

Proof. ✓ Proof The spectral theorem writes T as a unitary conjugate of the diagonal matrix of its Hermitian eigenvalues. Squaring commutes with this conjugation, and cyclicity of trace reduces the identity to the trace of the squared diagonal matrix. $\square$

Theorem 20.15.6.49 (Rank-one eigenvector resolution of a Hermitian matrix). Lean ✓ Stmt *Let A be a finite Hermitian matrix. Choose an orthonormal eigenbasis $(u_k)_k$ with eigenvalues $(\lambda_k)_k$, and let $E_k = |u_k\rangle\langle u_k|$ be the rank-one projection onto u_k . Then $A = \sum_k \lambda_k E_k$.*

Proof. ✓ Proof The finite-dimensional spectral theorem gives $A = U \text{diag}(\lambda_k) U^\dagger$, where the columns of U are the vectors u_k . Expanding the diagonal matrix as the sum of its rank-one coordinate projections gives the stated identity. $\square$

Theorem 20.15.6.50 (PSD trace-power rank-one criterion). Lean ✓ Stmt *Let T be a finite real square matrix. If T is positive semidefinite, normalized by $\text{tr}(T) = 1$, and has constant traces on all positive powers, then T has a rank-one factorization.*

Proof. ✓ Proof The spectral theorem diagonalizes the positive semidefinite matrix with non-negative real eigenvalues. The trace gives that their sum is 1, while the second trace moment gives that the sum of their squares is also 1. Hence exactly one eigenvalue is equal to 1 and all others vanish, so the diagonal form is rank one. Unitary conjugation preserves the existence of an outer-product factorization. $\square$

Theorem 20.15.6.51 (Idempotent trace-one criterion). Lean ✓ Stmt *Let T be a finite real square matrix. If $T^2 = T$ and $\text{tr}(T) = 1$, then there are real vectors a, b such that $T = ab^\top$.*

Proof. ✓ Proof For an idempotent endomorphism f , one has $\text{tr}(f) = \dim(\text{range}(f))$. Hence the range of T has dimension one. Choosing a nonzero vector a spanning it expresses every column of T as $b_j a$ for a scalar b_j , and therefore $T = ab^\top$. $\square$

Theorem 20.15.6.52 (Constant trace powers of an idempotent matrix). Lean ✓ Stmt *Let T be a finite real square matrix with $T^2 = T$. Then, for every positive integer N , one has $\text{tr}(T^N) = \text{tr}(T)$. This recovers the trace identity displayed in the proof of [CPGSV16, Appendix C.2, Lemma C.5] from idempotence.*

Proof. ✓ Proof Induction on N : for $N \geq 1$, one has $T^{N+1} = T^N T = T T = T$, so every positive power of T equals T . $\square$

Theorem 20.15.6.53 (Positive entries in every row and column of a primitive matrix). Lean ✓ Stmt *Let T be a primitive non-negative matrix. Then every row of T contains a strictly positive entry, and so does every column.*

Proof. ✓ Proof If row k of T vanishes, then, for every $m \geq 1$,

$$(T^m)_{k,k} = \sum_j T_{k,j} (T^{m-1})_{j,k} = 0,$$

contradicting the strict entrywise positivity of some power of T . Likewise, if column h of T vanishes, then, for every $m \geq 1$,

$$(T^m)_{h,h} = \sum_j (T^{m-1})_{h,j} T_{j,h} = 0,$$

again contradicting the strict positivity of some power of T . $\square$

Theorem 20.15.6.54 (Rank of a strictly positive idempotent). [Lean](#) [✓ Stmt](#) Let I be a nonempty finite index set, and let P be a real matrix indexed by $I \times I$. Suppose that $P^2 = P$ and $P_{i,j} > 0$ for every $i, j \in I$. Then $\text{rank}(P) = 1$.

Proof. [✓ Proof](#) Put $u = P\mathbf{1}$, so every coordinate of u is positive and $Pu = u$. For a fixed vector x satisfying $Px = x$, choose an index k at which x_i/u_i is maximal and put $c = x_k/u_k$. Then $y = cu - x$ is non-negative, $y_k = 0$, and $P_y = y$. Hence

$$0 = y_k = (Py)_k = \sum_j P_{k,j} y_j.$$

Since every $P_{k,j}$ is strictly positive, all coordinates of y vanish. Thus every fixed vector is proportional to u . The range of an idempotent is its fixed space, so it is one-dimensional. $\square$

Theorem 20.15.6.55 (Rank and factorization of a stationary primitive square). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let T be a primitive real $N \times N$ matrix, where $N \geq 1$. Multiplication by a positive scalar preserves primitivity. If $T^2 = T^3$, then $\text{rank}(T^2) = 1$. There are vectors $a, b \in \mathbb{R}^N$ such that

$$T^2 = ab^\top, \\ \sum_i a_i b_i = 1.$$

This finite-dimensional matrix lemma supports the argument sought at [\[CPGSV16, Appendix C.2, line 1613\]](#); it is not a theorem asserted there.

Proof. [✓ Proof](#) Primitivity gives $m \geq 1$ such that $(T^m)_{i,j} > 0$ for every i, j . Stabilization gives $T^{2m} = T^2$. Thus, for every i, j ,

$$(T^2)_{i,j} = (T^{2m})_{i,j} = \sum_k (T^m)_{i,k} (T^m)_{k,j} > 0.$$

The same stabilization identity gives $(T^2)^2 = T^2$. The preceding theorem applied to $P = T^2$ now gives the conclusion. Since an idempotent has trace equal to its rank, $\text{tr}(T^2) = 1$. Factoring its one-dimensional range gives $T^2 = ab^\top$, and taking the trace yields $\sum_i a_i b_i = 1$. $\square$

Theorem 20.15.6.56 (Idempotence of the sector trace matrix). [Lean](#) [Lean](#) [✓ Stmt](#) Let V be a module over a commutative ring, let $l_1, \dots, l_n \in V$, let $r_1, \dots, r_n$ be linear functionals on V , and let $T_{k,h} = (r_k | l_h)$ be the pairing matrix; for the tensors of [\[CPGSV16, Appendix C.2\]](#) this is the sector trace matrix. Assume the identity

$$\sum_k |l_k)(r_k| = \sum_{k,h} T_{k,h} |l_k)(r_h|,$$

whose right-hand side is the square of its left-hand side, so that the operator $M = \sum_k |l_k)(r_k|$ satisfies $M^2 = M$. If the vectors $l_1, \dots, l_n$ are linearly independent, then $T^2 = T$. The same conclusion holds if instead the functionals $r_1, \dots, r_n$ are linearly independent.

Proof. [✓ Proof](#) Evaluating $M^2 v = Mv$ at an arbitrary $v \in V$ gives

$$\sum_k ((r_k | Mv) - (r_k | v)) l_k = 0,$$

so linear independence of the l_k yields $(r_k| Mv) = (r_k|v)$ for all k and v . Taking $v = l_h$ and expanding $Ml_h = \sum_j T_{j,h} l_j$ turns the left-hand side into $\sum_j T_{k,j} T_{j,h} = (T^2)_{k,h}$ and the right-hand side into $T_{k,h}$. For linearly independent functionals, apply the same argument in the dual space, with the r_k as vectors and evaluation at l_k as functionals. The dual pairing matrix therefore satisfies $(T^\top)^2 = T^\top$. Taking transposes gives $T^2 = T$. $\square$

Equivalently, the zero-correlation-length identity is the algebraic equality

$$\sum_k |l_k)(r_k| = \sum_{k,h} T_{k,h} |l_k)(r_h|, \quad T_{k,h} = (r_k|l_h),$$

which is the displayed identity of [CPGSV16, Appendix C.2, lines 1490–1493].

Theorem 20.15.6.57 (Unconditional constant trace powers of the pairing matrix). Lean
Lean Lean ✓ Stmt *Let V be a real vector space, let $l_1, \dots, l_n \in V$, let $r_1, \dots, r_n$ be real linear functionals on V , and let $T_{k,h} = (r_k|l_h)$ be the pairing matrix, as in Theorem 20.15.6.56. Assuming only the identity $M^2 = M$ for the operator $M = \sum_k |l_k)(r_k|$, without assuming linear independence of $l_1, \dots, l_n$ or of $r_1, \dots, r_n$, one has the following identities: the first unconditionally, the second for every $N \geq 2$, and the third for every positive integer N :*

$$\begin{aligned} T^2 &= T^3, \\ T^N &= T^2, \\ \text{tr}(T^N) &= \text{tr}(T). \end{aligned}$$

Proof. ✓ Proof Multiplying the identity $M^2 = M$ on the left by the functional map and on the right by the coefficient map turns the two sides directly into T^2 and T^3 , with no independence hypothesis; induction then gives $T^N = T^2$ for every $N \geq 2$. For the trace identity, restrict the operator M to the finite-dimensional span of $l_1, \dots, l_n$: there it is an idempotent endomorphism of a finite free module, and cyclicity of the trace of a composition between that span and the coefficient space identifies $\text{tr}(T)$ with the trace of the restriction and $\text{tr}(T^2)$ with the trace of its square, which agree since the restriction is idempotent. $\square$

Theorem 20.15.6.58 (Rank-one factorization of the sector trace matrix). Lean ✓ Stmt *In the setting of Theorem 20.15.6.56 with real scalars and linearly independent vectors $l_1, \dots, l_n$, if in addition $\text{tr}(T) = 1$, then there are real vectors a, b such that $T = ab^\top$. This is the factorization $T_{k,h} = a_k b_h$ obtained in the proof of [CPGSV16, Appendix C.2, Lemma C.5] for the sector trace matrix, derived here from the displayed zero-correlation-length identity together with the linear independence of the l_k , instead of the Perron–Frobenius trace-power argument of the source.*

Proof. ✓ Proof Theorem 20.15.6.56 gives $T^2 = T$, and the idempotent trace-one criterion of Theorem 20.15.6.51 gives the factorization $T = ab^\top$. $\square$

Theorem 20.15.6.59 (PSD-corrected rank-one T criterion). Lean ✓ Stmt *Let T be a finite real square matrix. Assume that T is primitive, positive semidefinite, normalized by $\text{tr}(T) = 1$, and has constant traces on all positive powers. Then $T = ab^\top$ for some vectors a and b , with $a \cdot b = 1$.*

Proof. ✓ Proof The positive-semidefinite matrix criterion diagonalizes T , uses the first two trace moments to show that exactly one eigenvalue is nonzero, and hence gives the factorization. Taking traces and using $\text{tr}(ab^\top) = a \cdot b$ gives the normalization $a \cdot b = 1$. $\square$

Definition 20.15.6.60 (Sector tensors paired into the sector trace matrix). [Lean](#) [✓ Stmt](#) For a real $n \times n$ matrix T and a real vector space V , this structure consists of vectors $|l_0), \dots, |l_{n-1}) \in V$ and linear functionals $(r_0|, \dots, (r_{n-1}|$ on V satisfying the pairing identity $T_{k,h} = (r_k|l_h)$ of [CPGSV16, Appendix C.2] together with the zero-correlation-length identity

$$\sum_k |l_k)(r_k| = \sum_{k,h} T_{k,h} |l_k)(r_h|,$$

displayed in the proof of [CPGSV16, Lemma C.5]. The vectors are the closed sector tensors obtained there from the sector splitting of an injective simple tensor.

Theorem 20.15.6.61 (Idempotence of the sector trace matrix from the zero-correlation-length identity). [Lean](#) [Lean](#) [✓ Stmt](#) For sector tensors paired into T as in Definition 20.15.6.60, the operator $M = \sum_k |l_k)(r_k|$ satisfies $M^2 = M$. If in addition the vectors $|l_0), \dots, |l_{n-1})$ are linearly independent, then $T^2 = T$.

Proof. [✓ Proof](#) Substituting $T_{k,h} = (r_k|l_h)$ into the right-hand side of the zero-correlation-length identity gives, for every $v \in V$,

$$\sum_{k,h} T_{k,h} (r_h|v) |l_k) = \sum_{k,h} (r_k|l_h) (r_h|v) |l_k) = \sum_k (r_k|Mv) |l_k) = M^2v,$$

so $M^2 = M$. Linear independence of the $|l_k)$ then gives $T^2 = T$ by Theorem 20.15.6.56. $\square$

Theorem 20.15.6.62 (Unconditional constant trace powers of the sector trace matrix). [Lean](#) [Lean](#) [✓ Stmt](#) For sector tensors paired into T as in Definition 20.15.6.60, without any independence hypothesis on the closed sector tensors or the functionals, one has $T^2 = T^3$, and, for every positive integer N , $\text{tr}(T^N) = \text{tr}(T)$.

Proof. [✓ Proof](#) The operator identity $M^2 = M$ established in the proof of Theorem 20.15.6.61 gives $T^2 = T^3$ and the constant trace powers by Theorem 20.15.6.57, with no independence hypothesis. $\square$

Theorem 20.15.6.63 (Nonvanishing and independence of the sector tensors). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose the sector trace matrix T of Definition 20.15.6.60 is primitive. Then every closed sector tensor satisfies $|l_k) \neq 0$. If, in addition, each $|l_k)$ lies in its own member of an independent family of subspaces of V , then the vectors $|l_0), \dots, |l_{n-1})$ are linearly independent.

Proof. [✓ Proof](#) By Theorem 20.15.6.53, some entry of the k -th column of T is positive, and that entry is the pairing $(r_j|l_k)$, so $|l_k) \neq 0$. Nonzero vectors chosen from an independent family of subspaces, one from each member, are linearly independent. $\square$

Theorem 20.15.6.64 (Rank one from sector-pairing idempotence under linear independence). [Lean](#) [Lean](#) [✓ Stmt](#) Let T be a sector trace matrix with $\text{tr}(T) = 1$, paired from sector tensors satisfying the zero-correlation-length identity of Definition 20.15.6.60. If the closed sector tensors $|l_0), \dots, |l_{n-1})$ are linearly independent, then there are real numbers a_k, b_h such that

$$T_{k,h} = a_k b_h, \\ \sum_k a_k b_k = 1.$$

The same conclusion holds if, instead of the independence, T is primitive and the closed sector tensors lie in distinct members of an independent family of subspaces. In contrast with the positive-semidefinite criterion of Theorem 20.15.6.59, the constant trace powers are derived rather than assumed.

Proof. [✓ Proof](#) Theorem 20.15.6.61 gives $T^2 = T$, and Theorem 20.15.6.62 gives the constant trace powers unconditionally. The idempotent trace-one criterion of Theorem 20.15.6.51 yields the factorization $T = ab^\top$, and taking traces gives $\sum_k a_k b_k = \text{tr}(T) = 1$. When the independence is not assumed directly, Theorem 20.15.6.63 derives it from primitivity once each sector tensor lies in its own member of an independent family of subspaces. $\square$

20.15.7 Refinement and coarse-graining channels

Theorem 20.15.7.1 (Positivity and zero-weight neighboring operators). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) *The sector set and every middle-subspin space are nonempty. Each $\Omega_{k,h}$ is positive semidefinite, and $a_k b_h = 0$ implies $\Omega_{k,h} = 0$.*

Definition 20.15.7.2 (The three-site neighboring-state preparation $\mathcal{T}_1$). [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) On a sector pair with $a_k b_h \neq 0$, set $\widehat{\Omega}_{k,h} = (a_k b_h)^{-1} \Omega_{k,h}$. If $a_k b_h = 0$, positivity and vanishing trace imply $\Omega_{k,h} = 0$; choose any density operator on that summand and denote the resulting completed density by $\overline{\Omega}_{k,h}$. The sectorwise preparation is $X \mapsto X \otimes \overline{\Omega}_{k,h}$. Taking the orthogonal direct sum over the outer sectors (k, h) gives the preparation stage $\mathcal{T}_1$ of [CPGSV16, Appendix C.2, lines 1523–1535]. The zero-weight choice makes the source map total without imposing a nonzero-weight hypothesis.

Theorem 20.15.7.3 (Properties of the three-site neighboring densities). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The normalized density is positive semidefinite and has trace one on every active pair. The density chosen on an inactive pair is positive definite and has trace one. Hence $\overline{\Omega}_{k,h}$ is positive semidefinite with trace one for every pair, and agrees with $\widehat{\Omega}_{k,h}$ on active pairs. The corresponding sectorwise and controlled maps prepare these densities on their diagonal summands.*

Theorem 20.15.7.4 ($\mathcal{T}_1$ is trace-preserving completely positive). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
Each completed preparation is trace-preserving and completely positive. The same holds for their orthogonally controlled direct sum.

Proof. [✓ Proof](#) On an active pair, $\text{tr}(\widehat{\Omega}_{k,h}) = (a_k b_h)^{-1} a_k b_h = 1$; the chosen density on a zero-weight pair also has trace one. Preparation by a positive operator of trace one is trace-preserving and completely positive. The orthogonal control discards coherences between distinct outer-sector pairs and applies these preparations on the diagonal summands. $\square$

Definition 20.15.7.5 (The refinement channel $\mathcal{T}$). [Lean](#) [✓ Stmt](#) Define the channel from two physical sites to three physical sites by $\mathcal{T} = \mathcal{T}_2 \mathcal{T}_1 \mathcal{T}_0$. Thus $\mathcal{T}_0$ traces the neighboring subspins, $\mathcal{T}_1$ adjoins the completed three-site neighboring density, and $\mathcal{T}_2$ restores the three complete physical sectors. This is the refinement map in [CPGSV16, Appendix C.2, lines 1522–1545].

Theorem 20.15.7.6 ($\mathcal{T}$ is trace-preserving completely positive). [Lean](#) [✓ Stmt](#) *The refinement channel $\mathcal{T}$ is trace-preserving and completely positive.*

Proof. [✓ Proof](#) Each constituent map is trace-preserving and completely positive, and this property is preserved under composition. $\square$

Applied to the two-site closure, on the active (k, h) -block, $\mathcal{T}_0$ traces $B_k^R \otimes B_h^L$ and sends $\mathfrak{R}_2(\mathcal{K}_2(X))_{k,h}$ to $a_k b_h B_{k,h}(X)$ on $B_k^L \otimes B_h^R$. The preparation $\mathcal{T}_1$ then adjoins

$$\widehat{\Omega}_{k,h} = (a_k b_h)^{-1} \bigoplus_l (\eta_{k,l} \otimes \eta_{l,h}),$$

$$(B_k^L \otimes B_k^R) \otimes (B_l^L \otimes B_l^R) \otimes (B_h^L \otimes B_h^R).$$
$$\mathcal{H}_{k,h}^\Omega = \bigoplus_l (B_k^R \otimes B_l^L) \otimes (B_l^R \otimes B_h^L)$$
$$\mathcal{A}_2 \text{ --- } \mathcal{T}_0 \text{ --- } \mathcal{A}_\partial \text{ --- } \mathcal{T}_1 \text{ --- } \mathcal{A}_\Omega \text{ --- } \mathcal{T}_2 \text{ --- } \mathcal{A}_3$$
$$\mathcal{A}_2 \text{ --- } \mathcal{T} \text{ --- } \mathcal{A}_3$$
$$\left[R_{\Phi_3} \mathfrak{R}_3(\mathcal{K}_3(X)) \right]_{(k,h),(k,h)} = B_{k,h}(X) \otimes \Omega_{k,h}.$$

Proof. [✓ Proof](#) On the summand labelled by the middle sector l , this is the fixed-sector factorization $B_{k,h}(X) \otimes (\eta_{k,l} \otimes \eta_{l,h})$. Taking the direct sum over l gives the stated block. $\square$

Theorem 20.15.7.13 (Off-diagonal outer-sector entries of the three-site closure). [Lean](#) [✓ Stmt](#) Every matrix entry of $R_{\Phi_3} \mathfrak{R}_3(\mathcal{K}_3(X))$ between distinct outer-sector pairs $(k, h) \neq (p, q)$ vanishes.

Proof. [✓ Proof](#) Distinct outer-sector pairs differ either in their first sector or in their third sector. In the corresponding physical slice, the sector-coordinate tensor is block diagonal, so that factor vanishes. $\square$

Theorem 20.15.7.14 (Regrouped two-site closure block). [Lean](#) [✓ Stmt](#) For every virtual matrix X and outer-sector pair (k, h) ,

$$\left[R_{\Phi_2} \mathfrak{R}_2(\mathcal{K}_2(X)) \right]_{(k,h),(k,h)} = B_{k,h}(X) \otimes \eta_{k,h}.$$

Proof. [✓ Proof](#) This is the fixed-sector two-site factorization after identifying the sitewise sector coordinates with the regrouped boundary and neighboring coordinates. $\square$

Theorem 20.15.7.15 (Diagonal-block action of $\mathcal{T}_0$). [Lean](#) [✓ Stmt](#) For every two-site matrix Y , outer-sector pair (k, h) , and boundary indices a, b ,

$$[\mathcal{T}_0(Y)]_{(k,h,a),(k,h,b)} = \left[\text{tr}_{R_k \otimes L_h} \left([R_{\Phi_2} Y]_{(k,h),(k,h)} \right) \right]_{a,b}.$$

Proof. [✓ Proof](#) The controlled partial trace acts on a diagonal outer-sector block by the ordinary partial trace over its neighboring factor. $\square$

Theorem 20.15.7.16 (Partial trace of a factorized two-site block). [Lean](#) [✓ Stmt](#) If the (k, h) diagonal block of $R_{\Phi_2} Y$ is $B \otimes \eta_{k,h}$, then $[\mathcal{T}_0(Y)]_{(k,h),(k,h)} = a_k b_h B$.

Proof. [✓ Proof](#) The partial trace of $B \otimes \eta_{k,h}$ is $\text{tr}(\eta_{k,h})B = a_k b_h B$. $\square$

Theorem 20.15.7.17 (Preparation of a scaled boundary block). [Lean](#) [✓ Stmt](#) For every outer-sector pair (k, h) and boundary matrix B , the completed sectorwise preparation satisfies $\bar{\mathcal{T}}_{k,h}(a_k b_h B) = B \otimes \Omega_{k,h}$.

Proof. [✓ Proof](#) If $a_k b_h \neq 0$, the scalar cancels the normalization in $\hat{\Omega}_{k,h} = (a_k b_h)^{-1} \Omega_{k,h}$. If $a_k b_h = 0$, both sides vanish because $\Omega_{k,h} = 0$. $\square$

Theorem 20.15.7.18 (Diagonal-block action of $\mathcal{T}_1$). [Lean](#) [✓ Stmt](#) If the (k, h) diagonal block of a boundary matrix Y is $a_k b_h B$, then $[\mathcal{T}_1(Y)]_{(k,h),(k,h)} = B \otimes \Omega_{k,h}$.

Proof. [✓ Proof](#) On a diagonal outer-sector block, the controlled map is the completed sectorwise preparation, to which the preceding theorem applies. $\square$

Theorem 20.15.7.19 (Off-diagonal action of $\mathcal{T}_1$). [Lean](#) [✓ Stmt](#) The map $\mathcal{T}_1$ annihilates every matrix entry between distinct outer-sector pairs.

Proof. [✓ Proof](#) This is the orthogonal control in the definition of $\mathcal{T}_1$. $\square$

Theorem 20.15.7.20 (Reindexing action of $\mathcal{T}_2$). [Lean](#) [✓ Stmt](#) Let Φ_3 be the three-site regrouping. For every prepared matrix Y and three-site indices p, q , $[\mathcal{T}_2(Y)]_{p,q} = Y_{\Phi_3(p), \Phi_3(q)}$.

Proof. [✓ Proof](#) This is matrix reindexing along the inverse regrouping Φ_3^{-1} . $\square$

Theorem 20.15.7.21 (Refinement of the two-site closure). [Lean](#) [✓ Stmt](#) Assume the neighboring-operator trace factorization of Definition 20.15.3.32. For every virtual matrix X , in the canonical physical-sector coordinates selected by the isometry U , the refinement channel satisfies

$$\mathcal{T}(\mathfrak{R}_2(\mathcal{K}_2(X))) = \mathfrak{R}_3(\mathcal{K}_3(X)). \quad (20.60)$$

This is the refinement half of Proposition C.7 [CPGSV16, Appendix C.2, lines 1510–1516 and 1522–1545].

Proof. [✓ Proof](#) The regrouping contained in $\mathcal{T}_0$ changes $\mathfrak{R}_2$ into $\mathfrak{G}_2$. On the (k, h) diagonal block, the partial trace then gives

$$\mathcal{T}_0(\mathfrak{R}_2(\mathcal{K}_2(X)))_{k,h} = a_k b_h B_{k,h}(X).$$

On an active pair, $\mathcal{T}_1$ adjoins $(a_k b_h)^{-1} \Omega_{k,h}$, and hence

$$\mathcal{T}_1 \mathcal{T}_0(\mathfrak{R}_2(\mathcal{K}_2(X)))_{k,h} = B_{k,h}(X) \otimes \Omega_{k,h}.$$

If $a_k b_h = 0$, both sides vanish because $\Omega_{k,h} = 0$. After the canonical reassociation of the intermediate sector sum, these are precisely the diagonal blocks of $\mathfrak{G}_3(\mathcal{K}_3(X))$; all off-diagonal sector blocks vanish. Finally, the inverse regrouping $\mathcal{T}_2$ changes $\mathfrak{G}_3$ into $\mathfrak{R}_3$, which gives (20.60). $\square$

Theorem 20.15.7.22 (Coarse-graining of the three-site closure). [Lean](#) [Lean](#) [✓ Stmt](#) Assume the neighboring-operator trace factorization of Definition 20.15.3.32. For every virtual matrix X , in the canonical physical-sector coordinates selected by the isometry U , the coarse-graining channel satisfies

$$\mathcal{S}(\mathfrak{R}_3(\mathcal{K}_3(X))) = \mathfrak{R}_2(\mathcal{K}_2(X)). \quad (20.61)$$

Proof. [✓ Proof](#) On the (k, h) diagonal block, $\mathcal{S}_0$ traces $\Omega_{k,h}$ and gives $a_k b_h B_{k,h}(X)$. On an active sector pair, $\mathcal{S}_1((a_k b_h) B_{k,h}(X)) = B_{k,h}(X) \otimes \eta_{k,h}$. If $a_k b_h = 0$, positivity and zero trace give $\eta_{k,h} = 0$, so both sides of this identity vanish. Finally, $\mathcal{S}_2$ regroups the four factors into two physical sites. For distinct sector pairs $(k, h) \neq (p, q)$,

$$\begin{aligned} (\mathcal{S}_1 Y)_{(k,h),(p,q)} &= 0, \\ [\mathfrak{R}_2(\mathcal{K}_2(X))]_{(k,h),(p,q)} &= 0. \end{aligned}$$

These diagonal and off-diagonal identities give (20.61). $\square$

For the following construction, through Theorem 20.15.7.42, assume the neighboring-operator trace factorization of Definition 20.15.3.32. Write $q = \dim(\bigoplus_k L_k \otimes R_k)$ for the physical dimension of the sector-coordinate tensor $\widehat{\mathcal{K}}$.

Definition 20.15.7.23 (Refinement in ordinary physical coordinates). [Lean](#) [✓ Stmt](#) Transporting $\mathcal{T}$ through the canonical two-site and three-site coordinate identifications gives a channel $\widehat{\mathcal{T}} : M_{q^2}(\mathbb{C}) \rightarrow M_{q^3}(\mathbb{C})$, where all multiple physical indices are associated to the right.

Theorem 20.15.7.24 (The physical refinement map is a channel). [Lean](#) [✓ Stmt](#) The map $\widehat{\mathcal{T}}$ is trace-preserving and completely positive.

Proof. [✓ Proof](#) The coordinate identifications are unitary permutation channels. Composing them with a trace-preserving completely positive map preserves both properties. $\square$

Theorem 20.15.7.25 (Refinement identity in ordinary physical coordinates). [Lean](#) [✓ Stmt](#) For every virtual matrix X , $\widehat{\mathcal{T}}(\widehat{\mathcal{K}}_2(X)) = \widehat{\mathcal{K}}_3(X)$.

Proof. [✓ Proof](#) Transport the sector-coordinate identity through the inverse three-site coordinate identification. $\square$

Definition 20.15.7.26 (Physical coarse-graining). [Lean](#) [✓ Stmt](#) Transporting $\mathcal{S}$ through the canonical three-site and two-site coordinate identifications gives a channel $\widehat{\mathcal{S}} : M_{q^3}(\mathbb{C}) \rightarrow M_{q^2}(\mathbb{C})$.

Theorem 20.15.7.27 (The physical coarse-graining map is a channel). [Lean](#) [✓ Stmt](#) The map $\widehat{\mathcal{S}}$ is trace-preserving and completely positive.

Proof. [✓ Proof](#) This follows by composition with the two coordinate-permutation channels. $\square$

Theorem 20.15.7.28 (Coarse-graining identity in ordinary physical coordinates). [Lean](#) [✓ Stmt](#) For every virtual matrix X , $\widehat{\mathcal{S}}(\widehat{\mathcal{K}}_3(X)) = \widehat{\mathcal{K}}_2(X)$.

Proof. [✓ Proof](#) Transport the sector-coordinate identity through the inverse two-site coordinate identification. $\square$

Definition 20.15.7.29 (Localized physical refinement). [Lean](#) [✓ Stmt](#) Reassociating three right-associated sites as (12)3, applying $\widehat{\mathcal{T}} \otimes \text{id}$ to the first two, and reassociating the result back to 1(2(34)) defines the localized refinement channel $\widehat{\mathcal{T}}_{(12)3} : M_{q^3}(\mathbb{C}) \rightarrow M_{q^4}(\mathbb{C})$ on right-associated physical coordinates.

Theorem 20.15.7.30 (The localized physical refinement is a channel). [Lean](#) [✓ Stmt](#) The localized refinement map is trace-preserving and completely positive.

Proof. [✓ Proof](#) Tensoring a channel with the identity preserves complete positivity and trace, as do the two associativity permutations. $\square$

Theorem 20.15.7.31 (Localized refinement of physical closures). [Lean](#) [✓ Stmt](#) For every virtual matrix X , $(\widehat{\mathcal{T}} \otimes \text{id})(\widehat{\mathcal{K}}_3(X)) = \widehat{\mathcal{K}}_4(X)$, with the map understood in right-associated coordinates.

Proof. [✓ Proof](#) Fixing the last ket and bra indices turns the three-site closure into a two-site closure with the last tensor matrix absorbed into X . Apply the two-to-three-site refinement identity to this virtual matrix. $\square$

Definition 20.15.7.32 (Localized physical coarse-graining). [Lean](#) [✓ Stmt](#) Reassociating four right-associated sites as (123)4, applying $\widehat{\mathcal{S}} \otimes \text{id}$ to the first three, and reassociating the result back to 1(23) defines the localized coarse-graining channel $\widehat{\mathcal{S}}_{(123)4} : M_{q^4}(\mathbb{C}) \rightarrow M_{q^3}(\mathbb{C})$ on right-associated physical coordinates.

Theorem 20.15.7.33 (The localized physical coarse-graining is a channel). [Lean](#) [✓ Stmt](#) The localized coarse-graining map is trace-preserving and completely positive.

Proof. [✓ Proof](#) Tensor the physical coarse-graining channel with the identity and compose with the two associativity permutations. $\square$

Theorem 20.15.7.34 (Localized coarse-graining of physical closures). [Lean](#) [✓ Stmt](#) For every virtual matrix X , $(\widehat{\mathcal{S}} \otimes \text{id})(\widehat{\mathcal{K}}_4(X)) = \widehat{\mathcal{K}}_3(X)$ in right-associated coordinates.

Proof. ✓ Proof Fixing the last ket and bra indices identifies the four-site closure with a three-site closure having the last tensor matrix absorbed into X . Apply the three-to-two-site coarse-graining identity. $\square$

Definition 20.15.7.35 (Two-step physical refinement). Lean ✓ Stmt Let F be a physical-sector factorization whose neighboring operators satisfy Definition 20.15.3.32, and let $\widehat{\mathcal{K}}$ be its sector-coordinate tensor. Define

$$\begin{aligned}\widehat{\mathcal{T}}^2 &= \widehat{\mathcal{T}}_{(12)3} \circ \widehat{\mathcal{T}}, \\ \widehat{\mathcal{T}}^2 : M_{q^2}(\mathbb{C}) &\longrightarrow M_{q^4}(\mathbb{C}).\end{aligned}$$

Here the second application acts on the first two sites and leaves the third site unchanged. Thus the superscript denotes two successive overlapping applications, as in [CPGSV16, Appendix C.2, lines 1821–1825].

Theorem 20.15.7.36 (The two-step physical refinement is a channel). Lean ✓ Stmt The map $\widehat{\mathcal{T}}^2$ is trace-preserving and completely positive.

Proof. ✓ Proof It is the composition of the physical refinement channel and its localization on the first two sites. $\square$

Theorem 20.15.7.37 (Two-step refinement of the sector-coordinate tensor). Lean ✓ Stmt For every virtual matrix X , the sector-coordinate tensor satisfies $\widehat{\mathcal{T}}^2(\widehat{\mathcal{K}}_2(X)) = \widehat{\mathcal{K}}_4(X)$.

Proof. ✓ Proof The first refinement gives $\widehat{\mathcal{K}}_3(X)$; applying the localized refinement to its first two sites gives $\widehat{\mathcal{K}}_4(X)$. $\square$

Definition 20.15.7.38 (Two-step physical coarse-graining). Lean ✓ Stmt Define

$$\begin{aligned}\widehat{\mathcal{S}}^2 &= \widehat{\mathcal{S}} \circ \widehat{\mathcal{S}}_{(123)4}, \\ \widehat{\mathcal{S}}^2 : M_{q^4}(\mathbb{C}) &\longrightarrow M_{q^2}(\mathbb{C}).\end{aligned}$$

Here the first application acts on the first three sites and leaves the fourth site unchanged. The superscript again denotes successive overlapping applications.

Theorem 20.15.7.39 (The two-step physical coarse-graining is a channel). Lean ✓ Stmt The map $\widehat{\mathcal{S}}^2$ is trace-preserving and completely positive.

Proof. ✓ Proof It is the composition of the localized coarse-graining channel and the physical coarse-graining channel. $\square$

Theorem 20.15.7.40 (Two-step coarse-graining of the sector-coordinate tensor). Lean ✓ Stmt For every virtual matrix X , the sector-coordinate tensor satisfies $\widehat{\mathcal{S}}^2(\widehat{\mathcal{K}}_4(X)) = \widehat{\mathcal{K}}_2(X)$.

Proof. ✓ Proof The localized coarse-graining first gives $\widehat{\mathcal{K}}_3(X)$; the second coarse-graining gives $\widehat{\mathcal{K}}_2(X)$. $\square$

Theorem 20.15.7.41 (Blocked sector-coordinate tensor is a renormalization fixed point). Lean ✓ Stmt Let F be a physical-sector factorization whose neighboring operators satisfy Definition 20.15.3.32. Let $\widehat{\mathcal{K}}^{[2]}$ be the tensor obtained by blocking two adjacent sites of its sector-coordinate tensor $\widehat{\mathcal{K}}$. Then $\widehat{\mathcal{K}}^{[2]}$ is a renormalization fixed point in the sense of Definition 20.2.1. More precisely, after the canonical identifications of one blocked site with two original sites and two blocked sites with four original sites, the required channels are $\widehat{\mathcal{S}}^2$ and $\widehat{\mathcal{T}}^2$.

Proof. [✓ Proof](#) Decode one blocked index as a pair of physical indices and two blocked indices as four physical indices. The one-site and two-site closures of $\widehat{\mathcal{K}}^{[2]}$ thereby become respectively $\widehat{\mathcal{K}}_2(X)$ and $\widehat{\mathcal{K}}_4(X)$. Transporting the two channels through these permutation identifications preserves complete positivity and trace, and the two closure identities give the required equations in Definition 20.2.1. $\square$

Theorem 20.15.7.42 (Conditional two-site blocking theorem from neighboring trace factorization). [Lean](#) [✓ Stmt](#) Let F be a physical-sector factorization of $\mathcal{K}$ whose neighboring operators satisfy Definition 20.15.3.32. Thus they are positive semidefinite and there are real families $(a_k)_k, (b_k)_k$ with $\text{tr}(\eta_{k,h}) = a_k b_h$ and $\sum_k a_k b_k = 1$. Let $\mathcal{K}^{[2]}$ be the tensor obtained by blocking two adjacent sites of $\mathcal{K}$. Then $\mathcal{K}^{[2]}$ is a renormalization fixed point in the sense of Definition 20.2.1. This is the conditional channel construction of [CPGSV16, Proposition C.7] for one supplied factorization. It does not extend to the printed all-BNT implication [CPGSV16, Theorem 4.9, (iv) $\Rightarrow$ (v)]: condition (iv) constrains each BNT representative but not its raw repeated-copy weights, and that implication is refuted by Theorem 20.15.8.23. After condition (ii) has forced equal copy weights and their common values have been absorbed, these one-factorization channels become inputs to the projector-controlled assembly used by the viable implication (ii) $\Rightarrow$ (v).

Proof. [✓ Proof](#) Transport the refinement and coarse-graining channels for $\widehat{\mathcal{K}}^{[2]}$ through the two-site and four-site tensor powers of the physical isometry, as in (20.63). These unitary congruences preserve complete positivity and trace, while the corresponding closure identities carry the two fixed-point equations back to $\mathcal{K}^{[2]}$. $\square$

Theorem 20.15.7.43 (Arbitrary virtual boundaries of canonical block sums). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let a tensor be gauge equivalent, by an invertible virtual matrix G , to a finite block sum with component tensors A_s and weights μ_s . For an arbitrary virtual boundary X and an arbitrary word w , its closure is the sum of the component closures with boundaries given by the diagonal blocks of $G^{-1}XG$:

$$\text{tr}(B^w X) = \sum_s \mu_s^{|w|} \text{tr}(A_s^w (G^{-1}XG)_{ss}).$$

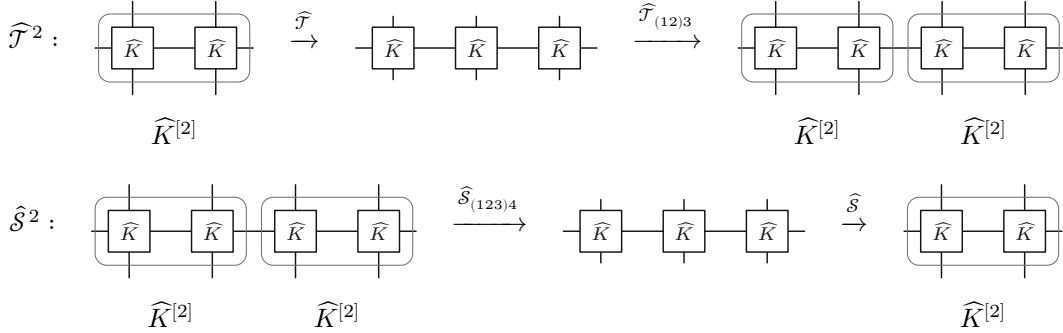


Figure 20.6: The superscripts in $\widehat{\mathcal{T}}^2$ and $\widehat{\mathcal{S}}^2$ denote successive overlapping applications. The enclosures identify the two-site and four-site endpoints with one and two sites of the blocked tensor. This is the channel construction in [CPGSV16, Appendix C.2, lines 1821–1825].

The conjugated boundary $G^{-1}XG$ is independent of the word length. This is the arbitrary-boundary form of the block calculation used in [CPGSV16, lines 186–195 and Appendix C.2, lines 1646–1665 and 1810–1825].

Proof. ✓ Proof A word of length $|w|$ in the block-sum tensor is block diagonal. Its s -th diagonal block is $\mu_s^{|w|} A_s^w$. On multiplying by an arbitrary boundary matrix, only the diagonal boundary blocks contribute to the trace, which gives the asserted sum before the gauge change.

Under the gauge G , the word A^w is conjugated to GA^wG^{-1} . Cyclicity of the trace moves the two gauge factors to the boundary and replaces X by $G^{-1}XG$. Combining these two identities proves the formula. Since this conjugation depends only on G and X , the transformed boundary is the same at every word length. $\square$

Theorem 20.15.7.44 (Absorbed-representative boundary decomposition). Lean Lean ✓ Stmt

Let the complete tensor B be exactly gauge equivalent, by an invertible virtual matrix G , to a sector decomposition: an arbitrary finite family of basis blocks A_j , $j = 1, \dots, g$, each carrying a copy multiplicity r_j and nonzero weights $\mu_{j,q}$. Suppose that the repeated-copy weights over each basis block are equal, with no equality required between different basis blocks, and absorb this common weight into its representative $\mathcal{K}_j$ as in Definition 20.15.1.7. Then, for every length N and every virtual boundary X , the physical closure of the complete tensor is the sum of the absorbed-representative closures, with the boundary of a representative given by the sum of the diagonal copy blocks of $G^{-1}XG$ lying over it:

$$B_N(X) = \sum_{j=1}^g (\mathcal{K}_j)_N \left(\sum_{q=1}^{r_j} (G^{-1}XG)_{(j,q),(j,q)} \right).$$

Neither the conjugation $X \mapsto G^{-1}XG$ nor the subsequent sum of diagonal copy blocks depends on N . In particular, the same representative boundary map applies at the two lengths entering the two channel equations of Definition 20.2.1. This is the arbitrary-boundary form of the block calculation used in [CPGSV16, lines 186–195 and Appendix C.2, lines 1646–1665 and 1810–1825].

Proof. ✓ Proof Apply Theorem 20.15.7.43 to the block sum indexed by the individual copies. Reindex this finite sum by a representative j and a copy label q . For fixed j , all copy weights are equal, so their common N -th power may be absorbed into the length N word of the representative tensor. The remaining dependence on q is linear in the diagonal boundary block. Summing over q therefore replaces these blocks by their sum, which is the stated representative boundary. Neither the conjugation $X \mapsto G^{-1}XG$ nor the subsequent sum of diagonal copy blocks depends on N . $\square$

Theorem 20.15.7.45 (Conditional composition of absorbed BNT channels). Lean Lean Lean

Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let K be exactly gauge equivalent, by an invertible virtual matrix G , to a canonical BNT sum. Suppose that the repeated-copy weights agree within each fixed representative and have been absorbed into tensors $(\mathcal{K}_s)_s$. Let $(P_s)_s$ be orthogonal projections on the one-site physical space such that

$$\sum_s P_s = I, \quad P_s P_t = 0 \quad (s \neq t).$$

Suppose also that, for every s and every pair of virtual indices β, α , the corresponding physical slice satisfies

$$P_s \mathcal{K}_s[\beta, \alpha] = \mathcal{K}_s[\beta, \alpha] = \mathcal{K}_s[\beta, \alpha] P_s.$$

If every blocked representative $\mathcal{K}_s^{[2]}$ is a renormalization fixed point via trace-preserving completely positive maps, then $K^{[2]}$ is also such a renormalization fixed point.

The global maps first measure the outer sector on the first original site of the blocked input. On one blocked site the corresponding projection is $P_s \otimes I$; on two blocked sites it acts on the first original site and as the identity on the remaining three. For every virtual boundary X , the boundary supplied to sector s is the sum of the diagonal copy blocks of $G^{-1}XG$ over that representative. The same boundary is used in the length-two and length-four identities. This is the conditional composition underlying [CPGSV16, Appendix C.2, lines 1646–1665 and 1810–1825]; it is not the printed implication (iv) $\Rightarrow$ (v) of Theorem 4.9.

Proof. ✓ Proof Write $\mathcal{C}_N(A, X)$ for the length- N closure of A with boundary X , and write $B_s(X)$ for the boundary supplied to representative s . At both relevant lengths the boundary decomposition gives

$$\mathcal{C}_N(K, X) = \sum_t \mathcal{C}_N(\mathcal{K}_t, B_t(X)), \quad N \in \{2, 4\}.$$

Let $\Phi_{s,N}$ be compression by P_s on the first site and by the identity on the remaining N sites. Two-sided physical support and $P_s P_t = 0$ for $s \neq t$ give the selection identity

$$\Phi_{s,N}(\mathcal{C}_{N+1}(\mathcal{K}_t, B_t(X))) = \begin{cases} \mathcal{C}_{N+1}(\mathcal{K}_t, B_t(X)), & s = t, \\ 0, & s \neq t. \end{cases}$$

Apply the representative channel to the surviving summand and sum over s . The projective resolution makes both resulting maps trace-preserving and completely positive. The boundary decomposition then identifies their outputs with the length-two and length-four closures of K carrying the original boundary. $\square$

Theorem 20.15.7.46 (Completion outside the absorbed BNT supports). Lean Lean Lean Lean
✓ Stmt Retain the hypotheses of Theorem 20.15.7.45, except that the pairwise orthogonal outer support projections are not required to sum to the identity. If every blocked absorbed representative is a renormalization fixed point via trace-preserving completely positive maps, then the blocked complete tensor is also such a fixed point.

Proof. ✓ Proof If the one-site physical space has dimension zero, all of its endomorphisms are equal. Hence $\sum_s P_s = I$, and Theorem 20.15.7.45 applies directly. Assume from now on that the physical dimension is positive. Fixed basis vectors witness that the two output spaces are nonempty; normalizing their identity matrices gives maximally mixed density matrices $\rho_2 \in M_{q^2}(\mathbb{C})$ and $\rho_4 \in M_{q^4}(\mathbb{C})$.

Let $Q = \sum_s P_s$, an orthogonal projection by pairwise orthogonality. For $N \in \{2, 4\}$ write $Q_N = Q \otimes \mathbb{1}^{\otimes(N-1)}$ for the lift of Q to the first original site of an N -site chain, and likewise $P_{s,N} = P_s \otimes \mathbb{1}^{\otimes(N-1)}$, so that $Q_N = \sum_s P_{s,N}$. Two-sided support of each absorbed representative carries over to the length-two and length-four closures of the complete tensor K ,

$$Q_2 K_2(X) Q_2 = K_2(X), \quad Q_4 K_4(X) Q_4 = K_4(X), \quad (20.62)$$

for every virtual boundary X .

By hypothesis every $\mathcal{K}_s^{[2]}$ carries trace-preserving completely positive maps $\mathcal{S}_s, \mathcal{T}_s$ witnessing its fixed-point property. Sum these channels after compression by the matching first-site projection, $\mathcal{S}_{\text{act}}(Y) = \sum_s \mathcal{S}_s(P_{s,4} Y P_{s,4})$ and $\mathcal{T}_{\text{act}}(Z) = \sum_s \mathcal{T}_s(P_{s,2} Z P_{s,2})$. Each sum is completely positive, and pairwise orthogonality of the P_s makes it preserve exactly the trace selected by Q ,

$$\text{tr}(\mathcal{S}_{\text{act}}(Y)) = \text{tr}(Q_4 Y), \quad \text{tr}(\mathcal{T}_{\text{act}}(Z)) = \text{tr}(Q_2 Z).$$

Since Q_N is Hermitian and idempotent, applying Theorem 21.2.6.1 to each active sum with $P = Q_N$ completes it on the orthogonal input complement,

$$\mathcal{S}_{\text{glob}}(Y) = \mathcal{S}_{\text{act}}(Y) + \text{tr}((I - Q_4)Y)\rho_2, \quad \mathcal{T}_{\text{glob}}(Z) = \mathcal{T}_{\text{act}}(Z) + \text{tr}((I - Q_2)Z)\rho_4,$$

and both global maps are trace-preserving and completely positive. By (21.49) and (20.62), $\mathcal{S}_{\text{glob}}(K_4(X)) = \mathcal{S}_{\text{act}}(K_4(X))$ and $\mathcal{T}_{\text{glob}}(K_2(X)) = \mathcal{T}_{\text{act}}(K_2(X))$ for every virtual boundary X , so the two channel equations for $\mathcal{S}_{\text{glob}}, \mathcal{T}_{\text{glob}}$ reduce to those of $\mathcal{S}_{\text{act}}, \mathcal{T}_{\text{act}}$ and hence, by construction of the active sums, to the representative equations. $\square$

Theorem 20.15.7.47 (Blocking channels from representative trace factorizations). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) *Let M be exactly gauge equivalent, by an invertible virtual matrix G , to a canonical BNT sum whose repeated-copy weights agree within each fixed representative and have been absorbed into tensors $(\mathcal{K}_s)_s$. Suppose that the simultaneous one-site tuples of the decomposition span the product of the virtual matrix algebras and that distinct absorbed representatives satisfy the unstarred layer orthogonality identities. If every $\mathcal{K}_s$ carries a physical-sector factorization with positive semidefinite neighboring operators $\eta_{k,h}^s$ and real families $(a_k^s)_k, (b_k^s)_k$ with*

$$\text{tr}(\eta_{k,h}^s) = a_k^s b_h^s, \quad \sum_k a_k^s b_k^s = 1,$$

then $M^{[2]}$ is a renormalization fixed point via trace-preserving completely positive maps. In particular, this conclusion holds whenever every representative carries a single-sector factorization whose sole neighboring operator is positive semidefinite of unit trace, since the coefficient families may then be taken identically one. The factorizations are assumed here; deriving them from the strong area law and zero correlation length is the remaining step of the source implication and stays open in Theorem 20.15.7.48.

Proof. [Proof](#) Positivity of the neighboring operators makes every absorbed representative generate positive semidefinite periodic operators, and one-site spanning makes it injective. With the unstarred layer identities, Corollary 20.15.8.17 gives pairwise orthogonal two-sided outer physical supports, realizing the projection onto each local subspace in the closing proof of the source. Each supplied trace factorization gives a blocked channel pair for its representative by Theorem 20.15.7.42. The projector-controlled combination of these pairs across the outer supports, completed trace-preservingly outside their sum, is Theorem 20.15.7.46. In the single-sector case the sole trace condition $\text{tr}(\eta^s) = 1$ realizes the required rank-one normalization with unit coefficient families. $\square$

Theorem 20.15.7.48 (Blocking channels after common-weight absorption). [Not ready](#) *Let K be a simple tensor in block-injective canonical form (biCF), equipped with a basis-of-normal-tensors decomposition and satisfying the strong area law and literal zero correlation length,*

$$\mathcal{T}_K^2 = \mathcal{T}_K.$$

Let $M = K^{[2]}$ be its two-site blocking. Then there are trace-preserving completely positive maps in both directions between the one-site and two-site closures of M . Equivalently, M is a renormalization fixed point via these two maps. This is implication (ii) $\Rightarrow$ (v) of [CPGSV16, Theorem 4.9 and Appendix C.2, lines 1810–1825]. The separate standing biCF and BNT hypotheses are imposed at lines 849–850 and restated at line 1628 of the same source. The source first uses zero correlation length at lines 1646–1665 to make repeated-copy weights independent of the copy label and absorbs their common value into each representative. The proof at lines 1815–1816 then

projects onto the mutually orthogonal outer physical sectors and invokes the one-representative channel construction. Thus a faithful proof must supply one blocked channel pair for every absorbed representative. Their combination through the outer physical projections, including the trace-preserving completion outside their sum, is Theorem 20.15.7.46. Given one neighboring trace factorization for every absorbed representative, the complete composition down to the blocked channel pair for M is Theorem 20.15.7.47. The remaining step is therefore the existence of these per-representative factorizations under the strong area law and zero correlation length.

$$\begin{aligned} U^{\otimes 4} \widehat{\mathcal{T}}^2 (U^\dagger)^{\otimes 2} &= \mathcal{T}, \\ U^{\otimes 2} \widehat{\mathcal{S}}^2 (U^\dagger)^{\otimes 4} &= \mathcal{S}. \end{aligned} \tag{20.63}$$

Conjugation by $U^{\otimes n}$ and $(U^\dagger)^{\otimes n}$ carries the sector-coordinate channels to the ordinary-coordinate channels. This combines Proposition C.7 with [CPGSV16, Appendix C.2, lines 1510–1563 and 1821–1825].

Tracing the ket and bra indices of a sector tensor gives the bond vector $|l_k\rangle$ or the bond functional $\langle r_k|$. Hence a neighboring pair gives $T_{k,h} = \langle r_k | l_h \rangle = \text{tr}(\eta_{k,h})$, as in [CPGSV16, Appendix C.2, lines 1474–1481].

20.15.8 Local simple-MPDO structure

Remark 20.15.8.1 (Normalization boundary in Lemma C.5). Lemma C.5 of [CPGSV16, Appendix C.2, lines 1484–1499] claims that an injective matrix product density operator tensor satisfying the strong area law and zero correlation length has real families $(a_k)_k, (b_k)_k$ with

$$\begin{aligned} T_{k,h} &= \text{tr}(\eta_{k,h}) = a_k b_h, \\ \sum_k a_k b_k &= 1. \end{aligned}$$

The lemma is proved inside Case I, where the tensor is also assumed to be an injective normal tensor. The raw four-sector tensor below has the source inverse-map factorization and no such families a, b , but it is not the normalized tensor of Case I: its doubled-index transfer satisfies $\mathcal{E}_{\mathcal{K}}^2 = (5/16)\mathcal{E}_{\mathcal{K}}$. Its normalized representative $A = (4/\sqrt{5})\mathcal{K}$ satisfies $\mathcal{B}_A^2 = (4/\sqrt{5})\mathcal{B}_A \neq \mathcal{B}_A$ and therefore fails the paper’s literal zero-correlation-length diagram. Thus it does not obstruct the all-sector coefficient theorem proved under the complete normal Case-I hypotheses. The remaining open step is the structural statement used by the following corollary.

Theorem 20.15.8.2 (SAL–ZCL rank-one normalization obstruction). [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [✓ Stmt](#) *There is an injective matrix product density operator tensor satisfying the strong area law and source zero correlation length whose source-selected inverse-map factorization has positive semidefinite neighboring operators and a primitive trace-one active trace matrix T , but for which there are no families a, b satisfying $T_{k,h} = a_k b_h$ for all k, h . Equivalently, for the corresponding closed sector tensors L, Q , one has $Q(1 - LQ)L = T - T^2 \neq 0$. Thus the rank-one conclusion in Lemma C.5 fails for this raw representative. The normal-tensor hypothesis imposed in Case I at source lines 1374–1381 is not included in this statement.*

Proof. [✓ Proof](#) The four physical sectors are one-dimensional. Their cyclic transition law is doubly stochastic. After tracing one site, conditioning on the middle physical label gives, at

every admissible cut,

$$\rho_{ABC} = \bigoplus_{k=0}^3 \frac{1}{4} \rho_{A,k}^L \otimes \rho_{k,C}^R.$$

Thus the reduced state has a four-sector quantum-Markov decomposition and the tensor satisfies the strong area law. Its physical-trace transfer is the idempotent matrix LQ , so it has source zero correlation length. The explicit Hayashi decomposition and inverse map recover the displayed tensors L, Q . The active trace matrix is primitive and has trace one, but direct multiplication gives $T^2 \neq T$. If $T = ab^\top$, then

$$T^2 = a(b^\top a)b^\top = \left(\sum_k a_k b_k \right) T = T,$$

where the last equality uses $\text{tr}(T) = 1$. Hence T has no rank-one factorization. $\square$

Theorem 20.15.8.3 (Blocked renormalization maps from two-to-four maps). [Lean](#) [✓ Stmt](#) *Let K be an MPO tensor. Suppose that trace-preserving completely positive maps S_{24} and T_{24} satisfy the following identities for every virtual matrix X :*

$$S_{24}[\mathcal{K}_4(X)] = \mathcal{K}_2(X), \quad T_{24}[\mathcal{K}_2(X)] = \mathcal{K}_4(X).$$

Then the two-site physical blocking $K^{[2]}$ is a renormalization fixed point.

Proof. [✓ Proof](#) Identify the two-site physical space of K with the one-site physical space of $K^{[2]}$, and identify the four-site physical space of K with the two-site physical space of $K^{[2]}$. These identifications merely permute orthonormal coordinates, so conjugating S_{24} and T_{24} by them preserves complete positivity and trace preservation. The two assumed closure identities then become precisely the two defining renormalization identities for $K^{[2]}$. $\square$

Theorem 20.15.8.4 (Two-site blocking preserves renormalization maps). [Lean](#) [✓ Stmt](#) *Suppose that a tensor K has trace-preserving completely positive maps S, T satisfying*

$$S[\mathcal{K}_2(X)] = \mathcal{K}_1(X), \quad T[\mathcal{K}_1(X)] = \mathcal{K}_2(X)$$

for every virtual matrix X . Then the two-site physical blocking $K^{[2]}$ has maps satisfying the same two identities and is therefore a renormalization fixed point.

Proof. [✓ Proof](#) Apply T twice to the first physical site, first passing from two sites to three and then from three sites to four. This gives a trace-preserving completely positive map from the two-site closure to the four-site closure. Applying S twice to the first two physical sites in the reverse direction gives the coarse-graining map. The canonical identifications of two neighboring sites with one blocked site are permutations of orthonormal coordinates and preserve both complete positivity and trace preservation. $\square$

Theorem 20.15.8.5 (Renormalization maps and normalization obstruction for the four-sector witness). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The four-sector tensor in Theorem 20.15.8.2 has trace-preserving completely positive maps S, T such that, for every virtual matrix X ,*

$$S[\mathcal{K}_2(X)] = \mathcal{K}_1(X), \quad T[\mathcal{K}_1(X)] = \mathcal{K}_2(X).$$

Its two-site physical blocking has maps satisfying the same two identities and is therefore also a renormalization fixed point. On the other hand, for the source-normalized scalar representative

$$A = \frac{4}{\sqrt{5}}\mathcal{K},$$

the two-site blocking $A^{[2]}$ is not a renormalization fixed point.

The last assertion concerns literal trace-preserving diagrammatic renormalization. The rescaled tensor obeys an idempotence relation up to a positive scalar, but it does not satisfy the paper's literal zero-correlation-length diagram. Consequently it is not, by itself, a counterexample to Theorem 4.9 of [CPGSV16]. The simple-biCF-BNT hypotheses of that theorem are not part of this theorem's statement.

Proof. ✓ Proof Let $\tau = QL$ be the neighboring transition matrix and set

$$f(i, j) = (i \bmod 2) + 2 \left\lfloor \frac{j}{2} \right\rfloor.$$

Direct multiplication of the sector matrices gives

$$S_i S_j = \tau_{ij} S_{f(i, j)}.$$

For each k , the transition weights on the fibre of f satisfy

$$\sum_{(i, j): f(i, j)=k} \tau_{ij} = 1.$$

The refinement channel has Kraus operators $\sqrt{\tau_{ij}}|i, j\rangle\langle f(i, j)|$; the coarse-graining channel has Kraus operators $|f(i, j)\rangle\langle i, j|$. Their identity resolutions give trace preservation, and the two displayed sector identities give the two required physical-closure identities. Applying each channel twice and regrouping two neighboring physical sites gives the maps for the blocked tensor, as in Theorem 20.15.8.4.

Let $P = \text{diag}(1, 0)$. The physical-trace transfer of the normalized representative is $(4/\sqrt{5})P$, while that of its two-site blocking is

$$\mathcal{B}_{A^{[2]}} = \frac{16}{5}P.$$

If $A^{[2]}$ admitted the two trace-preserving maps of Definition 4.1, trace preservation of the refinement identity would force $\mathcal{B}_{A^{[2]}}^2 = \mathcal{B}_{A^{[2]}}$. This is impossible because $P^2 = P$ and $16/5 \neq 1$. The relation $\mathcal{B}_A^2 = (4/\sqrt{5})\mathcal{B}_A$ is unchanged in form by a scalar normalization when its positive coefficient is allowed to vary. The literal diagram in the paper fixes that coefficient to one and is not unchanged. □

Theorem 20.15.8.6 (Pair-preserving obstruction for the four-sector witness). Lean Lean Lean ✓ Stmt For the inverse-map-selected four-sector factorization of the tensor in Theorem 20.15.8.5, let $C_{k, h} = \text{tr}(\eta_{k, h})$. Then

$$\begin{aligned} C^2 &= C^3, \\ C &\neq C^3. \end{aligned}$$

Consequently there is neither a trace-preserving family which sends each four-site neighboring operator $\Theta_{k, h}$ to $\eta_{k, h}$ while retaining (k, h) , nor such a family in the reverse direction.

Nevertheless, the two-site blocking of this tensor has trace-preserving completely positive renormalization maps. These maps are obtained from the label-coarsening rule

$$f(i, j) = (i \bmod 2) + 2 \left\lfloor \frac{j}{2} \right\rfloor,$$

so they mix or coarsen the selected labels rather than retaining every fixed outer-sector pair.

Proof. ✓ Proof For the selected factorization, $C = QL$. Since LQ is idempotent, associativity gives $C^2 = C^3$. On the other hand, the $(0, 1)$ entry of C is $1/8$, whereas that of C^2 is $1/4$. If $C = C^3$, then $C^2 = C^3$ would give $C = C^2$, a contradiction. Theorem 20.15.6.20 therefore excludes both pair-preserving families. The explicit Kraus maps in Theorem 20.15.8.5, applied twice and passed to blocked coordinates, give the remaining assertion. $\square$

Theorem 20.15.8.7 (Inverse-map factorization with a nonzero rectangular remainder). Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt *There is a four-sector, bond-dimension-two tensor with a refined Hayashi decomposition whose four sectors are one-dimensional and have weight $1/4$. Its three-site state is diagonal, with*

$$\rho(i, j, k; i', j', k') = \delta_{(i, j, k), (i', j', k')} \frac{1}{4} (QL)_{i, j} (QL)_{j, k}.$$

The normalized four-site tail is LQ . For this tail, the inverse-map construction recovers the closed sector tensors L and Q exactly. Its neighboring operators therefore coincide with those obtained directly from L and Q , while $Q(1 - LQ)L \neq 0$. The inverse-map construction is the one in [CPGSV16, Appendix C.2, lines 1413–1455].

Proof. ✓ Proof Write $S_i = L_{\bullet, i} Q_{i, \bullet}$ for the four physical slices, and let $f_i(a, b)$ be their dual coefficients, so that

$$\sum_{i=0}^3 f_i(a, b) S_i = E_{ab}.$$

With $T = QL$, $p_k = 1/4$, and $(LQ)_{00} = 1$, the two inverse-map contractions are

$$\begin{aligned} (LQ)_{00}^{-1} p_k \sum_{i=0}^3 f_i(0, \beta) T_{ik} &= \frac{1}{4} (4L_{\beta k}) = L_{\beta k}, \\ \sum_{i=0}^3 f_i(\alpha, 0) T_{ki} &= Q_{k\alpha}. \end{aligned}$$

Hence the neighboring operators are those obtained directly from L and Q , and direct multiplication gives $Q(1 - LQ)L \neq 0$. $\square$

Corollary 20.15.8.8 (Local sector structure from SAL and ZCL). Lean ✓ Stmt *Let $\mathcal{K}$ be an injective normal matrix product density operator tensor satisfying the strong area law and literal zero correlation length,*

$$\mathcal{T}_{\mathcal{K}}^2 = \mathcal{T}_{\mathcal{K}}.$$

Then there is a physical-sector factorization

$$\mathbb{C}^d \simeq \bigoplus_k (B_k^L \otimes B_k^R),$$

with

$$U\kappa_{\beta,\alpha}U^\dagger = \bigoplus_k (l_k)_\beta \otimes (r_k)_\alpha,$$

positive semidefinite neighboring operators $\eta_{k,h}$, and real families $(a_k)_k, (b_k)_k$ such that

$$\mathrm{tr}(\eta_{k,h}) = a_k b_h,$$

and

$$\sum_k a_k b_k = 1.$$

The orientation of U is the physical-sector-factorization convention displayed above. No additional closure property of the sector tensors is asserted. This is the structural corollary at [CPGSV16, Appendix C.2, lines 1503–1506].

Proof. [✓ Proof](#) Apply Theorem 20.15.6.40. Its factorization witness already contains the physical isometry, the direct-sum physical-slice formula, and positive neighboring operators. Theorem 20.15.6.40 gives the real-part identity $\mathrm{Re} \mathrm{tr}(\eta_{k,h}) = a_k b_h$. Positivity makes each neighboring operator Hermitian, so its trace is real and the displayed complex trace identity follows. $\square$

Theorem 20.15.8.9 (Positive neighboring sectors generate positive periodic operators). [Lean](#) [✓ Stmt](#) Suppose that an MPO tensor K admits a physical-sector factorization through an isometry U , with neighboring operators $\eta_{q,h}$. If $\eta_{q,h} \geq 0$ for every pair (q, h) , then for every $N \geq 1$,

$$\rho^{(N)}(K) \geq 0.$$

Proof. [✓ Proof](#) In sector coordinates, the periodic operator is block diagonal, and each block is a tensor product of positive neighboring operators. Hence every block, and therefore the full operator, is positive semidefinite. Conjugation by $U^{\otimes N}$ preserves positivity and returns $\rho^{(N)}(K)$. $\square$

Definition 20.15.8.10 (Unstarred orthogonality of BNT layers). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a family $\{\mathcal{K}_x\}_{x=0}^{g-1}$, write $\mathcal{K}_x[\beta, \alpha] \in M_d(\mathbb{C})$ for the physical matrix obtained by fixing the two virtual indices. The layers are orthogonal when, for $x \neq y$,

$$\mathcal{K}_y[\beta_y, \alpha_y] \mathcal{K}_x[\beta_x, \alpha_x] = 0 \tag{20.64}$$

for every choice of virtual indices. Equivalently, for all physical indices i, j ,

$$\sum_{k=0}^{d-1} (\mathcal{K}_y)_{\beta_y \alpha_y}^{ik} (\mathcal{K}_x)_{\beta_x \alpha_x}^{kj} = 0.$$

The factor order is exactly $\mathcal{K}_y \mathcal{K}_x$; no adjoint occurs. This is equation `AppKxKy=0` in [CPGSV16, Appendix C.2, lines 1634–1639].

$$= 0$$

Since k is active, choose α, u, v for which $r_{k,\alpha}(u, v) \neq 0$, and put $z = \overline{r_{k,\alpha}(u, v)}$. Evaluating (20.65) at the corresponding matrix entry and dividing by z yields

$$l_{k,\beta}^* = \sum_{\mu} \left(z^{-1} \sum_{\nu} c_{k,\beta,\alpha}(\mu, \nu) r_{k,\nu}(v, u) \right) l_{k,\mu}. \quad (20.66)$$

Interchanging the two factors gives the corresponding identity for $r_{k,\alpha}^*$. Thus both factor spans are closed under conjugate transpose. The joint column-support projections then absorb the factors on both sides. Their tensor product is the support projection of the product family, and finite-dimensional range decompositions give the two isometries. $\square$

Theorem 20.15.8.14 (Canonical restriction assembled from the active sector factors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let K be injective and let

$$UK[\beta, \alpha]U^* = \bigoplus_k l_{k,\beta} \otimes r_{k,\alpha}$$

be a physical-sector factorization whose neighboring operators are positive semidefinite. In every active sector choose isometries onto the joint column supports of the left and right factor families, and retain no coordinates in an inactive sector. Their direct sum, transported back by U^* , is an isometry

$$V : \mathbb{C}^e \longrightarrow \mathbb{C}^d$$

whose range projection is the canonical joint column support P_K of the physical slices of K . The restricted tensor $K_{\text{act}} = V^*KV$ is injective, satisfies

$$VK_{\text{act}}V^* = K,$$

and, in the dependent sector coordinates, has physical slices

$$K_{\text{act}}[\beta, \alpha] = \bigoplus_k ((V_k^L)^* l_{k,\beta} V_k^L) \otimes ((V_k^R)^* r_{k,\alpha} V_k^R).$$

This construction also applies when every sector is inactive: then $e = 0$, with no complementary summand added.

Proof. [✓ Proof](#) Take the tensor product of the two support isometries in each active sector and their dependent direct sum over all sectors, using an empty coordinate space in every inactive sector. Block multiplication gives orthonormal columns and the displayed compressed factorization. The range projection absorbs every physical slice. Conversely, every projection that absorbs all physical slices also absorbs the direct sum of their joint column supports. Hence this range projection is P_K . Two-sided absorption then gives exact re-embedding, and injectivity descends through the physical isometry. $\square$

Theorem 20.15.8.15 (Neighboring positivity and trace factorization on the canonical active restriction). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) In the setting of Theorem 20.15.8.14, the restricted tensor K_{act} has a physical-sector factorization indexed precisely by the active ambient sectors. For active sectors k, h , put

$$W_{k,h} = V_k^R \otimes V_h^L.$$

Its neighboring operator is

$$\eta_{k,h}^{\text{act}} = W_{k,h}^* \eta_{k,h} W_{k,h}.$$

In particular, every $\eta_{k,h}^{\text{act}}$ is positive semidefinite. If the active family is empty, the restricted factorization has no sectors and the assertion is vacuous.

Suppose moreover that the ambient neighboring operators admit real coefficients a_k, b_h such that

$$\begin{aligned} \text{tr}(\eta_{k,h}) &= a_k b_h, \\ \sum_k a_k b_k &= 1. \end{aligned}$$

For the active-sector correspondence $j \mapsto k(j)$, set $\tilde{a}_j = a_{k(j)}$ and $\tilde{b}_j = b_{k(j)}$. Then

$$\begin{aligned} \text{tr}(\eta_{j,l}^{\text{act}}) &= \tilde{a}_j \tilde{b}_l, \\ \sum_j \tilde{a}_j \tilde{b}_j &= 1. \end{aligned}$$

The identities being preserved are those of CPSV16 Theorem 4.9(iv), equations *tralktrrk* and *PsiPhi*, lines 864–889 [CPGSV16]. Their transport through the active physical support compression is an auxiliary construction, not an assertion of that source theorem; see [con26d].

Proof. ✓ **Proof** Remove the empty inactive fibers from the dependent direct sum and enumerate the remaining sector subtype by a finite ordinal. The compressed right factor of k and compressed left factor of h give

$$\sum_{\alpha} ((V_k^R)^* r_{k,\alpha} V_k^R) \otimes ((V_h^L)^* l_{h,\alpha} V_h^L) = W_{k,h}^* \eta_{k,h} W_{k,h}.$$

Positive semidefiniteness is preserved by this matrix congruence. The range projections of V_k^R and V_h^L satisfy the two-sided absorption identity

$$(P_k^R \otimes P_h^L) \eta_{k,h} = \eta_{k,h} = \eta_{k,h} (P_k^R \otimes P_h^L).$$

Cyclicity of trace therefore gives $\text{tr}(\eta_{k,h}^{\text{act}}) = \text{tr}(\eta_{k,h})$. For an inactive sector, $\eta_{k,k} = 0$, hence $a_k b_k = 0$. Consequently, splitting the normalized sum and reindexing its active part by $j \mapsto k(j)$ gives

$$\begin{aligned} 1 &= \sum_k a_k b_k \\ &= \sum_{\substack{k \\ \text{active}}} a_k b_k + \sum_{\substack{k \\ \text{inactive}}} a_k b_k \\ &= \sum_j \tilde{a}_j \tilde{b}_j + 0 \\ &= \sum_j \tilde{a}_j \tilde{b}_j. \end{aligned}$$

□

Theorem 20.15.8.16 (Orthogonal two-sided supports from unstarred layer orthogonality).

[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\{\mathcal{K}_x\}_{x=0}^{g-1}$ be injective MPO tensors, each of which generates positive semidefinite periodic operators at every positive length. If, for every pair $x \neq y$, the unstarred identities

$$\mathcal{K}_y[\beta_y, \alpha_y] \mathcal{K}_x[\beta_x, \alpha_x] = 0$$

hold for all virtual indices, then there are pairwise orthogonal projections $P_x \in M_d(\mathbb{C})$ such that

$$P_x \mathcal{K}_x[\beta, \alpha] = \mathcal{K}_x[\beta, \alpha] = \mathcal{K}_x[\beta, \alpha] P_x$$

for every x, β, α . Conversely, any pairwise annihilating family with these two-sided support identities satisfies the unstarred layer identities.

Proof. [✓ Proof](#) Let C_x be the matrix obtained by stacking the columns of all slices of $\mathcal{K}_x$, and let P_x be the orthogonal projection onto its column space. The left support identity is immediate. Theorem 20.15.8.12 gives the right support identity. For $x \neq y$, (20.64) and adjoint closure give

$$C_x^* C_y = 0, \quad P_x P_y = 0.$$

□

Corollary 20.15.8.17 (Supports of common-weight BNT representatives). [Lean](#) [✓ Stmt](#) Let a BNT decomposition have a common nonzero weight within each sector, and suppose that its one-site words span the product of the virtual matrix algebras. For every sector s , assume a physical-sector factorization with positive neighboring operators. If distinct absorbed representatives obey the unstarred layer identities, then there are pairwise orthogonal projections P_s such that

$$P_s \mathcal{K}_s[\beta, \alpha] = \mathcal{K}_s[\beta, \alpha] = \mathcal{K}_s[\beta, \alpha] P_s$$

for every s, β, α .

Proof. [✓ Proof](#) One-site spanning makes every absorbed representative injective. The positive neighboring operators make every representative an MPDO. Apply Theorem 20.15.8.16. □

Theorem 20.15.8.18 (Positive sector bonds on the printed physical supports). [Lean](#) [Lean](#) [✓ Stmt](#) Let K be injective and admit a physical-sector factorization with positive semidefinite neighboring operators. If an orthogonal projection P satisfies

$$PK[\beta, \alpha] = K[\beta, \alpha] = K[\beta, \alpha]P$$

for every β, α , then there is a positive two-site operator B whose periodic translates commute and such that

$$(P \otimes P)B(P \otimes P) = B, \\ \text{mpo}(K, N) = \prod_{j=0}^{N-1} \tau_j(B) \quad (N \geq 2).$$

Applied to pairwise orthogonal projections P_s , this construction gives an orthogonal family of positive commuting sector bonds for the tensors K_s .

Proof. ✓ Proof Restrict K to its canonical joint physical support Q by Theorems 20.15.8.14 and 20.15.8.15. The positive neighboring operators on the restriction give a positive commuting bond supported on $Q \otimes Q$, with the displayed periodic product. Minimality of the joint column support and the two-sided absorption by P give

$$PQ = Q = QP.$$

Hence support of the bond on $Q \otimes Q$ implies support on $P \otimes P$. Performing this construction in every orthogonal sector gives the final assertion. $\square$

Theorem 20.15.8.19 (The five BNT identities give the GSNNCH form). Lean Lean Lean

Lean Lean ✓ Stmt *Let M generate positive semidefinite periodic operators and have a BNT decomposition with representatives A_x of positive bond dimension, copy numbers $n_x \geq 1$, and arbitrary copy weights $\mu_{x,q}$. Suppose that the decomposition generates the same matrix product vectors as M at every positive length. Assume that the simultaneous one-site tuples $(A_x^i)_x$, as i ranges over the physical labels, span the product of the virtual matrix algebras. Assume also that, for distinct x, y ,*

$$A_x[\beta, \alpha]A_y[\beta', \alpha'] = 0$$

for all virtual indices. For every x , suppose that A_x has the physical-sector factorization

$$U_x A_x U_x^* = \bigoplus_k l_{x,k} \otimes r_{x,k},$$

with positive neighboring operators $\eta_{k,h}^x$ and real numbers a_k^x, b_k^x satisfying

$$\text{tr}(\eta_{k,h}^x) = a_k^x b_h^x, \quad \sum_k a_k^x b_k^x = 1.$$

Then the matrix product density operators generated by M have the GSNNCH form, with the BNT copy numbers n_x as their outer multiplicities. No equality among the copy weights is assumed.

Proof. ✓ Proof The physical-sector identities make every A_x an MPDO. The unstarred layer identities and one-site spanning therefore give pairwise orthogonal projections P_x supporting the raw representatives on both sides. Fix $N \geq 2$ and write

$$q_x(N) = \sum_{q=0}^{n_x-1} \mu_{x,q}^N, \quad \rho^{(N)}(M) = \sum_x q_x(N) \rho^{(N)}(A_x).$$

Sitewise compression by $P_x^{\otimes N}$ isolates $q_x(N) \rho^{(N)}(A_x)$. It is positive because $\rho^{(N)}(M)$ is positive. The operator $\rho^{(N)}(A_x)$ is positive and nonzero: the latter follows from positive bond dimension and one-site injectivity. Hence uniqueness of the positive scalar phase shows that $q_x(N)$ is a nonnegative real number.

Theorem 20.15.8.18 gives a supported positive commuting bond B_x with periodic product $\rho^{(N)}(A_x)$. Replace it at this chain length by

$$\widetilde{B}_{x,N} = \left(\frac{q_x(N)}{n_x} \right)^{1/N} B_x.$$

Then its multiplicity-weighted periodic product is exactly $q_x(N) \rho^{(N)}(A_x)$. The GSNNCH data may depend on N , so this construction proves the claimed form without copy independence. $\square$

Theorem 20.15.8.20 (Square-zero obstruction for arbitrary MPO tensors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *There is a two-label family of bond-dimension-one MPO tensors for which the unstarred layer identities hold, but no pairwise orthogonal one-site projections support both tensors on both sides. One may take both labels to have the single physical slice*

$$A = |0\rangle\langle 1|, \quad A \neq 0, A^2 = 0.$$

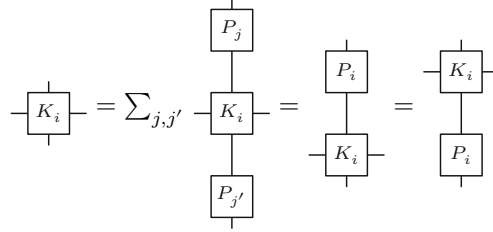
Proof. [✓ Proof](#) The equation $A^2 = 0$ gives both ordered layer products. If $P_0 P_1 = 0$ and $P_x A = A$ for both labels, then

$$A = P_0 A = P_0 P_1 A = 0,$$

contradicting $A \neq 0$. □

If the projections also satisfy $\sum_j P_j = \mathbb{1}$, then expanding this identity on both physical indices and using the sector projector relations leaves only the diagonal term:

$$\mathcal{K}_i = \sum_{j,j'} P_j \mathcal{K}_i P_{j'} = P_i \mathcal{K}_i = \mathcal{K}_i P_i.$$



Equivalently, each local tensor is supported on its own sector: $\mathcal{K}_i = P_i \mathcal{K}_i = \mathcal{K}_i P_i$. This is the calculation in [CPGSV16, Appendix C.2, lines 1745–1751].

Theorem 20.15.8.21 (Normal BNT representatives, biCF, exact assembly, and nonnilpotence in the Case-II example). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let A_0, A_1 be the bond-one tensors whose only nonzero doubled-physical entries are*

$$A_0^{00} = A_0^{11} = \frac{1}{\sqrt{2}}, \quad A_1^{22} = 1.$$

With $(\mu_0, \mu_1) = (1/\sqrt{2}, 1)$, they are normal and form a normalized BNT canonical decomposition. Their simultaneous word-tuple span is the full two-block direct sum already at length one, so the representative family is in biCF. The ambient tensor is

$$M^{ab} = \text{diag}(\mu_0 A_0^{ab}, \mu_1 A_1^{ab}),$$

both entrywise and as a doubled-index MPS tensor, and is in normalized BNT-refined horizontal form. Moreover, the doubled physical-trace transfer of neither representative is nilpotent.

Each representative occurs in exactly one copy. Thus independence of the weight from the copy label is vacuous within each representative and does not compare the two representatives or their displayed weights.

Proof. ✓ **Proof** The two ordinary transfer maps satisfy

$$\mathcal{E}_{A_0} = \mathcal{E}_{A_1} = \text{id}_{\mathbb{C}}.$$

Their disjoint physical supports exclude equivalence by a virtual gauge and a phase. Write $\mathbf{A}(i) = (A_0^i, A_1^i)$. For every $Y = (Y_0, Y_1) \in \mathbb{C} \oplus \mathbb{C}$, the two letters 0 and 8 give

$$Y = \frac{Y_0}{1/\sqrt{2}} \mathbf{A}(0) + Y_1 \mathbf{A}(8),$$

which proves the simultaneous one-letter span. Direct evaluation gives

$$M^{ab} = \text{diag}((1/\sqrt{2})A_0^{ab}, A_1^{ab}),$$

and hence the identity block gauge. Finally,

$$\begin{aligned} \mathcal{T}_{\text{phys}}(A_0) &= \frac{2}{\sqrt{2}}, \\ \mathcal{T}_{\text{phys}}(A_1) &= 1, \end{aligned}$$

so neither physical-trace transfer is nilpotent. □

Theorem 20.15.8.22 (Coefficient absorption need not preserve normality). Lean Lean Lean Lean ✓ **Stmt** Let A_0 have scalar entries $1/\sqrt{2}$ on doubled symbols $(0, 0)$ and $(1, 1)$, let A_1 have scalar entry 1 only on $(2, 2)$, and take $(\mu_0, \mu_1) = (1/\sqrt{2}, 1)$. Their weighted bond-diagonal assembly has

$$K^{00} = K^{11} = \text{diag}(1/2, 0), \quad K^{22} = \text{diag}(0, 1). \quad (20.67)$$

For every $N > 0$ and physical strings $\sigma, \tau \in \{0, 1, 2\}^N$,

$$\langle \sigma | \rho^{(N)}(K) | \tau \rangle = \delta_{\sigma, \tau} \left(2^{-N} \mathbf{1}_{\{\sigma_k \in \{0, 1\} \ \forall k\}} + \mathbf{1}_{\{\sigma_k = 2 \ \forall k\}} \right), \quad (20.68)$$

$$\text{tr } \rho^{(N)}(K) = 2 > 0. \quad (20.69)$$

Hence K generates positive semidefinite operators. It satisfies the strong area law and the literal zero-correlation-length equation

$$\mathcal{T}_K = I_2, \quad \mathcal{T}_K^2 = \mathcal{T}_K. \quad (20.70)$$

Its displayed BNT decomposition has the global unit-weight convention, its representatives form a biCF family, and K has simple canonical form at this normalized fixed representative. Nevertheless,

$$\rho(\mathcal{E}_{\mu_0 A_0}) = \frac{1}{2}, \quad \mu_0 A_0 \text{ is not normal.} \quad (20.71)$$

Copy-weight independence is only within the single copy of each representative; it does not equate μ_0 and μ_1 . Thus all conditions used in the Case-II passage are present, but the printed absorbed-normality inference is false [CPGSV16, lines 1628–1665 and 1740–1782]. This does not refute the asserted all-sector conclusion or Theorem 4.9.

Proof. ✓ Proof Set

$$\begin{aligned} P &= \text{diag}(1, 1, 0), & Q &= I - P = \text{diag}(0, 0, 1), \\ \omega_P &= \frac{1}{4}(P \otimes P), & \omega_Q &= Q \otimes Q. \end{aligned} \quad (20.72)$$

Equations (20.67) and (20.68) follow by multiplying the diagonal slices. There are 2^N binary strings and one terminal string, which gives (20.69). For a virtual matrix X , the one-site and two-site physical closures are

$$M_1(X) = \frac{X_{00}}{2}P + X_{11}Q, \quad M_2(X) = X_{00}\omega_P + X_{11}\omega_Q. \quad (20.73)$$

Define

$$\begin{aligned} \mathcal{S}(Y) &= \text{tr}_2(Y), \\ \mathcal{R}(Y) &= \text{tr}(PYP)\omega_P + \text{tr}(QYQ)\omega_Q. \end{aligned} \quad (20.74)$$

Both maps are trace-preserving and completely positive, and

$$\mathcal{S}[M_2(X)] = M_1(X), \quad \mathcal{R}[M_1(X)] = M_2(X). \quad (20.75)$$

These are the channel equations of Definition 4.1 [CPGSV16, lines 638–660]; with (20.69), they imply the strong area law. Summing the diagonal slices gives (20.70), the literal diagram of Definition 4.2 [CPGSV16, lines 735–739].

The transfer maps of the two unabsorbed representatives obey

$$\mathcal{E}_{A_0}(x) = \left(\frac{1}{2} + \frac{1}{2}\right)x = x, \quad \mathcal{E}_{A_1}(x) = x.$$

Hence both are normal. The letters 0 and 8 separately select the two scalar blocks, proving the simultaneous one-letter span and biCF. The weights satisfy $|\mu_s| \leq 1$ and $|\mu_1| = 1$, as required by the global canonical normalization at lines 224–246. Absorption nevertheless gives

$$\mathcal{E}_{\mu_0 A_0} = |\mu_0|^2 \mathcal{E}_{A_0} = \frac{1}{2} \text{id}, \quad \rho(\mathcal{E}_{\mu_0 A_0}) = \frac{1}{2}.$$

A normal tensor has transfer spectral radius one, proving (20.71) and the failure of the printed inference. □

Theorem 20.15.8.23 (Counterexample to the printed implication (iv) $\Rightarrow$ (v)). Lean ✓ Stmt
There is a one-letter matrix product density operator K whose BNT canonical presentation has one scalar normal representative $A = 1$ and two copies with raw weights

$$\mu_{0,0} = 1, \quad \mu_{0,1} = \frac{1}{2}.$$

The presentation satisfies the global unit-weight normalization, K generates positive semidefinite operators at every positive length, and the physical-trace transfer of A is nonnilpotent. Thus K satisfies the simplicity clause. The singleton BNT family has simultaneous one-letter span, so the presentation is in biCF. Moreover, A itself generates matrix product density operators, the distinct-layer equation in condition (iv) holds, and A has a physical-sector factorization with positive neighboring operator and normalized rank-one trace coefficients. Its two-site blocking also generates matrix product density operators. Nevertheless,

$$\neg \text{RFP}(K^{[2]}).$$

Consequently the literal implication (iv) $\Rightarrow$ (v) printed in [CPGSV16, Theorem 4.9] is false.

Proof. ✓ Proof The sole local matrix of K is the two-copy block diagonal matrix with eigenvalues 1 and $1/2$. Hence every periodic contraction equals $1 + 2^{-N} > 0$. All assertions about the scalar representative reduce to the one-by-one identity matrix; orthogonality is vacuous because there is only one BNT representative. If $K^{[2]}$ satisfied Definition 20.2.1, trace preservation would make its physical-trace transfer idempotent. Taking traces would then give

$$1 + 2^{-4} = 1 + 2^{-2},$$

a contradiction. □

Remark 20.15.8.24 (Printed simple-MPDO implication chain). For a simple tensor K in block-injective canonical form with a basis of normal tensors and $M = K^{[2]}$, [CPGSV16, Theorem 4.9] prints the chain (i) $\Rightarrow$ (ii), (ii) $\Leftrightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v) between the fixed-point condition, SAL with zero correlation length, the commuting Gibbs form with zero correlation length, the block-wise local structure, and the fixed-point condition for M . Theorem 20.15.8.22 refutes the printed absorbed-sector normality route for (ii) $\Rightarrow$ (iv) in Case II, and Theorem 20.15.8.23 refutes the literal implication (iv) $\Rightarrow$ (v). The viable channel route of Appendix C.2 is (ii) $\Rightarrow$ (v); its remaining source step is the existence of a physical-sector factorization for every absorbed representative, the outer-sector combination being complete.

Remark 20.15.8.25 (Printed mixed-state separation examples). The separations asserted in [CPGSV16, Examples 4.10–4.12, lines 897–942] are decided as follows. Example 4.10, read with one left and one right bit flip and $p = 1/4$ (Definition 20.15.8.26), has source zero correlation length and does not saturate the area law (Theorems 20.15.8.27 and 20.15.8.34); no claim is made about the printed universal statement in p . The literal finite periodic family of Example 4.11 fails both saturation and source zero correlation length (Theorems 20.15.8.48 and 20.15.8.42). The literal tensor of Example 4.12 is unnormalized, so its fixed-point and zero-correlation-length assertions fail for that representative (Theorem 20.15.8.59); the halved tensor satisfies the two channel equations (Theorem 20.15.8.60) without supplying the unit-weight convention, and both representatives lie outside the full GSNNCH class (Theorems 20.15.8.58 and 20.15.8.61).

Definition 20.15.8.26 (Corrected tensor for Example 4.10). Lean Lean Lean Lean Lean Lean ✓ Stmt Identify the local four-dimensional spin with two qubits and write $U = \sigma_x \otimes \sigma_x$. Starting from one Bell pair on every bond, apply independently at each site the channel

$$\mathcal{E}(X) = \frac{1}{4}UXU^* + \frac{3}{4}X.$$

Equivalently, use the local purification with Kraus amplitudes $\sqrt{3}/2$ for the identity and $1/2$ for the correlated flip. This is distinct from the repeated-left expression printed in the source.

Theorem 20.15.8.27 (Source zero correlation length of the corrected tensor). Lean Lean Lean Lean Lean Lean ✓ Stmt The corrected tensor is locally purified and hence positive at every chain length. It has source zero correlation length.

Proof. ✓ Proof Its physical-trace transfer is the nonzero projector

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Since $P^2 = P$, the tensor has source zero correlation length. □

Theorem 20.15.8.28 (One- through four-site spectra of the corrected tensor). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The normalized reduced states on one, two, three, and four consecutive sites have characteristic-polynomial root multisets*

$$\begin{aligned} L = 1 : & \quad \{0^{[0]}, (1/4)^{[4]}\}, \\ L = 2 : & \quad \{0^{[8]}, (5/32)^{[4]}, (3/32)^{[4]}\}, \\ L = 3 : & \quad \{0^{[48]}, (7/64)^{[4]}, (3/64)^{[12]}\}, \\ L = 4 : & \quad \{0^{[248]}, (41/128)^{[1]}, (15/128)^{[4]}, (9/128)^{[3]}\}. \end{aligned}$$

These spectra are consequences of the corrected $p = 1/4$ tensor and are not formulas stated in the source.

Proof. ✓ Proof For $1 \leq L \leq 4$, the proof constructs a bond-window weight family w_L and an isometry V_L such that

$$\rho_L = V_L \text{diag}(w_L) V_L^\dagger, \quad V_L^\dagger V_L = I.$$

The diagonal weights give the listed nonzero roots. The zero-root counts from the orthogonal complements are

$$4^1 - 2^2 = 0, \quad 4^2 - 2^3 = 8, \quad 4^3 - 2^4 = 48, \quad 4^4 - 2^4 = 240.$$

At four sites the eight odd-parity bond patterns have zero weight and supply the remaining eight zero roots. $\square$

Definition 20.15.8.29 (Correlated-flip weight families for Example 4.10). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) Place one maximally entangled qubit pair on each bond of the four-spin ring and apply, independently at every spin, the corrected correlated flip with flip probability one quarter. A joint flip configuration $s \in \{0, 1\}^4$ has probability

$$P(s) = \prod_{n=1}^4 \left(\frac{1}{4}\right)^{s_n} \left(\frac{3}{4}\right)^{1-s_n},$$

and its bond pattern $t \in \{0, 1\}^4$ is $t_n = s_n \oplus s_{n+1}$, with spin five identified with spin one. Write q for the pushforward of P to bond patterns. The weight family of a window of k consecutive spins, $1 \leq k \leq 3$, assigns to each configuration of its $k - 1$ interior bond labels and its two boundary qubits one quarter of the marginal probability of those interior labels; the four-spin window weight family is q itself.

Theorem 20.15.8.30 (Exact bond-pattern distribution for Example 4.10). The flip configuration distribution is a probability distribution on $\{0,1\}^4$. The bond-pattern distribution takes the value $41/128$ on the unflipped pattern, $15/128$ on each of the four patterns flipping two cyclically adjacent bonds, $9/128$ on the two patterns flipping two opposite bonds and on the fully flipped pattern, and zero on the eight patterns with an odd number of flipped bonds. Its one-bond marginal is $(5/8, 3/8)$, and its two-adjacent-bond marginal assigns $7/16$ to the doubly unflipped pair and $3/16$ to each of the three remaining pairs. All the window weight families are probability distributions.

Proof. ✓ Proof Each value is a finite sum of products of the factors $1/4$ and $3/4$ over the sixteen flip configurations, evaluated exactly. $\square$

Theorem 20.15.8.31 (Exact window entropies for Example 4.10). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\tilde{S}_k$ denote the entropy $\sum_x h(w(x))$ of the window weight family for k consecutive spins, with $h(x) = -x \log x$ and $\log$ the natural logarithm. Then

$$\begin{aligned}\tilde{S}_1 &= 2 \log 2, \\ \tilde{S}_2 &= 5 \log 2 - \frac{5}{8} \log 5 - \frac{3}{8} \log 3, \\ \tilde{S}_3 &= 6 \log 2 - \frac{7}{16} \log 7 - \frac{9}{16} \log 3, \\ \tilde{S}_4 &= 7 \log 2 - \frac{41}{128} \log 41 - \frac{57}{64} \log 3 - \frac{15}{32} \log 5.\end{aligned}$$

Divided by $\log 2$ these are the printed decimals 2, 2.9544, 3.8802, and 2.7839.

Proof. [✓ Proof](#) The window weight families list the values $(1/4)^{[4]}$; $(5/32)^{[4]}$, $(3/32)^{[4]}$; $(7/64)^{[4]}$, $(3/64)^{[12]}$; and $(41/128)^{[1]}$, $(15/128)^{[4]}$, $(9/128)^{[3]}$. Summing h over each family and expanding the logarithms of the rational values gives the displayed closed forms. $\square$

Theorem 20.15.8.32 (Positivity of the Example 4.10 logarithmic ratio). [Lean](#) [Lean](#) [✓ Stmt](#)
The exact scalar ratio from the four-site calculation satisfies

$$\log \left(\frac{2^{32} 7^7}{3^3 5^{20}} \right) > 0.$$

Proof. [✓ Proof](#) The numerator and denominator are the integers $2^{32} 7^7 = 3537090251849728$ and $3^3 5^{20} = 2574920654296875$, so the ratio is greater than one and its logarithm is strictly positive. $\square$

Theorem 20.15.8.33 (Exact mutual-information defect for Example 4.10). [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [✓ Stmt](#) For $\tilde{I}_1 = \tilde{S}_1 + \tilde{S}_3 - \tilde{S}_4$ and $\tilde{I}_2 = 2\tilde{S}_2 - \tilde{S}_4$,

$$\tilde{I}_2 - \tilde{I}_1 = \frac{1}{16} \log \left(\frac{2^{32} 7^7}{3^3 5^{20}} \right) > 0.$$

Proof. [✓ Proof](#) Since $\tilde{I}_2 - \tilde{I}_1 = 2\tilde{S}_2 - \tilde{S}_1 - \tilde{S}_3$, substituting the closed forms and collecting logarithms gives the displayed identity; positivity follows from the certified integer comparison. $\square$

Theorem 20.15.8.34 (Entropy obstruction for the corrected Example 4.10 tensor). [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For the corrected tensor at $p = 1/4$ on the four-site ring, let S_L be the entropy of L consecutive spins, with natural logarithms. Then

$$\begin{aligned}S_1 &= 2 \log 2, \\ S_2 &= 5 \log 2 - \frac{5}{8} \log 5 - \frac{3}{8} \log 3, \\ S_3 &= 6 \log 2 - \frac{7}{16} \log 7 - \frac{9}{16} \log 3, \\ S_4 &= 7 \log 2 - \frac{41}{128} \log 41 - \frac{57}{64} \log 3 - \frac{15}{32} \log 5.\end{aligned}$$

Consequently,

$$I_2 - I_1 = \frac{1}{16} \log \left(\frac{2^{32} 7^7}{3^3 5^{20}} \right) > 0.$$

The tensor has source zero correlation length and does not saturate the area law. These identities concern only the corrected fixed value $p = 1/4$.

Proof. [✓ Proof](#) The four root multisets give the displayed block entropies. Substitution into $I_L = S_L + S_{4-L} - S_4$ gives the exact logarithmic ratio, which is positive by Theorem [20.15.8.32](#). If the area law saturated, Lemma [20.7.6](#) would give $I_1 = I_2$, a contradiction. The source zero-correlation-length statement is Theorem [20.15.8.27](#). $\square$

Definition 20.15.8.35 (Effective binary-support model for Example 4.11). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The coefficient matrix extracted from the four diagonal neighboring operators printed in [\[CPGSV16, Example 4.11, lines 907–924\]](#) is

$$W = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

In the effective binary encoding, the supported label $k \in \{0, 1\}$ selects the ambient two-qubit basis vector $|k, k\rangle$. Let M be the cyclic-edge tensor for the ket–bra diagonal edge coefficient induced by W . For a positive cycle length N , define

$$c_N(\sigma) = \prod_{n \in \mathbb{Z}/N\mathbb{Z}} W_{\sigma_n, \sigma_{n+1}}, \quad Z_N = \sum_{\sigma} c_N(\sigma).$$

This is a physical-dimension-2 support model. The ambient physical-dimension-4 realization is obtained by the support inclusion defined below; the two physical spaces are not identified by an equivalence.

Theorem 20.15.8.36 (Diagonal formula and normalization of the binary-support model). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every positive N and binary words σ, τ ,

$$\text{mpo}(M, N)_{\sigma, \tau} = \delta_{\sigma, \tau} c_N(\sigma), \quad Z_N = \text{tr}(W^N) = \frac{3^N + 1}{2^N}.$$

These identities first concern the effective binary support model. The ambient definition below transports the operator and trace identities, but no spectrum, entropy, saturation of the area law, or zero correlation length assertion is made here.

Proof. [✓ Proof](#) The closed cyclic-edge identity reduces each matrix entry to a product of local edge coefficients. Ket–bra diagonality makes the product vanish unless $\sigma = \tau$, and in the diagonal case the product is $c_N(\sigma)$. Summing these cycle weights and applying the cyclic trace-power identity gives $Z_N = \text{tr}(W^N)$. Finally, the Hadamard basis diagonalizes W with eigenvalues $3/2$ and $1/2$, so cyclicity of the trace gives $\text{tr}(W^N) = (3/2)^N + (1/2)^N = (3^N + 1)/2^N$. $\square$

Definition 20.15.8.37 (Internal transition count). [Lean](#) [✓ Stmt](#) For a binary word $x = (x_1, \dots, x_L)$, let $t(x)$ be the number of indices $n \in \{1, \dots, L-1\}$ for which $x_n \neq x_{n+1}$.

Theorem 20.15.8.38 (Exact effective-binary finite-ring block probabilities). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $1 \leq L \leq N$ and let $x \in \{0, 1\}^L$. The effective-binary block state is diagonal, and its entry at x is

$$\mu_{N,L}(x) = \frac{2^L(3^{N-L+1} + (-1)^{t(x)})}{2^{t(x)+2}(3^N + 1)}.$$

This includes $L = 1$ and $L = N$. It is a project-derived consequence of the finite periodic model, not a formula stated in CPSV16.

Proof. ✓ Proof Substitution in the exact finite-ring block formula gives the two displayed values. Source zero correlation length would identify the normalized two-site marginal obtained from length four with that obtained from length three, contradicting $14/41 \neq 5/14$. Isometric support inclusion preserves these entries, so the same obstruction holds in the ambient realization. At length one, the trace-loop identity and the exact effective and ambient partition functions give

$$\begin{aligned}\mathrm{tr}(\mathrm{mpo}(M, 1)) &= \mathrm{tr}(\mathcal{T}_M^1) = \mathrm{tr}(\mathcal{T}_M) = 2, \\ \mathrm{tr}(\mathrm{mpo}(\widetilde{M}, 1)) &= \mathrm{tr}(\mathcal{T}_{\widetilde{M}}^1) = \mathrm{tr}(\mathcal{T}_{\widetilde{M}}) = 2.\end{aligned}$$

Hence both physical-trace transfers are nonzero. For either $X \in \{M, \widetilde{M}\}$, literal idempotence would give

$$\mathcal{T}_X \neq 0, \quad \mathcal{T}_X^2 = \mathcal{T}_X \implies \mathcal{T}_X^2 = \lambda \mathcal{T}_X \quad \text{with } \lambda = 1 > 0,$$

which the marginal instability $14/41 \neq 5/14$ already refutes. □

Theorem 20.15.8.43 (Exact length-four effective spectra). Lean Lean Lean Lean Lean Lean Lean ✓ Stmt *On the periodic ring of length four, let $\rho_L := \rho_L^{(4)}(M)$ be the effective-binary reduced state, and let χ_A denote the characteristic polynomial of a matrix A . Then*

$$\mathrm{roots}(\chi_{\rho_1}) = \{(1/2)^{[2]}\}, \tag{20.76}$$

$$\mathrm{roots}(\chi_{\rho_2}) = \{(14/41)^{[2]}, (13/82)^{[2]}\}, \tag{20.77}$$

$$\mathrm{roots}(\chi_{\rho_3}) = \{(10/41)^{[2]}, (4/41)^{[4]}, (5/82)^{[2]}\}, \tag{20.78}$$

$$\mathrm{roots}(\chi_{\rho_4}) = \{(8/41)^{[2]}, (2/41)^{[12]}, (1/82)^{[2]}\}.$$

Here $a^{[m]}$ denotes a root a of multiplicity m . These are the effective nonzero spectra, not the full ambient root multisets.

Proof. ✓ Proof The reduced-state formula is diagonal. For a binary word with t internal transitions, its diagonal probability is

$$p_{L,t} = \frac{2^L(3^{5-L} + (-1)^t)}{2^{t+2} \cdot 82}.$$

For example, at $L = 2$ and $t = 0$,

$$p_{2,0} = \frac{2^2(3^3 + 1)}{2^2 \cdot 82} = \frac{112}{328} = \frac{14}{41}.$$

Enumerating words of lengths one through four gives transition-count multiplicities (2), (2, 2), (2, 4, 2), and (2, 6, 6, 2), respectively. The same substitution for every transition count gives the displayed roots and multiplicities. □

Theorem 20.15.8.44 (Exact length-four ambient spectra). Lean Lean Lean Lean Lean Lean ✓ Stmt *For every $L \leq 4$, let $\rho_L := \rho_L^{(4)}(M)$ be the effective reduced state and $\tilde{\rho}_L := \rho_L^{(4)}(\widetilde{M})$ the ambient reduced state. Let χ_A denote the characteristic polynomial of A . Then*

$$\chi_{\tilde{\rho}_L}(X) = X^{4^L - 2^L} \chi_{\rho_L}(X).$$

For the positive block lengths $L = 1, 2, 3$, and 4 , the full ambient root multiset is the effective multiset together with zero of multiplicity 2, 12, 56, and 240, respectively.

Proof. ✓ Proof The reduced ambient state is $\tilde{\rho}_L = V^{\otimes L} \rho_L (V^{\otimes L})^*$. Applying characteristic-polynomial cyclicity to the rectangular factors gives

$$\begin{aligned}\chi_{V^{\otimes L} \rho_L (V^{\otimes L})^*}(X) &= X^{4^L - 2^L} \chi_{\rho_L (V^{\otimes L})^* V^{\otimes L}}(X) \\ &= X^{4^L - 2^L} \chi_{\rho_L}(X),\end{aligned}\tag{20.79}$$

where $(V^{\otimes L})^* V^{\otimes L} = I$. The ambient and effective dimensions are 4^L and 2^L , so their difference gives the stated zero multiplicity. $\square$

Theorem 20.15.8.45 (Exact length-four effective entropies). Lean Lean Lean Lean Lean ✓ Stmt *The effective binary tensor generates positive semidefinite periodic operators. On the ring of length four, write $S_L = S(\rho_L)$. With $h(x) = -x \log x$, where $\log$ is the natural logarithm,*

$$\begin{aligned}S_1 &= \log 2, \\ S_2 &= 2h(14/41) + 2h(13/82), \\ S_3 &= 2h(10/41) + 4h(4/41) + 2h(5/82), \\ S_4 &= 2h(8/41) + 12h(2/41) + 2h(1/82).\end{aligned}$$

Proof. ✓ Proof Every positive-length periodic operator is diagonal, and each diagonal entry is a product of entries of $W = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, hence is non-negative. For each L , summing $h(\lambda)$ over the roots and their multiplicities in Theorem 20.15.8.43 gives the displayed entropy. $\square$

Theorem 20.15.8.46 (Positivity of the Example 4.11 logarithmic ratio). Lean ✓ Stmt *The exact scalar ratio from the four-site calculation satisfies*

$$\log\left(\frac{41^{82}5^{50}}{2^{48}13^{52}7^{112}}\right) > 0.$$

Proof. ✓ Proof The numerator is strictly larger than the denominator, as follows by an exact integer calculation, so the ratio is greater than one and its logarithm is strictly positive. $\square$

Theorem 20.15.8.47 (Exact length-four mutual-information obstruction). Lean Lean Lean ✓ Stmt *For $I_L = S_L + S_{4-L} - S_4$,*

$$I_2 - I_1 = \frac{1}{82} \log\left(\frac{41^{82}5^{50}}{2^{48}13^{52}7^{112}}\right) > 0.$$

Consequently, the effective binary tensor does not satisfy saturation of the area law.

Proof. ✓ Proof Since $I_2 - I_1 = 2S_2 - S_1 - S_3$, expanding $h(x) = -x \log x$ and using the logarithm rules gives the displayed identity. The certified comparison $41^{82}5^{50} > 2^{48}13^{52}7^{112}$ makes its logarithm strictly positive. Saturation would instead require $I_1 = I_2$ on the four-site ring. $\square$

Theorem 20.15.8.48 (Ambient length-four saturation obstruction). Lean ✓ Stmt *The isometric inclusion of the effective binary tensor into the two-qubit one-site space does not satisfy saturation of the area law.*

Proof. ✓ Proof If the ambient tensor satisfied saturation, descent through the physical isometry would imply saturation for the effective binary tensor, contrary to the preceding theorem. $\square$

Definition 20.15.8.49 (Literal tensor in Example 4.12). [Lean](#) [Lean](#) [✓ Stmt](#) Let M be the tensor with $d = D = 2$ whose only nonzero components are

$$1 = M_{00}^{00} = M_{00}^{11} = M_{11}^{00} = -M_{11}^{11},$$

and let $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

Definition 20.15.8.50 (One-site Pauli sign). [Lean](#) [✓ Stmt](#) For a physical letter $a \in \{0, 1\}$, define its Pauli sign by

$$z_a = (\sigma_z)_{aa}, \quad z_0 = 1, \quad z_1 = -1.$$

Definition 20.15.8.51 (Configuration sign). [Lean](#) [✓ Stmt](#) For a binary configuration $\sigma = (\sigma_0, \dots, \sigma_{N-1})$, define

$$z(\sigma) = \prod_{k=0}^{N-1} z_{\sigma_k}.$$

This is the diagonal entry of $\sigma_z^{\otimes N}$ at σ .

Theorem 20.15.8.52 (Exact positive-length operator and positivity). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For every $N > 0$,

$$\rho^{(N)}(M) = I^{\otimes N} + \sigma_z^{\otimes N}, \quad \text{tr } \rho^{(N)}(M) = 2^N,$$

and $\rho^{(N)}(M)$ is positive semidefinite.

Proof. [✓ Proof](#) Both tensor powers are diagonal. On a binary configuration their sum has eigenvalue $1 + \prod_k z_k$, where each z_k is 1 or -1 ; hence every eigenvalue is either 0 or 2. Summing the signed term over all binary configurations gives zero when $N > 0$, so the trace is 2^N . $\square$

Theorem 20.15.8.53 (One-site loss and saturation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Tracing one site removes the $\sigma_z^{\otimes N}$ term. More generally, every nonempty proper normalized L -site reduced state is $2^{-L} I^{\otimes L}$. Therefore its entropy is $L \log 2$, the mutual information is independent of L , and M satisfies saturation of the area law.

Proof. [✓ Proof](#) The identity term contributes the dimension of the traced complement, while the signed term contains the vanishing one-site sum $1 + (-1)$. The entropy of the resulting maximally mixed state is $L \log 2$; the two block terms in the mutual information then add to $N \log 2$ independently of the cut. $\square$

Definition 20.15.8.54 (Four-site tripartition in Example 4.12). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For the state in [CPGSV16, Example 4.12, lines 932–938], at $N = 4$ regard the sites as $A = \{0\}$, $B = \{1, 3\}$, and $C = \{2\}$. Encode the two binary labels (b_1, b_3) as one four-dimensional label b , and write $z_a \in \{1, -1\}$ for the eigenvalue of σ_z at a . In the basis $(a, b, c) \leftrightarrow (a, b_1, c, b_3)$, let ρ_{ABC} be the normalized four-site state and set

$$w(a, b, c) = \frac{1 + z_a z_{b_1} z_c z_{b_3}}{16}.$$

Theorem 20.15.8.55 (Four-site failure of strong-subadditivity equality). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The normalized four-site state is diagonal with probability

1/8 on the eight even-parity configurations and zero on the eight odd-parity configurations. Its relevant marginals are

$$\rho_{AB} = \frac{I_8}{8}, \quad \rho_{BC} = \frac{I_8}{8}, \quad \rho_B = \frac{I_4}{4}.$$

Consequently,

$$S(ABC) = 3 \log 2, \quad S(B) = 2 \log 2, \quad S(AB) = S(BC) = 3 \log 2,$$

so $S(ABC) + S(B) = 5 \log 2 < 6 \log 2 = S(AB) + S(BC)$. Thus equality in strong subadditivity fails for this tripartition.

Proof. [✓ Proof](#) The diagonal coefficient is $(1 + z_a z_{b_1} z_c z_{b_3})/16$. Summing over either a or c cancels the signed term, giving $I_8/8$; summing over both gives $I_4/4$. There are eight nonzero coefficients, all equal to $1/8$, so the full entropy is $8[-(1/8) \log(1/8)] = 3 \log 2$. The three maximally mixed marginals have the displayed entropies, and $\log 2 > 0$ gives the strict inequality. $\square$

Theorem 20.15.8.56 (Positive commuting overlapping products are quantum Markov). [Lean](#)

[✓ Stmt](#) Let ρ_{ABC} be a trace-one operator, and let X_{AB} and Y_{BC} be positive semidefinite operators. Suppose that their natural lifts to ABC commute and that, after the natural reassociation of the three tensor factors,

$$\rho_{ABC} = (X_{AB} \otimes I_C)(I_A \otimes Y_{BC}).$$

Then ρ_{ABC} admits a quantum Markov decomposition on B .

Proof. [✓ Proof](#)

The spatial decomposition [Bei12, Lemma 2.1] gives a unitary decomposition $B = \bigoplus_q B_q^L \otimes B_q^R$ in which the two overlapping actions have the forms $R_q \otimes I_{B_q^R}$ and $I_{B_q^L} \otimes S_q$. Compression of the original positive operators shows that every R_q and S_q is positive semidefinite. Normalize the two factors in each summand and take $p_q = \text{tr}(R_q) \text{tr}(S_q)$. The factorization and the trace-one hypothesis imply $\sum_q p_q = 1$; zero factors give zero-weight summands. Thus the transformed state is the direct sum $\bigoplus_q p_q \widehat{R}_q \otimes \widehat{S}_q$, which is the quantum Markov form of [HJPW04, Theorem 6]. $\square$

Theorem 20.15.8.57 (Four-cycle GSNNCH states are quantum Markov). [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [✓ Stmt](#) Let ρ be a positive semidefinite trace-one operator on four d -dimensional sites, and suppose that ρ has the sector decomposition of Definition 4.8. Regroup the sites as $A = \{0\}$, $B = \{1, 3\}$, and $C = \{2\}$. Then the resulting tripartite state admits a quantum Markov decomposition on B . Consequently,

$$S(ABC) + S(B) = S(AB) + S(BC).$$

Proof. [✓ Proof](#) Write $B_j^{(x)}$ for the four translated bonds in sector x , n_x for its multiplicity, and $c > 0$ for the common normalization. Set

$$L = c \sum_x n_x B_3^{(x)} B_0^{(x)}, \quad R = \sum_x B_1^{(x)} B_2^{(x)}.$$

Orthogonality of the one-site sector projections makes each cross-sector term in LR vanish. Within one sector the translated bonds commute, so $(B_3^{(x)} B_0^{(x)})(B_1^{(x)} B_2^{(x)})$ is the sector product in Definition 4.8. Hence $LR = \rho$. Both L and R are positive semidefinite; since $LR = \rho$ is Hermitian, also $RL = LR$.

In the displayed tripartition, $L = X_{AB} \otimes I_C$ and $R = I_A \otimes Y_{BC}$ for positive semidefinite X_{AB} and Y_{BC} . The preceding theorem therefore gives a quantum Markov decomposition of ρ on B . Its direct-sum product form gives the asserted entropy equality. $\square$

Proof. ✓ **Proof** If M were GSNNCH, its normalized four-site state would satisfy the hypotheses of the preceding theorem and hence equality in strong subadditivity for $A = \{0\}$, $B = \{1, 3\}$, $C = \{2\}$. The generic regrouping agrees exactly with the binary coordinates used in the entropy calculation. The strict inequality proved above is therefore a contradiction. $\square$

$$\mathcal{T}_M = I + \sigma_z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{T}_M^2 = 2\mathcal{T}_M \neq \mathcal{T}_M.$$

Proof. ✓ Proof The transfer calculation is immediate from the two diagonal physical letters. Trace preservation of a refinement from one site to two sites would force the one-site and two-site traces to agree for every virtual boundary, and therefore force $\mathcal{T}_M^2 = \mathcal{T}_M$, contradicting the displayed calculation. $\square$

$$\widehat{M} = \frac{1}{2}M.$$
$$\rho^{(N)}(\widehat{M}) = 2^{-N} \rho^{(N)}(M) = \frac{\rho^{(N)}(M)}{\text{tr } \rho^{(N)}(M)}, \quad \text{tr } \rho^{(N)}(\widehat{M}) = 1.$$
$$C_p = |\pi(p)\rangle\langle p|, \quad R_p = 2^{-1/2}|p\rangle\langle\pi(p)|.$$
$$\mathcal{S}[\widehat{M}_2(X)] = \widehat{M}_1(X), \quad \mathcal{T}[\widehat{M}_1(X)] = \widehat{M}_2(X).$$

Proof. ✔ Proof Each parity fibre contains two pairs. Hence $\sum_p C_p^\dagger C_p = I_4$ and $\sum_p R_p^\dagger R_p = I_2$. The diagonal tensor products obey

while all off-diagonal physical letters vanish. Substitution into the two Kraus sums proves both equations entrywise. $\square$

Theorem 20.15.8.61 (The normalized representative of Example 4.12 is not GSNNCH). [Lean](#)

[Lean](#) [✓ Stmt](#) *The condition of Definition 4.8 constrains only the normalized density-operator family, and that family is unchanged by any nonzero complex rescaling of the tensor. The condition is therefore invariant under nonzero rescaling. In particular the normalized representative $\widehat{M} = \frac{1}{2}M$ of Example 4.12 is not a Gibbs state of a nearest-neighbor commuting Hamiltonian.*

Proof. [✓ Proof](#) For nonzero c and every chain length, the normalized density operator of cM equals that of M , so every clause of Definition 4.8 holds for cM exactly when it holds for M . Taking $c = \frac{1}{2}$ transports the obstruction proved for the printed tensor to $\widehat{M}$. $\square$

Theorem 20.15.8.62 (Bond-one physical-sector boundary). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [✓ Stmt](#) *Let K be an MPO tensor of virtual bond dimension one. If K satisfies the strong area law and literal zero correlation length, then there is a physical-sector factorization of K whose neighboring operator is positive semidefinite and whose trace has the normalized rank-one form*

$$\mathrm{tr}(\eta_{k,h}) = a_k b_h.$$

The factors are normalized by

$$\sum_k a_k b_k = 1.$$

Proof. [✓ Proof](#) Use one physical sector, with the full physical space as its left factor and a one-dimensional right factor. Its neighboring operator is the one-site periodic operator, so positivity follows from the strong area law. The physical-trace transfer is a nonzero scalar. Literal idempotence therefore makes it equal to one, and the neighboring trace is normalized by $a_0 = b_0 = 1$. $\square$

Theorem 20.15.8.63 (Physical sector structure of absorbed BNT representatives). [Not ready](#)

Let K be a simple tensor in block-injective canonical form (biCF), with BNT representatives A_s and canonical coefficients $\mu_{s,q}$. Suppose that K generates positive semidefinite periodic operators, satisfies the strong area law, and has literal zero correlation length,

$$\mathcal{T}_K^2 = \mathcal{T}_K.$$

Then $\mu_{s,q}$ is independent of q for each fixed s . Absorb this common value into A_s and write $\mathcal{K}_s = \mu_s A_s$. The tensors $\mathcal{K}_s$ have mutually orthogonal physical supports and each generates positive semidefinite periodic operators. For every s , there is a physical-sector factorization

$$\begin{aligned} \mathbb{C}^d &\simeq \bigoplus_k (B_{s,k}^L \otimes B_{s,k}^R), \\ U_s \kappa_{\beta,\alpha}^{(s)} U_s^\dagger &= \bigoplus_k (l_{s,k})_\beta \otimes (r_{s,k})_\alpha, \end{aligned}$$

whose neighboring operators $\eta_{k,h}^{(s)}$ are positive semidefinite. There are real families $(a_{s,k})_k, (b_{s,k})_k$ satisfying

$$\begin{aligned} \mathrm{tr}(\eta_{k,h}^{(s)}) &= a_{s,k} b_{s,h}, \\ \sum_k a_{s,k} b_{s,k} &= 1. \end{aligned}$$

This is implication (ii) $\Rightarrow$ (iv) of [CPGSV16, Theorem 4.9 and Appendix C.2, lines 1740–1782]. The separate standing biCF and BNT hypotheses are imposed at line 849 and restated at line 1628 of the same source.

Theorem 20.15.8.64 (Counterexample to the low-level absorbed-sector implication). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) There are tensors K, A of physical dimension four and virtual dimension two, and a scalar μ with $0 < |\mu| < 1$, such that $K = \mu A$, the tensor A is normal, and K is injective, satisfies the strong area law, and obeys

$$\mathcal{T}_K^2 = \mathcal{T}_K.$$

Nevertheless, no physical-sector factorization of K has positive neighboring operators whose traces are of the form $a_k b_h$ with $\sum_k a_k b_k = 1$. Hence these local analytic hypotheses do not by themselves imply the asserted factorization. This does not refute the all-sector assertion in [CPGSV16, Theorem 4.9 and Appendix C.2, lines 1647–1782], whose hypotheses include an ambient simple tensor in biCF and its canonical reconstruction.

Proof. [✓ Proof](#) Take

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -7 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} 1/4 & -3/100 \\ 1/4 & -1/100 \\ 1/4 & 1/100 \\ 1/4 & 3/100 \end{pmatrix}, \quad (20.80)$$

and let the four diagonal physical slices be the outer products of the corresponding columns of L and rows of R . Then $LR = \text{diag}(1, 0)$, whereas RL has rank two. The scalar slice and two simple-spectrum slices force every summand in any physical-sector factorization to have one-dimensional left and right factors. The resulting neighboring trace matrix has a nonzero two-by-two minor, so it cannot equal $(a_k b_h)_{k,h}$. The four slices span the full two-by-two matrix algebra, and their displayed scalar neighboring operators are positive. For the positive test matrix $H = \text{diag}(20, 1)$, direct calculation gives

$$H - \mathcal{E}_K^*(H) = \begin{pmatrix} 85/8 & -9/40 \\ -9/40 & 4697/5000 \end{pmatrix} > 0.$$

Pair this positive-definite gap with a positive-definite Perron eigenmatrix ρ of $\mathcal{E}_K$. The trace-adjoint identity gives

$$0 < \text{tr}((H - \mathcal{E}_K^*(H))\rho) = (1 - r) \text{tr}(H\rho).$$

Since $\text{tr}(H\rho) > 0$, the Perron value satisfies $r < 1$. Perron normalization and a canonical gauge therefore express the tensor as $K = \mu A$ with A normal and $0 < |\mu| = \sqrt{r} < 1$. $\square$

20.16 Single-bond commuting form from the local structure

Definition 20.16.1 (Translation-invariant two-site bond). [Lean](#) [✓ Stmt](#) This records one positive semidefinite two-site bond $B \geq 0$ whose translated copies commute pairwise on the periodic N -site chain at every chain length $N \geq 2$: $[B_{i,i+1}, B_{j,j+1}] = 0$.

Definition 20.16.2 (η -local single-bond commuting-form data). [Lean](#) [✓ Stmt](#) This defines the one-bond eta-local form constructed in [CPGSV16, Appendix C.2]: a two-site bond B as in Definition 20.16.1 together with, for every chain length $N \geq 2$, a constant $c > 0$ such that

$$\rho^{(N)}(M) = c \prod_{i=0}^{N-1} B_{i,i+1}.$$

The bond acts on the full two-site space.

Proof. ✓ **Proof** At length N , apply the commuting two-site bond and realization identity provided by the η -local structure. □

Theorem 20.16.7 (η -local structure gives a single-bond product). Lean Lean Lean ✓ **Stmt**
The same η -local structure gives a single-bond product witness on every finite periodic chain.

Proof. ✓ **Proof** Evaluate the local structure at each chain length $N \geq 2$. □

Theorem 20.16.8 (Single-bond product with doubled-index idempotence). Lean Lean ✓ **Stmt**
An η -local structure together with the doubled-index transfer condition gives the corresponding single-bond branch.

Proof. ✓ **Proof** Use the commuting-form witness carried by the η -local structure and combine it with the doubled-index transfer hypothesis. □

Theorem 20.16.9 (A commuting bond admits unitary block actions). Lean Lean ✓ **Stmt** *Let $\widehat{B}$ be a pair-indexed positive two-site bond whose periodic translates commute. There are positive dimensions $d_{l,q} = \dim H_{q,l}$ and $d_{r,q} = \dim H_{q,r}$, a unitary decomposition $H \cong \bigoplus_{q=0}^{K-1} H_{q,l} \otimes H_{q,r}$, and Hermitian operators R_q on $H \otimes H_{q,l}$ and S_q on $H_{q,r} \otimes H$ such that*

$$\begin{aligned} (\mathbb{1}_H \otimes U^*) \widehat{B} (\mathbb{1}_H \otimes U) &= \bigoplus_{q=0}^{K-1} (R_q \otimes \mathbb{1}_{H_{q,r}}), \\ (U^* \otimes \mathbb{1}_H) \widehat{B} (U \otimes \mathbb{1}_H) &= \bigoplus_{q=0}^{K-1} (\mathbb{1}_{H_{q,l}} \otimes S_q). \end{aligned}$$

The basis bijection for each sector enumerates the numerical indices in right-left order,

$$\bigsqcup_q (\{0, \dots, d_{r,q} - 1\} \times \{0, \dots, d_{l,q} - 1\}) \cong \{0, \dots, d - 1\}.$$

Writing $e(q, (r, s))$ for this basis bijection, the left and right spatial coordinates are respectively

$$\begin{aligned} (q, ((i, s), r)) &\mapsto (i, e(q, (r, s))), \\ (q, (s, (r, k))) &\mapsto (e(q, (r, s)), k). \end{aligned}$$

Hence the tensor factors in the displayed block actions occur in left-right order.

Proof. ✓ **Proof** The pair-indexed bond is Hermitian by positivity, and its two overlapping lifts commute by the common three-site identification. Apply Theorem [LTC26, “Explicit unitary block actions for commuting overlaps”]. □

Theorem 20.16.10 (Positive neighboring operators from the commuting bond). Lean Lean ✓ **Stmt** *Let B be a pair-indexed positive bond whose periodic translates commute. There are a unitary coordinate decomposition $H \cong \bigoplus_q H_{q,r} \otimes H_{q,l}$ and positive semidefinite operators $\eta_{q,h}$ on $H_{q,r} \otimes H_{h,l}$. After regrouping the two-site coordinates, the (q, h) -sector is $H_{q,l} \otimes (H_{q,r} \otimes H_{h,l}) \otimes H_{h,r}$, and in these regrouped coordinates*

$$(U^* \otimes U^*) B (U \otimes U) = \bigoplus_{q,h} \mathbb{1}_{H_{q,l}} \otimes \eta_{q,h} \otimes \mathbb{1}_{H_{h,r}}.$$

The unitary and the neighboring operators depend only on the fixed bond, and hence are independent of a chain length or bond position.

Proof. [✓ Proof](#) The same unitary block decomposition at both ends of B gives identity action on $H_{q,l}$ through one block identity and on $H_{h,r}$ through the other. For fixed basis indices $\ell_q^0 \in H_{q,l}$ and $r_h^0 \in H_{h,r}$, let $B' = (U^* \otimes U^*)B(U \otimes U)$. The neighboring operator has entries

$$(\eta_{q,h})_{(r,\ell),(r',\ell')} = B'_{(q,r,\ell_q^0;h,r_h^0,\ell),(q,r',\ell_q^0;h,r_h^0,\ell')}.$$

Thus $\eta_{q,h}$ is a principal compression of the positive operator B' , and is therefore positive semidefinite. $\square$

Theorem 20.16.11 (Cyclic product of a locally decomposed bond). [Lean](#) [✓ Stmt](#) Suppose that, in one fixed coordinate decomposition $H \cong \bigoplus_q H_{q,r} \otimes H_{q,l}$, after regrouping the two-site coordinates according to this decomposition, a two-site operator has the form

$$B = \bigoplus_{q,h} \mathbb{1}_{H_{q,l}} \otimes \eta_{q,h} \otimes \mathbb{1}_{H_{h,r}}.$$

For every $N \geq 2$, regroup the cyclic chain as

$$H^{\otimes N} \cong \bigoplus_{k: \mathbb{Z}/N\mathbb{Z} \rightarrow \{0, \dots, K-1\}} \bigotimes_{n \in \mathbb{Z}/N\mathbb{Z}} (H_{k_n, r} \otimes H_{k_{n+1}, l}).$$

Then, in these coordinates and with $k_N = k_0$,

$$\prod_{n=0}^{N-1} B_{n,n+1} = \bigoplus_k \bigotimes_{n=0}^{N-1} \eta_{k_n, k_{n+1}}.$$

At $N = 2$, the two translated windows have the opposite orders $(0, 1)$ and $(1, 0)$ and give the two oriented factors η_{k_0, k_1} and η_{k_1, k_0} .

Proof. [✓ Proof](#) In a fixed sector configuration, the cyclic regrouping sends the factors (r_n, l_n) to the edge factors (r_n, l_{n+1}) . The translate beginning at n acts by $\eta_{k_n, k_{n+1}}$ on the n th edge factor and by the identity on every other edge factor. Different sector configurations do not mix. Multiplying the translated operators therefore gives the stated direct sum of tensor products. $\square$

Theorem 20.16.12 (Scaled finite-chain cyclic neighboring-operator form). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\sigma^{(N)}(\mathcal{K})$ be realized, for every $N \geq 2$, by translates of one positive commuting bond. There are one unitary U , positive sector dimensions, and positive semidefinite operators $\eta_{q,h}$, all independent of N , such that for every $N \geq 2$ there is a constant $c_N > 0$ satisfying, with $k_N = k_0$,

$$E_N^*(U^*)^{\otimes N} \sigma^{(N)}(\mathcal{K}) U^{\otimes N} E_N = c_N \bigoplus_{k_0, \dots, k_{N-1}} \bigotimes_{n=0}^{N-1} \eta_{k_n, k_{n+1}}.$$

Here E_N is the cyclic regrouping from the edge factors $H_{k_n, r} \otimes H_{k_{n+1}, l}$ to the site factors $H_{k_n, r} \otimes H_{k_n, l}$. The factor c_N is retained from the proportionality relation in [CPGSV16, Appendix C.2, lines 1571–1576]. The passage at [CPGSV16, Appendix C.2, lines 1603–1605] suppresses this factor when invoking the coefficient-free equation sigmaNK2; the coefficient-free assertion is not part of the present theorem.

Proof. [✓ Proof](#) Choose the unitary and the neighboring operators from the decomposition of the fixed bond. Conjugation by $(U^*)^{\otimes N}$ carries the product of its translates to the product of the conjugated two-site bonds. The cyclic regrouping theorem changes this product into the displayed direct sum. Applying both transformations to $\sigma^{(N)}(\mathcal{K}) = c_N \prod_n B_{n,n+1}$ preserves the same positive scalar c_N . $\square$

Remark 20.16.13 (An arbitrary proportional presentation need not have unit scalar). The proportional commuting-bond form alone does not permit one common local normalization. Indeed, take physical dimension $d = 1$ and bond dimension $D = 2$, with sole matrix-product letter $M^{00} = I_2$, and take the two-site bond to be the scalar $B = 1$. Then

$$\sigma^{(N)} = \text{tr}(I_2^N) = 2 = c_N \prod_{n=0}^{N-1} B_{n,n+1}, \quad c_N = 2.$$

Any positive neighboring decomposition of the one-dimensional physical space has one sector and one scalar $a \geq 0$. A coefficient-free identity at lengths two and four would give $a^2 = 2$ and $a^4 = 2$, a contradiction. This example is not a normal injective block. More generally, the source does not claim that every proportional commuting-bond presentation can be made coefficient-free by one local rescaling. In the SAL-to-commuting-bond direction, the proof chooses the neighboring operators from the source tensor and obtains scalar one directly. This distinguishes the proportionality at [CPGSV16, Appendix C.2, lines 1571–1576] from the coefficient-free invocation at [CPGSV16, Appendix C.2, lines 1603–1605]. The source-selected normal representative is Corollary 20.15.3.62.

Definition 20.16.14 (Cyclic edge-weight tensor). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A scalar weight $w(i, j; i', j')$ on two consecutive ket–bra pairs determines a translation-invariant matrix-product tensor with virtual dimension d^2 . Its incoming virtual pair is constrained to equal the current physical ket–bra pair, while its outgoing pair records the following physical pair.

Theorem 20.16.15 (Closed cyclic edge-weight identity). [Lean](#) [✓ Stmt](#) For every positive length N , and with $\sigma_N = \sigma_0$ and $\tau_N = \tau_0$, the closed matrix-product operator of the cyclic edge-weight tensor is

$$\rho_{\sigma, \tau}^{(N)} = \prod_{n=0}^{N-1} w(\sigma_n, \tau_n; \sigma_{n+1}, \tau_{n+1}).$$

In the virtual trace there is precisely one potentially nonzero cyclic virtual configuration.

Definition 20.16.16 (Fixed tensor for a bond-product family). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A fixed tensor for a bond-product family consists of a positive virtual dimension and one matrix-product tensor C , independent of N , such that, for every $N \geq 2$,

$$\rho^{(N)}(C) = \prod_{n=0}^{N-1} B_{n,n+1}.$$

If $d > 0$, a cyclic scalar edge-weight expression for the entries of the product gives such a tensor with positive virtual dimension d^2 .

Theorem 20.16.17 (Fixed tensor representation of a commuting bond). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $B \geq 0$ be a two-site bond whose periodic translates commute. There is a positive integer R and a matrix-product tensor C of virtual dimension R , both independent of the chain length, such that for every $N \geq 2$,

$$\rho^{(N)}(C) = \prod_{n=0}^{N-1} B_{n,n+1}.$$

The construction gives a positive R independent of N . No normality, zero-correlation-length, or scalar normalization hypothesis is required.

Proof. ✓ Proof Choose Beigi's chain-independent one-site unitary and the positive neighboring operators $\eta_{q,h}$. For a common matrix-unit index set $\mathcal{A}$, write

$$\eta_{q,h}((r,l), (r',l')) = \sum_{a \in \mathcal{A}} R_a(q)_{r,r'} L_a(h)_{l,l'}.$$

Thus the contraction of the right factor at one site with the left factor at the next satisfies $\sum_{a \in \mathcal{A}} R_a(q) \otimes L_a(h) = \eta_{q,h}$. If V is the resulting unitary change to the two-site sector coordinates, this contraction gives $\widetilde{B} = V B V^\dagger$ and $B_{\text{phys}} = V^\dagger \widetilde{B} V = B$. For a nonzero physical space, $\mathcal{A}$ is nonempty and gives a positive virtual dimension. For the zero-dimensional physical space, take one virtual state and no physical sectors. The cyclic contraction formula then gives the product of the translates of B at every length $N \geq 2$. $\square$

Theorem 20.16.18 (Positive physical sectors of the selected fixed tensor). Lean Lean Lean Lean Lean Lean ✓ Stmt Let C be the selected fixed tensor of the preceding theorem. There are physical sectors $\mathbb{C}^d \cong \bigoplus_{q=0}^{K-1} H_q^L \otimes H_q^R$, a unitary U , and matrices $L_\beta(q)$ and $R_\alpha(q)$ such that, writing $C_{\beta,\alpha}$ for the physical matrix at fixed virtual indices,

$$U C_{\beta,\alpha} U^\dagger = \bigoplus_{q=0}^{K-1} L_\beta(q) \otimes R_\alpha(q).$$

The neighboring operators $\eta_{q,h} = \sum_a R_a(q) \otimes L_a(h)$ are positive semidefinite. If U_2 denotes the induced two-site change of coordinates, then

$$B = U_2^\dagger \left[\bigoplus_{q,h} \mathbb{1}_{H_q^L \otimes H_h^R} \otimes \eta_{q,h} \right] U_2.$$

Thus the physical-sector factorization belongs to the same tensor C whose closed operators are the fixed products of B .

Proof. ✓ Proof Take the physical-sector factorization retained with C . Its defining one-site identity is

$$U C_{\beta,\alpha} U^\dagger = \bigoplus_q L_\beta(q) \otimes R_\alpha(q).$$

Contracting the adjacent right and left factors gives $\eta_{q,h} = \sum_a R_a(q) \otimes L_a(h) \succeq 0$, where positivity is the retained neighboring-operator property. Finally, the retained bond identity is

$$B = U_2^\dagger \left[\bigoplus_{q,h} \mathbb{1}_{H_q^L \otimes H_h^R} \otimes \eta_{q,h} \right] U_2.$$

$\square$

Remark 20.16.19 (The scalar edge formula depends on the one-site coordinates). A cyclic scalar edge formula need not hold in the original physical basis. For $d = 2$, let X be the Pauli flip and set $B = \mathbb{1} + X \otimes X$. This bond is positive semidefinite and its periodic translates commute. For every $N \geq 2$,

$$\langle 0^N | \left(\prod_n B_{n,n+1} \right) | 0^N \rangle = 2.$$

At $N = 2$ the periodic product is $B^2 = 2B$; for $N \geq 3$, the empty edge set and the full cycle are the two terms with zero boundary. An original-coordinate scalar edge weight would make this entry equal to a^N . The cases $N = 2$ and $N = 3$ would give $a^2 = 2$ and $a^3 = 2$, a contradiction. Equation *sigmaNK2* in [CPGSV16, Appendix C.2, lines 1581–1589] is stated only after the one-site unitary coordinate change, as used in the theorem.

Theorem 20.16.20 (Eventual proportionality from a fixed product tensor). Lean ✓ Stmt *Suppose one fixed tensor C generates the products of the bond carried by an eta-local structure. Then the doubled-index tensors of M and C are eventually nonzero proportional. The proof uses only lengths $N \geq 2$ and therefore assumes no relation at length one.*

Proof. ✓ Proof At every $N \geq 2$, substitute the exact tensor representation into the positive realization $\rho^{(N)}(M) = c_N \prod_n B_{n,n+1}$, where $c_N > 0$, and read ket–bra pairs as doubled physical indices. □

Theorem 20.16.21 (A normal rescaling gives a geometric phase). Lean ✓ Stmt *Let A be the normal doubled-index source tensor of positive bond dimension D , and let C be a fixed tensor for its commuting-bond product. Suppose that sC is normal for some $s > 0$. Then there is a complex number ζ , with $|\zeta| = 1$, such that, for every $N \geq 2$,*

$$\rho^{(N)}(A) = (\zeta s)^N \prod_{n=0}^{N-1} B_{n,n+1}.$$

This is a conditional reduction: it does not assert that C , or any rescaling of C , is normal.

Proof. ✓ Proof Scaling C by s gives $V^{(N)}(sC)_\sigma = s^N V^{(N)}(C)_\sigma$. Thus, for all sufficiently large N , the proportionality $V^{(N)}(A)_\sigma = c_N V^{(N)}(C)_\sigma$ becomes

$$V^{(N)}(A)_\sigma = \frac{c_N}{s^N} V^{(N)}(sC)_\sigma, \quad \frac{c_N}{s^N} \neq 0.$$

The normal-tensor overlap dichotomy gives one $\zeta \in \mathbb{C}$, with $|\zeta| = 1$, such that, at every positive length,

$$V^{(N)}(A)_\sigma = \zeta^N V^{(N)}(sC)_\sigma = (\zeta s)^N V^{(N)}(C)_\sigma.$$

□

Theorem 20.16.22 (A normal rescaling forces the positive geometric law). Lean ✓ Stmt *Let A be the normal doubled-index source tensor of positive bond dimension D , let C be a fixed tensor for its commuting-bond product, and suppose that sC is normal for some $s > 0$. Then the unit phase is trivial. For every $N \geq 2$,*

$$\rho^{(N)}(A) = s^N \prod_{n=0}^{N-1} B_{n,n+1}.$$

Thus a positive normal rescaling of the fixed product tensor already supplies the positive geometric realization law.

Proof. ✓ Proof Normality gives $\lim_{N \rightarrow \infty} \langle V^{(N)}(A) | V^{(N)}(A) \rangle = 1$. Hence at two sufficiently large consecutive lengths, say N and $N + 1$, both the source vector and the fixed bond product are nonzero. Positivity of the realization gives positive numbers c_N, c_{N+1} satisfying $(\zeta s)^N = c_N$ and $(\zeta s)^{N+1} = c_{N+1}$. Consequently $\zeta s = c_{N+1}/c_N > 0$. Since $|\zeta| = 1$ and $s > 0$, one has $\zeta = 1$. □

Theorem 20.16.23 (Finite-chain normalization under nonzero proportionality). [Lean](#) [✓ Stmt](#) Let M and C be MPO tensors, possibly of different bond dimensions, and fix a chain length N . Write

$$\hat{\rho}^{(N)}(M) = \text{tr}[\rho^{(N)}(M)]^{-1} \rho^{(N)}(M),$$

with the inverse of zero understood to be zero. Suppose that, for some $c \in \mathbb{C} \setminus \{0\}$, the doubled-index matrix-product vectors satisfy $|V^{(N)}(M)\rangle = c|V^{(N)}(C)\rangle$. Then

$$\hat{\rho}^{(N)}(M) = \hat{\rho}^{(N)}(C).$$

The proportionality hypothesis is the finite-length form of the hypothesis in [CPGSV16, Theorem 2.10]. The conclusion is an auxiliary cancellation used in the comparison preceding [CPGSV16, Appendix C.2, lines 1597–1619].

Proof. [✓ Proof](#) Reading a doubled physical index as a ket–bra pair gives $\rho^{(N)}(M) = c\rho^{(N)}(C)$, and hence $\text{tr}[\rho^{(N)}(M)] = c \text{tr}[\rho^{(N)}(C)]$. Since $c \neq 0$,

$$(c \text{tr}[\rho^{(N)}(C)])^{-1} c\rho^{(N)}(C) = \text{tr}[\rho^{(N)}(C)]^{-1} \rho^{(N)}(C).$$

This identity also holds when the two proportional traces vanish. □

Theorem 20.16.24 (Finite-chain reduced marginals under nonzero proportionality). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 20.16.23, for every $L \leq N$ the reduced operators on the first L sites agree:

$$\text{tr}_{N-L}[\hat{\rho}^{(N)}(M)] = \text{tr}_{N-L}[\hat{\rho}^{(N)}(C)].$$

Proof. [✓ Proof](#) Apply the same finite-index identification and partial trace to both sides of Theorem 20.16.23. □

Theorem 20.16.25 (Eventual equality of normalized finite-chain operators). [Lean](#) [✓ Stmt](#) If the doubled-index tensors of M and C are eventually nonzero proportional, then, for every sufficiently large N ,

$$\hat{\rho}^{(N)}(M) = \hat{\rho}^{(N)}(C).$$

Proof. [✓ Proof](#) At each sufficiently large length, apply Theorem 20.16.23 to the nonzero proportionality scalar at that length. □

Theorem 20.16.26 (Eventual equality of fixed-size reduced marginals). [Lean](#) [✓ Stmt](#) If the doubled-index tensors of M and C are eventually nonzero proportional, then for each fixed L and every sufficiently large $N \geq L$,

$$\text{tr}_{N-L}[\hat{\rho}^{(N)}(M)] = \text{tr}_{N-L}[\hat{\rho}^{(N)}(C)].$$

This equality alone does not identify the virtual sector families of M and C , and therefore does not prove the rank-one separation or the Markov decomposition asserted in [CPGSV16, Appendix C.2, lines 1597–1619].

Proof. [✓ Proof](#) At each sufficiently large length, apply Theorem 20.16.24. □

Lemma 20.16.27 (Marginal stability under source zero correlation length). [Lean](#) [Lean](#) [✓ Stmt](#)
Let M have source zero correlation length. For every $N \geq 0$, the normalized N -site marginal obtained by tracing two sites from $\sigma^{(N+2)}(M)$ equals the normalized N -site marginal obtained by tracing one site from $\sigma^{(N+1)}(M)$.

This is the first replacement at [CPGSV16, Appendix C.2, line 1613]. It does not imply the rank-one trace factorization asserted later on that line.

Proof. [✓ Proof](#) Write $E = \mathcal{T}_M$ and choose $\lambda > 0$ such that $E^2 = \lambda E$. For physical configurations u, v of length N , the two marginal matrix elements are $\text{tr}(M^{u,v} E^2) / \text{tr}(E^{N+2})$ and $\text{tr}(M^{u,v} E) / \text{tr}(E^{N+1})$. The numerator and denominator on the left are respectively λ times those on the right, and source zero correlation length makes the denominator on the right nonzero. $\square$

Theorem 20.16.28 (Iterated marginal stability under zero correlation length). [Lean](#) [✓ Stmt](#)
Let M have source zero correlation length. For every $L \geq 0$, the normalized L -site marginal obtained by tracing three sites from $\sigma^{(L+3)}(M)$ equals the normalized L -site marginal obtained by tracing one site from $\sigma^{(L+1)}(M)$.

Proof. [✓ Proof](#) First retain $L+1$ sites and apply Lemma 20.16.27. Trace the last of these retained sites, and then apply the same lemma while retaining L sites. $\square$

Definition 20.16.29 (Fixed-sector suffix contraction). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let S be the finite set of physical sectors and let $k = (k_0, \dots, k_{L-1}) \in S^L$ be a retained sector word. For $R \geq 1$ and $x, y \in I_k$, define

$$\Gamma_k^{(R)}(x, y) = \sum_{t \in S^R} \sum_{z \in I_t} \Omega_{kt}(xz, yz),$$

where kt and xz denote concatenation of sector words and their corresponding fiber coordinates. The cases $R = 1, 2, 3$ are denoted by $\Gamma_k^{(1)}$, $\Gamma_k^{(2)}$, and $\Gamma_k^{(3)}$.

Theorem 20.16.30 (Contraction of a sector-chain fiber). [Lean](#) [✓ Stmt](#) Let $t = (t_0, \dots, t_m)$ be a nonempty sector word. For functions a on the left coordinate of t_0 and b on the right coordinate of t_m ,

$$\begin{aligned} \sum_{z \in I_t} a(z_0^L) \prod_{i=0}^{m-1} \eta_{t_i, t_{i+1}}((z_i^R, z_{i+1}^L), (z_i^R, z_{i+1}^L)) b(z_m^R) \\ = \left(\sum_u a(u) \right) \left(\prod_{i=0}^{m-1} \text{tr}(\eta_{t_i, t_{i+1}}) \right) \left(\sum_v b(v) \right). \end{aligned} \quad (20.81)$$

Proof. [✓ Proof](#) Write $z_i = (\ell_i, r_i)$. Finite distributivity separates the left and right coordinates:

$$\begin{aligned} \sum_{\ell, r} a(\ell_0) \prod_{i=0}^{m-1} \eta_{t_i, t_{i+1}}((r_i, \ell_{i+1}), (r_i, \ell_{i+1})) b(r_m) \\ = \left(\sum_{\ell_0} a(\ell_0) \right) \prod_{i=0}^{m-1} \left(\sum_{r_i, \ell_{i+1}} \eta_{t_i, t_{i+1}}((r_i, \ell_{i+1}), (r_i, \ell_{i+1})) \right) \left(\sum_{r_m} b(r_m) \right). \end{aligned} \quad (20.82)$$

The internal double sum is $\text{tr}(\eta_{t_i, t_{i+1}})$ for each i , which gives the asserted factorization. $\square$

Theorem 20.16.31 (Contraction of a nonempty suffix fiber). [Lean](#) [✓ Stmt](#) Let $k = (k_0, \dots, k_n)$ be a nonempty retained sector word and $t = (t_0, \dots, t_m)$ a nonempty suffix sector word. In open-edge coordinates $x = (x_{\text{int}}, x_{\partial})$ and $y = (y_{\text{int}}, y_{\partial})$,

$$\sum_{z \in I_t} \Omega_{kt}(xz, yz) = \left(\prod_{j=0}^{n-1} \eta_{k_j, k_{j+1}} \right) (x_{\text{int}}, y_{\text{int}}) \left(\text{tr}_R(\eta_{k_n, t_0}) \otimes \text{tr}_L(\eta_{t_m, k_0}) \right) (x_{\partial}, y_{\partial}) \prod_{i=0}^{m-1} \text{tr}(\eta_{t_i, t_{i+1}}). \quad (20.83)$$

Proof. [✓ Proof](#) Split the cyclic product into the retained internal edges, the two boundary edges, and the internal suffix edges. Apply Theorem 20.16.30 to the suffix coordinates. The two boundary sums are the indicated partial traces. $\square$

Theorem 20.16.32 (Sector decomposition of a suffix marginal). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $\widetilde{\mathcal{K}}$ be the tensor in physical-sector coordinates. For $L \geq 0$ and $R \geq 1$, the normalized L -site marginal obtained from a chain of length $L + R$ satisfies

$$\text{reindex}_{\Phi_L} \left(\text{tr}_R \left[\widehat{\rho}^{(L+R)}(\widetilde{\mathcal{K}}) \right] \right) = \text{tr} \left[\rho^{(L+R)}(\widetilde{\mathcal{K}}) \right]^{-1} \bigoplus_{k \in S^L} \Gamma_k^{(R)}.$$

In particular, this identity holds for $R = 1$, $R = 2$, and $R = 3$.

Proof. [✓ Proof](#) Decompose each discarded physical configuration uniquely as a suffix sector word $t \in S^R$ and a fiber coordinate $z \in I_t$. By Theorem 20.15.3.24, matrix elements with distinct retained sector words vanish. For a fixed retained word k , summing the diagonal entries of the discarded fiber gives

$$\sum_{t \in S^R} \sum_{z \in I_t} \Omega_{kt}(xz, yz) = \Gamma_k^{(R)}(x, y).$$

Multiplication by the inverse trace of the length- $L + R$ operator gives the stated normalization. $\square$

Definition 20.16.33 (Rank-one trace factorization for neighboring operators). [Lean](#) [✓ Stmt](#) A rank-one trace factorization of neighboring matrices $\eta_{q,h}$ consists of real numbers a_q, b_h such that

$$\begin{aligned} \text{tr}(\eta_{q,h}) &= a_q b_h, \\ \sum_q a_q b_q &= 1. \end{aligned}$$

The factorization is an explicit assumption; it is not deduced from zero correlation length. It is the factorization invoked at [CPGSV16, Appendix C.2, line 1613].

Lemma 20.16.34 (Separation of the two-boundary edge contraction). [Lean](#) [✓ Stmt](#) Suppose $\eta_{q,h}$ has a rank-one trace factorization. For fixed boundary sectors u, v , set

$$\begin{aligned} R_{u,q} &= \text{tr}_{H_{q,l}}(\eta_{u,q}), \\ L_{h,v} &= \text{tr}_{H_{h,r}}(\eta_{h,v}). \end{aligned}$$

Then

$$\sum_{q,h} \text{tr}(\eta_{q,h})(R_{u,q} \otimes L_{h,v}) = \left(\sum_q a_q R_{u,q} \right) \otimes \left(\sum_h b_h L_{h,v} \right).$$

This is the corrected edge contraction at [CPGSV16, Appendix C.2, lines 1613–1617].

Proof. ✓ **Proof** Substitute $\text{tr}(\eta_{q,h}) = a_q b_h$ and distribute the two finite sums. □

Theorem 20.16.35 (Conditional fourth-region boundary trace). Lean ✓ **Stmt** Let $\eta_{q,h}$ be arbitrary matrices on $H_{q,r} \otimes H_{h,l}$ with a rank-one trace factorization. For a nonempty retained sector word $k = (k_0, \dots, k_{L-1})$, write

$$\begin{aligned} P_k &= \bigotimes_{n=0}^{L-2} \eta_{k_n, k_{n+1}}, \\ R_{u,q} &= \text{tr}_{H_{q,l}}(\eta_{u,q}), \\ L_{h,v} &= \text{tr}_{H_{h,r}}(\eta_{h,v}). \end{aligned}$$

The edge contraction corresponding to tracing the two boundary sites of the longer cyclic chain is

$$\begin{aligned} &\bigoplus_k P_k \otimes \sum_{q,h} \text{tr}(\eta_{q,h})(R_{k_{L-1},q} \otimes L_{h,k_0}) \\ &= \bigoplus_k P_k \otimes \left(\sum_q a_q R_{k_{L-1},q} \right) \otimes \left(\sum_h b_h L_{h,k_0} \right). \end{aligned}$$

This is the edge-coordinate interpretation of the displayed fourth-region trace in [CPGSV16, Appendix C.2, lines 1606 and 1613–1617].

Proof. ✓ **Proof** Apply Lemma 20.16.34 for each retained sector word. Tensor with the unchanged product P_k and take the direct sum over k . □

Remark 20.16.36. The theorem applies, in particular, to the positive chain-independent neighboring family supplied by Theorem 20.16.12, once the rank-one trace factorization is given. It does not derive that factorization, identify this two-boundary contraction with a one-site marginal, or prove a Markov decomposition or saturation of the area law.

Theorem 20.16.37 (A nonminimal sector factorization need not be primitive). Lean ✓ **Stmt** There is an injective normal tensor $\mathcal{K}$, of physical dimension two and bond dimension one, which generates MPDOs and has source zero correlation length, together with one positive two-site bond B whose translates commute. For every $N \geq 2$,

$$\sigma^{(N)}(\mathcal{K}) = \prod_{i=0}^{N-1} B_{i,i+1},$$

while the trace matrix of a chosen two-sector Beigi factorization is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, T is not primitive.

This shows that primitivity cannot be inferred for an arbitrary selected factorization in the step suggested at [CPGSV16, Appendix C.2, line 1613]. The same tensor also has a one-sector factorization with positive semidefinite neighboring operator and primitive trace matrix. Hence the example does not exclude a source-faithful minimal or visible-sector selection, and it does not disprove the stated commuting-form-to-SAL implication.

Proof. [✓ Proof](#) Let $\mathcal{K}^{0,0} = 1$ and let all other physical entries vanish. The matrices $\{\mathcal{K}^{i,j}\}_{i,j=0}^1$ span $M_1(\mathbb{C})$. For every $X \in M_1(\mathbb{C})$, the doubled-index transfer $\mathcal{E}_{\mathcal{K}}$ and the physical-trace transfer $P_{\mathcal{K}}$ satisfy

$$\begin{aligned}\mathcal{E}_{\mathcal{K}}(X) &= X, \\ P_{\mathcal{K}} &= I_1, \\ P_{\mathcal{K}}^2 &= P_{\mathcal{K}}.\end{aligned}$$

Hence $\mathcal{K}$ is injective and normal and has source zero correlation length. Decompose the physical space nonminimally as $\mathbb{C} \oplus \mathbb{C}$. With $w_0 = 1$ and $w_1 = 0$, take both sector factors to be the scalar w_k in sector k . The neighboring operators are

$$\eta_{k,h} = w_k w_h I_1 = \delta_{k,0} \delta_{h,0} I_1 \succeq 0.$$

In the sector coordinates the corresponding physical bond is

$$\begin{aligned}B &= \sum_{k,h} \mathbb{1}_{H_L^{(k)}} \otimes \eta_{k,h} \otimes \mathbb{1}_{H_R^{(h)}} \\ &= \mathbb{1}_{H_L^{(0)}} \otimes I_1 \otimes \mathbb{1}_{H_R^{(0)}}.\end{aligned}$$

Since only the all-zero physical-index word contributes, the factorization identity is

$$\sigma^{(N)}(\mathcal{K}) = \prod_{i=0}^{N-1} B_{i,i+1}.$$

Consequently,

$$\begin{aligned}T_{k,h} &= \text{tr}(\eta_{k,h}) = \delta_{k,0} \delta_{h,0}, \\ T &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

For every $n \geq 1$, $T^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus no positive power has all entries strictly positive, so T is not primitive. For the alternative factorization, take one sector $B^L = \mathbb{C}^2$ and $B^R = \mathbb{C}$, with

$$\begin{aligned}l &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ r &= (1).\end{aligned}$$

Its sole neighboring operator is $\hat{\eta}_{0,0} = \text{diag}(1,0) \succeq 0$, and hence its trace matrix is $\hat{T} = (\text{tr}(\hat{\eta}_{0,0})) = (1)$. Every positive power of $\hat{T}$ equals (1) , so this one-sector trace matrix is primitive. $\square$

Definition 20.16.38 (Cyclic-active sectors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a physical-sector factorization with neighboring operators $\eta_{k,h}$, retain a sector k when there is a sector h such that $\eta_{k,h} \neq 0$ and $h \rightsquigarrow k$, where $\rightsquigarrow$ denotes reachability by nonzero neighboring operators. The first edge $k \rightarrow h$ is explicit, so the resulting closed walk has positive length even when the return path is empty. The set C of retained sectors is therefore exactly the set of vertices occurring in positive-length directed cycles, in the sense of [Bei12, Section III]. Define, for $k, h \in C$,

$$T_{k,h} = \text{Re tr}(\eta_{k,h}).$$

Proof. [✓ Proof](#) In the selected physical coordinates, the periodic operators of K at both lengths are positive nonzero scalar multiples of those of C . Cancel these scalars by Theorem 20.16.41, transport through the physical isometry, and apply source marginal stability. $\square$

Theorem 20.16.43 (Iterated selected fixed-product marginals). [Lean](#) [✓ Stmt](#) *Under the preceding hypotheses, for every $L > 0$, the normalized L -site reduced states of the selected fixed-product tensor obtained from chain lengths $L + 3$ and $L + 1$ are equal.*

Proof. [✓ Proof](#) Apply the two-step source marginal identity to K , transport both sides through the selected physical isometry, and cancel the two positive realization scalars. $\square$

Theorem 20.16.44 (Positive traces of the selected fixed product). [Lean](#) [✓ Stmt](#) *Let K have an eta-local structure and source zero correlation length. For every $N \geq 2$, the trace of the selected fixed-product periodic operator is a strictly positive real number.*

Proof. [✓ Proof](#) The selected neighboring operators are positive semidefinite, so their cyclic product is positive semidefinite. Its trace cannot vanish: positive proportionality would then force the trace of the source periodic operator to vanish, contrary to source zero correlation length. A nonzero positive semidefinite matrix has strictly positive real trace. $\square$

Theorem 20.16.45 (Stabilization of the support of an irreducible square). [Lean](#) [✓ Stmt](#) *Let T be a nonnegative irreducible real matrix on a finite index set. Suppose that*

$$(T^2)_{i,j} > 0 \iff (T^3)_{i,j} > 0$$

for every pair i, j . Then $(T^2)_{i,j} > 0$ for every pair i, j .

Proof. [✓ Proof](#) Right multiplication by T propagates equality of the positive supports of two consecutive powers. Hence all powers from the second onward have the same positive support. Irreducibility supplies a positive path from i to j . If its length is one, append a positive return path at j . The resulting path has length at least two, so support stabilization reduces it to a positive path of length two from i to j . $\square$

Theorem 20.16.46 (Strict positivity from adjacent cyclic-sector marginals). [Lean](#) [✓ Stmt](#) *Let F be a physical-sector factorization whose neighboring operators are positive semidefinite, and let T_C be the trace matrix on its cyclic-active sectors. Assume that every ordered pair of cyclic-active sectors is joined by a path of nonzero neighboring operators. Assume also that, for every $L > 0$, the normalized L -site marginals obtained from chain lengths $L + 2$ and $L + 1$ are equal, and that the closed operator has a positive real trace at every length $N \geq 2$. Then $(T_C^2)_{q,h} > 0$ for every pair of cyclic-active sectors q, h .*

Proof. [✓ Proof](#) Fix q, h and choose a retained sector path from h to q . Positivity makes its retained trace product $P_{h,q}$ nonzero. Tracing one suffix site and two suffix sites gives the respective block-trace coefficients $P_{h,q}(T_C^2)_{q,h}$ and $P_{h,q}(T_C^3)_{q,h}$. Equality of the normalized marginals and positivity of their normalizations show that $(T_C^2)_{q,h}$ vanishes exactly when $(T_C^3)_{q,h}$ vanishes. Recurrence makes T_C irreducible, so Theorem 20.16.45 applies. $\square$

Theorem 20.16.47 (Normalization from adjacent cyclic-sector marginals). [Lean](#) [✓ Stmt](#) *Let F be a physical-sector factorization whose neighboring operators are positive semidefinite, and let T_C be the trace matrix on its cyclic-active sectors. Assume that every ordered pair of cyclic-active sectors is joined by a path of nonzero neighboring operators. Assume also that, for every $L > 0$, the normalized L -site marginals obtained from chain lengths $L + 2$ and $L + 1$ are equal, and that*

the closed operator has a positive real trace Z_N at every length $N \geq 2$. Then there is a number $\lambda \geq 0$ such that

$$(\lambda^{-1}T_C)^2 = (\lambda^{-1}T_C)^3.$$

Proof. ✓ Proof By Theorem 20.16.46, every entry of T_C^2 is positive. For fixed q, h , choose a positive summand $(T_C)_{h,r}(T_C)_{r,q}$ of $(T_C^2)_{h,q}$. The retained three-sector word h, r, q has nonzero trace product. Comparing the two adjacent marginals at retained length three gives

$$(T_C^3)_{q,h} = \frac{Z_5}{Z_4}(T_C^2)_{q,h}.$$

The ratio is independent of q, h . Taking $\lambda = Z_5/Z_4$ proves the assertion.

Theorem 20.16.48 (Selected fixed-product square-cube identity). [Lean] [✓ Stmt] *Let K be an injective tensor with an eta-local structure and source zero correlation length. Let F be the positive physical-sector factorization of the selected fixed-product tensor, and let T_G be its trace matrix on cyclic-active sectors. There is a number $\lambda > 0$ such that*

$$(\lambda^{-1}T_G)^2 = (\lambda^{-1}T_G)^3.$$

Proof. ✓ **Proof** Express the original tensor in the physical coordinates of F . At every length $N \geq 2$, its closed operator is a positive scalar multiple of the selected closed operator. Source zero correlation length therefore gives the adjacent marginal equality for the selected tensor, and positivity gives positive selected traces. Theorem 20.16.40 supplies recurrence. Theorem 20.16.47 now gives the required normalization. $\square$

Theorem 20.16.49 (Markov cuts from the selected fixed product). Lean ✔ Stmt *Let K be an injective tensor with an eta-local structure and source zero correlation length. Express K in the one-site physical coordinates of the positive fixed-product factorization selected by the eta-local structure. For arbitrary $A, C \geq 0$, the normalized marginal on consecutive regions of lengths A , 1 , and C , obtained from the periodic chain of length $A + C + 2$, admits a Hayashi Markov decomposition.*

Proof. ✓ Proof The selected trace matrix is recurrent, and its square is strictly positive. After the positive normalization of Theorem 20.16.48, its square and cube agree. Thus the normalized matrix is primitive, with its second power as a positive power, and the latter factors as ab^T for strictly positive vectors a, b .

Transport the three-suffix/one-suffix marginal identity from the source tensor to the selected fixed product. In cyclic-active coordinates, its retained blocks are the unchanged interior neighboring products tensored with the boundary factors determined by a and b . Regrouping at the distinguished middle site gives the direct sum of normalized positive left and right path states. Their trace products are nonnegative and sum to one. Finally, positive proportionality identifies this selected normalized marginal with the marginal of K in the same physical coordinates. $\square$

Theorem 20.16.50 (Primitivity on cyclic-active sectors). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [**✓ Stmt**](#) Suppose that the virtual dimension is positive, the MPO tensor is injective, and every $\eta_{k,h}$ is positive semidefinite. Then C is nonempty and the matrix T is primitive. More precisely, T is irreducible and every $k \in C$ satisfies

$$\begin{aligned}(T^2)_{k,k} &> 0, \\ (T^3)_{k,k} &> 0.\end{aligned}$$

Proof. [✓ Proof](#) The virtual matrices of all physical sectors span the full matrix algebra. If $A = A_k^{xy}$ is nonzero, nondegeneracy of the trace pairing gives a virtual matrix $B = A_q^{uv}$ such that $\text{tr}(AB) \neq 0$. Cyclicity gives $\text{tr}(BA) = \text{tr}(AB) \neq 0$, and therefore $\eta_{k,q} \neq 0$ and $\eta_{q,k} \neq 0$. Thus a sector containing a nonzero virtual matrix lies on a directed two-cycle. Hence every virtual matrix outside C vanishes, and the virtual matrices belonging to C still span the full algebra.

If a proper directed cut separated two nonempty subsets C_1, C_2 of C , let V_{C_i} denote the span of the virtual matrices in C_i . Absence of an edge across the cut gives

$$\begin{aligned} V_{C_1} V_{C_2} &= \{0\}, \\ V_{C_1} + V_{C_2} &= M_D(\mathbb{C}). \end{aligned}$$

Both spaces contain a nonzero matrix. This contradicts the one-sided product obstruction for two nonzero subspaces whose sum is the full matrix algebra. Thus T is irreducible. The same trace pairing gives a two-edge return through each sector and closes every retained edge through a third retained sector. These yield the displayed positive diagonal entries of T^2 and T^3 ; the coprime return lengths imply primitivity. $\square$

Theorem 20.16.51 (Rank-one cyclic-active two-step coefficient). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 20.16.50, suppose that the physical-trace transfer has source zero correlation length. There is a number $\lambda > 0$ such that, with $\widehat{T} = \lambda^{-1}T$,

$$\begin{aligned} \widehat{T}^2 &= \widehat{T}^3, \\ \text{rank}(\widehat{T}^2) &= 1, \end{aligned}$$

and $\text{tr}(\widehat{T}^m) = \text{tr}(\widehat{T})$ for every $m \geq 1$. By matrix multiplication, the two-step coefficient is $(\widehat{T}^2)_{k,h} = \sum_{q \in C} \widehat{T}_{k,q} \widehat{T}_{q,h}$. The assertion neither identifies $\widehat{T}$ with $\widehat{T}^2$ nor implies that the unreduced Beigi trace matrix is primitive or rank one.

Proof. [✓ Proof](#) Set the left sector tensor equal to zero outside C . The preceding span argument shows that this does not change the physical tensor, and the neighboring operators with both indices in C are unchanged. The source zero-correlation-length relation, transported through this restricted factorization, gives $\widehat{T}^2 = \widehat{T}^3$ on the cyclic-active indices. By Theorem 20.16.50, $\widehat{T}$ is primitive. Its square is therefore a strictly positive idempotent matrix and has rank one. $\square$

Theorem 20.16.52 (Cyclic-active three-boundary trace coefficient). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Let C be the set of cyclic-active sectors and suppose that every neighboring operator is positive semidefinite. For $q, h \in C$,

$$\sum_{r \in C} \text{tr}(\eta_{q,r}) \text{tr}(\eta_{r,h}) = (T_C^2)_{q,h}.$$

More generally, if $B_{q,h}(X)$ denotes the outer boundary operator in the three-site closure, then

$$\sum_{r \in C} \text{tr}_{\text{mid}}(B_{q,h}(X) \otimes \eta_{q,r} \otimes \eta_{r,h}) = (T_C^2)_{q,h} B_{q,h}(X).$$

Here tr_{mid} traces both neighboring factors. No equality between T_C and T_C^2 is asserted.

For every $\lambda \in \mathbb{R}$, set $\widehat{T}_C = \lambda^{-1}T_C$. The same calculation with each fully traced edge multiplied by λ^{-1} gives

$$\sum_{r \in C} \lambda^{-1} \text{tr}(\eta_{q,r}) \lambda^{-1} \text{tr}(\eta_{r,h}) = (\widehat{T}_C^2)_{q,h}.$$

Proof. ✓ **Proof** Positivity makes each $\text{tr}(\eta_{a,b})$ real. Expanding matrix multiplication gives

$$(T_C^2)_{q,h} = \sum_{r \in C} (T_C)_{q,r} (T_C)_{r,h} = \sum_{r \in C} \text{tr}(\eta_{q,r}) \text{tr}(\eta_{r,h}).$$

The partial trace of the fixed-sector three-site closure is $\text{tr}(\eta_{q,r}) \text{tr}(\eta_{r,h}) B_{q,h}(X)$. Summing over $r \in C$ proves the second identity. Multiplying each of the two trace factors by λ^{-1} gives the normalized identity. $\square$

Theorem 20.16.53 (Fourth-region blocks from an explicit factorization). Lean ✓ **Stmt** *Let F be a physical-sector factorization with positive semidefinite neighboring operators. Suppose that $\lambda > 0$ and that functions a, b on the cyclic-active sectors satisfy*

$$(\lambda^{-1} T_C)^2 = ab^\top.$$

For every $n \geq 0$, reindex the $(n+1)$ -site reduced state obtained from the periodic chain of length $n+4$ by its retained open-edge coordinates. The result is the inverse trace of the length- $(n+4)$ periodic operator times the direct sum over retained sector words. A word outside cyclic support contributes zero; every cyclic-active word contributes its unchanged interior neighboring product tensored with the separated boundary factors determined by λ, a, b .

Proof. ✓ **Proof** Expand the reduced state as a direct sum of three-suffix contractions. Terms outside positive-length cyclic support vanish. For a retained cyclic-active word, the three traced suffix sites leave the interior product and the two-step boundary contraction. Substitute $(\lambda^{-1} T_C)^2 = ab^\top$ to separate its two boundary factors, retaining the normalization by the length- $(n+4)$ trace. $\square$

Theorem 20.16.54 (Cyclic-active fourth-region factorization). Lean ✓ **Stmt** *Let $\mathcal{K}$ be injective and have source zero correlation length, and suppose its neighboring operators are positive semidefinite. There are $\lambda > 0$ and strictly positive functions a, b on the cyclic-active sectors, normalized by $a \cdot b = 1$, such that every fourth-region marginal is a direct sum over retained sector words of the unchanged interior neighboring product tensored with the separated left and right boundary factors determined by a and b .*

Proof. ✓ **Proof** The additional marginal replacement leaves the retained interior product unchanged and produces the two-step coefficient on the cyclic-active sectors. Its positive rank-one factorization separates the two surviving boundary sums. All sector words leaving the cyclic-active support vanish before the two outer partial traces are evaluated. $\square$

Theorem 20.16.55 (Markov decomposition from an explicit fourth-region formula). Lean ✓ **Stmt** *Let F be a positive-dimensional physical-sector factorization whose neighboring operators are positive semidefinite. Fix $A, C \geq 0$, $\lambda \in \mathbb{R}$, and strictly positive functions a, b on the cyclic-active sectors. Suppose that the length- $(A+C+4)$ periodic operator has positive trace, the length- $(A+C+2)$ periodic operator has nonzero trace, and the normalized $(A+C+1)$ -site reduced state at length $A+C+2$ has the explicit fourth-region block formula determined by λ, a, b . Then its tripartite form on regions of lengths A , 1 , and C admits a Hayashi Markov decomposition.*

Proof. ✓ **Proof** Regroup every retained fourth-region block at the distinguished middle sector. Its positive left and right path factors are normalized by their traces, with fixed density matrices used only when a factor vanishes. The positive long-chain trace makes all block weights nonnegative. The nonzero short-chain trace makes the reduced state trace one, so these weights sum to one. The resulting direct sum is exactly the Hayashi decomposition in the sector-coordinate middle-site splitting. $\square$

Theorem 20.16.56 (Cyclic-active Markov decomposition at every cut). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 20.16.54, for arbitrary $A, C \geq 0$, the fourth-region marginal on consecutive regions of lengths $A, 1$, and C admits a Hayashi Markov decomposition. The middle site decomposes as the direct sum of its left and right sector factors. Each block is the tensor product of normalized positive left and right path states, and its probability is the product of their unnormalized traces divided by the positive periodic normalization.*

Proof. [✓ Proof](#) Separate the retained chain at the distinguished middle site. The retained neighboring factors split into the left and right path products, while the two cyclic endpoint factors become their respective boundary matrices. Normalize each positive block by its trace, using an arbitrary trace-one positive matrix when the block vanishes. The trace-one identity for the full marginal shows that the resulting non-negative block weights sum to one. $\square$

Theorem 20.16.57 (Physical-sector factorization and ZCL imply SAL). [Lean](#) [✓ Stmt](#) *Let $\mathcal{K}$ have positive bond dimension and be injective, and let F be a physical-sector factorization of $\mathcal{K}$. Suppose every neighboring operator $\eta_{q,h}^F$ is positive semidefinite. If the physical-trace transfer of $\mathcal{K}$ has source zero correlation length, then $\mathcal{K}$ saturates the area law.*

Proof. [✓ Proof](#) Positivity of the neighboring operators makes every nonempty sector-coordinate chain operator positive. Source zero correlation length makes its trace nonzero. Apply the preceding theorem with $A = m - 1$ and $C = N - m - 1$ at every admissible cut, then use the all-cut Markov criterion. Finally transport saturation through the physical isometry of F . $\square$

Theorem 20.16.58 (A translation-invariant commuting bond and ZCL imply SAL). [Lean](#) [✓ Stmt](#) *Let $\mathcal{K}$ be an injective tensor generating MPDOs. Suppose that there is one positive two-site bond B whose translates commute and, for every $N \geq 2$, a constant $c_N > 0$ such that*

$$\sigma^{(N)}(\mathcal{K}) = c_N \prod_{i=0}^{N-1} B_{i,i+1}.$$

If the physical-trace transfer of $\mathcal{K}$ has zero correlation length, then $\mathcal{K}$ saturates the area law.

This is the proposition at [CPGSV16, Appendix C.2, lines 1597–1619].

Proof. [✓ Proof](#) Express $\mathcal{K}$ in the one-site physical coordinates selected by the fixed bond. Positivity of every periodic operator is preserved by this isometry, and its trace is unchanged, hence nonzero at every positive length by source zero correlation length.

For every $1 \leq m \leq \lfloor N/2 \rfloor$, apply Theorem 20.16.49 with $A = m - 1$ and $C = N - m - 1$. Since $A + C + 2 = N$, the resulting decomposition is the quantum Markov decomposition required at the m -th cut. The all-cut criterion gives saturation of the area law in the selected physical coordinates. Finally, isometric invariance of the chain and marginal entropies transports saturation back to $\mathcal{K}$. $\square$

Theorem 20.16.59 (η -local structure gives the one-sector GSNNCH form). [Lean](#) [Lean](#) [✓ Stmt](#) *The single positive bond determined by the η -local structure gives the one-sector instance of the explicit GSNNCH decomposition.*

Proof. [✓ Proof](#) Apply the one-sector construction to the single-bond product at every chain length. $\square$

Theorem 20.16.60 (Injective SAL tensors generate GSNNCH states). [Lean](#) [✓ Stmt](#) *Every injective MPO tensor satisfying SAL generates Gibbs states of a nearest-neighbor commuting Hamiltonian: the normalized finite-chain states are density operators with the explicit sector*

form of Definition 20.13.1, realized through one sector. The one-sector form is the commuting product form in which [CPGSV16, Proposition C.8] states its conclusion for an injective tensor, combined here with the density normalization of [CPGSV16, Definition 4.8].

Proof. ✓ Proof Corollary 20.15.3.61 gives the η -local structure, whose single positive bond gives the one-sector form by Theorem 20.16.59. SAL includes the nonvanishing of every positive-length trace, so Theorem 20.13.46 applies. $\square$

Remark 20.16.61 (Positive physical-sector factorization). Suppose that K satisfies $\text{SAL}(K)$ and admits a physical-sector factorization whose neighboring operators $\eta_{k,h}$ are simultaneously positive semidefinite. The associated two-site bond is

$$B = \sum_{k,h} \mathbb{1}_{H_L^{(k)}} \otimes \eta_{k,h} \otimes \mathbb{1}_{H_R^{(h)}}, \quad (20.84)$$

$$B \geq 0,$$

and, for every $N \geq 2$,

$$\sigma^{(N)}(K) = c_N \prod_{n=1}^N B_{n,n+1}, \quad (20.85)$$

$$c_N > 0,$$

$$[B_{i,i+1}, B_{j,j+1}] = 0.$$

Under this hypothesis, Theorem 20.15.3.60 gives the η -local structure and proves (20.84)–(20.85) with $c_N = 1$. Theorem 20.16.6 then gives the commuting-form conclusion. Adjacent translates commute on every periodic chain of length at least three by Theorem 20.15.3.53, and the crossed two-site case is Theorem 20.15.3.54. Disjoint translates commute by Theorem 20.15.3.55. These cases give pairwise commutativity at every length $N \geq 2$ by Theorem 20.15.3.56. Given the physical-sector factorization, the all-length product identity is Theorem 20.15.3.57. The doubled-index transfer condition enters only in the subsequent RFP and single-bond branch conclusions.

Chapter 21

Mixed States: General Case and Algebraic Structure

This chapter continues the mixed-state renormalization analysis of Chapter 20 with the general case of [CPGSV16, Section 4]. The source first places the tensor in vertical canonical form and then introduces its closed-chain operators. The algebraic characterization identifies the diagonal χ -matrices governing the trace-power formula and ends in the fusion-isometry description.

21.1 Recall of the vertical canonical form and closed-chain operators

The general case begins with the vertical canonical form of [CPGSV16, Proposition 4.13]. In the notation of Definition 17.4.6, there is a coisometry U from the original physical space onto the retained nonzero sector space. Writing $B = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}$, it satisfies

$$\begin{aligned} UU^{\dagger} &= \mathbb{1}, \\ U\widetilde{M}U^{\dagger} &= B, \\ \widetilde{M} &= U^{\dagger}BU, \end{aligned}$$

where the matrices μ_{α} are positive and diagonal and the tensors $\{M_{\alpha}\}_{\alpha}$ form a basis of normal tensors. The matrix $U^{\dagger}U$ is the support projection in the original physical space; the reconstruction identity retains the zero-sector complement allowed by the source. This result was established in Theorem 17.4.112.

The source next defines $O_L(M_{\alpha})$ by contracting cyclically the BNT bond indices of L copies in the vertical reading. Write the vertical physical label as a pair (a, b) of horizontal bond indices and set $\widehat{M}_{\alpha}^{ab} = M_{\alpha}^{(a,b)}$. For $\mathbf{a} = (a_1, \dots, a_L)$ and $\mathbf{b} = (b_1, \dots, b_L)$, the contraction is

$$\begin{aligned} [O_L(M_{\alpha})]_{\mathbf{a}, \mathbf{b}} &= \text{tr}(\widehat{M}_{\alpha}^{a_1 b_1} \dots \widehat{M}_{\alpha}^{a_L b_L}) \\ &= [\rho^{(L)}(\widehat{M}_{\alpha})]_{\mathbf{a}, \mathbf{b}}. \end{aligned}$$

Thus $O_L(M_{\alpha}) = \rho^{(L)}(\widehat{M}_{\alpha})$, where the trace is over the BNT bond space and the remaining row and column indices are the horizontal bond strings $\mathbf{a}$ and $\mathbf{b}$. These are the operators entering the algebra clause of [CPGSV16, Theorem 4.14].

21.2 Algebra structure

Definition 21.2.1 (Algebra-structure data for an MPDO). [Lean](#) [✓ Stmt](#) A family of support $*$ -algebras $\{\mathcal{A}_n\}_{n \geq 0}$, one at every blocking size, equipped with blocking maps $m_n : \mathcal{A}_n \otimes \mathcal{A}_n \rightarrow \mathcal{A}_{2n}$ and inclusion maps $\iota_n : \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$ realized, inside the ambient bond-space matrix algebra, by

$$\begin{aligned} m_n(x, y) &= xy, \\ \iota_n(x) &= x. \end{aligned}$$

Definition 21.2.2 (Blocked fixed-point-algebra tower). [Lean](#) [✓ Stmt](#) An MPO tensor M satisfies the *blocked fixed-point-algebra tower* used here when there exist blocked fixed-point-algebra data such that, for every positive blocking size n ,

$$\mathcal{A}_n = \text{Fix}(E_n^\dagger),$$

where E_n is the blocked transfer map. This is a nontrivial algebraic condition, but it is still weaker than the full coefficient formulation of [CPGSV16, Theorem 4.14(ii)]; that formulation also includes the coefficient family $c_{\alpha, \beta, \gamma}^{(L)}$.

Theorem 21.2.3 (Doubled-index ZCL yields a stationary algebra tower under a faithful fixed point). [Lean](#) [✓ Stmt](#) Assume the doubled tensor is trace-preserving and the transfer map admits a positive-definite fixed point. If $E_1^2 = E_1$, then there is a stationary family of support algebras

$$\mathcal{A}_n = \text{Fix}(E_1^\dagger),$$

with multiplication $m_n(x, y) = xy$ and inclusion $\iota_n(x) = x$.

Proof. [✓ Proof](#) Under the trace-preserving normalization and the faithful fixed point, Wolf's fixed-point-algebra theorem gives the $*$ -subalgebra $\text{Fix}(E_1^\dagger) \subseteq M_D(\mathbb{C})$. Since E_1 is idempotent, $E_n = E_1^n = E_1$ for every $n \geq 1$, so $\text{Fix}(E_n^\dagger) = \text{Fix}(E_1^\dagger)$ at every positive blocking size, giving the stationary tower $\mathcal{A}_n = \text{Fix}(E_1^\dagger)$. $\square$

The two closure maps have the same local form:

$$M_\alpha(X) = \begin{array}{c} \text{---} \\ | \\ \boxed{X} \text{---} \boxed{M_\alpha} \\ | \\ \text{---} \end{array} \quad A_\alpha(X) = \begin{array}{c} \text{---} \\ | \\ \boxed{X} \text{---} \boxed{A_\alpha} \\ | \\ \text{---} \end{array}$$

In each cell, X is inserted on the virtual bond and the virtual indices are closed by tr ; the physical legs remain open. Their spans are the fixed-point algebras compared in [CPGSV16, Appendix C.4, lines 1974–1980].

Theorem 21.2.4 (Stabilized blocked adjoint fixed points yield algebra data). [Lean](#) [✓ Stmt](#) Assume the doubled tensor is trace-preserving and the transfer map admits a positive-definite fixed point. If, for every positive blocked size n ,

$$\text{Fix}(E_n^\dagger) = \text{Fix}(E_1^\dagger),$$

then the blocked fixed-point-algebra tower holds.

Proof. [✓ Proof](#) Take the constant tower $\mathcal{A}_n := \text{Fix}(E_1^\dagger)$ for every n . The stabilization hypothesis $\text{Fix}(E_n^\dagger) = \text{Fix}(E_1^\dagger)$ then gives $\mathcal{A}_n = \text{Fix}(E_n^\dagger)$ at every positive blocking size n , which is compatibility with M . $\square$

This does not give the full converse direction of [CPGSV16, Theorem 4.14]: the compatibility condition above only sees stabilized adjoint fixed-point algebras, not the coefficient/BNT data. In particular, [CPGSV16, Theorem 4.14(ii)] also requires the special diagonal matrices $\chi_{\alpha,\beta,\gamma}$. The coefficient data attached to the algebra tower are: a chosen basis of each support algebra, coordinate maps into a finite scalar family, and the corresponding multiplication / inclusion coefficient arrays.

Definition 21.2.5 (Blocked coefficients for the algebra tower). Lean ✓ Stmt For each blocked size n , fix a basis $\{b_i\}_{i \in I_n}$ of $\mathcal{A}_n$ over a finite index set I_n . The *blocked coefficients* of $x \in \mathcal{A}_n$ are the coordinates $(x_i)_{i \in I_n} \in \mathbb{C}^{I_n}$ with

$$x = \sum_{i \in I_n} x_i b_i.$$

Theorem 21.2.6 (Reconstruction from blocked coefficients). Lean ✓ Stmt *Reconstructing a coefficient family gives the corresponding basis expansion $\sum_i a_i b_i$ in $\mathcal{A}_n$.*

Proof. ✓ Proof This is exactly the finite-basis reconstruction formula for the chosen basis of $\mathcal{A}_n$. □

Definition 21.2.7 (Blocked multiplication coefficients). Lean ✓ Stmt For $i, j \in I_n$, the *blocked multiplication coefficients* are the blocked coefficients $(c_{i,j,k}^{(n)})_{k \in I_{2n}}$ of $m_n(b_i, b_j) \in \mathcal{A}_{2n}$:

$$m_n(b_i, b_j) = \sum_{k \in I_{2n}} c_{i,j,k}^{(n)} b_k.$$

Theorem 21.2.8 (Blocked multiplication reconstructs from its coefficients). Lean ✓ Stmt *In the ambient bond-space matrix algebra, $b_i b_j = \sum_{k \in I_{2n}} c_{i,j,k}^{(n)} b_k$.*

Proof. ✓ Proof Apply the reconstruction formula of Theorem 21.2.6 to the element $m_n(b_i, b_j) \in \mathcal{A}_{2n}$. □

Definition 21.2.9 (Blocked inclusion coefficients). Lean ✓ Stmt For $i \in I_n$, the *blocked inclusion coefficients* are the blocked coefficients $(d_{ij})_{j \in I_{n+1}}$ of $\iota_n(b_i) \in \mathcal{A}_{n+1}$:

$$\iota_n(b_i) = \sum_{j \in I_{n+1}} d_{ij} b_j.$$

Theorem 21.2.10 (Blocked inclusion reconstructs from its coefficients). Lean ✓ Stmt *Reconstructing the inclusion coefficients recovers the image of b_i under ι_n : $\sum_{j \in I_{n+1}} d_{ij} b_j = \iota_n(b_i)$.*

Proof. ✓ Proof Coefficient extraction and reconstruction are mutually inverse coordinate maps for the chosen basis of $\mathcal{A}_{n+1}$, so extracting the coefficients of $\iota_n(b_i)$ and reconstructing returns the same element. □

Theorem 21.2.11 (Compatible adjoint fixed points reconstruct from blocked coefficients). Lean ✓ Stmt *If the algebra tower is compatible with an MPO tensor M and $E_n^\dagger X = X$ for a positive blocking size n , then $X \in \mathcal{A}_n$ and, writing $(X_i)_{i \in I_n}$ for its blocked coefficients, $X = \sum_{i \in I_n} X_i b_i$.*

Proof. ✓ Proof Compatibility identifies $\mathcal{A}_n$ with the adjoint fixed-point algebra of E_n , so the fixed point first becomes an element of $\mathcal{A}_n$ and then Theorem 21.2.6 reconstructs it. □

Theorem 21.2.12 (Adjoint fixed points descend through the algebra tower). [Lean](#) [✓ Stmt](#) *If an MPO tensor satisfies the blocked fixed-point-algebra tower, then for every positive blocking size n and every X , $E_n^\dagger X = X$ implies $E_1^\dagger X = X$.*

Proof. [✓ Proof](#) Compatibility at size n places X in $\mathcal{A}_n$. The inclusion map ι_n lifts X into $\mathcal{A}_{n+1}$, and since ι_n is just an inclusion it does not change the underlying operator, so we continue to denote the lifted element by X . Thus compatibility at size $n+1$ yields $E_{n+1}^\dagger X = X$. Rewriting $E_{n+1} = E_n \circ E_1$ and applying $(A \circ B)^\dagger = B^\dagger \circ A^\dagger$ together with the hypothesis $E_n^\dagger X = X$ extracts $E_1^\dagger X = X$. $\square$

Lemma 21.2.13 (Reverse inclusion of adjoint fixed points). [Lean](#) [✓ Stmt](#) *For every n , $E_1^\dagger X = X$ implies $E_n^\dagger X = X$.*

Proof. [✓ Proof](#) Induction on n using $E_{n+1} = E_n \circ E_1$ and $(A \circ B)^\dagger = B^\dagger \circ A^\dagger$; the base case is $E_0 = \text{id}$. This step does not require the blocked fixed-point-algebra data. $\square$

Lemma 21.2.14 (Blocked powers of adjoint eigenvectors). [Lean](#) [✓ Stmt](#) *If $E = E_1$ and $E^\dagger X = \lambda X$, then $E_n^\dagger X = \lambda^n X$ for every n .*

Proof. [✓ Proof](#) The case $n = 0$ is $E_0 = \text{id}$. For the induction step, $E_{n+1} = E_n \circ E_1$, hence

$$E_{n+1}^\dagger X = E_1^\dagger E_n^\dagger X = E_1^\dagger (\lambda^n X) = \lambda^n E_1^\dagger X = \lambda^{n+1} X.$$

$\square$

Theorem 21.2.15 (Finite-order adjoint eigenvalues for algebra towers). [Lean](#) [✓ Stmt](#) *Assume the blocked fixed-point-algebra tower holds. If $X \neq 0$, $E_1^\dagger X = \lambda X$, and $\lambda^n = 1$ for some $n > 0$, then $\lambda = 1$.*

Proof. [✓ Proof](#) By Lemma 21.2.14, $E_n^\dagger X = \lambda^n X = X$. Theorem 21.2.12 gives $E_1^\dagger X = X$. Thus $\lambda X = X$, so $(\lambda - 1)X = 0$. Since $X \neq 0$, one obtains $\lambda = 1$. $\square$

Theorem 21.2.16 (Adjoint fixed-point equality for algebra towers). [Lean](#) [✓ Stmt](#) *If an MPO tensor satisfies the blocked fixed-point-algebra tower, then $\text{Fix}(E_n^\dagger) = \text{Fix}(E_1^\dagger)$ for every positive blocking size n .*

Proof. [✓ Proof](#) Combine Theorem 21.2.12 with Lemma 21.2.13. $\square$

Theorem 21.2.17 (Algebra data give the stationary fixed-point tower). [Lean](#) [✓ Stmt](#) *Under the trace-preserving normalization and a positive-definite fixed point, if M satisfies the blocked fixed-point-algebra tower, then the stationary tower $\mathcal{A}_n = \text{Fix}(E_1^\dagger)$ is itself compatible with M : for every positive blocking size n , $\text{Fix}(E_1^\dagger) = \text{Fix}(E_n^\dagger)$.*

Theorem 21.2.18 (A blocked fixed-point-algebra tower need not be ZCL). [Lean](#) [✓ Stmt](#) *There exists an MPO tensor of physical and bond dimension two such that the doubled tensor is trace-preserving, the transfer map has a positive-definite fixed point, and a blocked fixed-point-algebra tower, while the doubled-index transfer map is not idempotent.*

Proof. [✓ Proof](#) Take the phase-flip channel with Kraus operators $K_0 = \frac{3}{5}I$ and $K_1 = \frac{4}{5}Z$. Its transfer map satisfies

$$E(X)_{ij} = \begin{cases} X_{ij}, & i = j, \\ -\frac{7}{25}X_{ij}, & i \neq j. \end{cases}$$

Hence, for every $n > 0$ and $i \neq j$, $E^n(X)_{ij} = (-7/25)^n X_{ij}$. Since $(-7/25)^n \neq 1$ for every $n > 0$ and E is self-adjoint, $\text{Fix}((E^n)^\dagger) = \text{Fix}(E^\dagger)$. Thus the blocked fixed-point-algebra tower exists. On the other hand, $E^2(X)_{01} = \frac{49}{625} X_{01} \neq -\frac{7}{25} X_{01} = E(X)_{01}$. Therefore $E^2 \neq E$, so the doubled-index ZCL condition fails. $\square$

Remark 21.2.19 (Why the algebra tower does not imply doubled-index ZCL). Write $E = E_1$ for the one-site transfer map. The present algebra predicate gives $\text{Fix}((E^n)^\dagger) = \text{Fix}(E^\dagger)$ for every $n > 0$. Hence a vector X with $E^\dagger X = \lambda X$ and $\lambda^n = 1$ lies in $\text{Fix}(E^\dagger)$, so $(\lambda - 1)X = 0$ and $\lambda = 1$ by Theorem 21.2.15. It does not imply $E^2 = E$, as Theorem 21.2.18 shows. More generally, if Π_{diag} is the projection onto diagonal matrices and $0 < \varepsilon < 1$, set $E(X) = \varepsilon X + (1 - \varepsilon)\Pi_{\text{diag}}(X)$, so that

$$\begin{aligned} E^n(X) &= \Pi_{\text{diag}}(X) + \varepsilon^n(X - \Pi_{\text{diag}}(X)), \\ E^2 - E &= (\varepsilon^2 - \varepsilon)(\text{id} - \Pi_{\text{diag}}). \end{aligned}$$

The fixed-point algebra is diagonal for every $n > 0$, but $E^2 \neq E$.

The missing source step is the coefficient argument in [CPGSV16, Appendix C.4]. One needs the same-length product law

$$O_L(M_\alpha)O_L(M_\beta) = \sum_\gamma \text{tr}(\chi_{\alpha,\beta,\gamma}^L)O_L(M_\gamma) \quad (21.1)$$

and the length-one idempotent trace identity

$$m_\gamma = \sum_{\alpha,\beta} \text{tr}(\chi_{\alpha,\beta,\gamma})m_\alpha m_\beta. \quad (21.2)$$

The proof of [CPGSV16, Theorem 4.14] uses (21.1) and (21.2), positivity, and the simple-matrix lemma [CPGSV16, lines 1155–1163] to give

$$\begin{aligned} \{\nu_{\gamma,k}\}_k &= \{\mu_{\alpha,k_1}\mu_{\beta,k_2}\chi_{\alpha,\beta,\gamma,k_3}\}_{\alpha,\beta,k_1,k_2,k_3}, \\ \text{tr}(\nu_\gamma) &= m_\gamma. \end{aligned} \quad (21.3)$$

The reconstruction maps in [CPGSV16, lines 2065–2085] then use

$$\begin{aligned} \widetilde{R}_2 \left(\bigoplus_\gamma X_\gamma \right) &= \bigoplus_\gamma \frac{\nu_\gamma}{m_\gamma} \otimes X_\gamma, \\ \widetilde{R}_1 \left(\bigoplus_\gamma X_\gamma \right) &= \bigoplus_\gamma \frac{\mu_\gamma}{m_\gamma} \otimes X_\gamma. \end{aligned} \quad (21.4)$$

Thus $\text{tr}(\nu_\gamma) = m_\gamma$ and the defining equality $m_\gamma = \text{tr}(\mu_\gamma)$ normalize the factors ν_γ/m_γ and μ_γ/m_γ , respectively, in $T = \widetilde{R}_2 \widetilde{T} R_1$ and $S = \widetilde{R}_1 \widetilde{S} R_2$. The scalar Newton–Girard power-sum identity needed to obtain the trace-power form is already available as the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”]. To obtain the trace-power conclusion of [CPGSV16, Theorem 4.14(ii)], one must impose the BNT-label hypotheses as a separate condition, relate them to the one-site and two-site vertical canonical forms, and construct the maps in (21.4). Theorem 21.2.18 shows that the support-algebra condition alone does not imply the BNT-label hypotheses.

21.2.1 Diagonal χ -matrices and the trace-power formula

The following statements isolate the form of [CPGSV16, Theorem 4.14(ii)]: the structure coefficients of the length- L operator algebra are trace-powers of positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$. They state the coefficient identity and the elementary trace-power identity for diagonal matrices.

Definition 21.2.1.1 (Diagonal χ family). [Lean](#) [✓ Stmt](#) A family of diagonal complex matrices, indexed by ordered triples (α, β, γ) drawn from a common index type, consisting of a size $r_{\alpha,\beta,\gamma} \in \mathbb{N}$ and diagonal entries $\chi_{\alpha,\beta,\gamma,1}, \dots, \chi_{\alpha,\beta,\gamma,r_{\alpha,\beta,\gamma}} \in \mathbb{C}$ for each triple, giving the diagonal matrix

$$\chi_{\alpha,\beta,\gamma} = \text{diag}(\chi_{\alpha,\beta,\gamma,1}, \dots, \chi_{\alpha,\beta,\gamma,r_{\alpha,\beta,\gamma}}).$$

This is the matrix $\chi_{\alpha,\beta,\gamma}$ of [CPGSV16, Theorem 4.14(ii)].

Theorem 21.2.1.2 (Trace of χ^L equals the sum of L -th powers of entries). [Lean](#) [✓ Stmt](#) For every triple (α, β, γ) , if $\chi_{\alpha,\beta,\gamma}$ has size $r_{\alpha,\beta,\gamma}$, then for every $L \geq 0$,

$$\text{tr}(\chi_{\alpha,\beta,\gamma}^L) = \sum_{k=1}^{r_{\alpha,\beta,\gamma}} \chi_{\alpha,\beta,\gamma,k}^L,$$

where $\chi_{\alpha,\beta,\gamma,k}$ are the diagonal entries.

Proof. [✓ Proof](#) Diagonal matrices raise to powers entrywise and the trace of a diagonal matrix is the sum of its entries. $\square$

Definition 21.2.1.3 (Trace-power form of a structure-coefficient family). [Lean](#) [✓ Stmt](#) A binary compatibility predicate between an abstract structure-coefficient family $c_{\alpha,\beta,\gamma}^{(L)}$ and a diagonal χ family: the pair (c, χ) is said to be in *trace-power form* when

$$c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L) = \sum_{k=1}^{r_{\alpha,\beta,\gamma}} \chi_{\alpha,\beta,\gamma,k}^L$$

for every $L \geq 0$ and every triple (α, β, γ) , where $r_{\alpha,\beta,\gamma}$ is the size of $\chi_{\alpha,\beta,\gamma}$. This is the formal analog of the target identity of [CPGSV16, Theorem 4.14(ii)]; the existential version—“there exists χ for which (c, χ) has trace-power form”—is obtained by quantifying over χ .

21.2.2 The BNT-label algebra law and idempotent trace vector

Definition 21.2.2.1 (BNT-label coefficient family). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The same-length coefficient system $c_{\alpha,\beta,\gamma}^{(L)}$ from [CPGSV16, Theorem 4.14(ii)], indexed by fixed BNT labels α, β, γ . The coefficient is attached to the length- L product $O_L(M_\alpha)O_L(M_\beta)$, not to the blocked-basis multiplication $\mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathcal{A}_{2n}$. A diagonal χ -family canonically determines the coefficient family $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L) = \sum_r \chi_{\alpha,\beta,\gamma,r}^L$, as in [CPGSV16, Appendix C.4].

Definition 21.2.2.2 (BNT-label operator family). [Lean](#) [✓ Stmt](#) A BNT-label operator family records, for each chain length L , the operators $O_L(M_\alpha)$ indexed by the fixed BNT labels α . The ambient algebra may depend on L and is not identified with a chosen blocked support algebra. Write $\widehat{M}_\alpha^{ab} = M_\alpha^{(a,b)}$ for the vertical reading. The vertical indices of the L copies are contracted cyclically, while the horizontal legs remain open, as in [CPGSV16, lines 962–967]:

$$\begin{aligned} [O_L(M_\alpha)]_{\mathbf{a}, \mathbf{b}} &= \text{tr}(\widehat{M}_\alpha^{a_1 b_1} \dots \widehat{M}_\alpha^{a_L b_L}), \\ \mathbf{a} &= (a_1, \dots, a_L), \\ \mathbf{b} &= (b_1, \dots, b_L). \end{aligned}$$

Definition 21.2.2.3 (Same-length BNT product form). [Lean](#) [Lean](#) [✓ Stmt](#) A BNT-label operator family and a BNT-label coefficient system have same-length product form when, for every positive length L ,

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} c_{\alpha,\beta,\gamma}^{(L)} O_L(M_\gamma).$$

This is the product identity appearing in [CPGSV16, Theorem 4.14(ii)], stated independently of the blocked-basis multiplication $\mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathcal{A}_{2n}$.

Definition 21.2.2.4 (BNT-label trace-scalar family). [Lean](#) [✓ Stmt](#) A BNT-label trace-scalar family records the scalars $m_\alpha = \text{tr}(\mu_\alpha)$ indexed by the fixed BNT labels α , as in the idempotent condition of [CPGSV16, Theorem 4.14(ii)].

Definition 21.2.2.5 (BNT idempotent coefficient form). [Lean](#) [Lean](#) [✓ Stmt](#) A BNT-label trace-scalar family and a BNT-label coefficient system have idempotent coefficient form when, for every BNT label γ ,

$$m_\gamma = \sum_{\alpha,\beta} c_{\alpha,\beta,\gamma}^{(1)} m_\alpha m_\beta.$$

This is the idempotent condition in [CPGSV16, Theorem 4.14(ii)], stated separately from the trace-power formula.

Definition 21.2.2.6 (Blocked-basis BNT-label assignment). [Lean](#) [Lean](#) [✓ Stmt](#) The BNT-label assignment consists, for every positive blocked length n , of a source label map σ_n from the chosen basis of $\mathcal{A}_n$ and a target label map τ_n from the chosen basis of $\mathcal{A}_{2n}$ to the fixed BNT labels. It also gives the combined label map $\sigma_n \sqcup \tau_n$ on the disjoint union of these two blocked bases.

Definition 21.2.2.7 (BNT comparison with blocked coefficients). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) A comparison between the BNT-label coefficients and the chosen blocked-basis coefficients consists of a BNT-label assignment together with the equality

$$c_{i,j,k}^{(n)} = c_{\sigma_n(i),\sigma_n(j),\tau_n(k)}^{(n)}.$$

This is a blocked-basis comparison: the left hand side is expanded in the chosen basis of $\mathcal{A}_{2n}$, so the statement is not itself the same-length BNT product law.

Definition 21.2.2.8 (Positive-length BNT-label trace-power form). [Lean](#) [✓ Stmt](#) A BNT-label coefficient system and a diagonal χ -family are compatible when, for every positive length L ,

$$c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L).$$

The same matrix $\chi_{\alpha,\beta,\gamma}$ is used for all positive L .

Lemma 21.2.2.9 (BNT-label coefficient as a positive-length trace power). [Lean](#) [✓ Stmt](#) Under positive-length BNT-label trace-power form, for every $L > 0$, $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L)$.

Proof. [✓ Proof](#) Combine the defining positive-length trace-power identity with the trace formula for powers of a diagonal matrix. $\square$

Lemma 21.2.2.10 (Canonical BNT coefficients from a χ -family). [Lean](#) [✓ Stmt](#) The coefficient family canonically determined by a diagonal χ -family satisfies the positive-length trace-power condition with respect to the same χ -family.

Proof.  **Proof** This follows from the definition of the canonical coefficient family. □

Definition 21.2.2.11 (Positive BNT-label χ trace-power witness).    **Stmt** A positive BNT-label χ witness consists of a length-independent diagonal $\chi_{\alpha,\beta,\gamma}$ -family, positivity of every diagonal entry, and the positive-length trace-power identity for the BNT-label coefficients. This is the coefficient statement of [CPGSV16, Theorem 4.14(ii)]. If the coefficient family is chosen canonically from the same χ -family, positivity of the diagonal entries alone gives such a witness.

Lemma 21.2.2.12 (Positive BNT-label witness gives trace powers).   **Stmt** A positive BNT-label χ witness gives, for every $L > 0$, the formula $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L)$.

Proof.  **Proof** Apply the trace reformulation of the positive-length trace-power identity from the witness. □

Theorem 21.2.2.13 (Same-length BNT product with chi-trace coefficients).   **Stmt** If the same-length BNT product identity holds and the BNT-label coefficients come from a positive length-independent χ -witness, then for every $L > 0$,

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} \text{tr}(\chi_{\alpha,\beta,\gamma}^L)O_L(M_\gamma).$$

Proof.  **Proof** Substitute the positive-length trace-power formula into the same-length BNT product expansion. □

Theorem 21.2.2.14 (Same-length BNT product for canonical chi coefficients).   **Stmt** If the same-length BNT product identity holds with the coefficient family canonically determined by a diagonal χ -family, then for every $L > 0$,

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} \text{tr}(\chi_{\alpha,\beta,\gamma}^L)O_L(M_\gamma).$$

Proof.  **Proof** Substitute the canonical trace-power formula for the coefficients into the same-length BNT product expansion. □

Theorem 21.2.2.15 (BNT idempotent coefficients as chi traces).   **Stmt** If the BNT trace scalars satisfy the idempotent coefficient condition and the BNT-label coefficients come from a positive length-independent χ -witness, then

$$m_\gamma = \sum_{\alpha,\beta} \text{tr}(\chi_{\alpha,\beta,\gamma})m_\alpha m_\beta.$$

Proof.  **Proof** Substitute the length-one trace-power formula into the idempotent coefficient identity. □

Theorem 21.2.2.16 (BNT idempotent coefficients for canonical chi coefficients).   **Stmt** If the BNT trace scalars satisfy the idempotent coefficient condition with the coefficient family canonically determined by a diagonal χ -family, then

$$m_\gamma = \sum_{\alpha,\beta} \text{tr}(\chi_{\alpha,\beta,\gamma})m_\alpha m_\beta.$$

Proof.  **Proof** Substitute the length-one canonical trace formula into the idempotent coefficient identity. □

Lemma 21.2.2.17 (Blocked coefficients inherit BNT trace powers). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
If the chosen blocked-basis coefficients are compared with the fixed BNT-label coefficient system, and that BNT-label system has a positive length-independent χ -witness, then each positive-length blocked coefficient is the trace power of the corresponding BNT-label χ -matrix:

$$c_{i,j,k}^{(n)} = \text{tr}(\chi_{\sigma_n(i), \sigma_n(j), \tau_n(k)}^n).$$

Proof. [✓ Proof](#) Use the comparison equality to replace the blocked-basis coefficient by the corresponding BNT-label coefficient, and then apply the BNT-label trace-power formula. In the special case where the compared BNT-label coefficient family is already the canonical family determined by χ , the same conclusion follows directly from the canonical trace formula. $\square$

Definition 21.2.2.18 (Blocked χ family for multiplication coefficients). [Lean](#) [✓ Stmt](#) For each positive blocked size n and multiplication triple (i, j, k) , where i and j label basis elements of $\mathcal{A}_n$ and k labels a basis element of $\mathcal{A}_{2n}$, a size $r_{n,i,j,k} \in \mathbb{N}$ and diagonal entries $\chi_{n,i,j,k,1}, \dots, \chi_{n,i,j,k,r_{n,i,j,k}} \in \mathbb{C}$, giving the diagonal matrix

$$\chi_{n,i,j,k} = \text{diag}(\chi_{n,i,j,k,1}, \dots, \chi_{n,i,j,k,r_{n,i,j,k}}).$$

Unlike the uniform BNT-label family in [CPGSV16, Theorem 4.14(ii)], the size and diagonal entries may depend on the blocked length and on the chosen basis indices.

Lemma 21.2.2.19 (Trace of a blocked χ power). [Lean](#) [✓ Stmt](#) For each positive blocked size n , each multiplication triple (i, j, k) , and every exponent L , $\text{tr}(\chi_{n,i,j,k}^L) = \sum_r \chi_{n,i,j,k,r}^L$.

Proof. [✓ Proof](#) Diagonal matrices raise to powers entrywise and the trace of a diagonal matrix is the sum of its entries. $\square$

Definition 21.2.2.20 (Trace-power form for blocked multiplication coefficients). [Lean](#) [✓ Stmt](#)
The blocked multiplication coefficients are in trace-power form when, for every positive blocked size n and every multiplication triple (i, j, k) , the coefficient $c_{i,j,k}^{(n)}$ equals $\sum_r \chi_{n,i,j,k,r}^n$. Equivalently, by Lemma 21.2.2.19, it is the trace of the n -th power of the corresponding diagonal matrix. The exponent n is the source blocking length of the two factors in $\mathcal{A}_n$; the product is expanded in the basis of $\mathcal{A}_{2n}$.

Lemma 21.2.2.21 (Blocked coefficient as a trace power). [Lean](#) [Lean](#) [✓ Stmt](#) Under trace-power form for the blocked multiplication coefficients, for every positive blocked size n , $c_{i,j,k}^{(n)} = \text{tr}(\chi_{n,i,j,k}^n)$.

Proof. [✓ Proof](#) Combine the defining trace-power identity with the trace formula for powers of a diagonal matrix. $\square$

Definition 21.2.2.22 (Positive blocked χ trace-power witness). [Lean](#) [✓ Stmt](#) A positive blocked χ trace-power witness consists of a blocked χ family, positivity of all its diagonal entries, and the positive-length trace-power identity for the blocked multiplication coefficients. This is the blocked-basis analog of the positive diagonal matrices in [CPGSV16, Theorem 4.14(ii)], without requiring a single family indexed only by the BNT labels.

Theorem 21.2.2.23 (BNT witnesses give blocked χ witnesses). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose that the chosen blocked-basis coefficients are compared with a fixed BNT-label coefficient system, and that this BNT-label system has a positive length-independent χ -witness. Then the chosen blocked bases carry a positive blocked χ trace-power witness. The construction is obtained by pulling back the BNT-label $\chi_{\alpha,\beta,\gamma}$ -matrices along the source and target label maps.

Proof. [✓ Proof](#) Reindex the positive BNT-label χ -family along the blocked-basis label map. Positivity is preserved by reindexing, and the trace-power identity follows by substituting the blocked-basis comparison into the BNT-label trace-power formula. $\square$

Lemma 21.2.2.24 (Constant power sums have entries zero or one). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
Let $x_1, \dots, x_n$ be non-negative real numbers whose power sums $\sum_k x_k^L$ take the same value at every exponent $L \geq 1$. Then every x_k equals 0 or 1. The same holds for a finite family of complex numbers that are non-negative in the complex order. Nonnegativity cannot be dropped: the primitive cube roots of unity have power sums invariant under doubling the exponent, yet do not lie in $\{0, 1\}$.

Proof. [✓ Proof](#) If some entry exceeded one, its powers would grow without bound while all other summands stay non-negative, contradicting constancy of the power sums; hence every entry lies in $[0, 1]$. Comparing the exponents one and two gives $\sum_k x_k(1 - x_k) = 0$ with non-negative summands, so each summand vanishes. $\square$

Definition 21.2.2.25 (Length-independent coefficients). [Lean](#) [Lean](#) [✓ Stmt](#) A BNT-label coefficient system $c_{\alpha, \beta, \gamma}^{(L)}$ is *length independent* when $c_{\alpha, \beta, \gamma}^{(L)} = c_{\alpha, \beta, \gamma}^{(1)}$ for every positive chain length L and all labels; equivalently, the coefficients agree at any two positive lengths. This is the case singled out in the discussion following [CPGSV16, Theorem 4.14].

Theorem 21.2.2.26 (Length-independent coefficients are natural numbers). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Suppose a BNT-label coefficient system is in trace-power form $c_{\alpha, \beta, \gamma}^{(L)} = \text{tr}(\chi_{\alpha, \beta, \gamma}^L)$ with positive diagonal matrices $\chi_{\alpha, \beta, \gamma}$, and is length independent. Then every diagonal entry $\chi_{\alpha, \beta, \gamma, r}$ lies in $\{0, 1\}$, as stated in [CPGSV16, Section 4.5]; by positivity every entry equals one. Consequently, at every positive length, $c_{\alpha, \beta, \gamma}^{(L)} = r_{\alpha, \beta, \gamma} \in \mathbb{N}$.

Proof. [✓ Proof](#) For fixed labels, the trace-power form turns length independence into constancy of the power sums of the diagonal entries of $\chi_{\alpha, \beta, \gamma}$, so by Lemma 21.2.2.24 every entry equals 0 or 1, and positivity excludes 0. The trace of $\chi_{\alpha, \beta, \gamma}^L$ is then the number of diagonal entries. $\square$

Theorem 21.2.2.27 (Unit entries give length independence). [Lean](#) [Lean](#) [✓ Stmt](#) If a BNT-label coefficient system is in trace-power form with respect to a diagonal χ -family all of whose entries equal one, then for every $L > 0$, $c_{\alpha, \beta, \gamma}^{(L)} = r_{\alpha, \beta, \gamma}$, so the coefficient system is length independent.

Proof. [✓ Proof](#) The power sums of a family of ones do not depend on the exponent. $\square$

Remark 21.2.2.28 (Topological order and length dependence). Starting from the same-length product law and the idempotent condition, in the particular case of natural-number structure coefficients independent of the length, one recovers all known non-chiral topologically ordered phases together with a description of their excitations [BMW⁺17]. The renormalization fixed-point density operators correspond to the boundary theories of the renormalization fixed-point representatives of those phases, which yields a large family of non-trivial examples. The source asks whether there exist renormalization fixed points whose structure coefficients depend on the length [CPGSV16, Section 4.5]. The example below answers this existential question affirmatively for its displayed BNT presentation.

Theorem 21.2.2.29 (An RFP with length-dependent BNT coefficients). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) There is an MPDO M in literal CPSV canonical form that is a renormalization fixed

point and has a tensor-attached BNT presentation with one label and

$$c^{(L)} = \left(\sqrt{\frac{1}{2}} \right)^L.$$

In particular, $c^{(1)} \neq c^{(2)}$, so the displayed coefficient system depends on the chain length. This is an existential statement about the exhibited BNT presentation; it does not assert that length dependence is invariant under every possible presentation of M .

Proof. ✓ Proof The bond-two singleton tensor is positive on every nonempty chain, is in literal canonical form, and carries the displayed one-label BNT algebra clause. The reverse implication of Theorem 4.14 supplies the two trace-preserving completely positive renormalization maps. Finally,

$$\sqrt{\frac{1}{2}} \neq \frac{1}{2} = \left(\sqrt{\frac{1}{2}} \right)^2,$$

which proves the length dependence. □

Theorem 21.2.2.30 (A length-dependent coefficient family stable under every uniform block rescaling). Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt

Theorem 21.2.2.29 answers the question of [CPGSV16, lines 995–999] affirmatively, but its one-label block has a single diagonal entry $\sqrt{1/2}$, and rescaling that entry to 1 makes the displayed coefficient constant. The family constructed here has a block with two distinct positive diagonal entries, and its length dependence survives every uniform positive rescaling of the block. The four letter matrices are the scaled matrix units of $M_2(\mathbb{C})$

$$A^0 = \frac{4}{5}E_{00}, \quad A^1 = \frac{4}{5}E_{11}, \quad A^2 = \frac{3}{5}E_{01}, \quad A^3 = \frac{3}{5}E_{10}.$$

Identify the physical index set $\{0, 1, 2, 3\}$ with $\{0, 1\} \times \{0, 1\}$ and let R be the bond-four MPO tensor obtained by undoing the vertical reading of the letters $B^{ab} = A^a \otimes \overline{A^b}$:

$$R_{ab}^{pq} = \frac{25}{32} \left(A^a \otimes \overline{A^b} \right)_{p,q},$$

with (p, q) the physical indices and (a, b) the bond indices. Its physical-trace transfer is the rank-one projection

$$\mathcal{T}_R = \frac{25}{32} |t\rangle\langle t|, \quad t = \left(\frac{4}{5}, \frac{4}{5}, 0, 0 \right),$$

where $t_a = \text{tr}(A^a)$; it satisfies $\mathcal{T}_R^2 = \mathcal{T}_R$, the literal zero-correlation-length condition of [CPGSV16, Definition 4.2, lines 736–741]. The hand-defined one-label coefficient model is in trace-power form with $\chi = \text{diag}(1, \frac{7}{25})$:

$$c^{(L)} = \text{tr}(\chi^L) = 1 + \left(\frac{7}{25} \right)^L.$$

Hence $c^{(1)} \neq c^{(2)}$, so the family is not length independent. For every real number $s > 0$, the rescaled block $s\chi = \text{diag}(s, \frac{7s}{25})$ has coefficient

$$c_s^{(L)} = \text{tr}((s\chi)^L) = s^L \left(1 + \left(\frac{7}{25} \right)^L \right),$$

which is again not length independent: if c_s were constant in L , comparing the lengths 1, 2, and 3 would force $(1 + \lambda)(1 + \lambda^3) = (1 + \lambda^2)^2$ with $\lambda = \frac{7}{25}$, which is false. Unlike the single-entry block of Theorem 21.2.2.29, no uniform rescaling makes both diagonal entries equal to one.

The theorem establishes the obstruction for the displayed coefficient family and for every uniform positive rescaling of its diagonal matrix. The unscaled family is attached below to an explicit one-label vertical BNT presentation of R . No such tensor attachment is asserted here for the arbitrary rescaled families. Positivity, literal CPSV canonical form, and the renormalization fixed-point property of R are proved separately. The tensor is a project example motivated by the discussion in [CPGSV16, lines 995–1010]; it is not a tensor stated there.

Proof. ✓ Proof The (a, b) entry of $\mathcal{T}_R$ is $\frac{25}{32} \text{tr}(A^a) \text{tr}(A^b)$, and $\sum_a \text{tr}(A^a)^2 = 2 \cdot \left(\frac{4}{5}\right)^2 = \frac{32}{25}$, so $\mathcal{T}_R^2 = \mathcal{T}_R$. The two coefficient displays are the power sums of the diagonal entries of χ^L and $(s\chi)^L$. Write $\lambda = \frac{7}{25}$. If c_s were length independent, comparing lengths one and two and lengths two and three would give

$$s = \frac{1 + \lambda}{1 + \lambda^2}, \quad s = \frac{1 + \lambda^2}{1 + \lambda^3},$$

hence $(1 + \lambda)(1 + \lambda^3) = (1 + \lambda^2)^2$, that is, $\lambda(1 - \lambda)^2 = 0$. This fails at $\lambda = \frac{7}{25}$. □

Lemma 21.2.2.31 (Doubled-index letters and their rank-one factorization). Lean Lean ✓ Stmt Identify each physical index $p \in \{0, 1, 2, 3\}$ with a pair $(p_1, p_2) \in \{0, 1\}^2$, and similarly write $q = (q_1, q_2)$. The doubled-index letter at the paired index (p, q) is the MPO letter R^{pq} . Moreover, this bond matrix is the rank-one outer product

$$R_{ab}^{pq} = \frac{25}{32} A_{p_1 q_1}^a A_{p_2 q_2}^b,$$

or equivalently

$$R^{pq} = \frac{25}{32} u_{pq} v_{pq}^\top, \quad u_{pq}(a) = A_{p_1 q_1}^a, \quad v_{pq}(b) = A_{p_2 q_2}^b.$$

Proof. ✓ Proof The first identity is the definition of the doubled-index reading. Expanding the Kronecker-product definition of R and using that the entries of the four letters A^a are real gives the displayed outer product. □

Lemma 21.2.2.32 (Rank-one form of the dimer transfer map). Lean Lean ✓ Stmt Set

$$s = \left(\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5} \right), \quad F = \text{diag}(s_0^2, s_1^2, s_2^2, s_3^2),$$

$$\lambda(X) = \left(\frac{25}{32} \right)^2 \sum_{b=0}^3 s_b^2 X_{bb}.$$

Then the transfer map of R has rank-one form

$$\mathcal{E}_R(X) = \lambda(X)F, \quad \lambda(F) = \frac{337}{512}.$$

Proof. ✓ Proof By Lemma 21.2.2.31, the doubled-index letters are the matrices R^{pq} , each with the displayed outer-product factorization. Substituting these rank-one factors into the transfer-map sum makes every term proportional to F , and collecting the diagonal coefficients gives $\lambda(X)$. Substituting $X = F$ and evaluating the four entries gives $337/512$. □

Lemma 21.2.2.33 (One-site transfer quasi-idempotence of the dimer tensor). [Lean](#) [✓ Stmt](#) The transfer map of the dimer tensor R satisfies

$$\mathcal{E}_R^2 = \frac{337}{512} \mathcal{E}_R.$$

Proof. [✓ Proof](#) By Lemma 21.2.2.32,

$$\mathcal{E}_R(\mathcal{E}_R(X)) = \lambda(X)\lambda(F)F = \frac{337}{512} \mathcal{E}_R(X).$$

□

Theorem 21.2.2.34 (Positivity of the rescaling-stable coefficient family). [Lean](#) [Lean](#) [✓ Stmt](#) The closed operator $\rho^{(N)}(R)$ is positive semidefinite for every positive chain length N : R is an MPDO. Writing each bond matrix R^{pq} as a rank-one product of letter-column vectors scaled by $25/32$, the closed-chain trace of a length- N configuration (σ, τ) collapses entrywise to

$$\rho^{(N)}(R)_{\sigma\tau} = \left(\frac{25}{32}\right)^N \mathbb{1}[\text{cyclic bond match}](\sigma, \tau) \prod_n W(\sigma_n, \tau_n),$$

where the indicator is one exactly when both σ and τ satisfy the cyclic bond-matching condition (consecutive letters share their bond bit around the ring), and W is the local factor from Theorem 21.2.2.30 with eigenvalues $\{1, 7/25\}$. So $\rho^{(N)}(R)$ is $(25/32)^N$ times the Hadamard product of the rank-one bond-matching indicator with the pullback of the N -fold Kronecker power of W ; the Schur product theorem then gives positivity.

Proof. [✓ Proof](#) The entrywise factorization is proved by induction on the chain length, telescoping the ordinary matrix product of the rank-one bond matrices via the rank-one contraction identity

$$\text{tr}((u_1 v_1^\top) \cdots (u_N v_N^\top)) = (v_N^\top u_1) \prod_{n=1}^{N-1} (v_n^\top u_{n+1}),$$

where u_n, v_n are the letter-column vectors of the n -th bond matrix, and closing the trace at the boundary produces the cyclic bond-matching condition. Positivity is the Hadamard product of the rank-one bond-matching indicator (positive semidefinite as an outer product) with the pullback of the Kronecker power of W (positive semidefinite by congruence along an arbitrary reindexing), scaled by the nonnegative scalar $(25/32)^N$. □

Theorem 21.2.2.35 (Unitality of the explicit per-site transfer map). [Lean](#) [✓ Stmt](#) For the explicit four-letter tensor A , its per-site transfer map $\varphi(Y) = \sum_a A^a Y (A^a)^\dagger$ satisfies

$$\varphi(\mathbb{1}) = \mathbb{1}.$$

Proof. [✓ Proof](#) Evaluate the four matrix entries of $\varphi(\mathbb{1})$ from the displayed matrices A^a . The diagonal entries are one and the off-diagonal entries vanish. □

Theorem 21.2.2.36 (Traceless diagonal eigenvector of the explicit transfer map). [Lean](#) [✓ Stmt](#) For $Z = \text{diag}(1, -1)$, the same transfer map satisfies

$$\varphi(Z) = \frac{7}{25} Z.$$

Proof. ✓ Proof Evaluate $\sum_a A^a Z(A^a)^\dagger$ entrywise. Its off-diagonal entries vanish, while its diagonal entries are $7/25$ and $-7/25$. $\square$

Theorem 21.2.2.37 (Annihilation of the first off-diagonal matrix unit). Lean ✓ Stmt *For the matrix unit E_{01} , the same transfer map satisfies*

$$\varphi(E_{01}) = 0.$$

Proof. ✓ Proof Substitution of the four matrices A^a shows that every entry of $\sum_a A^a E_{01}(A^a)^\dagger$ is zero. $\square$

Theorem 21.2.2.38 (Annihilation of the second off-diagonal matrix unit). Lean ✓ Stmt *For the matrix unit E_{10} , the same transfer map satisfies*

$$\varphi(E_{10}) = 0.$$

Proof. ✓ Proof Substitution of the four matrices A^a shows that every entry of $\sum_a A^a E_{10}(A^a)^\dagger$ is zero. $\square$

Theorem 21.2.2.39 (Spectral identification of the local factor and the per-site transfer map).

Lean Lean ✓ Stmt *The two diagonal entries of χ are eigenvalues of the local factor W : the uniform vector is fixed (eigenvalue 1) and the alternating vector is scaled by $7/25$. The per-site transfer map $\varphi(Y) = \sum_a A^a Y(A^a)^\dagger$ has eigenvalues $\{1, 7/25, 0, 0\}$ with explicit eigenvectors: the identity is fixed, the traceless diagonal $\text{diag}(1, -1)$ is scaled by $7/25$, and the off-diagonal matrix units are annihilated. This identifies the spectral data of W and of φ with χ .*

Proof. ✓ Proof The uniform and alternating vectors are eigenvectors of the explicit 2×2 matrix W ,

$$W \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad W \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{7}{25} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and the identity, the traceless diagonal, and the two off-diagonal matrix units are eigenvectors of φ by direct expansion of the letter matrices,

$$\varphi(1) = 1, \quad \varphi(\text{diag}(1, -1)) = \frac{7}{25} \text{diag}(1, -1), \quad \varphi(E_{01}) = \varphi(E_{10}) = 0.$$

$\square$

Lemma 21.2.2.40 (Algebraic injectivity of $R^{\bullet\bullet}$). Lean ✓ Stmt *The 16 letters R^{pq} of the doubled-index tensor $R^{\bullet\bullet}$ span the full bond matrix algebra $M_4(\mathbb{C})$.*

Proof. ✓ Proof Each letter R^{pq} is, up to a nonzero scalar, one of the 16 matrix units of the bond algebra $\mathbb{C}^{4 \times 4}$: the outer-product structure of R^{pq} collapses to a single matrix-unit letter because each factor A^a, A^b of the vertical letters $B^{ab} = A^a \otimes \overline{A^b}$ is itself a scaled matrix unit of $\mathbb{C}^{2 \times 2}$. Hence the standard matrix-unit basis lies in the span of the letters. $\square$

Theorem 21.2.2.41 (The literal CPSV canonical form of R). Lean ✓ Stmt *The doubled-index tensor $A = R^{\bullet\bullet}$ (bond dimension 4, 16 letters R^{pq} , $p, q \in \{0, \dots, 3\}$) is in the canonical form of Definition 8.6.7 (CF): a single retained normal block occupying the full ambient bond dimension 4 ($r = 1$, $D_1 = 4$, $U = 1$), with weight $\mu = \sqrt{337/512}$, the unique positive scalar making the retained block's transfer map exactly idempotent (hence spectral-radius one).*

Proof. [✓ Proof](#) By Lemma 21.2.2.40, the 16 letters span the full bond matrix algebra, so A is algebraically injective, and no nontrivial invariant orthogonal projection exists.

The transfer map $\mathcal{E}_A(X) = \sum_{p,q} R^{pq} X (R^{pq})^\dagger$ is rank one: every image is a scalar multiple of a fixed diagonal matrix, and the scalar evaluated at that fixed matrix is exactly $337/512$. Rescaling A by $\mu^{-1} = \sqrt{512/337}$ makes the transfer map exactly idempotent, hence primitive with spectral radius one (all other eigenvalues of the rank-one map are exactly zero). Together with algebraic injectivity this gives a CPSV normal tensor, and $A = \mu \cdot (\mu^{-1}A)$ pointwise gives the reconstruction equation with the trivial (identity) coisometry. $\square$

Theorem 21.2.2.42 (Non-nilpotency of the retained horizontal block). [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Let $\widehat{R} = \mu^{-1}R^{\bullet\bullet}$ be the single retained normal block in the displayed canonical form, where $\mu = \sqrt{337/512}$. Then

$$\mathcal{T}_{\widehat{R}} = \mu^{-1}\mathcal{T}_R, \quad \mathcal{T}_R^2 = \mathcal{T}_R, \quad (\mathcal{T}_R)_{00} = \frac{1}{2}.$$

Consequently neither $\mathcal{T}_R$ nor $\mathcal{T}_{\widehat{R}}$ is nilpotent. Thus the retained basis element satisfies the non-nilpotency clause in simplicity.

This conclusion concerns only that clause for the displayed one-block presentation, and it makes no claim about other presentations or blockings.

Proof. [✓ Proof](#) A nilpotent idempotent is zero, but the $(0,0)$ entry of $\mathcal{T}_R$ is

$$\frac{25}{32} \text{tr}(A^0)^2 = \frac{25}{32} \left(\frac{4}{5}\right)^2 = \frac{1}{2},$$

so $\mathcal{T}_R$ is not nilpotent. The diagonal physical contraction is linear in the tensor, which gives $\mathcal{T}_{\widehat{R}} = \mu^{-1}\mathcal{T}_R$. Since $\mu > 0$, a nilpotent $\mathcal{T}_{\widehat{R}}$ would remain nilpotent after multiplication by μ , forcing $\mathcal{T}_R$ to be nilpotent. $\square$

Theorem 21.2.2.43 (Simplicity of the rescaling-stable dimer). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)
The dimer tensor R is simple. The positive blocking length is $L = 1$. After the canonical relabeling of the doubled physical alphabet, the basis of normal tensors has one member, namely the retained block $\widehat{R}$, and its physical-trace transfer is non-nilpotent.

Proof. [✓ Proof](#) Theorem 21.2.2.34 gives the MPDO condition. Choose the one-site blocking. Its doubled-index tensor is the canonical physical relabeling of $R^{\bullet\bullet}$. Refine the canonical form from Theorem 21.2.2.41 to its BNT sector presentation. The only retained representative is $\widehat{R}$, whose physical-trace transfer is non-nilpotent by Theorem 21.2.2.42. $\square$

Theorem 21.2.2.44 (The rescaling-stable tensor is a renormalization fixed point). [Lean](#) [✓ Stmt](#)
The tensor R of Theorem 21.2.2.30 is a renormalization fixed point in the sense of Definition 20.2.1. The refinement map $\mathcal{T}$ has two Kraus operators built from the eigenvectors of W , with amplitude $5/8$ on the uniform eigenvector and $\sqrt{7}/8$ on the alternating eigenvector. The coarse-graining map $\mathcal{S}$ has four Kraus operators: two share the amplitude $1/\sqrt{2}$ and the eigenvectors of W on the physical-index pairs satisfying the bond-matching condition, and two are a deterministic assignment on the pairs that fail it, chosen so the family resolves the identity. This proves the fixed-point condition directly. The following theorem constructs the corresponding one-label tensor-attached BNT algebra clause explicitly.

Proof. [✓ Proof](#) Identify each physical index $p \in \{0, 1, 2, 3\}$ with a pair of bits $(p', p'') \in \{0, 1\}^2$ as above, and write $i_1 * i_2 := (i'_1, i''_2)$ for the physical index gluing the first bit of i_1 to the second

bit of i_2 . The two-site physical operator is a gate-restricted, rescaled pullback of the one-site physical operator along $*$:

$$M_2(X)_{(i_1, i_2)(j_1, j_2)} = \frac{25}{32} \mathbb{1}[i_1'' = i_2'] \mathbb{1}[j_1'' = j_2'] W(i_1'', j_1'') M_1(X)_{i_1 * i_2, j_1 * j_2},$$

vanishing whenever either indicator fails. Write $e_0 = (1, 1)$, $e_1 = (1, -1)$ for the eigenvectors of W and $\lambda_0 = 1$, $\lambda_1 = 7/25$ for the matching eigenvalues (Theorem 21.2.2.39). The refinement map's two Kraus operators carry the amplitudes $\sqrt{25\lambda_s/64}$ on e_s , and reassemble the rescaled local factor from the eigendecomposition of W ,

$$\sum_{s \in \{0,1\}} \frac{25}{64} \lambda_s e_s(b) e_s(b') = \frac{25}{32} W(b, b'), \quad b, b' \in \{0, 1\},$$

which is exactly the coefficient displayed above. The coarse-graining map's two gated Kraus operators carry the shared amplitude $1/\sqrt{2}$ on e_0, e_1 , and their resolution of the identity on the bond-matching pairs is the single identity

$$\frac{1}{2} \sum_{s \in \{0,1\}} e_s(b) e_s(b') = \begin{cases} 1, & b = b', \\ 0, & b \neq b', \end{cases} \quad b, b' \in \{0, 1\} :$$

the case $b = b'$ is the trace-preserving normalization and the case $b \neq b'$ the off-diagonal cancellation between the two gated Kraus operators. The coarse-graining map's two ungated Kraus operators supply a deterministic assignment on the bond-mismatched pairs, where the two-site physical operator already vanishes by the displayed formula above; together the four Kraus operators resolve the identity on the full two-site domain. $\square$

Theorem 21.2.2.45 (Explicit one-label tensor-attached BNT algebra clause for R). [Lean](#) [Stmnt](#) *The tensor R has a one-label vertical BNT presentation. Its component has 16 physical letters, bond dimension 4, and*

$$M_0^{(a,b)} = A^a \otimes \overline{A^b}.$$

The 1×1 multiplicity matrix and the 2×2 positive coefficient matrix are

$$\mu_0 = \left(\frac{25}{32}\right), \\ \chi_{0,0,0} = \text{diag}\left(1, \frac{7}{25}\right).$$

The vertical tensor is reconstructed exactly by

$$\text{vertical}(R) = \frac{25}{32} M_0.$$

For every $L > 0$, its closed component operator satisfies

$$O_L(M_0)^2 = \left(1 + \left(\frac{7}{25}\right)^L\right) O_L(M_0) = \text{tr}(\chi_{0,0,0}^L) O_L(M_0).$$

At length one, $m_0 = \text{tr}(\mu_0) = 25/32$ and $c_{0,0,0}^{(1)} = 32/25$, so the idempotent identity is

$$\frac{25}{32} = \frac{32}{25} \left(\frac{25}{32}\right)^2.$$

Proof. ✓ Proof Each letter of M_0 is a nonzero multiple of a matrix unit, so the letters span $M_4(\mathbb{C})$. The tensor A is left-canonical; hence M_0 is left-canonical and algebraically normal, and therefore is a normal tensor. The identity coisometry gives both the forward vertical decomposition and its reconstruction.

The closed operator of $M_0 = A \otimes \bar{A}$ is the pure-state operator $|V^{(L)}(A)\rangle\langle V^{(L)}(A)|$. Its square is its self-overlap times itself. A direct transfer-map computation gives $\langle V^{(L)}(A)|V^{(L)}(A)\rangle = \text{tr}(W^L)$, while the Walsh–Hadamard diagonalization gives $\text{tr}(W^L) = \text{tr}(\chi_{0,0,0}^L) = 1 + (7/25)^L$. The length-one identity is the displayed scalar calculation. $\square$

Remark 21.2.2.46 (Vertical and horizontal contractions). Let $\text{rot}(\text{vertical}(R))$ denote the MPO obtained by separating the 16-valued physical index of the vertical tensor into two four-valued indices. Then

$$\rho^{(L)}(\text{rot}(\text{vertical}(R))) = \left(\frac{25}{32}\right)^L O_L(M_0).$$

This is a rotated vertical contraction, not the original horizontal tensor R , even though both physical and bond dimensions equal 4. At length one, $\rho^{(1)}(R)$ has rank two, whereas $O_1(M_0)$ has rank one, so these operators are not proportional. In particular, no identity $\rho^{(L)}(R) = (25/32)^L O_L(M_0)$ is asserted.

Theorem 21.2.2.47 (Closed-operator factorization and local diagonalization for R). Lean
Lean Lean Lean ✓ Stmt For every positive chain length N , the closed operator of R factors literally as

$$\rho^{(N)}(R) = \left(\frac{25}{32}\right)^N B W_N B^\top,$$

where B is the boundary partial isometry sending a chain of first bits a to the unique cyclic-bond-matching physical-index string reading a back through its first bond bit ($B^\top B = 1$), and $W_N(a, b) = \prod_n W(a_n, b_n)$ is the N -fold Kronecker power of W . The Walsh–Hadamard change of basis $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ diagonalizes W exactly:

$$H\chi H = 2W, \quad \chi = \text{diag}(1, \frac{7}{25}).$$

Thus the matrix χ diagonalizes the local factor W in the horizontal closed-operator formula. Theorem 21.2.2.45 identifies the same matrix with the coefficient matrix of an explicit one-label vertical BNT presentation. The factorization here still concerns the original horizontal operator $\rho^{(N)}(R)$; it is not the scalar reconstruction formula for the rotated vertical contraction.

Proof. ✓ Proof B is built from a two-sided inverse of the first-bit reading φ on cyclic-bond-matching strings: given a , the unique matching string reads back each bond bit from the two adjacent entries of a around the ring, and φ recovers a from it. Injectivity of this embedding gives $B^\top B = 1$, and rewriting the entrywise formula of Theorem 21.2.2.34 through it gives the displayed factorization. The Walsh–Hadamard identity $H\chi H = 2W$ is a direct 2×2 computation from the explicit entries of W and χ . $\square$

Theorem 21.2.2.48 (Uniform χ -trace-power form of the explicit one-label model). Lean
Lean Lean ✓ Stmt The hand-defined family $c_{0,0,0}^{(L)}$ is a positive BNT-label χ trace-power model (Definition 21.2.2.11) with $\chi = \text{diag}(1, \frac{7}{25})$. For every $L > 0$,

$$c^{(L)} = \text{tr}(\chi^L).$$

The matrix χ diagonalizes the local factor W in Theorem 21.2.2.47, so

$$\mathrm{tr}(\chi^L) = \mathrm{tr}(W^L).$$

Hence $c^{(L)} = \mathrm{tr}(W^L) = 1 + (7/25)^L$. Here W^L is the ordinary matrix power of the fixed 2×2 matrix W , not the Kronecker power W_L inside $\rho^{(L)}(R)$. Theorem 21.2.2.45 identifies this coefficient model with the explicit one-label vertical component. It does not identify $O_L(M_0)$ with the original horizontal operator $\rho^{(L)}(R)$.

Proof. ✓ Proof The coefficient family is canonically determined by χ , so the positive BNT-label χ trace-power witness applies directly; its trace reformulation (Lemma 21.2.2.12) gives $c^{(L)} = \mathrm{tr}(\chi^L)$ for every $L > 0$. For the second equality, the Walsh–Hadamard identity $H\chi H = 2W$ gives the intertwining relation $H\chi = WH$ after cancelling one factor of H using $HH = 2 \cdot 1$; this lifts to every power, $H\chi^L = W^L H$, by induction. Conjugating back through H and its half-inverse and applying the cyclic property of the trace gives $\mathrm{tr}(W^L) = \mathrm{tr}(\chi^L)$ for every L . □

Definition 21.2.2.49 (Linearly independent operator family). Lean Lean ✓ Stmt A BNT-label operator family is *linearly independent at length L* when the operators $O_L(M_\alpha)$ are linearly independent, and *eventually linearly independent* when this holds for every length beyond some threshold. Eventual linear independence is the operator-level form of the third defining property of a basis of normal tensors in [CPGSV16, Section 2.3], invoked for the labelled operators in [CPGSV16, Appendix C.4].

Theorem 21.2.2.50 (Uniqueness of the structure coefficients). Lean ✓ Stmt Two coefficient systems satisfying the same-length product law for the same operator family agree at every positive length at which the operators are linearly independent.

Proof. ✓ Proof Applying the same-length product law to both coefficient systems gives

$$\sum_{\delta} c_{\alpha, \beta, \delta}^{(L)} O_L(M_\delta) = O_L(M_\alpha) O_L(M_\beta) = \sum_{\delta} c'_{\alpha, \beta, \delta} O_L(M_\delta).$$

Subtracting yields $\sum_{\delta} (c_{\alpha, \beta, \delta}^{(L)} - c'_{\alpha, \beta, \delta}) O_L(M_\delta) = 0$, and linear independence forces every coefficient of the difference to zero. □

Definition 21.2.2.51 (Explicit transport of vertical BNT representatives). Lean ✓ Stmt Let $(O_L(\alpha))_{\alpha \in \Lambda}$ and $(O'_L(\alpha))_{\alpha \in \Lambda'}$ be two labelled operator families. An explicit vertical transport consists of a bijection $e : \Lambda' \rightarrow \Lambda$, nonzero scalars s_α , and algebra isomorphisms Φ_L between the ambient operator algebras such that

$$O'_L(\alpha) = s_\alpha^L \Phi_L(O_L(e(\alpha)))$$

for every positive length and every target label.

Definition 21.2.2.52 (Coefficient family transported by vertical normalization). Lean ✓ Stmt Given a label bijection $e : \Lambda' \rightarrow \Lambda$ and scalars s_α , the transported coefficient family is

$$\tilde{c}_{\alpha, \beta, \gamma}^{(L)} = \left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L c_{e(\alpha), e(\beta), e(\gamma)}^{(L)}.$$

Here division is totalized by the convention $z/0 = 0$.

Theorem 21.2.2.53 (Transport of the same-length product law). [Lean](#) [✓ Stmt](#) *An explicit vertical transport carries a same-length product law to the transported coefficient family.*

Proof. [✓ Proof](#) For every positive length L , multiplication of the two transported representatives and the source product law give

$$\begin{aligned} O'_L(\alpha)O'_L(\beta) &= (s_\alpha s_\beta)^L \Phi_L(O_L(e(\alpha))O_L(e(\beta))) \\ &= (s_\alpha s_\beta)^L \sum_{\delta} c_{e(\alpha), e(\beta), \delta}^{(L)} \Phi_L(O_L(\delta)). \end{aligned}$$

Reindexing by $\delta = e(\gamma)$ and using $s_\gamma \neq 0$ yields

$$\begin{aligned} O'_L(\alpha)O'_L(\beta) &= \sum_{\gamma} \left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L c_{e(\alpha), e(\beta), e(\gamma)}^{(L)} s_\gamma^L \Phi_L(O_L(e(\gamma))) \\ &= \sum_{\gamma} \tilde{c}_{\alpha, \beta, \gamma}^{(L)} O'_L(\gamma). \end{aligned}$$

□

Theorem 21.2.2.54 (Abstract covariance of vertically transported coefficients). [Lean](#) [✓ Stmt](#) *Suppose that source and target operator families satisfy same-length product laws and are related by an explicit vertical transport. At a positive length where the target operators are linearly independent, the target coefficient family equals the transported source coefficient family.*

Proof. [✓ Proof](#) The target product law and Theorem 21.2.2.53 give

$$\sum_{\gamma} c'_{\alpha, \beta, \gamma}^{(L)} O'_L(\gamma) = O'_L(\alpha)O'_L(\beta) = \sum_{\gamma} \tilde{c}_{\alpha, \beta, \gamma}^{(L)} O'_L(\gamma).$$

Linear independence of the target operators identifies the two coefficient families term by term.

□

Theorem 21.2.2.55 (Tensor-attached coefficient covariance). [Lean](#) [✓ Stmt](#) *Suppose that two tensor-attached vertical BNT presentations are related by an explicit vertical transport. At every positive length L at which the target operators are linearly independent, their structure coefficients satisfy*

$$c'_{\alpha, \beta, \gamma}^{(L)} = \left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L c_{e(\alpha), e(\beta), e(\gamma)}^{(L)}.$$

Theorem 21.2.2.59 shows why a bare equality of horizontal periodic families cannot replace the explicit transport hypothesis.

Proof. [✓ Proof](#) The two tensor-attached BNT clauses carry the product laws required by the abstract covariance theorem. Hence

$$c'_{\alpha, \beta, \gamma}^{(L)} = \tilde{c}_{\alpha, \beta, \gamma}^{(L)} = \left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L c_{e(\alpha), e(\beta), e(\gamma)}^{(L)}.$$

□

Theorem 21.2.2.56 (Abstract exact comparison under compatible normalization). [Lean](#) [✓ Stmt](#)
Under the hypotheses of the abstract covariance theorem, if $s_\alpha s_\beta = s_\gamma$, then

$$c'^{(L)}_{\alpha,\beta,\gamma} = c^{(L)}_{e(\alpha),e(\beta),e(\gamma)}.$$

Proof. [✓ Proof](#) The compatible normalization and $s_\gamma \neq 0$ give

$$\left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L = \left(\frac{s_\gamma}{s_\gamma} \right)^L = 1.$$

Substitution in the abstract covariance law gives the result. $\square$

Theorem 21.2.2.57 (Exact tensor-attached comparison under compatible normalization). [Lean](#) [✓ Stmt](#)
Under the hypotheses of the preceding theorem, if $s_\alpha s_\beta = s_\gamma$, then

$$c'^{(L)}_{\alpha,\beta,\gamma} = c^{(L)}_{e(\alpha),e(\beta),e(\gamma)}.$$

Proof. [✓ Proof](#) The tensor-attached exact comparison is the specialization

$$c'^{(L)}_{\alpha,\beta,\gamma} = \left(\frac{s_\alpha s_\beta}{s_\gamma} \right)^L c^{(L)}_{e(\alpha),e(\beta),e(\gamma)} = c^{(L)}_{e(\alpha),e(\beta),e(\gamma)},$$

where the last equality uses $s_\alpha s_\beta = s_\gamma$ and $s_\gamma \neq 0$. $\square$

Definition 21.2.2.58 (Horizontally gauge-equivalent tensors with distinct vertical normalizations). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ X &= \begin{pmatrix} 1 & -3/4 \\ 0 & 1 \end{pmatrix}, \\ Q &= XPX^{-1} = \begin{pmatrix} 1 & 3/4 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Define one-letter MPO tensors by $M_P^{00} = P$ and $M_Q^{00} = Q$. Equip them with one-label vertical BNT presentations whose normalized representatives and positive multiplicity weights are

$$\begin{aligned} A_P &= P, & m_P &= 1, \\ A_Q &= \frac{4}{5}Q, & m_Q &= \frac{5}{4}. \end{aligned}$$

The corresponding positive diagonal matrices in the algebra clause are $[1]$ and $[4/5]$. No horizontal canonical-form assumption is imposed on the sheared tensor M_Q .

Theorem 21.2.2.59 (Bare horizontal periodic equality does not determine vertical BNT coefficients). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The tensors M_P and M_Q are related by the nonunitary gauge X . For every positive length N , their unnormalized periodic operators are equal:*

$$\rho_N(M_P) = \rho_N(M_Q).$$

In particular, both tensors generate matrix product density operators. Their chosen vertical presentations nevertheless have structure coefficients

$$\begin{aligned} c_P^{(L)} &= 1, \\ c_Q^{(L)} &= \left(\frac{4}{5}\right)^L. \end{aligned}$$

Hence there is no relabelling of the singleton label sets for which, at every positive length,

$$c_Q^{(L)}(\alpha, \beta, \gamma) = c_P^{(L)}(e\alpha, e\beta, e\gamma).$$

Indeed, the two sides at length one are $4/5$ and 1 .

Proposition 4.13 and Theorem 4.14 of [CPGSV16] instead start with one horizontally canonical matrix product density operator and choose its vertical decomposition. The tensor M_Q above is not asserted to be a horizontal canonical representative. Thus the theorem excludes an additional cross-presentation conclusion under bare periodic equality; it does not alter the source's fixed-presentation characterization. It also leaves Theorem 21.2.2.50 unchanged, since that theorem compares two coefficient systems for the same vertical operator family.

Proof. ✓ Proof The identities $P^2 = P$, $Q^2 = Q$, and $\text{tr}(P) = \text{tr}(Q) = 1$ prove the periodic equality. The Frobenius norms of P and Q are 1 and $5/4$, respectively, which gives the displayed normalized representatives and multiplicity weights. Their products satisfy

$$\begin{aligned} A_P^2 &= A_P, \\ A_Q^2 &= \frac{4}{5}A_Q. \end{aligned}$$

Taking the L -fold vertical operators gives the two displayed coefficient families. Their values at $L = 1$ exclude the only possible relabelling. □

Theorem 21.2.2.60 (Length independence is not preserved by bare horizontal periodic equality). Lean Lean Lean ✓ Stmt The coefficients in the P presentation are length independent, whereas those in the Q presentation are not. More precisely,

$$\begin{aligned} c_P^{(L)} &= 1, \\ c_Q^{(1)} &= \frac{4}{5}, \\ c_Q^{(2)} &= \frac{16}{25}. \end{aligned}$$

Proof. ✓ Proof The first family is constant. The last two displayed values are unequal. □

Theorem 21.2.2.61 (Associativity constraint on the structure coefficients). Lean ✓ Stmt At every positive length L at which the operators are linearly independent, the structure coefficients of the same-length product law satisfy

$$\sum_{\delta} c_{\alpha, \beta, \delta}^{(L)} c_{\delta, \gamma, \epsilon}^{(L)} = \sum_{\delta} c_{\beta, \gamma, \delta}^{(L)} c_{\alpha, \delta, \epsilon}^{(L)}.$$

These are the restrictions that the associativity of the multiplication imposes on the structure coefficients, noted in [CPGSV16, Section 4.5].

Proof. ✓ **Proof** Applying the same-length product law twice in each order gives

$$\begin{aligned} (O_L(M_\alpha)O_L(M_\beta))O_L(M_\gamma) &= \sum_{\varepsilon'} \left(\sum_{\delta} c_{\alpha,\beta,\delta}^{(L)} c_{\delta,\gamma,\varepsilon'}^{(L)} \right) O_L(M_{\varepsilon'}), \\ O_L(M_\alpha)(O_L(M_\beta)O_L(M_\gamma)) &= \sum_{\varepsilon'} \left(\sum_{\delta} c_{\beta,\gamma,\delta}^{(L)} c_{\alpha,\delta,\varepsilon'}^{(L)} \right) O_L(M_{\varepsilon'}). \end{aligned}$$

Associativity of multiplication equates the two left-hand sides, so the right-hand sides agree; linear independence of the operators then forces the corresponding coefficient sums to agree for each ε' . $\square$

Theorem 21.2.2.62 (Associativity constraint in trace-power form). Lean ✓ **Stmt** *For a coefficient system satisfying the same-length product law for a BNT-label operator family and in trace-power form, at every positive length L at which the labelled operators are linearly independent, the associativity constraint is a restriction on the family of diagonal matrices:*

$$\sum_{\delta} \text{tr}(\chi_{\alpha,\beta,\delta}^L) \text{tr}(\chi_{\delta,\gamma,\epsilon}^L) = \sum_{\delta} \text{tr}(\chi_{\beta,\gamma,\delta}^L) \text{tr}(\chi_{\alpha,\delta,\epsilon}^L).$$

In [CPGSV16, Section 4.5] these restrictions, combined with the isometry statement, lead to a pentagon-like equation for the fusion isometries whose solutions come from unitary fusion categories.

Proof. ✓ **Proof** Substitute the trace-power form of the coefficients into the associativity constraint. $\square$

Theorem 21.2.2.63 (The dimer RFP has length-dependent displayed BNT coefficients). Lean ✓ **Stmt** *There is an MPDO M in literal CPSV canonical form that is a renormalization fixed point and has a tensor-attached BNT presentation whose structure coefficients depend on the positive chain length. One may take $M = R$ and the one-label presentation above, for which*

$$c_{0,0,0}^{(L)} = 1 + \left(\frac{7}{25} \right)^L.$$

This gives a normalization-free variant of the question after Theorem 4.14 for the displayed vertical presentation. The statement does not assert that length dependence is invariant under a change of presentation. It also does not supply the source's additional canonical normalization: every weight has modulus at most one and at least one has modulus one.

Proof. ✓ **Proof** The preceding results show that R is an MPDO in literal CPSV canonical form, construct its fixed-point maps and its tensor-attached one-label BNT clause, and identify the clause's coefficients with the displayed trace-power family. Its values at lengths one and two differ, so that family is not length independent. $\square$

Definition 21.2.2.64 (BNT algebra clause). Lean ✓ **Stmt** Fix a coefficient family $c_{\alpha,\beta,\gamma}^{(L)}$, BNT-label operators $O_L(M_\alpha)$, and trace scalars $m_\alpha = \text{tr}(\mu_\alpha)$. They satisfy the *BNT algebra clause* when there is a length-independent family of positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$ such that, for every $L > 0$, $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L)$, the same-length product law holds,

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} c_{\alpha,\beta,\gamma}^{(L)} O_L(M_\gamma),$$

and the trace scalars satisfy

$$m_\gamma = \sum_{\alpha, \beta} c_{\alpha, \beta, \gamma}^{(1)} m_\alpha m_\beta.$$

This is the algebra statement in [CPGSV16, Theorem 4.14(ii), labelled IV.13 in the source, lines 972–985], before comparison with a chosen blocked basis.

Definition 21.2.2.65 (BNT algebra clause of a vertical canonical decomposition). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Choose a vertical canonical decomposition

$$U \widetilde{M} U^\dagger = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha},$$

$$\mu_{\alpha} = \text{diag}(\omega_{\alpha,0}, \dots, \omega_{\alpha, r_{\alpha}-1}).$$

Separating the two factors of the doubled physical index of M_{α} gives an MPO tensor $\widehat{M}_{\alpha}$. Define

$$O_L(M_{\alpha}) = \text{mpo}(\widehat{M}_{\alpha}, L),$$

$$m_{\alpha} = \sum_q \omega_{\alpha,q} = \text{tr}(\mu_{\alpha}).$$

The chosen decomposition satisfies the tensor-attached BNT algebra clause when the tensors M_{α} form a basis of normal tensors and the BNT algebra clause holds for precisely these operators and these trace scalars. This is the data in [CPGSV16, Theorem 4.14(ii), lines 972–993, and Appendix C.4, lines 2046–2064]; it does not include a comparison with a two-site vertical canonical decomposition.

Theorem 21.2.2.66 (Terminal sector positivity from the tensor-attached algebra clause). [Lean](#) [✓ Stmt](#) Let M be a matrix product density operator with physical dimension d and bond dimension D , equipped with a tensor-attached BNT algebra clause, and let M_{γ} be any retained normal tensor. Separating its doubled physical index gives two physical legs of dimension D . Closing these legs gives the bond matrix

$$T_{\gamma} = \sum_{a=0}^{D-1} M_{\gamma}^{(a,a)}. \quad (21.5)$$

Then $T_{\gamma} \succeq 0$. This is a necessary consequence of the tensor-attached clause and positivity of the one-site operator; it does not require fusion maps or a renormalization fixed-point hypothesis.

Proof. [✓ Proof](#) Apply the retained physical coisometry to the positive one-site operator. For a positive diagonal coordinate q of the multiplicity matrix μ_{γ} , the corresponding principal submatrix is $(\mu_{\gamma})_{q,q} T_{\gamma}$. Principal submatrices remain positive semidefinite, and $(\mu_{\gamma})_{q,q} > 0$; division by this scalar proves $T_{\gamma} \succeq 0$. $\square$

Lemma 21.2.2.67 (Terminal transfers of a nonunitary similarity pair). [Lean](#) [Lean](#) [✓ Stmt](#) Let A have letters $\mathbb{1}, E_{00}, E_{01}, E_{10}$, let $X = \text{diag}(2, 1)$, and set $B^v = X A^v X^{-1}$. After separating the doubled physical index, their terminal transfer matrices are

$$T_A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$T_B = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Proof. ✓ Proof Sum the two diagonal physical letters $A^{(0,0)} + A^{(1,1)}$ and then use $B^v = XA^vX^{-1}$ with $X = \text{diag}(2, 1)$. □

Lemma 21.2.2.68 (The nonunitary similarity pair violates terminal positivity). Lean Lean
✓ Stmt Neither T_A nor T_B in Lemma 21.2.2.67 is positive semidefinite.

Proof. ✓ Proof A positive semidefinite matrix is Hermitian. In both matrices the $(0, 1)$ entry is zero, whereas the $(1, 0)$ entry is respectively 1 and $1/2$. Hence neither matrix is Hermitian. □

Theorem 21.2.2.69 (The nonunitary similarity pair is excluded from retained positive sectors).
Lean Lean ✓ Stmt Neither A nor B can occur, up to the bond-dimension identification, as a retained bond-two sector of a tensor-attached BNT algebra clause for a matrix product density operator.

Proof. ✓ Proof Any such retained sector has positive-semidefinite terminal transfer by Theorem 21.2.2.66. Lemma 21.2.2.68 gives the contradiction for both tensors. □

Lemma 21.2.2.70 (The nonunitary similarity pair has no positive isometric realization). Lean
✓ Stmt Let A have letters $\mathbb{1}, E_{00}, E_{01}, E_{10}$, let $X = \text{diag}(2, 1)$, and set $B^i = XA^iX^{-1}$. There are no $c > 0$ and $V \in M_2(\mathbb{C})$ satisfying $V^\dagger V = \mathbb{1}$ and $V^\dagger B^i V = cA^i$ for every letter i .

Proof. ✓ Proof The identity letter forces $c = 1$. The matrix $S = V^\dagger X$ then commutes with all four letters of A , which span $M_2(\mathbb{C})$, and hence S is scalar. Together with $V^\dagger V = \mathbb{1}$, this would make $X^\dagger X = \text{diag}(4, 1)$ scalar, a contradiction. □

Definition 21.2.2.71 (Virtual per-block linear extension). ✓ Stmt For injective families A_k and B_k with the same closed matrix product vectors, the per-block linear extension is the unique linear map $T_k : M_{D_k}(\mathbb{C}) \rightarrow M_{D_k}(\mathbb{C})$ satisfying $T_k(A_k^i) = B_k^i$ for every letter i .

Theorem 21.2.2.72 (Multiplicativity of the virtual per-block extension). ✓ Stmt The per-block extension satisfies $T_k(MN) = T_k(M)T_k(N)$ for all $M, N \in M_{D_k}(\mathbb{C})$.

Proof. ✓ Proof The letters of A_k span $M_{D_k}(\mathbb{C})$, and equality of the closed matrix product vectors gives the identity on products of spanning letters. □

Theorem 21.2.2.73 (Bijectivity of the virtual per-block extension). ✓ Stmt The per-block extension T_k is bijective.

Proof. ✓ Proof Multiplicativity makes the kernel an ideal of the simple matrix algebra. The map is nonzero: otherwise every letter of B_k would vanish, contradicting the nonvanishing trace pairing for the injective family. Thus the kernel is zero, and finite dimensionality gives surjectivity. □

Theorem 21.2.2.74 (Positive virtual per-block extensions are unitarily implemented). ✓ Stmt
 If T_k is positive, then there is a unitary matrix U_k such that $T_k(M) = U_k M U_k^\dagger$ for every $M \in M_{D_k}(\mathbb{C})$.

Proof. ✓ Proof The per-block map is a bijective algebra endomorphism. Positivity makes it preserve adjoints, so unitary Skolem–Noether implements it by conjugation with a unitary matrix. □

Theorem 21.2.2.75 (Closed matrix product vectors do not force a positive virtual extension). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) [Discussion: #5284](#) *Regard the tensors A and B of Lemma 21.2.2.70 as one-block families. They are injective and have equal closed matrix product vectors at every length, but their per-block linear extension is not positive. This concerns the virtual extension determined by the closed vectors; it does not negate a comparison of positive physical sector realizations, nor does it decide whether the tensor-attached marked comparison admits a positive-coefficient identity-dressed realization.*

Proof. [✓ Proof](#) If the per-block extension were positive, its unitary-implementation theorem would give a unitary U satisfying $B^i = UA^iU^\dagger$ for every letter i . Hence $U^\dagger B^i U = A^i$, giving the forbidden realization with $c = 1$ in Lemma 21.2.2.70. $\square$

Definition 21.2.2.76 (Bond-two singleton tensor and companion Bell projector). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $I = \{0, 1\}$ and let $E_{a,b}$ be the matrix units of $M_2(\mathbb{C})$. Define a tensor A by

$$A^{((a,b),(c,d))} = \begin{cases} E_{a,b}, & (a,b) = (c,d), \\ 0, & (a,b) \neq (c,d). \end{cases}$$

Its terminal matrix is

$$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Independently, define the companion Bell projector by

$$|\Phi\rangle = |0,0\rangle + |1,1\rangle,$$

and

$$P = |\Phi\rangle\langle\Phi|.$$

The tensor A is the retained vertical tensor of the ambient model in Definition 21.2.3.1. Theorem 21.2.3.4 identifies P with its two-site operator after the canonical reindexing. For $X = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, set $B^v = XA^vX^{-1}$ and define

$$J_X = XJX^{-1}.$$

Separately, define

$$P_X = (X \otimes X)P(X^{-1} \otimes X^{-1}).$$

Lemma 21.2.2.77 (Terminal-positive deformation and Bell comparator). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *The tensor A is injective and normal. The matrix X is invertible but not unitary, and A and B generate the same matrix product vectors at every positive length. Nevertheless,*

$$\sum_a A^{(a,a)} = \sum_a B^{(a,a)} = J \succeq 0, \\ J_X = J,$$

The matrix P is positive semidefinite while P_X is not Hermitian. Theorem 21.2.3.4 identifies P with the reindexed two-site operator of the ambient base model. It does not identify P_X with a two-site operator of a deformed matrix product density operator.

Proof. ✓ Proof The four diagonal physical letters of A are the matrix units, so A is injective. Similarity preserves every cyclic contraction. Direct multiplication gives $XJ = JX$, hence $J_X = J \succeq 0$, whereas $X^\dagger X \neq \mathbb{1}$. In the separate Bell calculation, the $(00, 01)$ and $(01, 00)$ entries of P_X are unequal, so P_X is not Hermitian. $\square$

Theorem 21.2.2.78 (Terminal and companion Bell Hermiticity force a scalar Gram matrix).

Lean Lean Lean Lean ✓ Stmt Let $Y \in \text{GL}(2, \mathbb{C})$. If the terminal-matrix similarity YJY^{-1} and the independent Bell-projector similarity $(Y \otimes Y)P(Y^{-1} \otimes Y^{-1})$ are Hermitian, then there is a real number $\omega > 0$ such that

$$Y^\dagger Y = \omega \mathbb{1}.$$

The comparisons in [CPGSV16, Appendix C.4, lines 2048–2057] and [CPGSV16, Proposition 4.13] motivate this finite-dimensional criterion; neither source states this model-specific theorem. Theorem 21.2.3.4 supplies the exact base-model interpretation of P , but the premise does not construct a corresponding deformed tensor-attached model.

Proof. ✓ Proof Put $G = Y^\dagger Y$, which is positive definite. Hermiticity of the terminal-matrix similarity gives $GJ = JG$. Hermiticity of the companion Bell-projector similarity gives $G_{00}G_{01} + G_{10}G_{11} = 0$. The commutator identity implies $G_{01} = G_{10}$ and $G_{00} = G_{11}$. Since $G_{00} > 0$, the remaining equation forces $G_{01} = G_{10} = 0$, and therefore $G = G_{00}\mathbb{1}$. $\square$

21.2.3 An all-length bond-two singleton base model

Definition 21.2.3.1 (Ambient singleton base tensor). Lean ✓ Stmt Let $I = \{0, 1\}$ and identify $I \times I$ with four bond coordinates. Define the bond-four MPO tensor M by

$$M^{i,j} = E_{(i,j),(i,j)}.$$

This test model is motivated by [CPGSV16, Appendix C.4, lines 2048–2057]; it is not a tensor stated there.

Theorem 21.2.3.2 (Exact vertical singleton identity). Lean ✓ Stmt The vertical tensor of M is exactly the matrix-unit singleton A from Definition 21.2.2.76.

Proof. ✓ Proof A doubled bond coordinate (a, b) selects $E_{a,b}$ when the two physical pair labels agree, and gives zero otherwise. $\square$

Theorem 21.2.3.3 (Positive-length ambient GHZ formula). Lean Lean Lean Lean Lean ✓ Stmt For every $N \geq 1$, let $|\text{GHZ}_N\rangle = |0\rangle^{\otimes N} + |1\rangle^{\otimes N}$. Then

$$\rho_N(M) = |\text{GHZ}_N\rangle\langle\text{GHZ}_N|.$$

Hence M is an MPDO tensor. The displayed operator is an unnormalized rank-one positive operator; it is not asserted to be an orthogonal projection. At $N = 0$, the closed MPO word instead gives $4\mathbb{1}$, whereas the empty GHZ outer product is $\mathbb{1}$; hence the positive-length restriction is sharp.

Proof. ✓ Proof The cyclic equality weights force the ket and bra words to be independently constant. The result is an outer product and is therefore positive semidefinite. $\square$

Theorem 21.2.3.4 (Exact two-site Bell reindexing). [Lean](#) [✓ Stmt](#) Let ε identify a binary word of length two with its ordered pair of letters, and put $|\Phi\rangle = |0,0\rangle + |1,1\rangle$. Then

$$\text{reindex}_{\varepsilon,\varepsilon}(\rho_2(M)) = |\Phi\rangle\langle\Phi|.$$

Thus the companion Bell matrix is exactly the two-site ambient operator in ordered-pair coordinates.

Proof. [✓ Proof](#) Specialize Theorem 21.2.3.3 to $N = 2$ and evaluate the coordinate identification. $\square$

Definition 21.2.3.5 (Physical-similarity deformation of the ambient model). [Lean](#) [✓ Stmt](#) For $X \in \text{GL}(2, \mathbb{C})$, define M_X by applying X to the ket index and X^{-1} to the bra index of M :

$$M_X^{i,j} = \sum_{a,b \in I} X_{i,a} (X^{-1})_{b,j} M^{a,b}.$$

This family is motivated by the sector comparison in [CPGSV16, Appendix C.4, lines 2048–2057], with the physical-sector argument of [CPGSV16, Proposition 4.13, lines 1898–1921] as a template. Neither passage states the deformation or the results below.

Theorem 21.2.3.6 (Vertical tensor under physical similarity). [Lean](#) [✓ Stmt](#) For every vertical letter v ,

$$\widetilde{M}_X^v = X A^v X^{-1}.$$

Proof. [✓ Proof](#) Left and right multiplication on the physical indices become matrix multiplication on the vertically viewed tensor. Now use Theorem 21.2.3.2. $\square$

Theorem 21.2.3.7 (Exact one-site physical-similarity identity). [Lean](#) [✓ Stmt](#) After identifying one-site words with their single letters in I , the one-site operator of M_X is

$$\text{reindex}(\rho_1(M_X)) = X J X^{-1}.$$

Proof. [✓ Proof](#) Expand the one-site contraction. The four diagonal physical letters sum to J , and the two physical multiplications give conjugation by X . $\square$

Theorem 21.2.3.8 (Exact two-site physical-similarity identity). [Lean](#) [✓ Stmt](#) Under the ordered-pair identification ε from Theorem 21.2.3.4,

$$\text{reindex}_{\varepsilon,\varepsilon}(\rho_2(M_X)) = (X \otimes X) P (X^{-1} \otimes X^{-1}).$$

Proof. [✓ Proof](#) Evaluate the two-site contraction in ordered-pair coordinates. Each ket index contributes X , each bra index contributes X^{-1} , and the undeformed contraction is P . $\square$

Theorem 21.2.3.9 (MPDO positivity forces a scalar physical Gram matrix). [Lean](#) [✓ Stmt](#) If M_X is a matrix product density operator, then there is a real number $\omega > 0$ such that

$$X^\dagger X = \omega \mathbb{1}.$$

Thus this natural physical-similarity family is excluded whenever $X^\dagger X$ is nonscalar.

Proof. [✓ Proof](#) Positivity of the one- and two-site operators implies Hermiticity. Apply the two exact reindexing identities and then Theorem 21.2.2.78. $\square$

These data reconstruct A and satisfy the positive- χ , product, and idempotent algebra conditions. At $L = 0$, however, $\widehat{Q}_0 = 2\mathbb{1}$, so $\widehat{Q}_0^2 = 4\mathbb{1} \neq 2\mathbb{1}$; the geometric algebra identity therefore also requires positive length.

Proof. ✓ Proof The identity $s^2 = 1/2$ gives left-canonical normalization. Scaling every local tensor contributes s^L to a length- L contraction, and s^{-1} cancels the one-site scaling in the weighted reconstruction. $\square$

Theorem 21.2.3.15 (Canonical retained-coordinate coisometry). Lean Lean Lean Lean Lean Lean ✓ Stmt The unique label and copy identify the retained coordinate with I . The corresponding permutation matrix W satisfies, for every $Z \in M_2(\mathbb{C})$,

$$WW^\dagger = \mathbb{1}.$$

Conjugation by W gives

$$WZW^\dagger = \text{reindex}(Z),$$

and the adjoint reconstructs the original matrix:

$$W^\dagger \text{reindex}(Z)W = Z.$$

Proof. ✓ Proof Each row and column of W contains one nonzero entry, equal to one. $\square$

Theorem 21.2.3.16 (Tensor-attached singleton BNT algebra clause). Lean Lean ✓ Stmt The ambient tensor M has a tensor-attached BNT algebra clause with one label, one copy, bond dimension two, normalized representative $\widehat{A}$, multiplicity weight s^{-1} , coefficient s^L , and retained-coordinate coisometry W . Its forward and reconstruction identities hold exactly for every vertical letter.

Definition 21.2.3.17 (BNT multiplicity-spectrum comparison). Lean ✓ Stmt Fix the one-site diagonal multiplicity matrices μ_α and the two-site diagonal multiplicity matrices ν_γ in the respective vertical canonical decompositions. A multiplicity-spectrum comparison records $\text{tr}(\mu_\alpha) = m_\alpha$ and, for every γ , equality with multiplicity between the entries of ν_γ and the products $\mu_{\alpha,i}\mu_{\beta,j}\chi_{\alpha,\beta,\gamma,k}$. This is the conclusion of the vertical BNT comparison and positive power-sum argument in [CPGSV16, Appendix C.4, lines 2048–2058]. The definition does not require this comparison to follow from a BNT algebra clause; constructing it from the MPDO tensor is a separate obligation.

Definition 21.2.3.18 (Source-derived two-site multiplicity spectrum). Lean ✓ Stmt For a tensor-attached BNT algebra clause, a source-derived two-site multiplicity spectrum consists of a vertical canonical decomposition

$$\widetilde{M^{(2)}} = \bigoplus_{\delta} \nu_{\delta} \otimes A_{\delta},$$

a bijection σ from the one-site sectors to its sectors such that M_γ and $A_{\sigma(\gamma)}$ generate the same matrix product vectors at every positive length, and, for every γ , the multiset equality

$$\{\nu_{\sigma(\gamma),r} : r\} = \{\mu_{\alpha,i}\mu_{\beta,j}\chi_{\alpha,\beta,\gamma,k} : \alpha, \beta, i, j, k\}.$$

This is the decomposition and comparison retained in [CPGSV16, Appendix C.4, lines 2046–2058].

Definition 21.2.3.19 (Exact invertible gauges for the source-derived two-site sectors). [Lean](#)

[✓ Stmt](#) The invertible-gauge sub-result for a source-derived two-site multiplicity spectrum additionally records, for every one-site sector γ , equality of bond dimensions $d_\gamma = d_{\sigma(\gamma)}$ and an invertible matrix $Z_\gamma \in \text{GL}(d_\gamma, \mathbb{C})$ such that, for every physical index i ,

$$A_{\sigma(\gamma)}^i = Z_\gamma M_\gamma^i Z_\gamma^{-1}.$$

Thus the matched normal tensors have no residual scalar phase. This is the exact invertible conjugacy retained from [CPGSV16, Appendix C.4, lines 2053–2057]. It does not include the further conclusion on line 2057 that Z_γ may be chosen unitary.

Theorem 21.2.3.20 (The two-site multiplicity spectrum from the BNT algebra clause). [Lean](#)

[Lean](#) [✓ Stmt](#) Let M be an MPDO whose doubled-index MPS tensor is in literal CPSV canonical form, equipped with a tensor-attached BNT algebra clause $\widetilde{M} = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha$. There is a vertical canonical decomposition of the two-site blocking,

$$\widetilde{M}^{(2)} = \bigoplus_\delta \nu_\delta \otimes A_\delta.$$

Its normal-tensor sectors are in bijection, via σ , with the sectors of the full-support one-site product expansion such that M_γ and $A_{\sigma(\gamma)}$ generate the same matrix product vectors at every positive length and, for every γ ,

$$\{\nu_{\sigma(\gamma),r} : r\} = \{\mu_{\alpha,i} \mu_{\beta,j} \chi_{\alpha,\beta,\gamma,k} : \alpha, \beta, i, j, k\} \quad (21.6)$$

as multisets. Thus this two-site decomposition and relabelling realize the multiplicity-spectrum comparison of Appendix C.4, lines 2046–2058 of [CPGSV16]. Moreover, the paired bond dimensions agree and there are invertible matrices Z_γ satisfying

$$A_{\sigma(\gamma)}^i = Z_\gamma M_\gamma^i Z_\gamma^{-1} \quad (21.7)$$

exactly. This is only the invertible-gauge sub-result of lines 2053–2057; no unitary choice of Z_γ follows from this statement.

Proof. [✓ Proof](#) Multiplying the two one-site decompositions and using the BNT algebra law expresses the two-site closed chain in the basis M_γ , with coefficient

$$\sum_{\alpha,\beta,i,j,k} (\mu_{\alpha,i} \mu_{\beta,j} \chi_{\alpha,\beta,\gamma,k})^L.$$

All these coefficients are positive. Hence every basis sector occurs, and the characterization of bases of normal tensors matches these sectors bijectively with those of the two-site vertical canonical decomposition. Normalization makes each matching scalar have modulus one, while positivity of two consecutive coefficients makes it positive; it is therefore one. The normal-tensor matching theorem also supplies equality of the paired bond dimensions and an invertible gauge; after the scalar is one, its gauge-phase relation is precisely the exact conjugacy in (21.7). Eventual linear independence of the basis vectors gives the corresponding equality of power sums. The finite positive power-sum identity then gives the multiset equality (21.6). $\square$

Theorem 21.2.3.21 (Gauge-dressed physical corner of a matched two-site sector). [Lean](#) [Lean](#)

[✓ Stmt](#) Suppose that the one-site and two-site sectors are related by the exact invertible gauges above. Let γ be a one-site sector matched with a two-site sector by the bijection σ , and identify

their bond dimensions. There are an isometry V_γ from the sector bond space into the blocked physical-index space and a scalar $c_\gamma > 0$ such that, for every vertical letter v ,

$$V_\gamma^\dagger \widetilde{M^{(2)}}^v V_\gamma = c_\gamma Z_\gamma M_\gamma^v Z_\gamma^{-1}. \quad (21.8)$$

If, in addition, M generates matrix product density operators, then for every pair of open bond indices and every pair of two-site tail words, the marked-chain coefficient of

$$(Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1}$$

equals the corresponding reflected-adjoint marked-chain coefficient in the adjoint two-site blocking. This is one of the two marked comparisons in Figures 7–8 of [CPGSV16, Proposition 4.13, lines 1909–1919], applied to the sector matching of Appendix C.4, lines 2048–2057.

Scope restriction (gauge-dressed corner): This theorem by itself constructs only the physical corner in (21.8). It does not construct a same-sided identity-dressed physical corner for M_γ and does not prove the common-target Gram identity

$$(Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1} = M_\gamma^v.$$

This restriction is documented in [con26n]. The mixed-prefix comparison below proves the Gram identity without constructing a same-sided raw corner for the two-site blocking.

Proof. ✓ Proof Choose the first positive-weight copy of the matched two-site sector. Its distinguished inclusion is isometric and its compression selects the corresponding weighted normal tensor. Substituting the exact sector conjugacy gives (21.8), with the selected weight as c_γ . The reflected marked-chain identity for this corner, conjugated on its open indices by $Z_\gamma^\dagger$ and $(Z_\gamma^{-1})^\dagger$, gives the stated equality of marked-chain coefficients; the gauge factors cancel on the reflected side. □

Theorem 21.2.3.22 (Raw physical corner of a matched one-site sector). Lean ✓ Stmt In the setting above, each sector γ admits an isometry $W_\gamma: \mathbb{C}^{d_\gamma} \rightarrow \mathbb{C}^d$ and a scalar $a_\gamma > 0$ such that

$$W_\gamma^\dagger \widetilde{M}^v W_\gamma = a_\gamma M_\gamma^v \quad (21.9)$$

for every vertical letter v . Here the bond space of M_γ is identified with that of its matched two-site sector. This is the distinguished positive-weight copy in the one-site vertical decomposition of [CPGSV16, Proposition 4.13, statement lines 943–951, and Appendix C.4, lines 2046–2057].

Proof. ✓ Proof Select the first copy of M_γ in the one-site vertical direct sum. Its inclusion is isometric. Compressing the reconstruction identity by this inclusion gives (21.9), with a_γ equal to the positive multiplicity weight of the selected copy. Transport along the equality of the paired bond dimensions. □

Theorem 21.2.3.23 (Reflected marked chains with a two-site initial block). Lean ✓ Stmt Let M generate matrix product density operators, let A be a vertical tensor, and suppose that an isometry V selects from the two-site blocking a positive multiple of an invertible conjugate of A :

$$V^\dagger \widetilde{M^{(2)}}^v V = b X A^v X^{-1}, \quad b > 0. \quad (21.10)$$

For every pair of open bond indices and every pair of equal-length words in the original physical alphabet, the marked-chain coefficient of

$$(X^\dagger X) A^v (X^\dagger X)^{-1}$$

followed by the corresponding unblocked tail of M equals the reflected-adjoint marked-chain coefficient of A followed by the reversed tail of $M^\dagger$. Thus the first two sites are compressed jointly, while all later sites remain unblocked.

Local fix (mixed one-site/two-site prefix): This is the common unblocked-tail comparison needed when [CPGSV16, Proposition 4.13, lines 1909–1919] is invoked in [CPGSV16, Appendix C.4, lines 2048–2057]. It is recorded in [con26n].

Proof. ✓ Proof Compress the first two physical sites of the density operator of length $N + 2$ by V . Hermiticity exchanges the open bond indices and the two physical words. Taking adjoints reverses only the remaining N one-site factors. Finally conjugate the open indices by $X^\dagger$ and $(X^{-1})^\dagger$. The left side becomes the Gram dressing, while the gauge factors cancel on the reflected side. $\square$

Theorem 21.2.3.24 (Oblique physical-letter realization of a matched two-site sector). Lean Lean Lean ✓ Stmt In the setting of the preceding theorem, choose V_γ and $c_\gamma > 0$ as in (21.8), and set

$$L_\gamma = V_\gamma (Z_\gamma^{-1})^\dagger, \quad R_\gamma = V_\gamma Z_\gamma. \quad (21.11)$$

Then, for every vertical letter v ,

$$L_\gamma^\dagger \widetilde{M^{(2)}}^v R_\gamma = c_\gamma M_\gamma^v, \quad (21.12)$$

$$R_\gamma^\dagger \widetilde{M^{(2)}}^v L_\gamma = c_\gamma (Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1}. \quad (21.13)$$

Write $\mathcal{R}(M_\gamma)^{rs}$ for the horizontal slice $v \mapsto (M_\gamma^v)_{rs}$. For open bond indices r, s and blocked physical indices i, j , define

$$f_{(r,s),(i,j)}^\gamma = c_\gamma^{-1} \overline{(L_\gamma)_{ir}} (R_\gamma)_{js}. \quad (21.14)$$

These coefficients realize every identity-dressed horizontal slice in the physical-letter span of the two-site blocking:

$$\sum_{i,j} f_{(r,s),(i,j)}^\gamma (M^{[2]})^{ij} = \mathcal{R}(M_\gamma)^{rs}. \quad (21.15)$$

This supplies the physical-letter span needed for the direct comparison of the two vertical forms at [CPGSV16, Appendix C.4, lines 2048–2057].

Scope restriction (oblique realization): In general $L_\gamma \neq R_\gamma$. Thus (21.12) is not a positive same-sided corner. At the marked-chain level, adjoint reflection exchanges the two maps together with the bond-pair and tail reversal. The reflected orientation is therefore governed by the Gram-dressed compression in (21.13), rather than by a second copy of the raw tensor. The theorem supplies the physical-letter realization needed for the identity-dressed mark, but it proves neither a common reflected target nor the Gram identity

$$(Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1} = M_\gamma^v.$$

This is the obstruction in the direct two-site argument. The mixed-prefix theorem below instead compares the one-site raw corner with the two-site gauge corner while retaining the same one-site tail.

Proof. ✓ Proof Taking adjoints in the left factor of (21.11) and substituting (21.8) gives

$$L_\gamma^\dagger \widetilde{M^{(2)}}^v R_\gamma = Z_\gamma^{-1} (V_\gamma^\dagger \widetilde{M^{(2)}}^v V_\gamma) Z_\gamma = c_\gamma M_\gamma^v.$$

Reversing the two maps instead gives (21.13). Finally, taking the (r, s) matrix entry of a general two-sided compression yields

$$[L^\dagger \widetilde{M^{(2)}}^v R]_{rs} = \sum_{i,j} \overline{L_{ir}} R_{js} (M^{[2]})_v^{ij}.$$

Divide by c_γ and use (21.12) to obtain (21.15). □

Theorem 21.2.3.25 (Conditional Gram identity from the positive-tail reflected target). Lean
Lean Lean Lean Lean Lean ✓ Stmt *In the setting of the preceding theorem, suppose in addition that the doubled-index tensor of M is in literal CPSV canonical form. Assume that, for every positive tail length, the identity-dressed marked chain equals the reflected-adjoint marked chain of M_γ in the adjoint two-site blocking. No equality for the empty tail is assumed. Then, for every vertical letter v ,*

$$(Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1} = M_\gamma^v \quad (21.16)$$

Consequently, there is a real number $\omega_\gamma > 0$ such that

$$Z_\gamma^\dagger Z_\gamma = \omega_\gamma \mathbb{1}. \quad (21.17)$$

Scope restriction (conditional reflected target): *This theorem assumes that the identity-dressed physical-letter family has the same reflected-adjoint target as the gauge-dressed marked chains at every positive tail length. The oblique construction alone cannot provide this identity: adjoint reflection exchanges L_γ and R_γ and is governed by the Gram-dressed compression (21.13). The mixed-prefix comparison below derives the target under the standing canonical-form and positivity assumptions. This distinction is recorded in [con26n].*

Proof. ✓ Proof The gauge-dressed corner gives a physical-letter coefficient family for the gauge-dressed mark and identifies its marked chains with the reflected target. The oblique theorem supplies a physical-letter coefficient family for the identity-dressed mark; the remaining hypothesis identifies its marked chains with the same target. Their marked traces therefore agree for every positive-length tail word. Literal CPSV canonical form passes to the two-site blocking, and the positive-tail form of the original-coordinate marked separation theorem, corresponding to [CPGSV16, Appendix C.3, lines 1835–1858], identifies the two open-bond matrices. This gives (21.16). Normality of M_γ then makes its commuting Gram matrix a positive real scalar multiple of the identity, proving (21.17). □

Theorem 21.2.3.26 (Normalization of conjugation by a scalar Gram matrix). Lean ✓ Stmt
If an invertible matrix Z satisfies $Z^\dagger Z = \omega \mathbb{1}$ with $\omega > 0$, then conjugation by $\omega^{-1/2} Z$ equals conjugation by Z .

Proof. ✓ Proof The Gram identity identifies Z^{-1} with $\omega^{-1} Z^\dagger$; the scalar factors cancel in the conjugation. □

Theorem 21.2.3.27 (Conditional unitary conjugacy from the positive-tail reflected target).
Lean Lean ✓ Stmt *Under the hypotheses of the preceding conditional theorem, define*

$$U_\gamma = \omega_\gamma^{-1/2} Z_\gamma.$$

Then U_γ is unitary and, for every physical index i ,

$$A_{\sigma(\gamma)}^i = U_\gamma M_\gamma^i U_\gamma^\dagger. \quad (21.18)$$

More generally, the same conclusion follows from any exact invertible sector gauge whose Gram matrix is a positive scalar multiple of the identity. This is the inverse-square-root normalization used in [CPGSV16, Proposition 4.13, lines 1903–1908], applied at Appendix C.4, lines 2053–2057.

Scope restriction (conditional reflected target): The physical-letter coefficients are supplied by Theorem 21.2.3.24, while their common positive-tail reflected target is an assumption of this theorem. Appendix C.4, lines 2048–2057, does not itself spell out the star-compatible comparison for the raw representative. The mixed-prefix theorem below supplies it under the standing assumptions, as recorded in [con26n].

Proof. ✓ Proof The preceding theorem gives $Z_\gamma^\dagger Z_\gamma = \omega_\gamma \mathbb{1}$ with $\omega_\gamma > 0$. Therefore $U_\gamma^\dagger U_\gamma = \mathbb{1}$. The two scalar factors introduced by U_γ and $U_\gamma^\dagger$ cancel, using the Gram identity to identify the adjoint factor with Z_γ^{-1} . Thus conjugation by U_γ equals conjugation by Z_γ , and the exact sector conjugacy gives (21.18). $\square$

Theorem 21.2.3.28 (Mixed-prefix Gram identity and unitary sector conjugacy). Lean Lean Lean ✓ Stmt Let M be an MPDO whose doubled-index tensor is in literal CPSV canonical form, and let the one-site and two-site sectors be paired by the exact gauges Z_γ above. Then, for every sector γ and every vertical letter v ,

$$(Z_\gamma^\dagger Z_\gamma) M_\gamma^v (Z_\gamma^\dagger Z_\gamma)^{-1} = M_\gamma^v. \quad (21.19)$$

Consequently there is a real number $\omega_\gamma > 0$ such that

$$Z_\gamma^\dagger Z_\gamma = \omega_\gamma \mathbb{1}, \quad (21.20)$$

and there is a unitary U_γ satisfying

$$A_{\sigma(\gamma)}^v = U_\gamma M_\gamma^v U_\gamma^\dagger. \quad (21.21)$$

The identity-dressed marked chains therefore also have the positive-tail reflected target used in the conditional statements above.

Local fix (mixed one-site/two-site prefix): The proof compares a one-site raw corner with a two-site gauge corner while retaining the same unblocked tail. This supplies the common comparison omitted when [CPGSV16, Proposition 4.13, lines 1909–1919] is invoked in [CPGSV16, Appendix C.4, lines 2048–2057]. The distinction from the unsuccessful direct two-site comparison is recorded in [con26n].

Proof. ✓ Proof Put $G_\gamma = Z_\gamma^\dagger Z_\gamma$. The two-site initial-block identity applied to the gauge corner identifies the marked chains of $G_\gamma M_\gamma G_\gamma^{-1}$ with the reflected-adjoint chains of M_γ , followed by an arbitrary unblocked tail of $M^\dagger$. The ordinary one-site reflected identity applied to the raw corner gives exactly the same target for M_γ .

Both marked tensors are linear combinations of the one-site physical letters. Indeed, the raw mark is obtained from the pair (W_γ, W_γ) , whereas the Gram-dressed mark is obtained from

$$(W_\gamma G_\gamma^\dagger, W_\gamma G_\gamma^{-1}).$$

Their marked traces agree against every positive-length tail. The original-coordinate form of Lemma L therefore identifies the two marked tensors and proves (21.19). Normality of M_γ gives (21.20), and normalizing Z_γ by $\omega_\gamma^{-1/2}$ gives the unitary in (21.21). $\square$

Lemma 21.2.3.29 (Idempotence for the canonical chi coefficients). [Lean](#) [✓ Stmt](#) *The length-one idempotent law of a BNT algebra clause also holds for the coefficient family determined by its diagonal chi matrices:*

$$m_\gamma = \sum_{\alpha, \beta} \text{tr}(\chi_{\alpha, \beta, \gamma}) m_\alpha m_\beta.$$

Proof. [✓ Proof](#) The chi representation at length one gives $c_{\alpha, \beta, \gamma}^{(1)} = \text{tr}(\chi_{\alpha, \beta, \gamma})$. Substitution into the idempotent law $m_\gamma = \sum_{\alpha, \beta} c_{\alpha, \beta, \gamma}^{(1)} m_\alpha m_\beta$ gives the stated identity. $\square$

Definition 21.2.3.30 (Multiplicity-spectrum comparison from eventual power sums). [Lean](#) [✓ Stmt](#) Suppose that every entry of the one-site multiplicity matrices, every entry of the two-site multiplicity matrices, and every entry of the matrices $\chi_{\alpha, \beta, \gamma}$ is positive. Suppose also that $\sum_i \mu_{\alpha, i} = m_\alpha$ for every α . If, for every γ and all sufficiently large L ,

$$\sum_r \nu_{\gamma, r}^L = \sum_{\alpha, \beta, i, j, k} (\mu_{\alpha, i} \mu_{\beta, j} \chi_{\alpha, \beta, \gamma, k})^L,$$

then the entries on the two sides agree as multisets. Together with the one-site trace identity $\sum_i \mu_{\alpha, i} = m_\alpha$, the one-site and two-site multiplicity entries and their dimensions define a BNT multiplicity-spectrum comparison.

Positivity makes all entries nonzero. Eventual equality of two finite sums of geometric sequences gives, for every $L \geq 1$,

$$\sum_r \nu_{\gamma, r}^L = \sum_{\alpha, \beta, i, j, k} (\mu_{\alpha, i} \mu_{\beta, j} \chi_{\alpha, \beta, \gamma, k})^L.$$

The finite power-sum identity determines both the cardinality and the multiset of entries.

Theorem 21.2.3.31 (Multiplicity trace normalization from the spectrum comparison and BNT algebra clause). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Suppose that the BNT algebra clause holds and that the one-site and two-site vertical multiplicity spectra have been compared as above. Then every two-site multiplicity matrix has the prescribed trace:*

$$\text{tr}(\nu_\gamma) = \sum_{\alpha, \beta, i, j, k} \mu_{\alpha, i} \mu_{\beta, j} \chi_{\alpha, \beta, \gamma, k} = \sum_{\alpha, \beta} c_{\alpha, \beta, \gamma}^{(1)} m_\alpha m_\beta = m_\gamma. \quad (21.22)$$

This is the normalization used for the preparation matrix ν_γ/m_γ in the algebra-to-RFP implication of [CPGSV16, Appendix C.4, lines 2058–2064].

Proof. [✓ Proof](#) Sum the multiset equality of the two spectra. For each α, β, γ , the resulting product sum satisfies

$$\sum_{i, j, k} \mu_{\alpha, i} \mu_{\beta, j} \chi_{\alpha, \beta, \gamma, k} = \left(\sum_i \mu_{\alpha, i} \right) \left(\sum_j \mu_{\beta, j} \right) \left(\sum_k \chi_{\alpha, \beta, \gamma, k} \right).$$

The identities

$$\begin{aligned} \sum_i \mu_{\alpha, i} &= m_\alpha, \\ \sum_j \mu_{\beta, j} &= m_\beta, \\ \sum_k \chi_{\alpha, \beta, \gamma, k} &= c_{\alpha, \beta, \gamma}^{(1)} \end{aligned}$$

$$\mathrm{tr}(\nu_\gamma) = \sum_{\alpha, \beta} m_\alpha m_\beta c_{\alpha, \beta, \gamma}^{(1)} = m_\gamma,$$

where the final equality is the length-one idempotent law. $\square$

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} c_{\alpha,\beta,\gamma}^{(L)} O_L(M_\gamma),$$
$$m_\gamma = \sum_{\alpha, \beta} c_{\alpha, \beta, \gamma}^{(1)} m_\alpha m_\beta.$$
$$c_{i,j,k}^{(n)} = c_{\sigma_n(i), \sigma_n(j), \tau_n(k)}^{(n)}.$$

Theorem 21.2.3.33 (Source BNT coefficient and product formulas). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [✓ Stmt](#) Let $c_{\alpha,\beta,\gamma}^{(L)}$ be the coefficient family and let $\chi_{\alpha,\beta,\gamma}$ be the positive diagonal matrices carried by the theorem data. For every $L > 0$,

where c^χ is the canonical coefficient family determined by χ , and every diagonal entry of every $\chi_{\alpha,\beta,\gamma}$ is strictly positive. Consequently, for every $L > 0$, the same-length product identity reads

$$m_\gamma = \sum_{\alpha, \beta} \text{tr}(\chi_{\alpha, \beta, \gamma}) m_\alpha m_\beta.$$

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At every positive length, the BNT-label coefficient family also agrees with the canonical coefficient family determined by these χ -matrices. Hence both families are represented by the χ -trace formulas (21.25) and (21.26). If σ_n and τ_n are the source and target label maps for a positive blocked length n , then the blocked-basis coefficients satisfy

$$c_{i,j,k}^{(n)} = c_{\sigma_n(i),\sigma_n(j),\tau_n(k)}^{(n)} = \text{tr}(\chi_{\sigma_n(i),\sigma_n(j),\tau_n(k)}^n).$$

Proof. [✓ Proof](#) Unpack the existential witness and apply the corresponding witness-level equations. $\square$

Theorem 21.2.3.38 (BNT theorem witnesses give positive blocked χ witnesses). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) An existential BNT-label theorem witness gives a positive blocked-basis χ trace-power witness for the chosen algebra-structure data: a blocked χ -family with positive diagonal entries such that, for every positive blocked length n ,

$$\begin{aligned} \chi_{n,i,j,k} &= \chi_{\sigma_n(i),\sigma_n(j),\tau_n(k)}, \\ c_{i,j,k}^{(n)} &= \text{tr}(\chi_{n,i,j,k}^n). \end{aligned}$$

The blocked-basis χ -family is the pullback of the BNT-label χ -family along the source and target comparison maps σ_n, τ_n . The proposition-level nonempty form of the witness gives the same existential blocked-basis family, stated either through this trace-power predicate or through the coefficient equation it unfolds to.

Proof. [✓ Proof](#) Apply the corresponding theorem-data consequence carried by the witness. $\square$

Corollary 21.2.3.39 (Pulled-back blocked coefficient formula). [Lean](#) [Lean](#) [✓ Stmt](#) For every positive blocked length n , the blocked-basis coefficients obtained from an existential BNT-label theorem witness are traces of powers of the BNT-label χ -matrices selected by the source and target maps. Equivalently, they are traces of powers of the pulled-back blocked-basis χ -matrices:

$$c_{i,j,k}^{(n)} = \text{tr}(\chi_{\sigma_n(i),\sigma_n(j),\tau_n(k)}^n) = \text{tr}(\chi_{n,i,j,k}^n).$$

Proof. [✓ Proof](#) Apply the trace-power identity of the positive blocked-basis χ -witness obtained from the existential witness. $\square$

21.2.4 Direct-sum sector conjugations

Definition 21.2.4.1 (Direct-sum unitary conjugations). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Work in the setting of Definition 21.2.3.19. Let

$$\mathcal{A}_1 = \bigoplus_{\gamma} M_{d_{\gamma}}(\mathbb{C}), \quad \mathcal{A}_2^{\sigma} = \bigoplus_{\gamma} M_{\tilde{d}_{\sigma(\gamma)}}(\mathbb{C}) \quad (21.27)$$

be the one-site sector algebra and the two-site sector algebra relabelled by the given sector matching. Suppose that $d_{\gamma} = \tilde{d}_{\sigma(\gamma)}$ and that there is a unitary U_{γ} on each matched sector such that

$$A_{\sigma(\gamma)}^i = U_{\gamma} M_{\gamma}^i U_{\gamma}^* \quad (21.28)$$

for every physical index i . Denote by ι_γ the coordinate identification induced by the equality of the two bond dimensions. Define mutually inverse linear maps

$$\tilde{T}: \mathcal{A}_1 \longrightarrow \mathcal{A}_2^\sigma, \quad [\tilde{T}(X)]_\gamma = U_\gamma \iota_\gamma (X_\gamma) U_\gamma^*, \quad (21.29)$$

$$\tilde{S}: \mathcal{A}_2^\sigma \longrightarrow \mathcal{A}_1, \quad [\tilde{S}(Y)]_\gamma = \iota_\gamma^{-1} (U_\gamma^* Y_\gamma U_\gamma). \quad (21.30)$$

These are the converse-direction middle conjugations introduced in [CPGSV16, Appendix C.4, lines 2053–2080]. The existence of the unitaries in (21.28) is an assumption here. They are distinct from the transported direct-sum maps in [CPGSV16, Appendix C.4, lines 1955–1997]. No maps between the ambient full matrix algebras are included here.

Theorem 21.2.4.2 (Properties of the direct-sum unitary conjugations). Lean Lean Lean Lean Lean ✓ Stmt *The maps $\tilde{T}$ and $\tilde{S}$ are completely positive and preserve the total trace on the direct sums. They are mutually inverse:*

$$\tilde{S}\tilde{T} = \text{id}_{\mathcal{A}_1}, \quad \tilde{T}\tilde{S} = \text{id}_{\mathcal{A}_2^\sigma}. \quad (21.31)$$

They also carry the matched tensor letters in both directions:

$$\begin{aligned} \tilde{T}((M_\gamma^i)_\gamma) &= (A_{\sigma(\gamma)}^i)_\gamma, \\ \tilde{S}((A_{\sigma(\gamma)}^i)_\gamma) &= (M_\gamma^i)_\gamma. \end{aligned} \quad (21.32)$$

Proof. ✓ Proof

On each summand, conjugation by U_γ and the coordinate identification ι_γ are completely positive and trace-preserving. Their direct sums therefore have the same properties. Unitarity gives

$$U_\gamma^* U_\gamma = I, \quad U_\gamma U_\gamma^* = I, \quad (21.33)$$

which proves (21.31) sector by sector. The first identity in (21.32) is precisely (21.28); the second follows by applying the inverse map. $\square$

21.2.5 Ambient sector retractions and normalized embeddings

Definition 21.2.5.1 (Ambient sector retractions and normalized embeddings). Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let

$$\mathcal{A}_1 = \bigoplus_\gamma M_{d_\gamma}(\mathbb{C}), \quad \mathcal{A}_2^\sigma = \bigoplus_\gamma M_{\tilde{d}_{\sigma(\gamma)}}(\mathbb{C})$$

be the one-site sector algebra and the relabelled two-site sector algebra. Their retained physical spaces are

$$\mathcal{K}_1 = \bigoplus_\gamma (\mathbb{C}^{r_\gamma} \otimes \mathbb{C}^{d_\gamma}), \quad \mathcal{K}_2^\sigma = \bigoplus_\gamma (\mathbb{C}^{\tilde{r}_{\sigma(\gamma)}} \otimes \mathbb{C}^{\tilde{d}_{\sigma(\gamma)}}).$$

Let $W_1: \mathbb{C}^d \rightarrow \mathcal{K}_1$ and $W_2: \mathbb{C}^{d^2} \rightarrow \mathcal{K}_2^\sigma$ be the corresponding coisometries, so that $W_i W_i^\dagger = I$. Write $\Pi_{1,\gamma}$ for the projection onto the γ -summand of $\mathcal{K}_1$ and $\Pi_{2,\gamma}^\sigma$ for the projection onto the $\sigma(\gamma)$ -summand of $\mathcal{K}_2^\sigma$. Let

$$\kappa_*: M_{d^2}(\mathbb{C}) \longrightarrow M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$$

be the canonical relabelling of a pair of physical indices. For the positive diagonal multiplicity matrices μ_γ and $\nu_{\sigma(\gamma)}$, set

$$m_\gamma = \text{tr}(\mu_\gamma) = \text{tr}(\nu_{\sigma(\gamma)}). \quad (21.34)$$

Define

$$R_1: M_d(\mathbb{C}) \longrightarrow \mathcal{A}_1, \quad [R_1(X)]_\gamma = \text{tr}_{\text{mult}} \left(\Pi_{1,\gamma} W_1 X W_1^\dagger \Pi_{1,\gamma} \right), \quad (21.35)$$

$$R_2: M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \longrightarrow \mathcal{A}_2^\sigma, \quad [R_2(Y)]_\gamma = \text{tr}_{\text{mult}} \left(\Pi_{2,\gamma}^\sigma W_2 \kappa_*^{-1}(Y) W_2^\dagger \Pi_{2,\gamma}^\sigma \right), \quad (21.36)$$

$$\widetilde{R}_1: \mathcal{A}_1 \longrightarrow M_d(\mathbb{C}), \quad \widetilde{R}_1((X_\gamma)_\gamma) = W_1^\dagger \left(\bigoplus_\gamma \frac{\mu_\gamma}{m_\gamma} \otimes X_\gamma \right) W_1, \quad (21.37)$$

$$\widetilde{R}_2: \mathcal{A}_2^\sigma \longrightarrow M_d(\mathbb{C}) \otimes M_d(\mathbb{C}), \quad \widetilde{R}_2((Y_\gamma)_\gamma) = \kappa_* \left[W_2^\dagger \left(\bigoplus_\gamma \frac{\nu_{\sigma(\gamma)}}{m_\gamma} \otimes Y_\gamma \right) W_2 \right]. \quad (21.38)$$

In Appendix C.4 of [CPGSV16], lines 2067–2068 define R_1 by (21.35), and lines 2078–2079 define R_2 by (21.36). Lines 2081–2083 give the normalized map $\widetilde{R}_1$ in (21.37). Lines 2058–2071 give $\widetilde{R}_2$, including its common denominator, in (21.38). These four definitions use only the two vertical decompositions, their sector relabelling, and (21.34).

Theorem 21.2.5.2 (Complete positivity of the ambient sector maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [StmT](#) Represent a sector family by its block-diagonal matrix. For a map whose domain is a sector algebra, first extract the diagonal sector blocks from an arbitrary matrix. With these canonical full-matrix representatives, each of R_1 , R_2 , $\widetilde{R}_1$, and $\widetilde{R}_2$ is completely positive.

Proof. ✓ Proof Each raw retraction is the composition of compression by W_i , a unitary relabelling of coordinates, sectorwise partial trace, and block-diagonal inclusion. Each normalized embedding is the composition of diagonal-block extraction, preparation of the positive matrix μ_γ/m_γ or $\nu_{\sigma(\gamma)}/m_\gamma$, coordinate relabelling, and expansion by $W_i^\dagger$. Every constituent has a Kraus representation, and compositions of such maps are completely positive. $\square$

Theorem 21.2.5.3 (Trace of the ambient sector maps). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For arbitrary ambient matrices $X \in M_d(\mathbb{C})$ and $Y \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$,

$$\mathrm{Tr}_{\mathrm{sec}}(R_1(X)) = \mathrm{tr}(W_1 X W_1^\dagger), \quad (21.39)$$

$$\mathrm{Tr}_{\mathrm{sec}}(R_2(Y)) = \mathrm{tr}(W_2 \kappa_*^{-1}(Y) W_2^\dagger). \quad (21.40)$$

If X and Y are positive semidefinite, then

$$\mathrm{Tr}_{\mathrm{sec}}(R_1(X)) \leq \mathrm{tr}(X), \quad \mathrm{Tr}_{\mathrm{sec}}(R_2(Y)) \leq \mathrm{tr}(Y). \quad (21.41)$$

In the other direction, for $Z \in \mathcal{A}_1$ and $Z' \in \mathcal{A}_2^\sigma$,

$$\mathrm{tr}(\widetilde{R}_1(Z)) = \mathrm{Tr}_{\mathrm{sec}}(Z), \quad \mathrm{tr}(\widetilde{R}_2(Z')) = \mathrm{Tr}_{\mathrm{sec}}(Z'). \quad (21.42)$$

Hence the canonical full-matrix representatives of $\widetilde{R}_1$ and $\widetilde{R}_2$ are trace-preserving completely positive maps.

The inequalities in (21.41) may be strict. Indeed, $W_i^\dagger W_i$ is the projection onto the retained nonzero sectors and need not be the identity on the ambient space. A positive matrix supported on the discarded orthogonal complement is annihilated by the corresponding compression. Thus R_1 and R_2 are not asserted to preserve the trace on all ambient positive matrices, and no term acting on the discarded complement is included.

Proof. ✓ Proof Sectorwise partial trace and block-diagonal inclusion preserve the total trace, which gives (21.39) and (21.40). For $P_i = W_i^\dagger W_i$, cyclicity of trace gives

$$\mathrm{tr}(W_i Z W_i^\dagger) = \mathrm{tr}(P_i Z) \leq \mathrm{tr}(Z)$$

when $Z \succeq 0$. The relabelling κ_* preserves positivity and trace, proving (21.41). Finally, each normalized multiplicity matrix has trace one, and $W_i W_i^\dagger = I$ gives $\mathrm{tr}(W_i^\dagger B W_i) = \mathrm{tr}(B)$. This proves (21.42); complete positivity is supplied by the preceding theorem. $\square$

Theorem 21.2.5.4 (Retraction identities for the ambient sector maps). Lean Lean Lean
✓ Stmt The raw retractions are left inverses of their normalized embeddings:

$$R_1 \widetilde{R}_1 = \mathrm{id}_{\mathcal{A}_1}, \quad R_2 \widetilde{R}_2 = \mathrm{id}_{\mathcal{A}_2^*}. \quad (21.43)$$

No identity is asserted for either reverse composite on the full ambient matrix algebra.

Proof. ✓ Proof Compressing an expansion by $W_i^\dagger$ gives the original retained block-diagonal matrix because $W_i W_i^\dagger = I$. On every sector,

$$\mathrm{tr}_{\mathrm{mult}} \left(\frac{\mu_\gamma}{m_\gamma} \otimes X_\gamma \right) = X_\gamma, \quad \mathrm{tr}_{\mathrm{mult}} \left(\frac{\nu_{\sigma(\gamma)}}{m_\gamma} \otimes Y_\gamma \right) = Y_\gamma,$$

where the second equality uses $\mathrm{tr}(\nu_{\sigma(\gamma)}) = m_\gamma$. The two coordinate relabellings in the two-site composite cancel. These identities prove (21.43). $\square$

Theorem 21.2.5.5 (Action on the weighted tensor letters). Lean Lean Lean Lean Lean
Lean ✓ Stmt For every vertical letter v , the one-site maps satisfy

$$R_1(\widetilde{M}_v) = (m_\gamma M_\gamma^v)_\gamma, \quad \widetilde{R}_1((m_\gamma M_\gamma^v)_\gamma) = \widetilde{M}_v. \quad (21.44)$$

The relabelled two-site maps satisfy

$$\begin{aligned} R_2(\kappa_*(\widetilde{M}^{[2]}_v)) &= (m_\gamma A_{\sigma(\gamma)}^v)_\gamma, \\ \widetilde{R}_2((m_\gamma A_{\sigma(\gamma)}^v)_\gamma) &= \kappa_*(\widetilde{M}^{[2]}_v). \end{aligned} \quad (21.45)$$

The first and second one-site identities combine the one-site weighted sector form of [CPGSV16, Appendix C.4, lines 2048–2051] with, respectively, [CPGSV16, Appendix C.4, lines 2067–2068] and [CPGSV16, Appendix C.4, lines 2081–2083]. The first two-site identity combines the matched weighted sector form of [CPGSV16, Appendix C.4, lines 2048–2064] with the map R_2 of [CPGSV16, Appendix C.4, lines 2078–2079]. The second combines the same sector form with the normalized map $\widetilde{R}_2$ of [CPGSV16, Appendix C.4, lines 2070–2071].

Proof. ✓ Proof The vertical reconstruction identities give

$$\begin{aligned}\widetilde{M}_v &= W_1^\dagger \left(\bigoplus_{\gamma} \mu_{\gamma} \otimes M_{\gamma}^v \right) W_1, \\ \widetilde{M}^{[2]}_v &= W_2^\dagger \left(\bigoplus_{\gamma} \nu_{\sigma(\gamma)} \otimes A_{\sigma(\gamma)}^v \right) W_2.\end{aligned}$$

Compression removes the two outer coisometries, and partial trace over the multiplicity coordinate multiplies the simple-sector letter by $\text{tr}(\mu_{\gamma}) = m_{\gamma}$ or $\text{tr}(\nu_{\sigma(\gamma)}) = m_{\gamma}$. This proves the first identity in each of (21.44) and (21.45). Preparing the normalized multiplicity matrix cancels the factor m_{γ} and the reconstruction identities give the two reverse formulas. $\square$

21.2.6 Trace-preserving completions and physical channels

Theorem 21.2.6.1 (Trace-preserving completion outside a projection). Lean Lean Lean Lean

Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt

Let $P \in M_m(\mathbb{C})$ be an orthogonal projection, let $\rho \in M_n(\mathbb{C})$ be positive semidefinite with $\text{tr}(\rho) = 1$, and let $\mathcal{E}: M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be completely positive. Define the trace-and-prepare map and its complementary restriction by

$$\mathcal{D}_{\rho}(X) = \text{tr}(X)\rho, \quad \mathcal{C}_{P,\rho}(X) = \mathcal{D}_{\rho}((I - P)X(I - P)^{\dagger}). \quad (21.46)$$

Both maps are completely positive, and $\mathcal{D}_{\rho}$ is trace-preserving. If

$$\text{tr}(\mathcal{E}(X)) = \text{tr}(PX) \quad (21.47)$$

for every $X \in M_m(\mathbb{C})$, then the completion

$$\hat{\mathcal{E}}(X) = \mathcal{E}(X) + \mathcal{C}_{P,\rho}(X) \quad (21.48)$$

is trace-preserving and completely positive. It agrees with $\mathcal{E}$ on every matrix supported by P :

$$PXP = X \implies \hat{\mathcal{E}}(X) = \mathcal{E}(X). \quad (21.49)$$

Proof. ✓ Proof

The map $\mathcal{D}_{\rho}$ factors as state preparation, interchange of the two tensor factors, and partial trace. The map $X \mapsto (I - P)X(I - P)^{\dagger}$ has a single Kraus operator. Thus the two maps in (21.46) are completely positive. Since $P = P^{\dagger} = P^2$,

$$\text{tr}(\mathcal{C}_{P,\rho}(X)) = \text{tr}((I - P)X).$$

Adding this identity to (21.47) gives $\text{tr}(\hat{\mathcal{E}}(X)) = \text{tr}(X)$. Finally, $PXP = X$ implies $(I - P)X(I - P) = 0$, which proves (21.49). $\square$

Theorem 21.2.6.2 (Physical channels from unitary sector conjugacies). Lean Lean Lean

Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean Lean

Lean Lean Lean Lean ✓ Stmt Suppose that the one-site and two-site sector decompositions are paired by a relabelling σ and unitaries U_{γ} satisfying

$$A_{\sigma(\gamma)}^i = U_{\gamma} M_{\gamma}^i U_{\gamma}^{\dagger} \quad (21.50)$$

for every sector γ and every vertical letter i . With the maps of the preceding two subsections, set

$$\mathcal{T}_0 = \widetilde{R}_2 \circ \widetilde{T} \circ R_1, \quad \mathcal{S}_0 = \widetilde{R}_1 \circ \widetilde{S} \circ R_2. \quad (21.51)$$

Let

$$P_1 = W_1^\dagger W_1, \quad P_2 = \kappa_*(W_2^\dagger W_2), \quad Q_i = I - P_i. \quad (21.52)$$

The matrices P_1 and P_2 are orthogonal projections. If $d > 0$, put $\tau_1 = d^{-1}I_d$ and $\tau_2 = d^{-2}I_{d^2}$, and define

$$\begin{aligned} \mathcal{T}(X) &= \mathcal{T}_0(X) + \text{tr}(Q_1 X Q_1) \tau_2, \\ \mathcal{S}(Y) &= \mathcal{S}_0(Y) + \text{tr}(Q_2 Y Q_2) \tau_1. \end{aligned} \quad (21.53)$$

If $d = 0$, take both maps to be zero; their domain and codomain matrix algebras then consist only of zero. Then $\mathcal{T}$ and $\mathcal{S}$ are trace-preserving completely positive maps on the full physical matrix algebras. For every virtual matrix Z , they satisfy

$$\mathcal{T}[M_1(Z)] = M_2(Z), \quad \mathcal{S}[M_2(Z)] = M_1(Z). \quad (21.54)$$

Hence M is a renormalization fixed point in the sense of Definition 20.2.1.

The compositions in (21.51) are the formulas of [CPGSV16, Appendix C.4, lines 2065–2085]. Those formulas determine the action after compression to the retained physical sectors. With rectangular coisometries, the literal composites annihilate the discarded zero-sector complements and fail to preserve trace there. Definition 4.1 nevertheless requires maps on the full physical algebras [CPGSV16, Definition 4.1, lines 638–660]. The source does not specify a trace-restoring modification on the discarded complements. Formula (21.53) makes one such choice when the physical space is nonzero. Replacing τ_1, τ_2 by any density matrices gives the same action on the physical operators generated by M . The conclusion is conditional on the unitaries in (21.50); their existence is not asserted here.

Proof. ✓ Proof Complete positivity of the four outer maps and of the two direct-sum conjugations gives complete positivity of $\mathcal{T}_0$ and $\mathcal{S}_0$. Their traces are

$$\text{tr}(\mathcal{T}_0(X)) = \text{tr}(P_1 X), \quad \text{tr}(\mathcal{S}_0(Y)) = \text{tr}(P_2 Y). \quad (21.55)$$

Suppose first that $d > 0$. Since Q_i is a projection and τ_i has trace one, the corresponding complementary term is completely positive and has trace $\text{tr}(Q_i X Q_i) = \text{tr}(Q_i X)$. Adding (21.55) and using $P_i + Q_i = I$ proves trace preservation of the completed maps. If $d = 0$, every ambient matrix is zero, so the zero maps are completely positive and trace preserving.

The reconstruction identities for the two vertical decompositions imply

$$P_1 M_1(Z) P_1 = M_1(Z), \quad P_2 M_2(Z) P_2 = M_2(Z). \quad (21.56)$$

Thus both complementary terms vanish on the corresponding physical operators. The retraction formulas extract the families $(m_\gamma M_\gamma^i)_\gamma$ and $(m_\gamma A_{\sigma(\gamma)}^i)_\gamma$; unitary conjugation carries these families into one another, and the normalized restorations recover the one-site and two-site letters. Linearity in the virtual matrix gives (21.54). $\square$

Theorem 21.2.6.3 (Renormalization fixed point from positive-tail reflected sector targets).

Lean Lean ✓ Stmt Let the one-site and blocked two-site BNT decompositions have the source-derived exact sector pairing, including the matched multiplicity spectrum, and suppose that M

generates matrix product density operators and that its doubled-index tensor is in literal CPSV canonical form. Assume, for every sector γ , every positive tail length, every pair of open bond indices, and every pair of two-site tail words, that the identity-dressed marked-chain coefficient equals the corresponding reflected-adjoint marked-chain coefficient of M_γ in the adjoint two-site blocking, with both tail words reversed on the reflected-adjoint side. This is the additional comparison needed to reproduce the Figure 8 argument in [CPGSV16, Proposition 4.13, lines 1903–1921] at its invocation in Appendix C.4, line 2057. Then there are trace-preserving completely positive maps $\mathcal{T}$ and $\mathcal{S}$ satisfying the identities in (21.54) for every virtual matrix X . Hence M is a renormalization fixed point in the sense of Definition 20.2.1. The target is an explicit hypothesis of this theorem. It is derived from the BNT algebra clause by the mixed-prefix comparison in Theorem 21.2.3.28.

Proof. ✓ Proof For each sector, Theorem 21.2.3.27 supplies a unitary U_γ satisfying (21.18), which is precisely (21.50). Choose these unitaries simultaneously over the finite set of sectors. Theorem 21.2.6.2 then gives $\mathcal{T}$ and $\mathcal{S}$ together with (21.54). $\square$

Theorem 21.2.6.4 (BNT algebra clause implies a renormalization fixed point). Lean Lean
✓ Stmt Let M be an MPDO whose doubled-index tensor is in literal CPSV canonical form. If M has a tensor-attached BNT algebra clause, then there are trace-preserving completely positive maps

$$\mathcal{T}: M_d(\mathbb{C}) \longrightarrow M_{d^2}(\mathbb{C}), \quad \mathcal{S}: M_{d^2}(\mathbb{C}) \longrightarrow M_d(\mathbb{C})$$

such that, for every virtual matrix X ,

$$\mathcal{T}[M_1(X)] = M_2(X), \quad \mathcal{S}[M_2(X)] = M_1(X). \quad (21.57)$$

Thus M is a renormalization fixed point in the sense of Definition 20.2.1. This is implication (ii) $\Rightarrow$ (i) of [CPGSV16, Theorem 4.14, lines 972–993].

Local fix (mixed one-site/two-site prefix): The common marked comparison needed to normalize the exact sector gauges is supplied by Theorem 21.2.3.28, as recorded in [con26n].

Local fix (zero-sector complement): The physical maps are completed on the discarded complements as recorded in [con26z].

Proof. ✓ Proof The tensor-attached algebra clause determines the two-site vertical decomposition, the sector relabelling, and exact invertible gauges. Theorem 21.2.3.28 normalizes these gauges simultaneously to unitaries. Apply Theorem 21.2.6.2 to obtain the two trace-preserving completely positive maps and (21.57). $\square$

21.3 Transfer retracts and fusion coisometries

The first definitions concern a linear-algebraic factorization of a doubled-index blocked transfer map. At a fixed blocked size $n \geq 1$, such a factorization exists precisely when E_n is idempotent. Requiring it at every positive blocked size is equivalent to idempotence of E_1 . This is distinct from the physical renormalization fixed-point condition in Definition 20.2.1 and [CPGSV16, Definition 4.1, lines 638–660], which requires trace-preserving completely positive maps on the physical indices. The physical maps constructed in Appendix C.4 likewise act on physical indices [CPGSV16, lines 1974–2085]. The active-support fusion coisometries of [CPGSV16, Theorem 4.14(ii)–(iii), lines 972–991], and their stronger full-support specialization, are recorded at the end of this section, together with the derivation of the same-length product law from them.

Definition 21.3.1 (Transfer-retract datum). [Lean](#) [✓ Stmt](#) A *transfer-retract datum* at blocked size n for an MPO tensor consists of a subspace $\mathcal{A}_n \subseteq M_D(\mathbb{C})$ together with linear maps

$$\begin{aligned} T_n : M_D(\mathbb{C}) &\longrightarrow \mathcal{A}_n, & S_n : \mathcal{A}_n &\longrightarrow M_D(\mathbb{C}), \\ T_n \circ S_n &= \text{id}_{\mathcal{A}_n}, & S_n \circ T_n &= E_n, \end{aligned} \quad (21.58)$$

where E_n denotes the blocked transfer map.

Theorem 21.3.2 (Transfer-retract data imply transfer-map idempotence). [Lean](#) [✓ Stmt](#) A *one-site transfer-retract datum implies doubled-index transfer-map idempotence*.

Proof. [✓ Proof](#) By (21.58),

$$E_1^2 = S_1 T_1 S_1 T_1 = S_1 (T_1 S_1) T_1 = S_1 T_1 = E_1.$$

Since E_1 is the original transfer map, this is exactly doubled-index transfer-map idempotence. $\square$

Theorem 21.3.3 (One-site transfer-retract criterion). [Lean](#) [✓ Stmt](#) A *one-site transfer-retract datum exists if and only if the doubled-index MPO transfer map is idempotent*.

Proof. [✓ Proof](#) The forward implication is the retract calculation $E_1^2 = S_1 T_1 S_1 T_1 = S_1 T_1$. Conversely, if E_1 is idempotent, then it factors through its range, giving the required one-site datum. $\square$

Theorem 21.3.4 (Transfer retracts at positive blocked sizes). [Lean](#) [✓ Stmt](#) If the doubled-index transfer map is idempotent, then transfer-retract data exist at every positive blocked size.

Proof. [✓ Proof](#) Every positive blocked transfer map equals a positive power of the idempotent one-site map and is therefore idempotent. Factoring it through its range gives the transfer-retract datum. $\square$

Definition 21.3.5 (Operator product of MPO tensors). [Lean](#) [✓ Stmt](#) The *product tensor* of two MPO tensors contracts the bra index of the first factor with the ket index of the second and takes the tensor product of the bond spaces:

$$(M \cdot N)^{ik} = \sum_j M^{ij} \otimes N^{jk}. \quad (21.59)$$

In the vertical reading of [CPGSV16, Section 4.5] this is the tensor obtained by joining two vertically read tensors along the original horizontal bond, the tensor written $M_\alpha M_\beta$ in [CPGSV16, Theorem 4.14(iii)].

Definition 21.3.6 (Canonical reassociation of three bond spaces). [Lean](#) [Lean](#) [✓ Stmt](#) For three bond spaces, let

$$A_{D_1, D_2, D_3} : \mathbb{C}^{D_1} \otimes (\mathbb{C}^{D_2} \otimes \mathbb{C}^{D_3}) \longrightarrow (\mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2}) \otimes \mathbb{C}^{D_3}$$

be the coordinate pullback induced by canonical reassociation, defined on elementary tensors by $A_{D_1, D_2, D_3}(x_1 \otimes (x_2 \otimes x_3)) = (x_1 \otimes x_2) \otimes x_3$.

Theorem 21.3.7 (Associativity of the product tensor). [Lean](#) [Lean](#) [✓ Stmt](#) The letters of the two parenthesizations of a triple product are related by the canonical bond reassociation. Equivalently, for every pair of physical indices i, k ,

$$((M \cdot N) \cdot P)^{ik} A_{D_1, D_2, D_3} = A_{D_1, D_2, D_3}(M \cdot (N \cdot P))^{ik}.$$

Proof. ✓ Proof Expand both product tensors. Reassociation preserves every elementary tensor factor, while exchanging the order of the two finite sums gives the same contraction over the two intermediate physical indices. $\square$

Lemma 21.3.8 (Word evaluation of the product tensor). Lean ✓ Stmt *The word evaluation of the product tensor along a configuration pair (σ, τ) expands as a sum over the contracted middle configurations ρ of tensor products of word evaluations:*

$$(M \cdot N)^{\sigma\tau} = \sum_{\rho} M^{\sigma\rho} \otimes N^{\rho\tau}. \quad (21.60)$$

Proof. ✓ Proof Induction on the length, using the mixed-product property of the tensor product to merge one letter at a time. $\square$

Theorem 21.3.9 (The operator family is multiplicative). Lean ✓ Stmt *For every chain length L , the operator generated by the product tensor is the product of the generated operators: $O_L(M \cdot N) = O_L(M)O_L(N)$.*

Proof. ✓ Proof Take traces in (21.60): the trace of a tensor product is the product of the traces, and the sum over middle configurations is the entrywise form of the operator product. $\square$

Lemma 21.3.10 (Word products of isometrically conjugated letters). Lean ✓ Stmt *For an isometry U (a rectangular matrix with $U^\dagger U = 1$) and a nonempty word of square matrices $F_1, \dots, F_L$,*

$$(UF_1U^\dagger)(UF_2U^\dagger) \cdots (UF_LU^\dagger) = UF_1F_2 \cdots F_LU^\dagger.$$

Proof. ✓ Proof The inner factors $U^\dagger U$ telescope to the identity. $\square$

Lemma 21.3.11 (Word products of block-diagonal tensor-product letters). Lean ✓ Stmt *For block-diagonal letters whose block of index γ is the tensor product of a fixed matrix X_γ with a varying letter $G_{\gamma,l}$,*

$$\prod_{l=1}^L \bigoplus_{\gamma} X_\gamma \otimes G_{\gamma,l} = \bigoplus_{\gamma} X_\gamma^L \otimes \prod_{l=1}^L G_{\gamma,l}.$$

Proof. ✓ Proof Blocks multiply blockwise and tensor products multiply factorwise, so the fixed factors accumulate into the matrix power X_γ^L . $\square$

Lemma 21.3.12 (Blockwise conjugation and restriction). Lean Lean ✓ Stmt *Let $F = \bigoplus_a F_a$ and $G = \bigoplus_a G_a$. Then $FGF^\dagger = \bigoplus_a F_a G_a F_a^\dagger$. Moreover, if r is a bijective reindexing of the row space of a matrix A , then $A_r X A_r^\dagger = (A X A^\dagger)_{r,r}$, where A_r denotes the corresponding row reindexing.*

Proof. ✓ Proof The first identity follows because block-diagonal matrices multiply blockwise. The second follows by commuting bijective row reindexing with multiplication and adjunction. $\square$

Lemma 21.3.13 (Cancellation of a nonempty identity factor). Lean Lean Lean Lean Lean ✓ Stmt *Let $D > 0$. The map $A \mapsto A \otimes I_D$ is injective and preserves matrix products. Consequently, $(F \otimes I_D)(G \otimes I_D) = I$ implies $FG = I$. In particular, orthonormal rows or columns of $F \otimes I_D$ imply the corresponding orthonormality of F .*

Proof. [✓ Proof](#) Evaluate the amplified identity on one basis vector of the nonzero factor. Multiplicativity of the Kronecker product gives the product and adjoint specializations. $\square$

Definition 21.3.14 (Fusion-isometry family). [Lean](#) [✓ Stmt](#) A *fusion-isometry family* over a finite set of labels consists of a family of tensors M_γ with a common physical dimension and per-label bond dimensions, diagonal matrices $\chi_{\alpha,\beta,\gamma}$ with positive entries, and isometries $U_{\alpha,\beta}$ such that, site by site,

$$U_{\alpha,\beta}(M_\alpha M_\beta)^{ij}U_{\alpha,\beta}^\dagger = \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij}. \quad (21.61)$$

This is the full-support specialization of the fusion statement in [CPGSV16, Theorem 4.14(iii)]; the tensors are the vertically read basis of normal tensors from [CPGSV16, Proposition 4.13], so the physical dimension is the bond dimension of the original tensor. Statement (iii) of the source theorem additionally asserts the idempotent identity for the trace scalars of the vertical decomposition, recorded separately by Definition 21.2.2.5.

Scope restriction (full-support fusion family): The displayed family assumes $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$ on the whole product bond space. The unrestricted source statement permits a common zero corner and instead gives a coisometry onto the active direct sum, together with exact reconstruction. This distinction is documented in [con26j].

Definition 21.3.15 (Active-support fusion coisometry family). [Lean](#) [Lean](#) [✓ Stmt](#) An *active-support fusion coisometry family* consists of the same labelled tensors and positive diagonal matrices, together with maps $U_{\alpha,\beta}$ satisfying $U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1$, the forward fusion identity

$$U_{\alpha,\beta}(M_\alpha M_\beta)^{ij}U_{\alpha,\beta}^\dagger = G_{\alpha,\beta}^{ij},$$

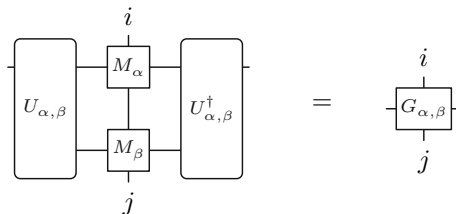
and exact reconstruction

$$(M_\alpha M_\beta)^{ij} = U_{\alpha,\beta}^\dagger G_{\alpha,\beta}^{ij} U_{\alpha,\beta}. \quad (21.62)$$

Zero-dimensional retained sums are permitted. If in addition $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$ for every pair, this family gives the preceding full-support fusion-isometry family.

Local fix (Figure-11 fusion coisometry): The source uses the retained-row orientation of Proposition 4.13. Thus its fusion maps are coisometries onto the active sectors; exact reconstruction records the discarded common zero corner. This convention is documented in [con26j].

For fixed labels α, β , put $G_{\alpha,\beta}^{ij} = \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij}$. Then [CPGSV16, Theorem 4.14(iii), equation (89), lines 986–991] gives



Lemma 21.3.16 (Weighted zipper identities on the active support). [Lean](#) [Lean](#) [✓ Stmt](#) An *active-support fusion coisometry* satisfies

$$\begin{aligned} U_{\alpha,\beta}(M_\alpha M_\beta)^{ij} &= G_{\alpha,\beta}^{ij} U_{\alpha,\beta}, \\ (M_\alpha M_\beta)^{ij} U_{\alpha,\beta}^\dagger &= U_{\alpha,\beta}^\dagger G_{\alpha,\beta}^{ij}. \end{aligned}$$

Proof. ✓ Proof Substitute exact reconstruction (21.62) into the left-hand sides and use $U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1$. □

Theorem 21.3.17 (Weighted zipper identities). Lean Lean ✓ Stmt For every pair of labels and physical indices, write $G_{\alpha,\beta}^{ij} = \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M_\gamma^{ij}$. The fusion isometry satisfies the two chi-weighted zipper identities

$$\begin{aligned} U_{\alpha,\beta}(M_\alpha M_\beta)^{ij} &= G_{\alpha,\beta}^{ij} U_{\alpha,\beta}, \\ (M_\alpha M_\beta)^{ij} U_{\alpha,\beta}^\dagger &= U_{\alpha,\beta}^\dagger G_{\alpha,\beta}^{ij}. \end{aligned}$$

The corresponding identities in [BMW⁺17, Equation (21)] have identity matrices on the multiplicity spaces in place of the positive diagonal matrices χ . Their unweighted form requires the length-independent integer specialization and is not a consequence of the present hypotheses. Even after that specialization, comparison of the two triple-fusion orders and final-label separation are still needed to construct an invertible F -move.

Proof. ✓ Proof Multiply the fusion identity (21.61) on the right by $U_{\alpha,\beta}$ and on the left by $U_{\alpha,\beta}^\dagger$, respectively, and use $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$. □

Theorem 21.3.18 (Length-independent zipper identities and completeness). Lean Lean Lean Lean ✓ Stmt Suppose the structure coefficients have positive trace-power form and are independent of the positive chain length. Then $\chi_{\alpha,\beta,\gamma} = 1_{r_{\alpha,\beta,\gamma}}$ for every triple of labels. If

$$H_{\alpha,\beta}^{ij} = \bigoplus_\gamma 1_{r_{\alpha,\beta,\gamma}} \otimes M_\gamma^{ij},$$

the fusion isometries satisfy

$$\begin{aligned} U_{\alpha,\beta}(M_\alpha M_\beta)^{ij} &= H_{\alpha,\beta}^{ij} U_{\alpha,\beta}, \\ (M_\alpha M_\beta)^{ij} U_{\alpha,\beta}^\dagger &= U_{\alpha,\beta}^\dagger H_{\alpha,\beta}^{ij}, \\ (M_\alpha M_\beta)^{ij} &= U_{\alpha,\beta}^\dagger H_{\alpha,\beta}^{ij} U_{\alpha,\beta}. \end{aligned} \tag{21.63}$$

These are the identity-weight specialization of Equations (19)–(21) in [BMW⁺17, Section 3]. The MPDO fusion identity appears in [CPGSV16, Theorem 4.14(iii), lines 986–991]; its identity-weight specialization follows from the length-independent case at [CPGSV16, line 1010].

Proof. ✓ Proof Length independence makes every positive diagonal entry of $\chi_{\alpha,\beta,\gamma}$ equal to one. Substitute the resulting identity matrices into Theorem 21.3.17. For completeness, insert $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$ on both sides of the product letter and apply the fusion identity between the two inserted factors, yielding (21.63). □

Theorem 21.3.19 (Common positive blocking of the BNT fusion isometries). Lean Lean Lean Lean Lean ✓ Stmt Suppose that the labelled doubled-index tensors form the basis of normal tensors, and that their fusion-isometry family has positive trace-power coefficients independent of the positive chain length. Then there is one integer $L > 0$ such that the blocked tensors $M_\gamma^{[L]}$ span their full product matrix algebra in one letter. Hence every $M_\gamma^{[L]}$ is injective. For this same L , write

$$\begin{aligned} P_{\alpha,\beta}^{IJ} &= (M_\alpha^{[L]} M_\beta^{[L]})^{IJ}, \\ H_{\alpha,\beta}^{IJ} &= \bigoplus_\gamma 1_{r_{\alpha,\beta,\gamma}} \otimes (M_\gamma^{[L]})^{IJ}, \end{aligned}$$

where $r_{\alpha,\beta,\gamma}$ is the dimension of the specialized chi space. The original fusion isometry satisfies

$$U_{\alpha,\beta} P_{\alpha,\beta}^{IJ} U_{\alpha,\beta}^\dagger = H_{\alpha,\beta}^{IJ}, \quad (21.64)$$

the two blocked zipper identities

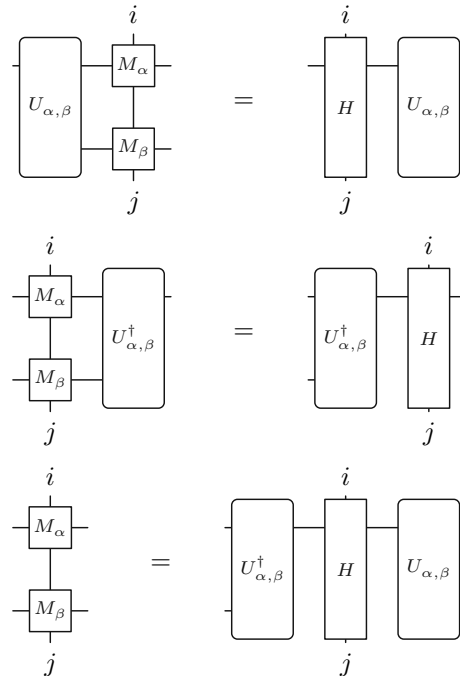
$$\begin{aligned} U_{\alpha,\beta} P_{\alpha,\beta}^{IJ} &= H_{\alpha,\beta}^{IJ} U_{\alpha,\beta}, \\ P_{\alpha,\beta}^{IJ} U_{\alpha,\beta}^\dagger &= U_{\alpha,\beta}^\dagger H_{\alpha,\beta}^{IJ}, \end{aligned}$$

and the literal reconstruction

$$P_{\alpha,\beta}^{IJ} = U_{\alpha,\beta}^\dagger H_{\alpha,\beta}^{IJ} U_{\alpha,\beta}.$$

Proof. ✓ Proof Choose the positive block length supplied by simultaneous BNT blocking. The canonical pairing of the blocked ket and bra words transfers the one-letter product-algebra span to the blocked MPO tensors, and projection to each labelled factor gives injectivity. Along a nonempty block, the products of the conjugated fusion letters telescope using only $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$. Length independence replaces every chi matrix by the identity on its own chi space. The two zipper identities and reconstruction then follow by multiplying the blocked fusion identity (21.64) by $U_{\alpha,\beta}$ or $U_{\alpha,\beta}^\dagger$ on the appropriate side. $\square$

The three rows below are, respectively, the two unweighted zipper identities and their local reconstruction consequence. The symbol H denotes $H_{\alpha,\beta}^{ij}$, and the open physical legs carry the indices i, j . The reconstruction uses only $U_{\alpha,\beta}^\dagger U_{\alpha,\beta} = 1$; it does not assert $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = 1$, surjectivity, or completeness of the total fusion space. The zipper orientation is that of [BMW⁺17, Section 3, Equations (19)–(21)]; the MPDO identity is [CPGSV16, Theorem 4.14(iii), lines 986–991], with identity weights in the length-independent case at [CPGSV16, line 1010].



Definition 21.3.20 (Operator family of a fusion-isometry family). [Lean](#) [Lean](#) [✓ Stmt](#) The tensors of a fusion-isometry family generate the concrete BNT-label operator family $O_L(M_\gamma)$, and the positive diagonal matrices give the positive trace-power witness for the coefficient family $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L)$.

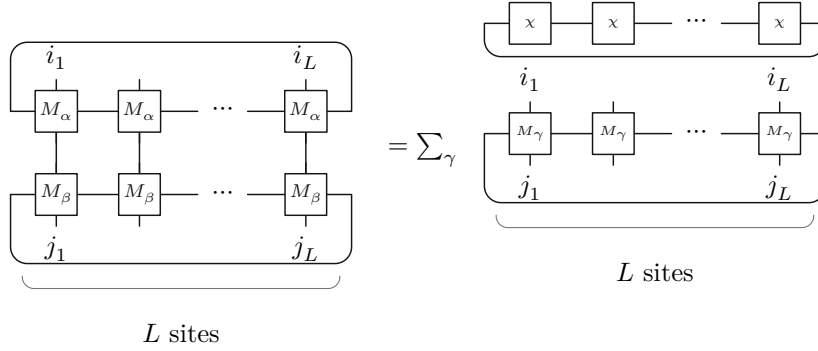


Figure 21.1: Closing an L -site chain of the local fusion identity. On the left, the bra legs of the M_α operator are contracted with the ket legs of the M_β operator. On the right, the multiplicity loop and the M_γ loop are disjoint; the former contributes $\text{tr}(\chi_{\alpha,\beta,\gamma}^L)$. Thus the diagram is the trace-power product law of [CPGSV16, Theorem 4.14(ii)–(iii), equations (87) and (89), lines 962–991].

Theorem 21.3.21 (Fusion families give the same-length product law). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) An active-support fusion coisometry family, and hence also a full-support fusion-isometry family, satisfies the same-length product law with the trace-power coefficients of its diagonal matrices. For every chain length $L \geq 1$,

$$O_L(M_\alpha)O_L(M_\beta) = \sum_{\gamma} \text{tr}(\chi_{\alpha,\beta,\gamma}^L) O_L(M_\gamma). \quad (21.65)$$

This derives the product law of [CPGSV16, Theorem 4.14(ii)] from the isometry statement of [CPGSV16, Theorem 4.14(iii)]. The active-support formulation is the one used in [CPGSV16, Appendix C.4, lines 2020–2029].

Proof. [✓ Proof](#) The product of the operators is generated by the product tensor. For a nonempty word W of letters of the product tensor, the conjugated letters multiply to the conjugated word, and the closed trace is unchanged. For an active-support family, exact reconstruction writes the word as $U_{\alpha,\beta}^\dagger W U_{\alpha,\beta}$, and cyclicity of trace together with $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = 1$ removes the outer maps. In the full-support case one may equivalently use the isometric orientation: $\text{tr}(U_{\alpha,\beta}^\dagger W U_{\alpha,\beta}) = \text{tr}(W U_{\alpha,\beta}^\dagger U_{\alpha,\beta}) = \text{tr}(W)$. By the fusion identity the conjugated letters are the block-diagonal tensor-product letters of the weighted direct sum. Their word product splits block by block with the diagonal factors accumulating into $\chi_{\alpha,\beta,\gamma}^L$; taking the trace block by block and factor by factor yields the coefficients $\text{tr}(\chi_{\alpha,\beta,\gamma}^L)$ in (21.65). $\square$

Definition 21.3.22 (BNT algebra clause derived from fusion isometries). [Lean](#) [✓ Stmt](#) Suppose the fusion-isometry identity holds with positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$, and suppose the trace scalars satisfy the length-one idempotent law for the corresponding trace-power coefficients. For

$L > 0$, the resulting BNT algebra clause has coefficients $c_{\alpha,\beta,\gamma}^{(L)} = \text{tr}(\chi_{\alpha,\beta,\gamma}^L)$ and records positivity of every diagonal entry. The product law and the assumed idempotent condition are, respectively,

$$\begin{aligned} O_L(M_\alpha)O_L(M_\beta) &= \sum_{\gamma} c_{\alpha,\beta,\gamma}^{(L)} O_L(M_\gamma), \\ m_\gamma &= \sum_{\alpha,\beta} c_{\alpha,\beta,\gamma}^{(1)} m_\alpha m_\beta. \end{aligned}$$

This construction realizes implication (iii) $\Rightarrow$ (ii) of [CPGSV16, Theorem 4.14]; it does not assert either of the implications involving the renormalization fixed-point condition.

Definition 21.3.23 (Active-sector fusion for a vertical canonical decomposition). [Lean](#) [Lean](#)

[✓ Stmt](#) Choose a vertical canonical decomposition

$$U\widetilde{M}U^\dagger = \bigoplus_{\alpha} \mu_{\alpha} \otimes M_{\alpha}.$$

The fusion clause for this decomposition consists of positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$ and coisometries $U_{\alpha,\beta}$ onto the active product sectors, satisfying

$$U_{\alpha,\beta}(M_{\alpha}M_{\beta})^{ij}U_{\alpha,\beta}^\dagger = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_{\gamma}^{ij}. \quad (21.66)$$

They obey $U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1$, and the active direct sum reconstructs the product tensor:

$$(M_{\alpha}M_{\beta})^{ij} = U_{\alpha,\beta}^\dagger \left(\bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_{\gamma}^{ij} \right) U_{\alpha,\beta}.$$

The trace scalars $m_{\alpha} = \text{tr}(\mu_{\alpha})$ satisfy the length-one idempotent identity for the coefficients determined by these same diagonal matrices. This is statement (iii) of [CPGSV16, Theorem 4.14, lines 986–993], attached to the chosen decomposition of [CPGSV16, Proposition 4.13, lines 943–951].

Theorem 21.3.24 (Fusion clause from paired decompositions with supplied fusion matrices and coisometries). [Lean](#) [✓ Stmt](#) Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be positive vertical canonical decompositions of an MPO tensor and its two-site blocking. Suppose their normal sectors are identified by a bijection σ and unitary conjugacies with the corresponding multiplicity-trace ratios. Suppose also that positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$ and coisometries $U_{\alpha,\beta}$ are supplied, with $U_{\alpha,\beta}U_{\alpha,\beta}^\dagger = 1$, and satisfy both

$$\begin{aligned} U_{\alpha,\beta}(M_{\alpha}M_{\beta})^{ij}U_{\alpha,\beta}^\dagger &= \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_{\gamma}^{ij}, \\ (M_{\alpha}M_{\beta})^{ij} &= U_{\alpha,\beta}^\dagger \left(\bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes M_{\gamma}^{ij} \right) U_{\alpha,\beta}. \end{aligned}$$

Then the simultaneous blocked representations yield the missing length-one idempotent trace-scalar law and complete the tensor-attached fusion clause. This completes the fusion-clause construction extracted from [CPGSV16, Appendix C.4, lines 2001–2046].

Proof. [✓ Proof](#) The sector correspondence expresses the blocked closed-chain operator in the one-site normal-tensor basis. Multiplication of the one-site vertical decomposition gives a second expression in the same basis. At every sufficiently large length, linear independence equates their coefficients. Equality of the resulting positive power sums identifies the multiplicity multisets, and summing their entries gives $m_{\gamma} = \sum_{\alpha,\beta} \text{tr}(\chi_{\alpha,\beta,\gamma}) m_{\alpha} m_{\beta}$. $\square$

Theorem 21.3.25 (Literal CPSV renormalization fixed points have active-sector fusion). [Lean](#)

[✓ Stmt](#) Let M generate matrix product density operators, and suppose that its doubled-index MPS tensor is in literal CPSV canonical form. If the one-site and two-site physical closures are related in both directions by trace-preserving completely positive maps as in Definition 4.1, then a vertical canonical decomposition of M has the active-sector fusion clause. In particular, its positive diagonal fusion matrices satisfy

$$m_\gamma = \sum_{\alpha, \beta} \text{tr}(\chi_{\alpha, \beta, \gamma}) m_\alpha m_\beta.$$

This is the corrected active-support formulation of implication (i) $\Rightarrow$ (iii) in [CPGSV16, Theorem 4.14, lines 972–993], rather than the unqualified printed statement.

Scope restriction (active product BNT): For each fixed product pair, only its active normal corners occur; an absent label has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): A fixed product pair may have an empty active family, so no unsupported normal sector is added. See [con26q].

Local fix (Figure-11 fusion coisometry): Each fusion map is a coisometry onto the active direct sum, and its adjoint reconstructs the whole product tensor, including any discarded common zero corner. See [con26j].

Proof. [✓ Proof](#) Choose the literal one-site and two-site vertical decompositions. Transporting the two physical maps through them identifies their normal sectors by trace-ratio unitary conjugacies. The literal positive-fusion theorem gives the positive diagonal matrices, coisometries, and exact reconstruction on every active product sector. The paired-decomposition theorem then gives the tensor-attached fusion clause. $\square$

Theorem 21.3.26 (Horizontal-form specialization of active-sector fusion). [Lean](#) [✓ Stmt](#) Let M be a matrix product density operator in normalized BNT-refined horizontal form. If the one-site and two-site physical closures are related in both directions by trace-preserving completely positive maps as in Definition 4.1, then a vertical canonical decomposition of M has the active-sector fusion clause above. In particular, its positive diagonal fusion matrices satisfy the length-one idempotent trace-scalar identity.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis. See [con26y].

Proof. [✓ Proof](#) Choose the one-site and two-site vertical canonical decompositions supplied by the normalized BNT-refined horizontal form. The renormalization maps identify their normal sectors and give positive fusion coisometries. Apply the paired-decomposition theorem to obtain the tensor-attached fusion clause. $\square$

Theorem 21.3.27 (Active-sector fusion implies the tensor-attached algebra clause). [Lean](#) [Lean](#)

[✓ Stmt](#) Suppose the active-sector fusion clause holds for a chosen vertical canonical decomposition. Then the algebra clause holds for the same decomposition. For every positive chain length L , its coefficients are $c_{\alpha, \beta, \gamma}^{(L)} = \text{tr}(\chi_{\alpha, \beta, \gamma}^L)$, determined by the same diagonal matrices. Thus statement (iii) implies statement (ii) of [CPGSV16, Theorem 4.14, lines 972–993]. No assertion concerning the renormalization fixed-point condition is made here.

Proof. [✓ Proof](#) Iterating the fusion identity along L sites and taking the closed trace gives $O_L(M_\alpha)O_L(M_\beta) = \sum_\gamma \text{tr}(\chi_{\alpha, \beta, \gamma}^L) O_L(M_\gamma)$. Writing $m_\alpha = \text{tr}(\mu_\alpha)$, the assumed length-one idempotent identity is $m_\gamma = \sum_{\alpha, \beta} \text{tr}(\chi_{\alpha, \beta, \gamma}) m_\alpha m_\beta$. These are the product and idempotent identities of the algebra clause. Retaining the chosen vertical decomposition and the same diagonal matrices gives the tensor-attached algebra clause. $\square$

Theorem 21.3.28 (Printed algebra and fusion characterization). Not ready For a canonical-form tensor generating matrix product density operators, [CPGSV16, Theorem 4.14] states that the fixed-point condition, the tensor-attached BNT algebra clause, and the displayed fusion isometry clause are equivalent.

The fixed-point condition is equivalent to the tensor-attached algebra clause. For the fusion clause, an absent product sector must have multiplicity zero, and the retained-row map is a coisometry with exact reconstruction. Under this active-support interpretation, the three conditions are equivalent by Theorem 21.3.29. Without these qualifications, the displayed fusion clause is not well-defined on empty product sectors and has the wrong isometry orientation.

Theorem 21.3.29 (BNT characterizations of MPDO renormalization fixed points). Lean Lean

✓ Stmt Let M be an MPDO whose doubled-index tensor is in literal CPSV canonical form. Consider the following conditions:

- (i) M is a renormalization fixed point;
- (ii) M has a tensor-attached BNT algebra clause;
- (iii_{act}) M has the active-support BNT fusion clause.

Then

$$(i) \iff (ii), \quad (i) \iff (iii_{\text{act}}). \quad (21.67)$$

The first equivalence is the equivalence of (i) and (ii) in [CPGSV16, Theorem 4.14, lines 972–985]. The second is the corrected active-support form corresponding to the printed conditions (i) and (iii), rather than an assertion of the unrestricted printed condition (iii).

Scope restriction (active product BNT): Only nonzero product corners occur in (iii_{act}); a sector absent from a fixed product pair has zero fusion multiplicity. This is recorded in [con26c].

Local fix (fixed-pair support and coisometry): A fixed pair may have empty active support, and the retained-row fusion map is a coisometry. These points are recorded in [con26q] and [con26j].

Local fix (mixed one-site/two-site prefix): The implication (ii) $\Rightarrow$ (i) uses the marked comparison recorded in [con26n].

Local fix (zero-sector complement): The physical maps in (ii) $\Rightarrow$ (i) are completed on the discarded complements as recorded in [con26z].

Proof. ✓ Proof A renormalization fixed point has the active-support fusion clause by Theorem 21.3.25. Such a fusion clause gives the tensor-attached algebra clause by Theorem 21.3.27. Finally, the algebra clause gives a renormalization fixed point by Theorem 21.2.6.4. These three implications prove both equivalences in (21.67). $\square$

The associativity of the algebra product in [CPGSV16, Equation (87)] constrains the diagonal matrices $\chi_{\alpha,\beta,\gamma}$ and, through the isometry statement, induces a compatibility equation between the fusion isometries of the two ways of bracketing a triple product, $(M_\alpha M_\beta)M_\gamma$ and $M_\alpha(M_\beta M_\gamma)$ [CPGSV16, lines 995–999]. The source attributes the construction of that equation, and the classification of its solutions, to [BMW⁺17]. The next definitions and theorems first record the tensor-level content of the two bracketings and then compare the resulting full direct sums. Restriction of that comparison to a fixed final label is a separate step.

Definition 21.3.30 (Iterated fusion isometry of a triple product, left bracketing). Lean ✓ Stmt For labels α, β, γ , the *left iterated fusion isometry* $U_{\alpha,\beta,\gamma}^L$ is obtained by applying $U_{\alpha,\beta}$ (tensored

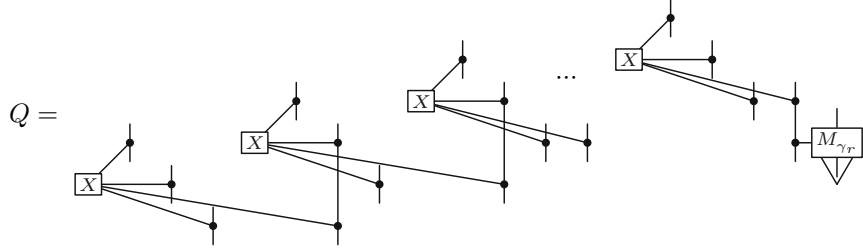


Figure 21.2: The recursive contraction defining the structure operator Q , with terminal factor $\text{tr}(M_{\gamma_r})$ [CPGSV16, lines 999–1008].

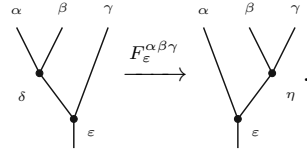


Figure 21.3: The two ways of fusing three labels to a fixed final label ϵ , using the printed- F row/column convention of [BMW⁺17, Section 3.4]. The solid fusion trees are the left- and right-bracketed maps constructed below. The arrow is the change of basis from left- to right-bracketed multiplicity coordinates. The results below first construct this map and then prove its two-sided unitarity under the simultaneous-selector and positive-bond-dimension hypotheses.

with the identity on the bond space of M_γ) to fuse the first two tensors, then applying $U_{\delta,\gamma}$, for each label δ appearing in the resulting direct sum, to fuse the outcome with the third tensor. Its codomain is indexed by pairs of labels (δ, ϵ) together with the multiplicity indices of $\chi_{\alpha,\beta,\delta}$ and $\chi_{\delta,\gamma,\epsilon}$.

Theorem 21.3.31 (The left iterated fusion isometry is an isometry). Lean ✓ Stmt *The left iterated fusion isometry satisfies $(U_{\alpha,\beta,\gamma}^L)^\dagger U_{\alpha,\beta,\gamma}^L = 1$.*

Proof. ✓ Proof The composite of two isometries is an isometry. □

Theorem 21.3.32 (The left iterated fusion isometry conjugates a triple product onto a doubly weighted direct sum). Lean ✓ Stmt *Site by site,*

$$\begin{aligned} & U_{\alpha,\beta,\gamma}^L ((M_\alpha M_\beta) M_\gamma)^{ij} (U_{\alpha,\beta,\gamma}^L)^\dagger \\ &= \bigoplus_{\delta} \bigoplus_{\epsilon} \chi_{\alpha,\beta,\delta} \otimes (\chi_{\delta,\gamma,\epsilon} \otimes M_\epsilon^{ij}). \end{aligned}$$

Proof. ✓ Proof Apply the fusion identity for $U_{\alpha,\beta}$ to the first two tensors, summed against the physical index contracting with the third tensor. Regroup the result over δ and use blockwise conjugation to obtain $\chi_{\alpha,\beta,\delta}$ tensored with a letter of $M_\delta M_\gamma$. The fusion identity for $U_{\delta,\gamma}$ gives the second direct sum. □

Definition 21.3.33 (Iterated fusion isometry of a triple product, right bracketing). Lean ✓ Stmt For labels α, β, γ , the *right iterated fusion isometry* $U_{\alpha,\beta,\gamma}^R$ is obtained by applying $U_{\beta,\gamma}$, tensored with the identity on the bond space of M_α , and then applying $U_{\alpha,\delta}$ for every intermediate label δ . Its codomain is indexed by the intermediate and final labels (δ, ϵ) and by the multiplicity

indices of $\chi_{\beta,\gamma,\delta}$ and $\chi_{\alpha,\delta,\varepsilon}$. This is the right-bracketed composite in the associativity equation of [BMW⁺17, Section “Associativity and the pentagon equation”], before the change of basis by an F -matrix.

Theorem 21.3.34 (The right iterated fusion isometry is an isometry). [Lean](#) [✓ Stmt](#) *The right iterated fusion isometry satisfies $(U_{\alpha,\beta,\gamma}^R)^\dagger U_{\alpha,\beta,\gamma}^R = 1$.*

Proof. [✓ Proof](#) The two successive fusion maps are isometries on each intermediate sector, so their block-diagonal composite is an isometry. $\square$

Theorem 21.3.35 (The right iterated fusion isometry conjugates a triple product onto a doubly weighted direct sum). [Lean](#) [✓ Stmt](#) *Site by site,*

$$\begin{aligned} U_{\alpha,\beta,\gamma}^R (M_\alpha (M_\beta M_\gamma))^{ij} (U_{\alpha,\beta,\gamma}^R)^\dagger \\ = \bigoplus_{\delta} \bigoplus_{\varepsilon} \chi_{\beta,\gamma,\delta} \otimes (\chi_{\alpha,\delta,\varepsilon} \otimes M_\varepsilon^{ij}). \end{aligned}$$

Proof. [✓ Proof](#) Apply the fusion identity for $U_{\beta,\gamma}$ to $M_\beta M_\gamma$. In the summand indexed by δ , exchange the first multiplicity factor with the bond space of M_α . Blockwise conjugation regroups the contraction as $\chi_{\beta,\gamma,\delta} \otimes (M_\alpha M_\delta)^{ij}$. The fusion identity for $U_{\alpha,\delta}$ gives the sum over ε . $\square$

Definition 21.3.36 (Full comparison of the two triple-fusion orders). [Lean](#) [✓ Stmt](#) The canonical comparison from the right-associated full direct sum to the left-associated full direct sum is

$$C_{\alpha,\beta,\gamma} = U_{\alpha,\beta,\gamma}^L A_{D_\alpha, D_\beta, D_\gamma} (U_{\alpha,\beta,\gamma}^R)^\dagger. \quad (21.68)$$

Both intermediate labels and the final label remain in the direct sums. This definition does not select a fixed final label and makes no invertibility assertion. Its orientation is from the right-associated sum to the left-associated sum; consequently it is adjoint-oriented with respect to the printed- F row/column convention used here for [BMW⁺17, Section 3.4].

Lemma 21.3.37 (Left range projection of the full triple-fusion comparison). [Lean](#) [✓ Stmt](#) *The product of the full comparison with its adjoint is the range projection of the left iterated fusion isometry:*

$$C_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}^\dagger = U_{\alpha,\beta,\gamma}^L (U_{\alpha,\beta,\gamma}^L)^\dagger.$$

The right side is not asserted to be the identity. Surjectivity of the left iterated fusion map requires completeness of the fusion decomposition, corresponding to the invertible total fusion matrix in [BMW⁺17, lines 181–191].

Proof. [✓ Proof](#) Substitute (21.68). The isometry identity $(U^R)^\dagger U^R = 1$ and the two-sided inverse identities for the permutation matrix A leave $U^L (U^L)^\dagger$. $\square$

Lemma 21.3.38 (Right range projection of the full triple-fusion comparison). [Lean](#) [✓ Stmt](#) *The product in the opposite order is the range projection of the right iterated fusion isometry:*

$$C_{\alpha,\beta,\gamma}^\dagger C_{\alpha,\beta,\gamma} = U_{\alpha,\beta,\gamma}^R (U_{\alpha,\beta,\gamma}^R)^\dagger.$$

The right side is not asserted to be the identity. Surjectivity of the right iterated fusion map likewise requires completeness of the fusion decomposition.

Proof. [✓ Proof](#) Substitute (21.68). The isometry identity $(U^L)^\dagger U^L = 1$ and the two-sided inverse identities for A leave $U^R (U^R)^\dagger$. $\square$

Theorem 21.3.39 (The full triple-fusion comparison intertwines the unweighted direct sums). Lean ✓ Stmt Suppose the positive trace-power coefficients are independent of the positive chain length. For every pair of physical indices i, k , let

$$D_{ik}^L = \bigoplus_{\delta, \varepsilon} 1_{r_{\alpha, \beta}^{\delta}} \otimes (1_{r_{\delta, \gamma}^{\varepsilon}} \otimes M_{\varepsilon}^{ik}),$$

and define D_{ik}^R by the right-associated multiplicities $r_{\beta, \gamma}^{\delta}$ and $r_{\alpha, \delta}^{\varepsilon}$. Then

$$D_{ik}^L C_{\alpha, \beta, \gamma} = C_{\alpha, \beta, \gamma} D_{ik}^R.$$

This is an identity on the full direct sums. It neither extracts a fixed-final F -transformation nor proves invertibility or the pentagon identity.

Proof. ✓ Proof Length independence makes every positive diagonal factor equal to the identity. Substitute the left and right triple-fusion identities into (21.68), cancel the two isometries, and apply Theorem 21.3.7. □

Definition 21.3.40 (Common word selectors for the final labels). Lean ✓ Stmt For some common length S , suppose that for each final label ε there are coefficients $c_{\varepsilon}(w)$, indexed by the words w of length S in the doubled physical alphabet, such that

$$\begin{aligned} \sum_{|w|=S} c_{\varepsilon}(w) M_{\varepsilon}^w &= 1, \\ \sum_{|w|=S} c_{\varepsilon}(w) M_{\varepsilon'}^w &= 0 \quad \text{if } \varepsilon' \neq \varepsilon. \end{aligned}$$

This is the finite-word form of the simultaneous inverse relation $B_d^+ B_{d'} = \delta_{d, d'} 1$ used in the fixed-channel argument of [BMW⁺17].

Theorem 21.3.41 (Pair fusion completeness from positive-length selectors). Lean ✓ Stmt Suppose the positive trace-power coefficients are independent of the positive chain length, and suppose that common selectors exist at a positive length. Then every pair fusion isometry is surjective: $U_{\alpha, \beta} U_{\alpha, \beta}^{\dagger} = 1$. The common selectors are the finite-word form of the simultaneous inverse used in [BMW⁺17, line 269].

Proof. ✓ Proof For a word w of the positive selector length, the identity $U_{\alpha, \beta}^{\dagger} U_{\alpha, \beta} = 1$ gives $D^w = U_{\alpha, \beta} (M_{\alpha} M_{\beta})^w U_{\alpha, \beta}^{\dagger}$, where $D = \bigoplus_{\gamma} 1_{r_{\alpha, \beta}^{\gamma}} \otimes M_{\gamma}$. Hence the range projection $P = U_{\alpha, \beta} U_{\alpha, \beta}^{\dagger}$ satisfies $P D^w = D^w$. Summing the selector polynomials over the final label gives the identity on the full direct sum, and therefore $P = 1$. □

Theorem 21.3.42 (Pair fusion completeness from a source BNT). Lean Lean ✓ Stmt Suppose that the labelled tensors in the fusion family form a basis of normal tensors and that the positive trace-power coefficients are independent of the positive chain length. Then every pair fusion isometry is surjective: $U_{\alpha, \beta} U_{\alpha, \beta}^{\dagger} = 1$. In particular, final-label selectors and coisometry are conclusions, not additional hypotheses. This is a derived consequence of the common BNT word span in [CPGSV16, lines 317–345], the fusion identity at lines 986–993, and length independence at lines 995–1010.

Proof. ✓ Proof The basis-of-normal-tensors argument gives a positive common word length at which the labelled word evaluations span the full product matrix algebra. Choosing the tuple which is the identity in one labelled block and zero in every other block gives common final-label selectors. Apply Theorem 21.3.41. □

Definition 21.3.43 (One fusion layer of the topological projector). [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [✓ Stmt](#) Let P_γ be operators on the bond space of the vertically read tensor M_γ . Define one fusion layer of the recursive operator Q by

$$Q_{\alpha,\beta} = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes P_\gamma. \quad (21.69)$$

The source terminal matrices are $P_\gamma = \text{tr}(M_\gamma)$ after spectral refinement, where the trace closes the horizontal operator leg and leaves the bond indices. Before spectral refinement, the fusion identity implies

$$U_{\alpha,\beta} \text{tr}(M_\alpha M_\beta) U_{\alpha,\beta}^\dagger = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes \text{tr}(M_\gamma). \quad (21.70)$$

Indeed, summing the forward fusion identity over equal horizontal operator indices and distributing matrix multiplication over this finite sum gives (21.70). This is one pairwise step in the recursion of [CPGSV16, lines 999–1010].

Scope restriction (one fusion layer): This entry does not assert the arbitrary-chain recursion described below. That recursion, the density decomposition, and the physical commuting Gibbs decomposition for $L \geq 2$ are given in Theorems 21.3.46, 21.3.51, and 21.3.62, respectively. The source boundary at $L = 1$ is recorded in Remark 21.3.65.

Theorem 21.3.44 (Projection property of one fusion layer). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)
[✓ Stmt](#) Let P_γ be self-adjoint idempotents on the terminal bond space of M_γ . Suppose that the matrices $\chi_{\alpha,\beta,\gamma}$ have the positive-length trace-power form of the label-coefficient family and that this family is independent of the positive chain length. Then

$$Q_{\alpha,\beta} = \bigoplus_{\gamma} 1_{r_{\alpha,\beta}^\gamma} \otimes P_\gamma$$

is self-adjoint and idempotent. This is the single recursive fusion step in [CPGSV16, lines 999–1010]. Since the active fusion map satisfies $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = I$, the transported operator $U_{\alpha,\beta}^\dagger Q_{\alpha,\beta} U_{\alpha,\beta}$ is also an orthogonal projection on the product bond space.

Proof. [✓ Proof](#) The trace-power hypothesis and length independence give $\chi_{\alpha,\beta,\gamma} = 1$. Hence

$$\begin{aligned} Q_{\alpha,\beta}^\dagger &= \bigoplus_{\gamma} (1 \otimes P_\gamma^\dagger) = Q_{\alpha,\beta}, \\ Q_{\alpha,\beta}^2 &= \bigoplus_{\gamma} (1 \otimes P_\gamma^2) = Q_{\alpha,\beta}. \end{aligned}$$

Finally, the coisometry identity $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = I$ and Theorem [LTC26, “Coisometric transport of a star projection”] show that $U_{\alpha,\beta}^\dagger Q_{\alpha,\beta} U_{\alpha,\beta}$ is an orthogonal projection. $\square$

Definition 21.3.45 (Recursive fusion histories and their sequential circuit). [Lean](#) [Lean](#) [Lean](#)
[Lean](#) [Lean](#) [✓ Stmt](#) Fix an initial label α and append labels successively along a finite chain. For total chain length $N \geq 1$, the appended-label list has length $N - 1$; the symbols W_N and Q_N below refer to this total chain length. A fusion history records, at every step, the preceding final label, the new final label, and an index in the corresponding multiplicity space. If h ends at $\gamma(h)$, let

$$w(h) = \prod_r (\chi_{\gamma_{r-1}, \beta_r, \gamma_r})_{k_r, k_r}. \quad (21.71)$$

For terminal bond operators P_γ , define

$$Q_N(P) = \bigoplus_h w(h) P_{\gamma(h)}. \quad (21.72)$$

The associated sequential map is obtained by tensoring the preceding map with the identity on the newly appended bond and then applying the active pairwise fusion coisometry in every preceding-history block. This is the recursively typed operator and circuit in [CPGSV16, lines 999–1010].

Theorem 21.3.46 (Fixed-label recursive fusion identity). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#) For every initial label and every finite list of appended labels, let $N \geq 1$ be one plus the length of that list. The sequential fusion map W_N is a retained-row coisometry. Letter by letter, it carries the left-associated product tensor to the recursive direct sum:

$$W_N M_{\alpha,\beta}^{i,j} W_N^\dagger = Q_N((M_\gamma^{i,j})_\gamma). \quad (21.73)$$

Consequently, after closing the horizontal operator index,

$$W_N \operatorname{tr}(M_{\alpha,\beta}) W_N^\dagger = Q_N((\operatorname{tr}(M_\gamma))_\gamma). \quad (21.74)$$

Exact reconstruction gives the reverse identities

$$M_{\alpha,\beta}^{i,j} = W_N^\dagger Q_N((M_\gamma^{i,j})_\gamma) W_N, \quad (21.75)$$

$$\operatorname{tr}(M_{\alpha,\beta}) = W_N^\dagger Q_N((\operatorname{tr}(M_\gamma))_\gamma) W_N. \quad (21.76)$$

If the positive trace-power coefficients are independent of the positive chain length and the terminal matrices are self-adjoint idempotents, then this recursive operator is self-adjoint and idempotent.

Local fix (Figure-11 fusion coisometry): The source embedding $\tilde{U}_N$ is $W_N^\dagger$, the adjoint of the retained-row coisometry used above. Exact reverse fusion uses the active-support reconstruction of Appendix C.4, lines 2020–2029. See [con26j].

Scope restriction (fixed initial label): This theorem concerns one chosen initial label and list of appended labels. The following theorem performs the vertical-canonical assembly over all labels and multiplicity coordinates. The commutator remains outside the present result; see [con26f].

Proof. [✓ Proof](#) The base case has no appended labels and represents the one-site chain; its fusion history is unique. For the induction step, tensor the preceding coisometry with the identity on the new bond, apply the induction hypothesis, and then apply the pairwise fusion identity in each history block. Reindexing the nested direct sum adjoins the new final label and multiplicity coordinate to the preceding history. Summing (21.73) over equal physical indices gives (21.74). Under length independence every accumulated coefficient $w(h)$ is one, so self-adjointness and idempotence hold block by block. $\square$

Lemma 21.3.47 (Sitewise physical-coordinate congruence). [Lean](#) [✓ Stmt](#) Let V be a one-site matrix, not necessarily square. Applying $V^{\otimes N}$ to both physical legs of the length- N operator generated by M is the operator generated by the tensor whose one-site physical matrices are conjugated by V .

Proof. [✓ Proof](#) Expand a matrix entry as a sum over the original ket and bra configurations. The coefficient factors site by site, and the remaining virtual product is the word evaluation of the original tensor. $\square$

Theorem 21.3.48 (Positivity of the terminal transfer matrices). [Lean](#) [✓ Stmt](#) Let M be a matrix product density operator, and choose a vertical canonical decomposition with basis tensors M_γ and positive diagonal multiplicity matrices μ_γ . Then every terminal transfer matrix $T_\gamma = \text{tr}(M_\gamma)$ is positive semidefinite. This supplies the positivity needed for the spectral decomposition invoked in [CPGSV16, lines 1010–1016]; it asserts neither that T_γ is a projection nor the subsequent spectral, Hamiltonian, or Gibbs decomposition.

Local fix (terminal trace orientation): The trace closes the horizontal operator leg and leaves the two bond indices of M_γ open. See [con26f].

Proof. [✓ Proof](#) The fusion clause gives the tensor-attached algebra clause for the same vertical decomposition. Apply Theorem 21.2.2.66. $\square$

Definition 21.3.49 (All-label recursive density operator). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a chain of positive length N , choose at every site a BNT label α_j and a diagonal coordinate q_j of μ_{α_j} . Equivalently, putting $L = N - 1$, the chain has length $L + 1 = N$ and contains L appended labels after the initial site. Define

$$m(\alpha, q) = \prod_{j=0}^{N-1} (\mu_{\alpha_j})_{q_j, q_j}. \quad (21.77)$$

The first label is the initial label of the recursive fusion, while the remaining labels are appended in the physical order $\alpha_1, \dots, \alpha_{N-1}$. The recursive specification records this sequence in the reverse order $(\alpha_{N-1}, \dots, \alpha_1)$, so removing its first entry removes the final physical site. This reversal affects only the recursive notation: the tensor product and the sequential fusion coisometry remain in the physical order $\alpha_0, \dots, \alpha_{N-1}$. Let $W_{N,c}$ be the resulting retained-row sequential coisometry for the configuration c , and set $Q_{N,c} = Q_N((\text{tr}(M_\gamma))_\gamma)$. The corresponding density block is $m(\alpha, q)W_{N,c}^\dagger Q_{N,c} W_{N,c}$. Thus the embedding $\tilde{U}$ in the source formula is $W_{N,c}^\dagger$. The all-label recursive density operator is the direct sum of these blocks over every sitewise label and diagonal coordinate; denote it by R_N . On the same direct-sum coordinates, define

$$\begin{aligned} A_N &= \bigoplus_c m(c) I_c, \\ T_N &= \bigoplus_c W_{N,c}^\dagger Q_{N,c} W_{N,c}, \\ R_N &= A_N T_N. \end{aligned} \quad (21.78)$$

Its scalar coefficients are precisely the selected diagonal coefficients of the factor $\mu^{\otimes N}$ in [CPGSV16, line 999]; the identities mentioned there act on the complementary tensor factors of each block.

Theorem 21.3.50 (All-label density decomposition from an active-sector fusion clause). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Fix an active-sector fusion clause for a vertical canonical decomposition of M . For every $N > 0$, its retained-row vertical coisometry V and all-label recursive density operator R_N satisfy

$$\begin{aligned} V^{\otimes N} \rho^{(N)}(M) (V^{\otimes N})^\dagger &= R_N, \\ ((V^\dagger)^{\otimes N}) R_N V^{\otimes N} &= \rho^{(N)}(M). \end{aligned}$$

Scope restriction (active product BNT): only nonzero product corners occur in the fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction. See [con26j].

Local fix (rectangular vertical map): the reverse equality uses exact retained-sector reconstruction rather than a square-unitary identity. See [con26z].

Proof. ✓ Proof In retained coordinates, the one-site entries are block diagonal in the label and multiplicity coordinates. Within each configuration, multiply the scalar weights and apply the forward and adjoint recursive fusion identities. Summing over all configurations gives the forward equality; exact sitewise retained-sector reconstruction gives the reverse equality. $\square$

Theorem 21.3.51 (All-label density decomposition of an RFP MPDO). Lean ✓ Stmt Let M be a matrix product density operator in normalized BNT-refined horizontal form and suppose that it satisfies the renormalization fixed-point condition. There is one vertical canonical decomposition and one compatible family of fusion coisometries such that, for every $N > 0$, the rectangular vertical coisometry V satisfies the two exact equations

$$V^{\otimes N} \rho^{(N)}(M) (V^{\otimes N})^\dagger = R_N, \quad (21.79)$$

$$((V^\dagger)^{\otimes N}) R_N V^{\otimes N} = \rho^{(N)}(M). \quad (21.80)$$

No additional compatibility hypothesis is imposed. The statement is restricted to positive lengths and does not assert the commutator at line 1001, length independence, or the subsequent spectral and Gibbs decomposition.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Local fix (rectangular vertical map): The second equality uses exact reconstruction from the retained nonzero sectors of Proposition 4.13 and Appendix C.4, lines 2020–2029, not a square-unitary identity. See [con26z].

Proof. ✓ Proof Under the normalized BNT-refined horizontal hypothesis, the vertical canonical-form theorem gives a rectangular coisometry onto the retained nonzero vertical sectors, together with exact reconstruction from those sectors. In the retained coordinates, one-site matrix entries are block diagonal in the label and multiplicity coordinate. Hence a closed chain vanishes between different sitewise configurations. Within one configuration, the scalar entries multiply to $m(\alpha, q)$, while the remaining horizontal matrices form the fixed-label product tensor. Apply (21.73) and (21.75), then sum over all configurations. The local reconstruction identity, applied at every site, gives (21.80). $\square$

Corollary 21.3.52 (All-label density-factor commutator from an active-sector fusion clause).

Lean Lean Lean Lean ✓ Stmt For any active-sector fusion clause and every $N > 0$, its all-label recursive density operator factors as

$$\begin{aligned} R_N &= A_N T_N, \\ A_N T_N &= T_N A_N. \end{aligned}$$

Thus the multiplicity-weight factor commutes with the reconstructed recursive factor. No length-independence or spectral hypothesis is used.

Proof. ✓ Proof In every copy configuration, A_N is a scalar multiple of the identity on the corresponding simple-bond space, so it commutes with the coisometrically reconstructed recursive factor. Taking the direct sum gives both identities. $\square$

Corollary 21.3.53 (All-label density-factor commutator). [Lean](#) [✓ Stmt](#) Let M be a matrix product density operator in normalized BNT-refined horizontal form and suppose that it satisfies the renormalization fixed-point condition. One choice of the vertical canonical decomposition and fusion coisometries satisfies, for every $N > 0$,

$$R_N = A_N T_N, \quad (21.81)$$

$$A_N T_N = T_N A_N. \quad (21.82)$$

Equivalently, $[A_N, T_N] = 0$, as asserted in [CPGSV16, lines 1000–1002]. No additional compatibility hypothesis is imposed. This statement makes no length-independence, terminal spectral, projection, Hamiltonian, or Gibbs-state assertion.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Proof. [✓ Proof](#) In every sitewise copy configuration, the factor A_N is the scalar $m(c)$ times the identity on the corresponding simple-bond space. It therefore commutes with $W_{N,c}^\dagger Q_{N,c} W_{N,c}$. Multiplication of the two direct sums proves (21.81) and (21.82). The renormalization fixed-point condition supplies the same fusion family used in the density decomposition. $\square$

Definition 21.3.54 (Terminal spectral refinement for an active-sector fusion clause). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M generate matrix product density operators, and fix an active-sector fusion clause. For every terminal label γ , close the horizontal operator leg of M_γ to obtain a positive semidefinite bond matrix K_γ . Its spectral indices are pairs $s = (\gamma, k)$; write λ_s for the corresponding eigenvalue and E_s for the rank-one spectral projection. Recursively transport this family through every fusion history and the adjoint sequential coisometry, and denote the resulting all-label operator at positive length N by $P_s^{(N)}$.

Scope restriction (active product BNT): only nonzero product corners occur in the fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction. See [con26j].

Definition 21.3.55 (Terminal spectral refinement data). [Lean](#) [✓ Stmt](#) For an MPDO in normalized BNT-refined horizontal form satisfying the renormalization fixed-point condition, fix the fusion clause selected by that construction. For every terminal label γ , close the horizontal operator leg of M_γ to obtain a matrix K_γ on its bond space. Its spectral indices are pairs $s = (\gamma, k)$ with $0 \leq k < D_\gamma$. Write λ_s for the corresponding eigenvalue and E_s for the rank-one spectral projection, extended by zero on terminal labels different from γ . Recursively transport this terminal family through every fusion history, then through the adjoint sequential fusion coisometry, and assemble over all sitewise label and multiplicity configurations. Denote the resulting all-label operator at positive chain length N by $P_s^{(N)}$.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Theorem 21.3.56 (Terminal spectral projectors from an active-sector fusion clause). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let M generate matrix product density operators, fix an active-sector fusion clause, and suppose that its positive trace-power coefficients are independent

of the positive chain length. Then every terminal eigenvalue λ_s is non-negative. For every $N > 0$, the operators $P_s^{(N)}$ are pairwise orthogonal projections and

$$\begin{aligned} T_N &= \sum_s \lambda_s P_s^{(N)}, \\ A_N P_s^{(N)} &= P_s^{(N)} A_N, \\ R_N &= \sum_s \lambda_s A_N P_s^{(N)}. \end{aligned}$$

This is the terminal spectral decomposition of [CPGSV16, lines 1003–1012]; it makes no Hamiltonian or Gibbs-state assertion.

Scope restriction (active product BNT): only nonzero product corners occur in the fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction. See [con26j].

Proof. ✓ Proof Positivity of the physical density operator makes each terminal bond matrix positive semidefinite. Its spectral theorem supplies the non-negative eigenvalues and orthogonal rank-one projections. Length independence makes every retained fusion coefficient equal to one, so recursive coisometric transport preserves orthogonality and the spectral sum. The generic density-factor commutator gives the final two identities. $\square$

Theorem 21.3.57 (Terminal spectral projectors for a length-independent RFP MPDO). Lean

✓ Stmt Let M be a matrix product density operator in normalized BNT-refined horizontal form satisfying the renormalization fixed-point condition. Fix internally the fusion clause supplied by that construction, and suppose that its positive trace-power coefficients are independent of the positive chain length. Then K_γ is positive semidefinite for every terminal label, so $\lambda_s \geq 0$. Write the total positive chain length as $N = n + 1$, where $n \in \mathbb{N}$ is its predecessor. For every $N > 0$,

$$\begin{aligned} (P_s^{(N)})^\dagger &= P_s^{(N)}, \\ (P_s^{(N)})^2 &= P_s^{(N)}. \end{aligned}$$

For distinct spectral indices s and t ,

$$P_s^{(N)} P_t^{(N)} = 0.$$

The recursive factor and multiplicity weights satisfy

$$T_N = \sum_s \lambda_s P_s^{(N)}, \tag{21.83}$$

$$A_N P_s^{(N)} = P_s^{(N)} A_N. \tag{21.84}$$

Consequently,

$$R_N = \sum_s \lambda_s A_N P_s^{(N)}. \tag{21.85}$$

The length-independence assumption here is clause-relative: it is stated on the coefficients of the fixed clause selected from the renormalization fixed-point construction, not on an arbitrary fusion clause.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

This theorem stops at [CPGSV16, lines 1003–1012]; it does not assert the commuting nearest-neighbor Hamiltonian or Gibbs-state conclusion beginning at line 1013.

Proof. ✓ Proof Positivity of the one-site density operator passes to every fixed BNT block; division by its strictly positive multiplicity weight shows that each K_γ is positive semidefinite. Apply the finite-dimensional spectral theorem to obtain the non-negative eigenvalues and rank-one orthogonal projections. Length independence makes every retained fusion coefficient equal to one, so the recursive direct sum preserves idempotence and pairwise orthogonality. Write $Q_{N,c,s}$ for the recursive spectral block in copy configuration c . The coisometry identity $W_{N,c}W_{N,c}^\dagger = 1$ gives

$$\begin{aligned}(W_{N,c}^\dagger Q_{N,c,s} W_{N,c})^\dagger &= W_{N,c}^\dagger Q_{N,c,s} W_{N,c}, \\ (W_{N,c}^\dagger Q_{N,c,s} W_{N,c})^2 &= W_{N,c}^\dagger Q_{N,c,s} W_{N,c}.\end{aligned}$$

Thus transport through the adjoint sequential coisometry preserves the projector identities. Linearity gives (21.83). Finally, A_N is scalar on every copy-configuration block, hence it commutes with each $P_s^{(N)}$ and gives (21.85). $\square$

Definition 21.3.58 (Retained-coordinate commuting Gibbs form above one site). Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let M generate matrix product density operators, fix an active-sector fusion clause, and let μ be its strictly positive diagonal multiplicity matrix in the retained one-site coordinates. Define $h = \text{diag}(-\log \mu) \otimes I$ on two adjacent sites. For a chain of length $L = N + 2$, let $H_L = \sum_{i=1}^L h_{i,i+1}$, with periodic indices.

The terminal spectral family has cardinality at most the physical one-site dimension d . Fix an injection of its indices into $\{1, \dots, d\}$ and extend the remaining eigenvalues and projectors by zero.

Scope restriction (retained vertical coordinates): the Hamiltonian and projectors act on the retained nonzero vertical sectors. Applying the adjoint retained-row map reconstructs the physical density operator, but does not provide a Hamiltonian or projectors on the original physical space. This restriction is recorded in [con26r].

Scope restriction (positive chains of length at least two): the local interaction is defined on two spins, so these data are indexed by $L = N + 2$. The length-one boundary is recorded in [con26i].

Theorem 21.3.59 (Active-sector fusion gives a retained-coordinate commuting Gibbs decomposition above one site). Lean Lean Lean Lean Lean Lean Lean Lean Lean ✓ Stmt Let M generate matrix product density operators, and fix an active-sector fusion clause for a vertical canonical decomposition of M . Suppose that the trace-power coefficients of its positive diagonal fusion matrices are independent of the positive chain length. There are non-negative numbers $\lambda_1, \dots, \lambda_d$, independent of the chain length. For every $L \geq 2$, the same two-site matrix h is Hermitian, its periodic translates commute pairwise, and

$$H_L = \sum_{j=1}^L h_{j,j+1}, \tag{21.86}$$

$$e^{-H_L} = \mu^{\otimes L}. \tag{21.87}$$

There are orthogonal projections $P_1^{(L)}, \dots, P_d^{(L)}$ such that

$$[P_i^{(L)}, e^{-H_L}] = 0, \quad (21.88)$$

$$R_L = \sum_{i=1}^d \lambda_i P_i^{(L)} e^{-H_L}. \quad (21.89)$$

Applying the adjoint retained-row map at every site reconstructs $\rho^{(L)}(M)$ from the right-hand side of (21.89).

Scope restriction (active product BNT): only nonzero product corners occur in the fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family, and no unsupported normal sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction of the product tensor. See [con26j].

Scope restriction (retained vertical coordinates): the formula is an equality for R_L on the retained vertical space, not the literal physical-space equality asserted in [CPGSV16, lines 1013–1016]. Exact reconstruction by the adjoint retained-row map is weaker than the existence of physical projectors and a finite physical Hamiltonian. See [con26r].

Scope restriction (positive chains of length at least two): the conclusion is proved for $L = N + 2$. Definition 4.8 of [CPGSV16] defines h on two spins but does not state a length-one convention; see [con26i].

Proof. ✓ **Proof** Each diagonal entry of μ is strictly positive, so its negative logarithm is real. The translated terms are diagonal, and each site contributes once to their sum. Exponentiating gives (21.87).

Positivity of the multiplicities embeds one terminal mode for every terminal spectral index into the retained coordinates. The rank of the retained-row coisometry is at most d , so the terminal family has at most d members. Extend the eigenvalues and spectral projectors by zero along an injection into $\{1, \dots, d\}$. The added zero projectors preserve self-adjointness, idempotence, commutation, and the spectral sum. Combine that sum with (21.87) and the exact adjoint reconstruction of the retained density operator. $\square$

Theorem 21.3.60 (Retained-coordinate commuting Gibbs decomposition above one site).

Lean ✓ **Stmt** Let M be a matrix product density operator in normalized BNT-refined horizontal form satisfying the renormalization fixed-point condition. Select the active-sector fusion clause supplied by that condition, and suppose that the trace-power coefficients of this same clause are independent of the positive chain length. Then the retained-coordinate commuting Gibbs decomposition of Theorem 21.3.59 holds for every $L = N + 2$.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Scope restriction (retained vertical coordinates): the Hamiltonian and projectors act on the retained nonzero vertical sectors. Applying the adjoint retained-row map reconstructs the physical density operator, but does not provide a Hamiltonian or projectors on the original physical space. See [con26r].

Scope restriction (positive chains of length at least two): the conclusion is proved for $L = N + 2$; no length-one two-site convention is asserted. See [con26i].

Proof. ✓ Proof Select the active-sector fusion clause supplied by the renormalization fixed-point condition and apply Theorem 21.3.59 with the assumed length independence of its trace-power coefficients. $\square$

Theorem 21.3.61 (Active-sector fusion gives a physical-coordinate commuting Gibbs decomposition above one site). Lean ✓ Stmt *Let M generate matrix product density operators, and fix an active-sector fusion clause for a vertical canonical decomposition of M . Suppose that the trace-power coefficients of its positive diagonal fusion matrices are independent of the positive chain length. There are non-negative numbers $\lambda_1, \dots, \lambda_d$ and one fixed Hermitian two-site matrix $\hat{h}$, all independent of the chain length. For every $L \geq 2$, let $\widehat{H}_L = \sum_{j=1}^L \hat{h}_{j,j+1}$. The translates of $\hat{h}$ commute pairwise, and there are orthogonal projections $\widehat{P}_1^{(L)}, \dots, \widehat{P}_d^{(L)}$ on the physical chain such that*

$$\begin{aligned} [\hat{h}_{j-1,j}, \hat{h}_{j,j+1}] &= 0, \\ [\widehat{P}_i^{(L)}, e^{-\widehat{H}_L}] &= 0, \\ \rho^{(L)}(M) &= \sum_{i=1}^d \lambda_i \widehat{P}_i^{(L)} e^{-\widehat{H}_L}. \end{aligned} \tag{21.90}$$

This is the fusion-clause implication in the proof of [CPGSV16, lines 999–1016].

Scope restriction (active product BNT): only nonzero product corners occur in the fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family, and no unsupported normal sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction of the product tensor. See [con26j].

Local fix (physical complement): the retained one-site energy is compressed to physical coordinates and extended by zero energy on the orthogonal complement. The retained projectors are transported through the adjoint sitewise coisometry. See [con26r].

Scope restriction (positive chains of length at least two): the conclusion is proved for $L = N + 2$. Definition 4.8 of [CPGSV16] defines the local interaction on two spins but gives no length-one convention. See [con26i].

Proof. ✓ Proof Put V for the retained-row coisometry and k for the retained logarithmic multiplicity energy. The physical energy $K = V^\dagger k V$ is Hermitian and

$$e^{-K} = V^\dagger e^{-k} V + (I - V^\dagger V). \tag{21.91}$$

Taking the fixed two-site term $K \otimes I$, its periodic translates act on separate one-site factors, hence commute, and their exponential is the tensor power of (21.91).

For the sitewise coisometry $W = V^{\otimes L}$, transport each retained projector by $\widehat{P}_i^{(L)} = W^\dagger P_i^{(L)} W$. The identity $WW^\dagger = I$ preserves self-adjointness, idempotence, and pairwise orthogonality. The one-site intertwining identities transport the retained commutator to the physical Gibbs factor. Applying the same identities term by term to (21.89), and then using exact adjoint reconstruction, gives (21.90). $\square$

Theorem 21.3.62 (Physical-coordinate commuting Gibbs decomposition above one site). Lean ✓ Stmt *Let M be a matrix product density operator in normalized BNT-refined horizontal form*

satisfying the renormalization fixed-point condition. Select the active-sector fusion clause supplied by that condition, and suppose that the trace-power coefficients of this same clause are independent of the positive chain length. Then the physical-coordinate commuting Gibbs decomposition of Theorem 21.3.61 holds for every $L = N + 2$.

Scope restriction (BNT-refined horizontal form): the normalized BNT-refined horizontal hypothesis is stronger than the literal CPSV canonical-form hypothesis.

Local fix (physical complement): the retained one-site energy is compressed to physical coordinates and extended by zero energy on the orthogonal complement. The retained projectors are transported through the adjoint sitewise coisometry. See [con26r].

Scope restriction (positive chains of length at least two): the conclusion is proved for $L = N + 2$; no length-one two-site convention is asserted. See [con26i].

Proof. ✓ **Proof** Select the active-sector fusion clause supplied by the renormalization fixed-point condition and apply Theorem 21.3.61 with the assumed length independence of its trace-power coefficients. □

Definition 21.3.63 (Selected literal CPSV RFP fusion clause). Lean ✓ Stmt Let M generate matrix product density operators, suppose that its doubled-index MPS tensor is in literal CPSV canonical form, and suppose that M satisfies the renormalization fixed-point condition. Choose one active-sector BNT fusion tensor clause furnished by these hypotheses and denote it by $\mathcal{F}_{\text{CPSV}}(M)$.

Scope restriction (active product BNT): only nonzero product corners occur in the selected fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family, and no unsupported normal sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction of the product tensor. See [con26j].

Theorem 21.3.64 (Literal CPSV physical-coordinate commuting Gibbs decomposition above one site). Lean ✓ Stmt Let M generate matrix product density operators, suppose that its doubled-index MPS tensor is in literal CPSV canonical form, and suppose that M satisfies the renormalization fixed-point condition of Definition 4.1. Select the active-sector fusion clause supplied by these three hypotheses, and assume that the trace-power coefficients of this same clause are independent of the positive chain length.

There are non-negative numbers $\lambda_1, \dots, \lambda_d$ and one fixed Hermitian two-site matrix $\hat{h}$, all independent of the chain length. For every $L = N + 2$, let $\widehat{H}_L = \sum_{j=1}^L \hat{h}_{j,j+1}$. The translates of $\hat{h}$ commute pairwise, and there are pairwise orthogonal projections $\widehat{P}_1^{(L)}, \dots, \widehat{P}_d^{(L)}$ on the physical chain such that

$$\begin{aligned} [\hat{h}_{j-1,j}, \hat{h}_{j,j+1}] &= 0, \\ [\widehat{P}_i^{(L)}, e^{-\widehat{H}_L}] &= 0, \\ \rho^{(L)}(M) &= \sum_{i=1}^d \lambda_i \widehat{P}_i^{(L)} e^{-\widehat{H}_L}. \end{aligned}$$

This is the physical-coordinate conclusion of [CPGSV16, lines 1013–1016], under the literal canonical form and renormalization fixed-point hypotheses of [CPGSV16, Theorem 4.14, lines 972–993].

Scope restriction (active product BNT): only nonzero product corners occur in the selected fusion clause. A normal label absent from a fixed product pair has zero fusion multiplicity. See [con26c].

Local fix (Figure-11 fixed-pair support): a fixed product pair may have an empty active family, and no unsupported normal sector is inserted. See [con26q].

Local fix (Figure-11 fusion coisometry): the retained-row fusion map is a coisometry onto the active direct sum, and its adjoint gives exact reconstruction of the product tensor. See [con26j].

Local fix (physical complement): the retained one-site energy is compressed to physical coordinates and extended by zero energy on the orthogonal complement. The retained projectors are transported through the adjoint sitewise coisometry. See [con26r].

Scope restriction (positive chains of length at least two): the conclusion is proved exactly for $L = N + 2$. Definition 4.8 of [CPGSV16] defines the local interaction on two spins but gives no length-one convention. See [con26i].

Proof. ✓ Proof Choose the active-sector fusion clause supplied by Theorem 21.3.25. Its trace-power coefficients are length independent by hypothesis. Apply Theorem 21.3.61. $\square$

Remark 21.3.65 (Printed length-one domain boundary). Definition 4.8 of [CPGSV16] lets a local operator act on the first two spins and translates it around the periodic chain. Thus its construction has domain $L \geq 2$. The periodic identification $L + 1 = 1$ identifies site labels, but supplies neither a map from two-site operators to one-site operators nor a multiplicity for the coincident edge.

Theorem 4.15 is printed without an explicit lower bound on L , but its $L = 1$ instance is therefore undefined without an author or erratum convention. The source construction is defined for $L \geq 2$, and no length-one theorem is asserted. The physical-coordinate theorem above proves the result on this domain; no $L = 1$ statement follows without an additional convention.

Definition 21.3.66 (Full-support specialization of one fusion layer). Lean Lean ✓ Stmt For a full-support fusion family, let P_γ act on the bond space of M_γ . Define

$$Q_{\alpha,\beta}^{\text{fs}} = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes P_\gamma,$$

$$\widehat{Q}_{\alpha,\beta}^{\text{fs}} = U_{\alpha,\beta}^\dagger Q_{\alpha,\beta}^{\text{fs}} U_{\alpha,\beta}.$$

The terminal contraction is the same as in the source operator. The additional condition is that the active support is the whole product bond space.

Scope restriction (full-support fusion family): Full support is not an additional hypothesis of the arbitrary-chain source statement. This distinction is documented in [con26f].

Theorem 21.3.67 (Projection property of the full-support specialization). Lean Lean Lean ✓ Stmt Suppose that the labelled tensors form a basis of normal tensors and the positive trace-power coefficients are independent of the positive chain length. If each P_γ is a self-adjoint idempotent on the bond space, then $Q_{\alpha,\beta}^{\text{fs}}$ is a self-adjoint idempotent and $\widehat{Q}_{\alpha,\beta}^{\text{fs}}$ is an orthogonal projection.

Proof. ✓ Proof Length independence gives $\chi_{\alpha,\beta,\gamma} = 1$, hence $(Q_{\alpha,\beta}^{\text{fs}})^2 = Q_{\alpha,\beta}^{\text{fs}}$ and $(Q_{\alpha,\beta}^{\text{fs}})^\dagger = Q_{\alpha,\beta}^{\text{fs}}$. The basis-of-normal-tensors completeness theorem gives $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = 1$, so

$$(\widehat{Q}_{\alpha,\beta}^{\text{fs}})^2 = U_{\alpha,\beta}^\dagger Q_{\alpha,\beta}^{\text{fs}} (U_{\alpha,\beta} U_{\alpha,\beta}^\dagger) Q_{\alpha,\beta}^{\text{fs}} U_{\alpha,\beta} = \widehat{Q}_{\alpha,\beta}^{\text{fs}}.$$

$\square$

Theorem 21.3.68 (Left iterated fusion completeness). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem 21.3.41, $U_{\alpha,\beta,\gamma}^L (U_{\alpha,\beta,\gamma}^L)^\dagger = 1$.

Proof. [✓ Proof](#) Pair fusion completeness gives $\sum_\delta r_{a,b}^\delta D_\delta = D_a D_b$. Applying this equality first to (δ, γ) and then to (α, β) shows that the source and target dimensions of $U_{\alpha,\beta,\gamma}^L$ coincide. A square isometry is surjective. $\square$

Theorem 21.3.69 (Right iterated fusion completeness). [Lean](#) [✓ Stmt](#) Under the same hypotheses, $U_{\alpha,\beta,\gamma}^R (U_{\alpha,\beta,\gamma}^R)^\dagger = 1$.

Proof. [✓ Proof](#) The right-associated dimension calculation is

$$\sum_{\delta,\varepsilon} r_{\beta,\gamma}^\delta r_{\alpha,\delta}^\varepsilon D_\varepsilon = D_\alpha (D_\beta D_\gamma).$$

The right iterated fusion map is therefore a square isometry. $\square$

Theorem 21.3.70 (Right adjoint inverse for the full triple-fusion comparison). [Lean](#) [✓ Stmt](#) Under the same hypotheses, the comparison of the full direct sums satisfies $C_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}^\dagger = 1$.

Proof. [✓ Proof](#) The left range identity reduces the product to $U^L (U^L)^\dagger$, which is the identity by Theorem 21.3.68. $\square$

Theorem 21.3.71 (Left adjoint inverse for the full triple-fusion comparison). [Lean](#) [✓ Stmt](#) Under the same hypotheses, $C_{\alpha,\beta,\gamma}^\dagger C_{\alpha,\beta,\gamma} = 1$. This proves invertibility on the full direct sums. It does not yet prove invertibility of a fixed-final multiplicity matrix.

Proof. [✓ Proof](#) Apply the right range identity and Theorem 21.3.69. $\square$

Theorem 21.3.72 (Vanishing between distinct final sectors). [Lean](#) [✓ Stmt](#) Suppose the positive trace-power coefficients are independent of the positive chain length and common word selectors exist. If $\varepsilon \neq \varepsilon'$, then the corner of the full comparison from the right final sector ε' to the left final sector ε vanishes: $P_\varepsilon^L C_{\alpha,\beta,\gamma} P_{\varepsilon'}^R = 0$. The assertion is conditional on the displayed word-selector identities. It neither states that a diagonal corner is invertible nor identifies one with an F -matrix, and it does not assert the pentagon identity.

Proof. [✓ Proof](#) Fix one left multiplicity index and one right multiplicity index. Taking the corresponding bond-space corner of the full letterwise identity gives $M_\varepsilon^{ij} X = X M_{\varepsilon'}^{ij}$. Induction on the word w gives $M_\varepsilon^w X = X M_{\varepsilon'}^w$. Multiply these identities by $c_\varepsilon(w)$ and sum over all words of length S . The two selector identities give $X = 0$. Since the multiplicity indices were arbitrary, the whole off-diagonal final-sector corner vanishes. $\square$

Definition 21.3.73 (Fixed-final multiplicity spaces). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) Fix a final label ε . The multiplicity spaces of the two bracketings are

$$\begin{aligned} \mathcal{H}_\varepsilon^L &= \bigoplus_\delta \mathbb{C}^{\dim \chi_{\alpha,\beta,\delta}} \otimes \mathbb{C}^{\dim \chi_{\delta,\gamma,\varepsilon}}, \\ \mathcal{H}_\varepsilon^R &= \bigoplus_\delta \mathbb{C}^{\dim \chi_{\beta,\gamma,\delta}} \otimes \mathbb{C}^{\dim \chi_{\alpha,\delta,\varepsilon}}. \end{aligned}$$

These spaces have the bracketing pattern of the (e, μ, ν) and (f, λ, σ) index spaces in the printed- F row/column convention of [BMW⁺17, Section 3.4]. Identifying the dimensions of the positive diagonal matrices with the fusion multiplicities of that equation requires the length-independent integer specialization and is not asserted here. The corresponding fixed-final row spaces are $\mathcal{H}_\varepsilon^L \otimes \mathbb{C}^{D_\varepsilon}$ and $\mathcal{H}_\varepsilon^R \otimes \mathbb{C}^{D_\varepsilon}$.

Theorem 21.3.74 (Equality of fixed-final multiplicity dimensions). [Lean](#) [Stmnt](#) Suppose that the positive trace-power coefficients are independent of the positive chain length and that the labelled operators are eventually linearly independent. Then, for each final label ε ,

$$\dim \mathcal{H}_\varepsilon^L = \sum_{\delta} \dim \chi_{\alpha,\beta,\delta} \dim \chi_{\delta,\gamma,\varepsilon} = \sum_{\delta} \dim \chi_{\beta,\gamma,\delta} \dim \chi_{\alpha,\delta,\varepsilon} = \dim \mathcal{H}_\varepsilon^R.$$

Consequently a matrix between the two fixed-final multiplicity spaces is square. This is the numerical associativity statement of [\[BMW⁺17, lines 237–241\]](#); it does not assert completeness of the fusion maps or invertibility of the fixed-final comparison.

Proof. [Proof](#) Choose a positive length beyond the linear-independence threshold. Associativity of the same-length product law gives equality of the two sums of products of structure coefficients. Length independence and positivity identify every coefficient with the size of its corresponding diagonal χ -matrix. The two sums are precisely the cardinalities of the displayed dependent sums. $\square$

Definition 21.3.75 (Triple-fusion maps with fixed final label). [Lean](#) [Lean](#) [Stmnt](#) The maps $U_{\alpha,\beta,\gamma;\varepsilon}^L$ and $U_{\alpha,\beta,\gamma;\varepsilon}^R$ are obtained by restricting the two iterated fusion isometries to the summands with final label ε . Their adjoints, after the canonical reassociation of the triple bond space, have the orientation and bracketing pattern of the weighted analogs of the two sides in the printed- F row/column convention of [\[BMW⁺17, Section 3.4\]](#).

Theorem 21.3.76 (Left triple fusion at a fixed final label). [Lean](#) [Stmnt](#) For every final label ε , the left-associated restriction satisfies the positive-diagonal weighted identity

$$U_{\alpha,\beta,\gamma;\varepsilon}^L ((M_\alpha M_\beta) M_\gamma)^{ij} (U_{\alpha,\beta,\gamma;\varepsilon}^L)^\dagger = \bigoplus_{\delta} \chi_{\alpha,\beta,\delta} \otimes (\chi_{\delta,\gamma,\varepsilon} \otimes M_\varepsilon^{ij}).$$

Proof. [Proof](#) Restrict Theorem [21.3.32](#) to the rows and columns whose final label is ε . Reindexing the conjugating matrix commutes with conjugation, and the outer block-diagonal matrix retains precisely the displayed ε -blocks. $\square$

Theorem 21.3.77 (Right triple fusion at a fixed final label). [Lean](#) [Stmnt](#) For every final label ε , the right-associated restriction satisfies the positive-diagonal weighted identity

$$U_{\alpha,\beta,\gamma;\varepsilon}^R (M_\alpha (M_\beta M_\gamma))^{ij} (U_{\alpha,\beta,\gamma;\varepsilon}^R)^\dagger = \bigoplus_{\delta} \chi_{\beta,\gamma,\delta} \otimes (\chi_{\alpha,\delta,\varepsilon} \otimes M_\varepsilon^{ij}).$$

Proof. [Proof](#) Restrict Theorem [21.3.35](#) to the rows and columns whose final label is ε . Reindexing the conjugating matrix commutes with conjugation, and the outer block-diagonal matrix retains precisely the displayed ε -blocks. $\square$

Theorem 21.3.78 (Injective fixed-final intertwiners). [Lean](#) [Stmnt](#) Suppose that M_ε is injective. Let C be a matrix from the right fixed-final-sector space to the left fixed-final-sector space. Under the canonical identifications with $\mathcal{H}_\varepsilon^L \otimes \mathbb{C}^{D_\varepsilon}$ and $\mathcal{H}_\varepsilon^R \otimes \mathbb{C}^{D_\varepsilon}$, assume that, for every pair of physical indices i, j ,

$$(1_{\mathcal{H}_\varepsilon^L} \otimes M_\varepsilon^{ij})C = C(1_{\mathcal{H}_\varepsilon^R} \otimes M_\varepsilon^{ij}).$$

Then there is a rectangular matrix $F : \mathcal{H}_\varepsilon^R \rightarrow \mathcal{H}_\varepsilon^L$ such that $C = F \otimes 1_{D_\varepsilon}$. This conclusion is conditional on the supplied matrix C . It does not assert the existence or invertibility of C or F , and it does not identify the dimensions of the positive diagonal matrices χ with the fusion multiplicities in the printed- F row/column convention of [\[BMW⁺17, Section 3.4\]](#).

Proof. ✓ Proof Apply Lemma 2.1.6 to the family $\{M_\varepsilon^{ij}\}_{i,j}$. This family spans the full matrix algebra by injectivity, and the assumed relation is precisely the amplified intertwining identity of that lemma. $\square$

Theorem 21.3.79 (Conditional fixed-final Kronecker form and separation). Lean ✓ Stmt *Suppose the positive trace-power coefficients are independent of the positive chain length and common word selectors separate the final labels. Fix labels $\alpha, \beta, \gamma, \varepsilon$, and suppose that M_ε is injective at the present blocking. Write $C_{\alpha, \beta, \gamma}^{\varepsilon, \varepsilon'} = P_\varepsilon^L C_{\alpha, \beta, \gamma} P_{\varepsilon'}^R$ for the corner from the right final sector ε' to the left final sector ε . There is a rectangular matrix $F_\varepsilon : \mathcal{H}_\varepsilon^R \rightarrow \mathcal{H}_\varepsilon^L$ such that, after the canonical index identifications, $C_{\alpha, \beta, \gamma}^{\varepsilon, \varepsilon} = F_\varepsilon \otimes 1_{D_\varepsilon}$. For every $\varepsilon' \neq \varepsilon$, $C_{\alpha, \beta, \gamma}^{\varepsilon, \varepsilon'} = 0$. The selector identities and injectivity at the present blocking are additional hypotheses here. The theorem neither asserts that F_ε is square or invertible nor identifies it with the printed F -matrix of [BMW⁺ 17, Section 3.4]; no pentagon identity is asserted.*

Proof. ✓ Proof Taking the ε - ε corner of the full comparison identity gives, for every pair i, j ,

$$(1_{\mathcal{H}_\varepsilon^L} \otimes M_\varepsilon^{ij}) C^{\varepsilon, \varepsilon} = C^{\varepsilon, \varepsilon} (1_{\mathcal{H}_\varepsilon^R} \otimes M_\varepsilon^{ij}).$$

Injectivity and the amplified commutant result give $C^{\varepsilon, \varepsilon} = F_\varepsilon \otimes 1_{D_\varepsilon}$. The common word selectors give $C^{\varepsilon, \varepsilon'} = 0$ whenever $\varepsilon' \neq \varepsilon$ by Theorem 21.3.72. $\square$

Theorem 21.3.80 (Unitarity of each fixed-final comparison corner). Lean Lean ✓ Stmt *Suppose the positive trace-power coefficients are independent of the positive chain length and common selectors exist at a positive length. For every final label ε , write $C_\varepsilon = P_\varepsilon^L C_{\alpha, \beta, \gamma} P_\varepsilon^R$. Then*

$$C_\varepsilon C_\varepsilon^\dagger = 1, \tag{21.92}$$

$$C_\varepsilon^\dagger C_\varepsilon = 1. \tag{21.93}$$

The conclusion is conditional on the simultaneous selector identities. It is the fixed-output-channel invertibility consequence of the simultaneous inverse used in [BMW⁺ 17, lines 247–277].

Proof. ✓ Proof Order each full direct sum first by its final label. In the product $CC^\dagger$, the ε -diagonal corner is $\sum_{\varepsilon'} C^{\varepsilon, \varepsilon'} (C^{\varepsilon, \varepsilon'})^\dagger$. Every term with $\varepsilon' \neq \varepsilon$ vanishes, while $CC^\dagger = 1$. Hence $C_\varepsilon C_\varepsilon^\dagger = 1$. Taking instead the ε -diagonal corner of $C^\dagger C = 1$ gives the second identity. $\square$

Theorem 21.3.81 (Unitarity of the fixed-final multiplicity matrix). Lean ✓ Stmt *Assume the hypotheses of Theorem 21.3.80. Suppose in addition that M_ε is injective at the present blocking and that $D_\varepsilon > 0$. There is a matrix $F_\varepsilon : \mathcal{H}_\varepsilon^R \rightarrow \mathcal{H}_\varepsilon^L$ such that*

$$C_\varepsilon = F_\varepsilon \otimes 1_{D_\varepsilon},$$

$$F_\varepsilon F_\varepsilon^\dagger = 1,$$

$$F_\varepsilon^\dagger F_\varepsilon = 1.$$

Thus F_ε is invertible. Its orientation is opposite to the printed- F row/column convention of [BMW⁺ 17, Section 3.4]; no pentagon identity is asserted.

Proof. ✓ Proof Injectivity gives $C_\varepsilon = F_\varepsilon \otimes 1_{D_\varepsilon}$. Substituting this expression into (21.92) and (21.93) gives

$$(F_\varepsilon F_\varepsilon^\dagger) \otimes 1_{D_\varepsilon} = 1,$$

$$(F_\varepsilon^\dagger F_\varepsilon) \otimes 1_{D_\varepsilon} = 1.$$

Since $D_\varepsilon > 0$, Lemma 21.3.13 removes the identity factor and gives the two asserted identities. $\square$

Definition 21.3.82 (Simultaneous MPO block left inverse). [Lean](#) [✓ Stmt](#) Let $M_c^{il} \in M_{D_c}(\mathbb{C})$ be a finite labelled MPO tensor family. A simultaneous block left inverse consists of coefficients $C_{(c,x,y),(i,l)}$ such that

$$\sum_{i,l} C_{(c,x,y),(i,l)} (M_d^{il})_{x'y'} = \delta_{cd} \delta_{xx'} \delta_{yy'}.$$

This is the common block inverse used in [CPGSV16, Appendix C.2, lines 1666–1676] and [BMW⁺17, lines 269–277].

Theorem 21.3.83 (Simultaneous inverse from a one-letter product-algebra span). [Lean](#) [Lean](#) [✓ Stmt](#) Let $A_c^q \in M_{D_c}(\mathbb{C})$ be a finite labelled tensor family. If the one-letter tuples $q \mapsto (A_c^q)_c$ span the full product algebra $\prod_c M_{D_c}(\mathbb{C})$, then there are coefficients $C_{(c,x,y),q}$ such that

$$\sum_q C_{(c,x,y),q} (A_d^q)_{x'y'} = \delta_{cd} \delta_{xx'} \delta_{yy'}.$$

For an MPO tensor the letter is the ket–bra pair $q = (i, k)$, so the same coefficients form the common block-letter left inverse used in [BMW⁺17, lines 269–277]. The full product-algebra span is obtained after the common blocking described in [BMW⁺17, lines 427–431].

Proof. [✓ Proof](#) Fix (c, x, y) and consider the product-algebra element which is the matrix unit E_{xy} in block c and zero in every other block. By the spanning hypothesis it is a linear combination of the one-letter tuples. Reading its (x', y') entry in block d gives the displayed identity. Reindexing the doubled physical letter by the ket–bra pair gives the MPO form. $\square$

Definition 21.3.84 (Complete zipper fusion family). [Lean](#) [✓ Stmt](#) Let Λ be a finite set. A complete zipper fusion family consists of positive bond dimensions D_a , MPO blocks $B_a^{ij} \in M_{D_a}(\mathbb{C})$, fusion multiplicities N_{ab}^c , and matrices

$$\begin{aligned} X_{ab,\mu}^c &: \mathbb{C}^{D_c} \longrightarrow \mathbb{C}^{D_a} \otimes \mathbb{C}^{D_b}, \\ X_{ab,\mu}^{c+} &: \mathbb{C}^{D_a} \otimes \mathbb{C}^{D_b} \longrightarrow \mathbb{C}^{D_c}. \end{aligned}$$

The matrices $X_{ab,\mu}^{c+}$ are independently chosen left inverses, not adjoints. For all labels and multiplicity indices,

$$X_{ab,\nu}^{d+} X_{ab,\mu}^c = \delta_{cd} \delta_{\mu\nu} \mathbb{1}_{D_c}. \quad (21.94)$$

The opposite product $P_{ab} = \sum_{c,\mu} X_{ab,\mu}^c X_{ab,\mu}^{c+}$ is the projector onto the product-MPO support; it need not be the identity on the ambient bond space. Every product-MPO letter is reconstructed on this support:

$$B_{ab}^{ik} = \sum_{c,\mu} X_{ab,\mu}^c B_c^{ik} X_{ab,\mu}^{c+}. \quad (21.95)$$

Writing $B_{ab}^{ik} = \sum_j B_a^{ij} \otimes B_b^{jk}$, the two zipper identities are

$$B_{ab}^{ik} X_{ab,\mu}^c = X_{ab,\mu}^c B_c^{ik}, \quad (21.96)$$

$$X_{ab,\mu}^{c+} B_{ab}^{ik} = B_c^{ik} X_{ab,\mu}^{c+}. \quad (21.97)$$

Each block is injective, and the labelled block-letter matrix has a common left inverse: its contraction with B_d^{ij} is $\delta_{cd} E_{xy}$ on the block selected by (c, x, y) . These are the fusion-tensor, reconstruction, zipper, and simultaneous inverse relations of [BMW⁺17, lines 161, 181–200, and 269–277].

$$\text{selector, } S > 0 : \sum_{|w|=S} c_\varepsilon(w) M_{\varepsilon'}^w = \delta_{\varepsilon, \varepsilon'} 1_{D_\varepsilon}$$

$$\text{sector cut, } \varepsilon \neq \varepsilon' : \quad \begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \varepsilon \\ \underbrace{\hspace{1.5cm}} \\ U_{\alpha, \beta, \gamma; \varepsilon}^L \end{array} \xleftarrow{C_{\alpha, \beta, \gamma}^{\varepsilon, \varepsilon'}} \begin{array}{c} \alpha \quad \beta \quad \gamma \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \varepsilon' \\ \underbrace{\hspace{1.5cm}} \\ U_{\alpha, \beta, \gamma; \varepsilon'}^R \end{array} = 0$$

$$\begin{aligned} \text{diagonal corner : } \mathcal{H}_\varepsilon^L \otimes \mathbb{C}^{D_\varepsilon} &\xleftarrow{F_\varepsilon \otimes 1_{D_\varepsilon}} \mathcal{H}_\varepsilon^R \otimes \mathbb{C}^{D_\varepsilon} \\ \Rightarrow F_\varepsilon F_\varepsilon^\dagger &= 1, \quad F_\varepsilon^\dagger F_\varepsilon = 1. \end{aligned}$$

Figure 21.4: The fixed-final comparison in the graphical order of [BMW⁺17, equation (pentagon3) and lines 269–277]. At a positive common length, the simultaneous word selector is the finite-word counterpart of the inverse B_d^+ in the source: it retains the chosen final label and annihilates every other one. Thus the comparison has no off-diagonal final-label corners. On the surviving corner, injectivity gives $C_\varepsilon = F_\varepsilon \otimes 1_{D_\varepsilon}$; two-sided unitarity of C_ε and $D_\varepsilon > 0$ give two-sided unitarity of F_ε . This is the associativity mechanism described in [CPGSV16, lines 995–1010]. No pentagon coherence identity is asserted here.

Theorem 21.3.85 (Complete zipper family from a source BNT). [Lean] [Lean] [Lean] [✓ Stmt]
Suppose that the labelled tensors of a length-independent fusion family form a source basis of normal tensors. There is one common positive blocking at which they determine a complete zipper fusion family. Its bond dimensions are the original labelled bond dimensions, its fusion multiplicities are $N_{\alpha\beta}^\gamma = \dim \chi_{\alpha, \beta, \gamma}$, its synthesis map is the adjoint of $U_{\alpha, \beta}$, and its analysis map is $U_{\alpha, \beta}$, after the canonical identification $\mathbb{C}^{D_\alpha D_\beta} \cong \mathbb{C}^{D_\alpha} \otimes \mathbb{C}^{D_\beta}$. In particular, the two zipper identities, literal reconstruction, injectivity, and the simultaneous block-letter inverse are conclusions; none is an additional hypothesis.

Proof. [✓ Proof] Length independence makes every chi matrix the identity. Use the common positive BNT blocking for the labelled tensors and transport the fusion equation through it. Pair completeness gives $U_{\alpha, \beta} U_{\alpha, \beta}^\dagger = \mathbb{1}$, so analysis followed by synthesis is the identity. The other three matrix identities are the transported zipper identities (21.96) and (21.97) and the reconstruction identity (21.95). The common one-letter product-algebra span supplies both block injectivity and the simultaneous block-letter inverse. These data are exactly those of Definition 21.3.84. $\square$

Definition 21.3.86 (One-letter final-block selector). [Lean] [✓ Stmt] Fix a final label d . Summing the diagonal matrix-unit coefficients of the simultaneous block inverse defines

$$s_d(i, j) = \sum_{x=0}^{D_d-1} C_{(d, x, x), (i, j)}.$$

Here $C_{(d, x, y), (i, j)}$ denotes the coefficient of the simultaneous block inverse corresponding to the matrix unit E_{xy} in block d .

Lemma 21.3.87 (Final-block selection identity). [Lean](#) [Lean](#) [✓ Stmt](#) *The one-letter selector satisfies*

$$\sum_{i,j} s_d(i,j) B_e^{ij} = \begin{cases} \mathbb{1}_{D_d}, & e = d, \\ 0, & e \neq d. \end{cases}$$

Thus one physical letter selects the identity on the chosen final block and annihilates every other final block.

Proof. [✓ Proof](#) Apply the simultaneous block inverse entrywise and sum over the diagonal matrix units of block d . If $e = d$, their sum is $\mathbb{1}_{D_d}$; if $e \neq d$, every term vanishes. $\square$

Theorem 21.3.88 (Printed F-move and inverse). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Fix $a, b, c, d \in \Lambda$. There is an invertible matrix*

$$F_d^{abc} : \bigoplus_e \mathbb{C}^{N_{ab}^e} \otimes \mathbb{C}^{N_{ec}^d} \longrightarrow \bigoplus_f \mathbb{C}^{N_{bc}^f} \otimes \mathbb{C}^{N_{af}^d}$$

whose rows are indexed by (f, λ, σ) and whose columns are indexed by (e, μ, ν) . Its entries satisfy

$$(X_{ab,\mu}^e \otimes \mathbb{1}) X_{ec,\nu}^d = \sum_{f,\lambda,\sigma} (F_d^{abc})_{e\mu\nu}^{f\lambda\sigma} (\mathbb{1} \otimes X_{bc,\lambda}^f) X_{af,\sigma}^d. \quad (21.98)$$

The matrix in the opposite orientation is its two-sided inverse. This is the printed F-move of [BMW⁺17, Section 3.4].

Proof. [✓ Proof](#) Form the full left- and right-associated synthesis matrices L and R and their analysis matrices L^+ and R^+ . Applying (21.95) twice to the two associative decompositions gives

$$B_{abc}^{il} = L D_L^{il} L^+ = R D_R^{il} R^+. \quad (21.99)$$

Contracting (21.99) with the one-letter final-block selector from Definition 21.3.86, the two direct sums D_L and D_R both contract to the identity. Consequently,

$$\begin{aligned} P &:= LL^+ = RR^+, \\ L^+ L &= R^+ R = \mathbb{1}. \end{aligned} \quad (21.100)$$

Hence $R^+ L$ changes left-tree coordinates into right-tree coordinates and

$$R(R^+ L) = (RR^+) L = PL = (LL^+) L = L.$$

Its opposite-oriented comparison is $L^+ R$. Using (21.100),

$$\begin{aligned} (R^+ L)(L^+ R) &= R^+ P R = \mathbb{1}, \\ (L^+ R)(R^+ L) &= L^+ P L = \mathbb{1}. \end{aligned}$$

The zipper identities (21.96) and (21.97) show that $R^+ L$ intertwines the direct sums of the final-block letters. The simultaneous block inverse annihilates its corners with different final labels. On the d -diagonal corner, injectivity gives $(R^+ L)_d = F_d^{abc} \otimes \mathbb{1}_{D_d}$. Restricting $R(R^+ L) = L$ to this corner gives (21.98). The corresponding fixed-final corner of $L^+ R$, together with $(R^+ L)(L^+ R) = \mathbb{1} = (L^+ R)(R^+ L)$, gives amplified inverse identities. Lemma 21.3.13 removes the nonempty bond-space identity factor and yields the two inverse identities for F_d^{abc} . $\square$

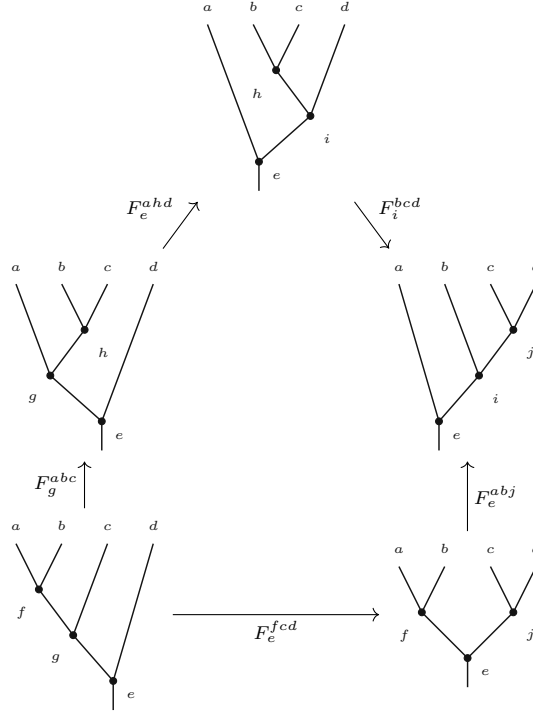


Figure 21.6: The five complete zipper fusion trees of [BMW⁺17, Section 3.4, equation (33)]. Each arrow is a printed F -move, with right-tree coordinates indexing rows and left-tree coordinates indexing columns.

denote the canonical reassociation. Define

$$\begin{aligned} r_1 &= r_{A,B,C} \times \mathbb{1}_D, \\ r_2 &= r_{A,B \times C,D}, \\ r_3 &= \mathbb{1}_A \times r_{B,C,D}, \\ s_1 &= r_{A \times B,C,D}, \\ s_2 &= r_{A,B,C \times D}. \end{aligned}$$

The three-edge and two-edge reassociations from $((A \times B) \times C) \times D$ to $A \times (B \times (C \times D))$ coincide:

$$r_3 \circ r_2 \circ r_1 = s_2 \circ s_1. \quad (21.102)$$

If $P(r)_{x,y} = 1$ when $r(x) = y$ and is zero otherwise, then the corresponding coordinate-permutation matrices satisfy

$$P(r_1)P(r_2)P(r_3) = P(s_1)P(s_2). \quad (21.103)$$

These are equalities of Cartesian bond-coordinate identifications only; they are distinct from the fusion-multiplicity identity in Theorem 21.3.90.

Proof. ✓ Proof Every element $((a,b),c),d)$ is sent by either composite to $(a,(b,(c,d)))$. This proves (21.102). The identity $P(v \circ u) = P(u)P(v)$ then turns (21.102) into (21.103). $\square$

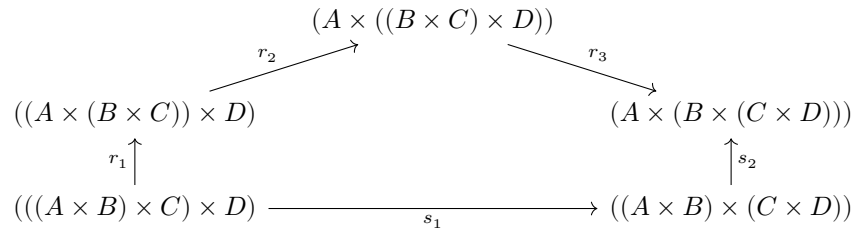


Figure 21.7: The five parenthesizations of four bond-coordinate factors. The upper path is the three-edge composite $r_3 \circ r_2 \circ r_1$, and the lower path is the two-edge composite $s_2 \circ s_1$. This figure depicts only canonical Cartesian reassociation. It does not contain fusion multiplicity spaces or F -matrices.

Chapter 22

Periodic MPS: irreducible form and the periodic Fundamental Theorem

This chapter develops the periodic equal-case Fundamental Theorem of MPS (Theorem 22.2.2.1) following [DICCSPG17], and gives its two applications: the symmetry corollary lifting a physical on-site symmetry of a tensor in irreducible form II to a virtual Z -gauge, and the equivalence between p -refinability of the matrix-product-vector family and p -divisibility of the transfer map.

22.1 Periodic irreducible form and physical-index rotation

Definition 22.1.1 (Spectrally periodic irreducible tensor). [Lean](#) [✓ Stmt](#) An irreducible MPS tensor A is *spectrally m -periodic* if $m \geq 1$, the spectral radius of its transfer map is one, and

$$\text{Spec}_{\text{per}}(\mathcal{E}_A) = \{\mu \in \mathbb{C} : \mu^m = 1\}.$$

No trace-preserving normalization is imposed. This is the form of the periodic blocks before the similarity transformation in [DICCSPG17, lines 248–261 and 313–332].

Theorem 22.1.2 (A spectrally periodic tensor has positive bond dimension). [Lean](#) [✓ Stmt](#) If A is spectrally periodic, then its bond dimension satisfies $D \geq 1$.

Proof. [✓ Proof](#) In bond dimension zero the matrix space and its endomorphism algebra are zero-dimensional, so the transfer map has spectral radius zero. This contradicts the spectral-radius-one hypothesis. $\square$

Lemma 22.1.3 (Peripheral spectrum is invariant under bond-space similarity). [Lean](#) [✓ Stmt](#) Let $C \in M_D(\mathbb{C})$ be invertible and let E be a linear map on $M_D(\mathbb{C})$. If

$$E_C(X) = C^{-1}E(CXC^\dagger)(C^\dagger)^{-1},$$

then E_C and E have the same peripheral spectrum.

Proof. [✓ Proof](#) Conjugation by $X \mapsto CXC^\dagger$ gives a linear equivalence of the matrix space. Transporting eigenvectors through this equivalence preserves every eigenvalue and hence the unit-circle eigenvalue set. $\square$

Definition 22.1.4 (Left-canonical periodic tensor). [Lean](#) [✓ Stmt](#) A left-canonical irreducible MPS tensor A of bond dimension D is m -periodic (for some $m \geq 1$) if the peripheral spectrum of its transfer map $\mathcal{E}_A$ is exactly the m -th roots of unity $\{e^{2\pi i k/m} : 0 \leq k < m\}$. This is the normalized form-II convention used later in the source. A 1-periodic tensor is exactly an irreducible tensor with primitive transfer map (the corresponding definition in the companion quantum-channel volume [\[LTC26, “Primitive map”\]](#)).

Theorem 22.1.5 (Perron normalization of spectrally periodic tensors). [Lean](#) [✓ Stmt](#) Let A be spectrally m -periodic. There is a positive definite matrix σ such that

$$B^i = \sigma^{1/2} A^i \sigma^{-1/2}$$

is a left-canonical m -periodic tensor. The similarity is pure, so it leaves every matrix-product vector unchanged and introduces no scalar weight into the normalization.

Proof. [✓ Proof](#) Apply Theorem 9.2.2. The transfer map of B is similar to that of A , so Lemma 22.1.3 preserves the prescribed set of m -th roots of unity. $\square$

Definition 22.1.6 (Replacement of the representative family). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let

$$A^i = \bigoplus_{j \in J} (R_j \otimes A_j^i)$$

be a multiplicity-bearing sector decomposition. For tensors B_j with the same bond dimension as A_j , its representative replacement is

$$A_B^i = \bigoplus_{j \in J} (R_j \otimes B_j^i).$$

Thus the index set, bond dimensions, multiplicities, and every entry of the diagonal matrices R_j remain unchanged. In particular, for every length N and every representative j , the grouped coefficient remains

$$\text{tr}(R_j^N) = \sum_q \mu_{j,q}^N.$$

Theorem 22.1.7 (Assembly of representative similarities). [Lean](#) [✓ Stmt](#) Suppose that, for every $j \in J$, there is an invertible matrix X_j such that

$$B_j^i = X_j A_j^i X_j^{-1}.$$

Then the two multiplicity-bearing tensors are related by the pure similarity

$$A_B^i = X A^i X^{-1}, \quad X = \bigoplus_{j \in J} (I_{r_j} \otimes X_j).$$

Proof. [✓ Proof](#) Repeat X_j on every copy indexed by a diagonal entry of R_j , and take their block-diagonal direct sum. On the (j, q) block, conjugation by this matrix sends $\mu_{j,q} A_j^i$ to $\mu_{j,q} B_j^i$. $\square$

Theorem 22.1.8 (Sectorwise Perron normalization). [Lean](#) [✓ Stmt](#) Suppose every representative A_j is spectrally m_j -periodic. There are left-canonical m_j -periodic representatives B_j , each purely similar to A_j , such that the two full multiplicity-bearing tensors are purely similar. Every multiplicity matrix R_j is unchanged.

Proof. ✓ Proof Apply the positive Perron gauge separately to every representative and assemble these gauges with Theorem 22.1.7. Since every spectral radius is already one, no representative is rescaled. $\square$

Theorem 22.1.9 (A periodic tensor has positive bond dimension). Lean ✓ Stmt *If A is periodic, then its bond dimension satisfies $D \geq 1$.*

Proof. ✓ Proof The prescribed peripheral spectrum contains the eigenvalue 1. In bond dimension zero the matrix space is zero-dimensional and admits no eigenvector, which is a contradiction. $\square$

Definition 22.1.10 (Irreducible form). Lean ✓ Stmt A tensor A admits an *irreducible-form representation* if it generates the same matrix-product-vector family as the block-diagonal tensor

$$\hat{A}^i = \bigoplus_{j \in J} \mu_j A_j^i,$$

where each block A_j is a left-canonical m_j -periodic tensor. The defining relation is $|V^{(N)}(A)\rangle = |V^{(N)}(\hat{A})\rangle$ for every $N \geq 0$, rather than literal equality of the tensors A and $\hat{A}$. In particular, it includes the empty word and hence equality of the bond-dimension traces. The source comparison is at positive physical lengths and permits the irreducible-form representative to have smaller bond dimension [DICCSPG17, lines 266–271]. Following [DICCSPG17, Section II], the phases are fixed by the convention $\mu_j > 0$, so the weights are positive real scalars. This is a normalized matrix-product-vector representation of the source irreducible form, whose displayed decomposition is a literal tensor equality and whose periodic blocks are only required to have irreducible transfer maps of spectral radius one. It extends the normal canonical form (Definition 8.9.2), which requires the stronger condition that every block is 1-periodic.

Remark 22.1.11 (Normal canonical form gives irreducible form). Lean Lean ✓ Stmt Let $A^i = \bigoplus_j \mu_j A_j^i$ be a normal canonical block family with nonzero complex weights. Replacing μ_j by $|\mu_j|$ and A_j by $(\mu_j/|\mu_j|)A_j$ gives the positive-weight irreducible form convention and does not change the MPV family. The corresponding irreducible-form witness has all periods equal to 1. This is the period-one part of the statement in [DICCSPG17, Section II] that normal tensors are periodic tensors with period 1.

Remark 22.1.12 (Irreducible form allows higher periods). Not ready The converse containment is not part of the implication above. Irreducible form permits periodic blocks with period larger than 1; for instance, the antiferromagnetic example has period 2.

Physical-index rotation and its MPV expansion are recorded in Section G.1. The compatibility results below supply the physical action used in the periodic analysis of [DICCSPG17].

Theorem 22.1.13 (Transfer map preserved by physical rotation). Lean ✓ Stmt *If $uu^\dagger = \mathbb{1}$, then the transfer map is invariant under physical-index rotation: $\mathcal{E}_{A_u} = \mathcal{E}_A$.*

Proof. ✓ Proof This is the unitary freedom of Kraus representations from [Wol12, Theorem 2.1(4)]. $\square$

Theorem 22.1.14 (Left-canonical preserved by physical rotation). Lean ✓ Stmt *If $uu^\dagger = \mathbb{1}$ and A is left-canonical, then A_u is left-canonical.*

Proof. ✓ Proof Expand $\sum_i (B^i)^\dagger B^i$, where $B^i = \sum_j u_{ij} A_j^i$, swap sums, apply orthogonality $u^\dagger u = \mathbb{1}$, and collapse to $\sum_j (A_j^\dagger)^\dagger A_j = \mathbb{1}$. $\square$

Theorem 22.1.15 (Irreducibility preserved by physical rotation). [Lean](#) [✓ Stmt](#) If $uu^\dagger = \mathbb{1}$ and A is irreducible, then A_u is irreducible.

Proof. [✓ Proof](#) Contrapositive: any nontrivial invariant projection P for the rotated tensor yields $(\mathbb{1} - P)A^k P = 0$ for all k via the orthogonality of u , contradicting irreducibility of A . $\square$

Theorem 22.1.16 (Periodicity preserved by physical rotation). [Lean](#) [✓ Stmt](#) If $uu^\dagger = \mathbb{1}$ and A is periodic with period m , then A_u is periodic with the same period.

Proof. [✓ Proof](#) Assembles transfer-map invariance, left-canonical preservation, and irreducibility preservation. $\square$

Theorem 22.1.17 (Equality of MPV families preserved by physical rotation). [Lean](#) [✓ Stmt](#) If $\mathcal{V}(A) = \mathcal{V}(B)$, then $\mathcal{V}(A_u) = \mathcal{V}(B_u)$.

Proof. [✓ Proof](#) Induction on the word length with a strengthened hypothesis carrying an arbitrary prefix matrix, reducing each step to the hypothesis via word concatenation. $\square$

Theorem 22.1.18 (Physical rotation distributes over block-diagonal composition). [Lean](#) [✓ Stmt](#) The rotated block-diagonal tensor satisfies

$$\left(\bigoplus_k \mu_k A_k \right)_u = \bigoplus_k \mu_k (A_k)_u.$$

Proof. [✓ Proof](#) Pointwise matrix-entry computation using linearity of the block-diagonal embedding and reindexing. $\square$

Theorem 22.1.19 (Irreducible form preserved by physical rotation). [Lean](#) [✓ Stmt](#) If $uu^\dagger = \mathbb{1}$ and A is in irreducible form, then A_u is in irreducible form with the same block structure and rotated blocks.

Proof. [✓ Proof](#) Each block remains periodic by Theorem 22.1.16, equality of MPV families transfers via Theorem 22.1.17, and block-diagonal compatibility follows from Theorem 22.1.18. $\square$

Corollary 22.1.20 (Periodic symmetry yields finite-order Z -gauge equivalence). [Lean](#) [✓ Stmt](#) Assume the periodic equal-case theorem: tensors in irreducible form II with equal MPV families are Z -gauge equivalent. Let A be in irreducible form II , and suppose $B := A_M$ is also in irreducible form II with $\mathcal{V}(A) = \mathcal{V}(B)$. Then there are $m > 0$, an invertible matrix Y , and a matrix $Z \in M_D(\mathbb{C})$ such that, for every physical index i ,

$$[Z, A^i] = 0, \quad Z^m = \mathbb{1}_D, \quad ZA^i = YB^iY^{-1}.$$

Thus A and B are related by a genuine $\mathbb{Z}_m$ virtual gauge, not merely by a scalar root of unity.

Proof. [✓ Proof](#) Apply the assumed periodic equal-case theorem to A and B . Its Z -gauge conclusion supplies m , Y , and Z with the three displayed identities by Definition 22.1.1.7. $\square$

22.1.1 The Z -gauge degree of freedom

Definition 22.1.1.1 (Entrywise periodic Z -gauge ratio). [Lean](#) [✓ Stmt](#) Given complex multiplicity entries μ, ν , define the entrywise Z -gauge ratio by $z(\mu, \nu) := \mu/\nu$.

Lemma 22.1.1.2 (Matched powers imply root-of-unity ratio). [Lean](#) [✓ Stmt](#) If $\mu^m = \nu^m$ and $\nu \neq 0$, then $z(\mu, \nu)^m = 1$.

Lemma 22.1.1.3 (Ratio rescales denominator back to numerator). [Lean](#) [✓ Stmt](#) If $\nu \neq 0$, then $z(\mu, \nu)\nu = \mu$.

Definition 22.1.1.4 (Diagonal periodic Z -gauge matrix). [Lean](#) [✓ Stmt](#) For matched multiplicity entries $(\mu_i)_i$ and $(\nu_i)_i$ on a finite index type, define the diagonal matrix $Z := \text{diag}(z(\mu_i, \nu_i))_i$.

Lemma 22.1.1.5 (Diagonal Z -gauge has finite order under matched powers). [Lean](#) [✓ Stmt](#) If $\mu_i^m = \nu_i^m$ and $\nu_i \neq 0$ for all i , then $Z^m = \mathbb{1}$.

Lemma 22.1.1.6 (Diagonal Z -gauge transports diagonal multiplicities). [Lean](#) [✓ Stmt](#) Under $\nu_i \neq 0$ for all i , one has $Z \text{diag}(\nu) = \text{diag}(\mu)$.

Definition 22.1.1.7 (Z -gauge equivalence). [Lean](#) [✓ Stmt](#) Let A be an m -periodic irreducible tensor of bond dimension D . A global finite-order intertwining relation is given by a matrix $Z \in M_D(\mathbb{C})$ satisfying:

1. Z commutes with every tensor matrix: $[A^i, Z] = 0$ for all physical indices i , and
2. $Z^m = \mathbb{1}_D$.

Two tensors A, B are *Z -gauge equivalent* if there exists an invertible Y and a Z -gauge transformation Z for A such that $ZA^i = YB^iY^{-1}$ for all i . The source equal-case theorem uses the finer block-multiplicity diagonal gauges Z_j from Theorem 22.2.2.1; those multiplicity-space matrices are not part of this global relation.

Remark 22.1.1.8 (Z -gauge is trivial for period-1 blocks). [Lean](#) [✓ Stmt](#) When $m = 1$, the condition $Z^1 = \mathbb{1}$ forces $Z = \mathbb{1}$, so Z -gauge equivalence reduces to ordinary gauge equivalence (Definition 2.3.1). The Z -gauge is therefore a genuine new degree of freedom that appears only for $m > 1$.

Theorem 22.1.1.9 (Blocking a $\mathbb{Z}_m$ -gauge gives an ordinary gauge). [Lean](#) [✓ Stmt](#) If A and B are $\mathbb{Z}_m$ -gauge equivalent, then their m -blocked tensors $A^{[m]}$ and $B^{[m]}$ are gauge equivalent in the ordinary sense.

Proof. [✓ Proof](#) Write the $\mathbb{Z}_m$ -gauge relation as $ZA^i = YB^iY^{-1}$ with $Z^m = \mathbb{1}$ and $[A^i, Z] = 0$. For a blocked letter, commute one factor of Z past each physical letter in the associated length- m word. The total prefactor becomes $Z^m = \mathbb{1}$, so the entire blocked word is related by the same ordinary gauge Y . $\square$

Corollary 22.1.1.10 (Blocking a $\mathbb{Z}_m$ -gauge preserves the MPV family). [Lean](#) [✓ Stmt](#) If A and B are $\mathbb{Z}_m$ -gauge equivalent, then the blocked tensors $A^{[m]}$ and $B^{[m]}$ have the same MPV family.

Proof. [✓ Proof](#) Apply Theorem 22.1.1.9 and then the gauge invariance of MPV families. $\square$

22.2 Periodic Fundamental Theorem

The theorems in Chapter 10 all require that each block has a *primitive* transfer map, i.e. peripheral spectrum exactly $\{1\}$. This is achieved by blocking the original tensor by a common period. [DICCSPG17] develops a framework that avoids this blocking step, giving a fundamental theorem that applies directly to periodic tensors.

22.2.1 Common-period blocking lemmas

Definition 22.2.1.1 (LCM period for a finite block family). Lean ✓ Stmt Given a finite family of blocking periods (p_i) indexed by a finite set, their least common multiple serves as the shared blocking period.

Definition 22.2.1.2 (Common-period blocked family). Lean ✓ Stmt Given a finite family of tensors (B_i) with designated periods (p_i) , block each B_i to the shared period $\text{lcm}(p_i)$, so every block lives in the same physical space.

Theorem 22.2.1.3 (Common-period blocking preserves transfer-map primitivity). Lean ✓ Stmt *If each tensor B_i has a primitive transfer map after blocking by its own positive period p_i , then blocking the entire family to the common LCM period preserves primitivity of each member's transfer map.*

Proof. ✓ Proof Each p_i divides $\text{lcm}(p_i)$. Apply the divisibility monotonicity theorem for primitive blocked transfer maps to each family member separately. □

22.2.2 Statement of the periodic Fundamental Theorem

The following is the equal-case theorem of [DICCSPG17, lines 643–656]. The statements recorded here are conditional components of that theorem: the final comparison still assumes periodic-overlap and power-equality hypotheses, so the literature theorem has not yet been obtained here as a single unconditional result.

Theorem 22.2.2.1 (Periodic Fundamental Theorem). Not ready *Let A and B be MPS tensors in irreducible form. Write*

$$A^i = \bigoplus_{j \in J} (R_j \otimes A_j^i),$$

$$B^i = \bigoplus_{k \in K} (S_k \otimes B_k^i),$$

where $\{A_j\}_{j \in J}$ and $\{B_k\}_{k \in K}$ are bases of periodic tensors, R_j and S_k are diagonal multiplicity matrices of sizes $r_j \times r_j$ and $s_k \times s_k$, and A_j^i and B_k^i act on bond spaces of dimensions D_j and D_k . Suppose that $\mathcal{V}(A) = \mathcal{V}(B)$ (equal MPV families for all lengths N).

Then $|J| = |K|$ and there is a bijection $\pi : J \rightarrow K$ such that A_j and $B_{\pi(j)}$ are m_j -periodic for the same period m_j . Moreover, for each $j \in J$ there exists:

1. an invertible matrix Y_j of size $D_{\pi(j)} \times D_j$, and
2. a diagonal unitary matrix Z_j of size $r_j \times r_j$ satisfying $Z_j^{m_j} = \mathbb{1}_{r_j}$,

such that $B_{\pi(j)}^i = Y_j A_j^i Y_j^{-1}$ for every physical index i , and, after reordering the diagonal entries of the multiplicity matrix $S_{\pi(j)}$, one has $Z_j R_j = S_{\pi(j)}$. Equivalently, setting $Z = \bigoplus_j Z_j \otimes \mathbb{1}_{D_j}$ gives an invertible matrix Y such that $ZA^i = YB^iY^{-1}$ for all i , with $[A^i, Z] = 0$ and $\mathcal{V}(A) = \mathcal{V}(ZA)$.

Remark. The proof is conditional on the periodic-overlap dichotomy of [DCCSPG17, lines 589–609], which is left here as a separate hypothesis.

Theorem 22.2.2.2 (Diagonal multiplicity gauge from equal powers). Lean ✓ Stmt Let $(\mu_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ be two finite lists of nonzero multiplicity entries. If $\mu_i^m = \nu_i^m$ for every $i \in I$, then there exists a diagonal matrix Z such that

$$\begin{aligned} Z^m &= \mathbb{1}, \\ Z \operatorname{diag}(\nu) &= \operatorname{diag}(\mu). \end{aligned}$$

Proof. ✓ Proof Take $Z = \operatorname{diag}(\mu_i/\nu_i)_{i \in I}$ and apply the two diagonal identities from Lemmas 22.1.1.5 and 22.1.1.6. □

Theorem 22.2.2.3 (Scalar multiplicity gauge from power sums). Lean ✓ Stmt Let $(\mu_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ be two finite lists of multiplicity entries. Suppose that the ν_i are nonzero, that $\mu_i^m = \nu_i^m$ for every $i \in I$, and that $\sum_{i \in I} \mu_i^k = \sum_{i \in I} \nu_i^k$ for every positive integer k . Then the two lists determine the same multiset, and there is a diagonal Z satisfying $Z^m = \mathbb{1}$ and $Z \operatorname{diag}(\nu) = \operatorname{diag}(\mu)$.

Proof. ✓ Proof Apply Theorem 22.2.2.2 to obtain Z , and apply the Newton–Girard power-sum identity to recover the multiplicity multiset. □

Theorem 22.2.2.4 (Conditional matching of bases of periodic tensors). Lean ✓ Stmt Let A and B be in irreducible form, and suppose their bases of periodic tensors contain no repeated pair of distinct basis elements. If the periodic-overlap hypothesis holds for the two bases, then the two bases have the same cardinality and are matched by a bijection whose matched elements are repeated blocks.

Proof. ✓ Proof Apply the conditional block-matching theorem to the two bases extracted from the irreducible-form representations. □

Theorem 22.2.2.5 (Conditional scalar equal-case component). Lean ✓ Stmt Let A and B be in irreducible form, and suppose their bases of periodic tensors contain no repeated pair of distinct basis elements. Assume the periodic-overlap hypothesis and the corresponding power equality for the scalar multiplicity entries. Then the bases are matched by a bijection, and each matched pair has a scalar diagonal $Z_j \in M_1(\mathbb{C})$ with

$$\begin{aligned} Z_j^{m_j} &= \mathbb{1}, \\ Z_j \operatorname{diag}(\mu_{A,j}) &= \operatorname{diag}(\mu_{B,\pi(j)}). \end{aligned}$$

The two scalar multiplicity entries also determine the same singleton multiset.

Proof. ✓ Proof First use Theorem 22.2.2.4 to match the bases of periodic tensors. For each matched block, apply Theorem 22.2.2.3 to the singleton list of multiplicity entries. □

22.2.3 Periodic overlap dichotomy

Theorem 22.2.3.1 (Sector scalars from a product tensor identity). Lean ✓ Stmt Let $z \in \mathbb{C}$ be nonzero. For each sector $v = 0, \dots, m-1$, let $\{A_v^i\}_i$ and $\{B_v^i\}_i$ be families of complex matrices with the same rectangular size. Suppose that for every choice of letters $\sigma = (\sigma_v)_v$,

$$\bigotimes_{v=0}^{m-1} A_v^{\sigma_v} = z \bigotimes_{v=0}^{m-1} B_v^{\sigma_v}.$$

Suppose also that for each v there is a reference letter i_v and a matrix entry (r_v, c_v) with $(B_v^{i_v})_{r_v, c_v} \neq 0$. Then there are scalars $\kappa_v \in \mathbb{C}$ such that

$$\prod_{v=0}^{m-1} \kappa_v = z,$$

$$A_v^i = \kappa_v B_v^i$$

for every sector v and every letter i .

This is the scalar extraction used in [DICCSPG17, Appendix A, lines 1072–1080], after the contraction with the maps F_u and Ω_u has produced the displayed product-tensor identity.

Proof. ✓ Proof For each sector choose the coordinate functional at the nonzero reference entry and normalize it so that it takes the value 1 on $B_v^{i_v}$. Applying the tensor product of these functionals to the reference tensor identity gives $\prod_v \kappa_v = z$. To prove the proportionality in sector v , replace only the v -th functional by the coordinate functional at an arbitrary entry (r, c) and apply the tensor identity with $\sigma_v = i$ and all other letters fixed at their references. The product over the other sectors is nonzero because its product with κ_v is z , so cancellation gives $(A_v^i)_{r, c} = (\kappa_v B_v^i)_{r, c}$ for every entry. □

Theorem 22.2.3.2 (Product-one sector scalars after choosing a root). Lean ✓ Stmt Let $m > 0$, let $z \in \mathbb{C}$ be nonzero, and choose $\xi \in \mathbb{C}$ with $\xi^m = z$. For each sector $v = 0, \dots, m-1$, let $\{A_v^i\}_i$ and $\{B_v^i\}_i$ be families of complex matrices with the same rectangular size. Suppose that for every choice of letters $\sigma = (\sigma_v)_v$,

$$\bigotimes_{v=0}^{m-1} A_v^{\sigma_v} = z \bigotimes_{v=0}^{m-1} B_v^{\sigma_v}.$$

Suppose also that each sector has a reference letter i_v and a matrix entry (r_v, c_v) with $(B_v^{i_v})_{r_v, c_v} \neq 0$. Then there are scalars $\kappa_v \in \mathbb{C}$ such that

$$\prod_{v=0}^{m-1} \kappa_v = 1,$$

$$A_v^i = \kappa_v \xi B_v^i$$

for every sector v and every letter i .

This is the normalization in [DICCSPG17, Appendix A, lines 1072–1080]: after the preceding translation step has made the scalar independent of the sector label, choosing a common m -th root of that scalar leaves a product-one family of sector scalars.

Proof. ✓ Proof Apply Theorem 22.2.3.1 to obtain scalars μ_v with $\prod_v \mu_v = z$ and $A_v^i = \mu_v B_v^i$. Since $m > 0$, the identities $\xi^m = z$ and $z \neq 0$ imply $\xi \neq 0$. Put $\kappa_v = \mu_v / \xi$. Then $\prod_v \kappa_v = (\prod_v \mu_v) / \xi^m = z / z = 1$, and $A_v^i = (\kappa_v \xi) B_v^i$. □

Theorem 22.2.3.3 (Left-canonical scalar normalization). Lean ✓ Stmt Let A have positive bond dimension, and let $c \in \mathbb{C}$. If both A and the scalar multiple cA are left-canonical, then $|c| = 1$.

Proof. ✓ Proof The two left-canonical equations give

$$\sum_i (A^i)^\dagger A^i = \mathbb{1},$$

$$\sum_i (cA^i)^\dagger (cA^i) = \mathbb{1}.$$

Substituting the first equation into the second gives $(\bar{c}c)\mathbb{1} = \mathbb{1}$. Since the bond dimension is positive, comparison of any diagonal entry gives $\bar{c}c = 1$, hence $|c| = 1$. $\square$

Theorem 22.2.3.4 (Left-canonical sector scalar normalization). [Lean](#) [✓ Stmt](#) *Let A and B be left-canonical tensors with the same positive bond dimension. Suppose that $|\xi| = 1$ and that $A^i = (\kappa\xi)B^i$ for every physical index i . Then $|\kappa| = 1$.*

This is the scalar normalization in [DICCSPG17, Appendix A, lines 1082–1084], after the sectorwise proportionality relation has been obtained.

Proof. [✓ Proof](#) By Theorem 22.2.3.3, applied to B and the scalar $\kappa\xi$, one has $|\kappa\xi| = 1$. Since $|\xi| = 1$, the multiplicativity of the complex norm gives $|\kappa| = 1$. $\square$

Definition 22.2.3.5 (Off-diagonal sector decomposition). [Lean](#) [Lean](#) [✓ Stmt](#) A family of compressed sector tensors $(C_k)_{k=0}^{m-1}$ on corner bond spaces forms an *off-diagonal decomposition* of the blocked tensor $A^{(m)}$ if there exist orthogonal projections $P_0, \dots, P_{m-1}$ on the ambient bond space such that $\sum_k P_k = \mathbb{1}$, the projections satisfy the cyclic-shift relation $\mathcal{E}_A^\dagger(P_{k+1}) = P_k$ with the index $k+1$ taken cyclically modulo m , each P_k commutes with every blocked letter $P_k A_i^{(m)} = A_i^{(m)} P_k$, and $V^{(N)}(C_k)_\sigma = \text{tr}(P_k(A^{(m)})^\sigma)$ for every word σ , and each block carries a $*$ -algebra isomorphism $\varphi_k : M_{\dim C_k}(\mathbb{C}) \xrightarrow{\sim} P_k \cdot M_D(\mathbb{C}) \cdot P_k$ (multiplicative and adjoint-preserving) intertwining the compressed adjoint transfer map with the sector adjoint transfer map on the corner of P_k . We shall sometimes keep these projections and corner isomorphisms as specified witnesses; forgetting the witnesses gives the displayed decomposition. This is the inverse cyclic indexing of the off-diagonal convention in [DICCSPG17, Appendix A]. There the same adjoint transfer map is written E^* , and the source projectors satisfy $E^*(P_u) = P_{u+1}$ together with $A^i = \sum_u P_u A^i P_{u+1}$. Here P_k corresponds to the source sector $u = -k \pmod{m}$, which turns the source shift into the displayed adjoint-transfer relation.

Theorem 22.2.3.6 (Single-site cyclic grading). [Lean](#) [✓ Stmt](#) *Let A be left-canonical, and let $P_0, \dots, P_{m-1}$ be orthogonal projections on the bond space. If the adjoint transfer map satisfies $\mathcal{E}_A^\dagger(P_{k+1}) = P_k$ for all k , with indices taken modulo m , then $P_{k+1}A^i = A^i P_k$ for every physical letter i . This is the single-site grading behind [DICCSPG17, eq. Aoffdiag]; the compressed blocks of [DICCSPG17, eq. Auprop] use the inverse cyclic indexing convention of Definition 22.2.3.5.*

Proof. [✓ Proof](#) Put $K_i = (A^i)^\dagger$ and let Φ_K be the Kraus map of the family K . Since A is left-canonical, Φ_K is unital. The projection identities and the shift relation give

$$\Phi_K(P_{k+1}^\dagger P_{k+1}) = \Phi_K(P_{k+1})^\dagger \Phi_K(P_{k+1}),$$

because both sides are equal to P_k . The equality case in the Kadison–Schwarz inequality therefore gives $P_{k+1}K_i^\dagger = K_i^\dagger \Phi_K(P_{k+1})$. Substituting $K_i^\dagger = A^i$ and $\Phi_K(P_{k+1}) = P_k$ gives $P_{k+1}A^i = A^i P_k$. $\square$

Theorem 22.2.3.7 (Off-diagonal reconstruction). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 22.2.3.6, assume in addition that $\sum_{u=0}^{m-1} P_u = \mathbb{1}$. Then every physical letter decomposes as $A^i = \sum_{u=0}^{m-1} P_{u+1} A^i P_u$. This is [DICCSPG17, eq. Aoffdiag], written in the same inverse cyclic indexing convention.*

Proof. [✓ Proof](#) Insert $\sum_u P_u = \mathbb{1}$ on the right of A^i , then use Theorem 22.2.3.6 and $P_u^2 = P_u$ term by term. $\square$

Theorem 22.2.3.8 (Word cyclic grading). Lean ✓ Stmt Under the hypotheses of Theorem 22.2.3.6, a word $w = i_1 \dots i_\ell$ satisfies $P_{k+\ell} A^w = A^w P_k$. Thus the single-site grading accumulates one cyclic step for each letter of the word.

Proof. ✓ Proof Induct on the length of w , using Theorem 22.2.3.6 for the first letter and the induction hypothesis for the remaining word. $\square$

Definition 22.2.3.9 (Corner-transition letters). Lean ✓ Stmt Let $P_0, \dots, P_{m-1}$ be cyclic sector projections for a tensor A . The one-site corner-transition letters of [DICCSPG17, Appendix A] are $A_u^i = P_u A^i P_{u+1}$, with the index $u+1$ taken cyclically modulo m .

Definition 22.2.3.10 (Repeated corner products). Lean ✓ Stmt For a word $\mathbf{i} = (i_1, \dots, i_\ell)$, define

$$F_u^{\mathbf{i}} = A_u^{i_1} A_{u+1}^{i_2} \dots A_{u+\ell-1}^{i_\ell}.$$

The empty word gives $F_u^\emptyset = P_u$, and a one-letter word gives $F_u^{(i)} = A_u^i$. This is the notation of [DICCSPG17, Appendix A]; in the Case 3 contraction the length is $\ell = mN_0$.

Lemma 22.2.3.11 (Concatenation of corner products). Lean ✓ Stmt If the sector maps P_u are orthogonal projections, then for any two words $\mathbf{i}$ and $\mathbf{j}$, $F_u^{\mathbf{ij}} = F_u^{\mathbf{i}} F_{u+|\mathbf{i}|}^{\mathbf{j}}$, where indices are taken cyclically modulo m .

Proof. ✓ Proof Induct on the word $\mathbf{i}$. For the empty word the identity is $P_u F_u^{\mathbf{j}} = F_u^{\mathbf{j}}$, which follows from the idempotent part of the orthogonal-projection hypothesis. For a nonempty word, remove the first letter and apply the induction hypothesis at the next cyclic sector. $\square$

Lemma 22.2.3.12 (Repeated-block matrix identity). Lean ✓ Stmt Let $P_0, \dots, P_{m-1}$ and $Q_0, \dots, Q_{m-1}$ be orthogonal projections, and suppose the m -block product identity

$$A_u^{i_1} \dots A_{u+m-1}^{i_m} = c_u B_{u+q}^{i_1} \dots B_{u+q+m-1}^{i_m}$$

holds for every base sector u and every length- m word. Then for every word W of length km with $k \geq 1$, $F_u^W = c_u^k \widehat{F}_{u+q}^W$, where $\widehat{F}$ is the repeated corner product built from B . Each of the k successive length- m blocks is anchored at the same base sector u , because a full cycle of unit shifts is trivial, $m \cdot 1 = 0$ in $\mathbb{Z}_m$; hence every block contributes the same scalar c_u .

Proof. ✓ Proof Induct on k . The base case $k = 1$ is the hypothesis. For the inductive step split W into its first length- m block and the remaining length- km word, apply the concatenation law, and use that the junction shift $m \cdot 1 = 0$ returns to the base sector before applying the block identity and the induction hypothesis. $\square$

Lemma 22.2.3.13 (Right-inverse contraction). Lean ✓ Stmt Suppose the coefficients Ω_{k+1} recover each inserted matrix X_k from the repeated corner product F_{k+1} over length- L words: $\sum_{\mathbf{j}} (\Omega_{k+1}(X_k))_{\mathbf{j}} F_{k+1}^{\mathbf{j}} = X_k$ ([DICCSPG17, eq. Omegauprop] at X_k). Then for any sector indices σ ,

$$\sum_{\rho} \left(\prod_k (\Omega_{k+1}(X_k))_{\rho_k} \right) \prod_k A_k^{\sigma_k} F_{k+1}^{\rho_k} = \prod_k A_k^{\sigma_k} X_k.$$

Summing the concatenated chain against these coefficients collapses each repeated product F_{k+1} to its inserted matrix X_k . The recovery is needed only at the inserted matrices X_k , which in the Case-3 contraction are the corner-supported tensors that Ω_{k+1} inverts.

Proof. ✓ Proof Expand each $A_k^{\sigma_k} X_k = \sum_{\mathbf{j}} (\Omega_{k+1}(X_k))_{\mathbf{j}} A_k^{\sigma_k} F_{k+1}^{\mathbf{j}}$ using the right inverse, apply the noncommutative distributivity identity

$$\prod_k \left(\sum_{\mathbf{j}} f_{k,\mathbf{j}} \right) = \sum_{\rho} \prod_k f_{k,\rho_k},$$

and pull the scalar coefficients $\prod_k (\Omega_{k+1}(X_k))_{\rho_k}$ out of each ordered product. □

Lemma 22.2.3.14 (Inverse cyclic reindexing to the off-diagonal convention). Lean ✓ Stmt *Let A be left-canonical, $\sum_i A_i^\dagger A_i = 1$, with orthogonal sector projectors P_k satisfying the adjoint shift $\mathcal{E}^*(P_{k+1}) = P_k$, which gives the inverse-indexed grading $P_{k+1} A^i = A^i P_k$. After the inverse reindexing $P_k \mapsto P_{-k}$ the projectors satisfy the off-diagonal convention $P_{-k} A^i = A^i P_{-(k+1)}$ of [DICCSPG17, eq. Auprop], under which the corner-transition products are nonzero.*

Proof. ✓ Proof Apply the grading at sector $-(k+1)$ and simplify the cyclic index. □

Lemma 22.2.3.15 (Telescoping of a corner product). Lean ✓ Stmt *Under the off-diagonal convention $P_k A^i = A^i P_{k+1}$, with the P_k orthogonal projections, the repeated corner-transition product collapses through its intermediate projectors:*

$$F_u^{\mathbf{i}} = P_u A^{\mathbf{i}} P_{u+|\mathbf{i}|},$$

where $A^{\mathbf{i}}$ is the ordinary word product and the right projector records the accumulated sector offset.

Proof. ✓ Proof Induct on the word. The head junction $P_u A^i P_{u+1} = P_u A^i$ collapses by the shift and idempotency; the induction hypothesis handles the tail, and the accumulated offset realigns. □

Lemma 22.2.3.16 (Blocked letter as a diagonal corner). Lean ✓ Stmt *Under the off-diagonal convention $P_k A^i = A^i P_{k+1}$, with the P_k orthogonal projections, the corner-transition product of a single blocked letter is the diagonal blocked corner of [DICCSPG17, eq. Cu]: $F_u^{(I)} = P_u (A^{[m]})^I P_u$. Combined with the corner-isomorphism letter identity, this exhibits the compressed sector letter as a corner-transition product.*

Proof. ✓ Proof A blocked letter is a word of length exactly m ; the accumulated offset $m \cdot 1 = 0$ vanishes, so the telescoping gives the diagonal corner. □

Definition 22.2.3.17 (Flattening bijection on word indices). Lean ✓ Stmt *The bijection between blocked words of length L over the alphabet of d^m blocked letters and single-site words of length mL over the alphabet of d letters, $[d^m]^L \simeq [d]^{mL}$, is obtained from the iterated blocking bijection. Here $[q]^r$ denotes the set of words of length r over a q -letter alphabet.*

Lemma 22.2.3.18 (Flattening bijection agrees with word flattening). Lean ✓ Stmt *If $W \in [d^m]^L$ is a blocked word and $\widehat{W} \in [d]^{mL}$ is its image under the flattening bijection, then $\widehat{W}$ is exactly the concatenation of the L decoded length- m blocks of W .*

Proof. ✓ Proof The decoding bijection identifies a word-indexing function with its ordered list of letters, and the iterated blocking bijection concatenates the decoded length- m blocks in order. □

Lemma 22.2.3.19 (Corner product of a flattened word). [Lean] [✓ Stmt] Let P_k be orthogonal projections satisfying the off-diagonal convention $P_k A^i = A^i P_{k+1}$, and let φ_k be the multiplicative corner isomorphisms with $\varphi_k(\bar{A}_k^I) = P_k(A^{[m]})^I P_k$. Then for a nonempty blocked word W , with flattening $\widehat{W}$, the corner-transition product is the corner image of the word evaluation,

$$F_k^{\widehat{W}} = \varphi_k(\bar{A}_k^W),$$

the content of [DICCSPG17, eq. Fu] read through φ .

Proof. [✓ Proof] Induct on W . The head blocked letter collapses by the blocked-letter diagonal corner to $\varphi_k(\bar{A}_k^I)$, and multiplicativity of φ reassembles the letters:

$$F_k^{\widehat{W}} = \prod_{I \in W} \varphi_k(\bar{A}_k^I) = \varphi_k\left(\prod_{I \in W} \bar{A}_k^I\right) = \varphi_k(\bar{A}_k^W).$$

The empty word is excluded so the induction stays inside the corner. □

Lemma 22.2.3.20 (Ambient corner right inverse from the sector right inverse). [Lean] [✓ Stmt] From a sector right inverse Ω recovering every sector word product $\bar{A}_k^W$ over blocked words $W \in [d^m]^L$ with $L > 0$ ([DICCSPG17, eq. Omegauprop]) one obtains an ambient corner right inverse $\hat{\Omega}$ over single-site words of length mL . For every corner-supported Y , that is, $P_k Y P_k = Y$,

$$\sum_{\mathbf{j} \in [d]^{mL}} (\hat{\Omega}_k(Y))_{\mathbf{j}} F_k^{\mathbf{j}} = Y.$$

Proof. [✓ Proof] Set $\hat{\Omega}_k(Y) = \Omega_k(\varphi_k^{-1}(Y))$ for corner-supported Y . Applying φ_k to the sector recovery and using the corner product of a flattened word,

$$\sum_W (\hat{\Omega}_k(Y))_{\widehat{W}} F_k^{\widehat{W}} = \varphi_k\left(\sum_W (\Omega_k(\varphi_k^{-1}(Y)))_W \bar{A}_k^W\right) = \varphi_k(\varphi_k^{-1}(Y)) = Y.$$

□

Lemma 22.2.3.21 (Sector contraction). [Lean] [✓ Stmt] Feeding the ambient corner right inverse into the right-inverse contraction yields the contracted product-tensor identity [DICCSPG17, eq. resultprop]. For corner-supported inserted matrices X_k ,

$$\sum_{\rho} \left(\prod_k (\hat{\Omega}_{k+1}(X_k))_{\rho_k} \right) \prod_k A_k^{\sigma_k} F_{k+1}^{\rho_k} = \prod_k A_k^{\sigma_k} X_k.$$

Proof. [✓ Proof] Apply the right-inverse contraction with the ambient corner right inverse in place of the abstract right inverse. □

Theorem 22.2.3.22 (Cyclic-sector single-site grading). [Lean] [✓ Stmt] If a periodic tensor admits a cyclic sector decomposition, then for every k and every physical letter i there exists a family $P_0, \dots, P_{m-1}$ such that each P_u is an orthogonal projection,

$$P_u^\dagger = P_u, \quad P_u^2 = P_u, \\ \sum_{u=0}^{m-1} P_u = \mathbb{1},$$

and $P_{k+1} A^i = A^i P_k$.

Proof. ✓ Proof Choose the projection family contained in the cyclic sector decomposition. It satisfies

$$\begin{aligned} P_u^\dagger &= P_u, & P_u^2 &= P_u, \\ \sum_u P_u &= \mathbb{1}, \\ \mathcal{E}_A^\dagger(P_{u+1}) &= P_u. \end{aligned}$$

Apply Theorem 22.2.3.6. □

Theorem 22.2.3.23 (Cyclic-sector off-diagonal reconstruction). Lean ✓ Stmt *If a periodic tensor admits a cyclic sector decomposition, then for every physical letter i there exists a family $P_0, \dots, P_{m-1}$ such that each P_u is an orthogonal projection,*

$$\begin{aligned} P_u^\dagger &= P_u, & P_u^2 &= P_u, \\ \sum_{u=0}^{m-1} P_u &= \mathbb{1}, \end{aligned}$$

and

$$A^i = \sum_{u=0}^{m-1} P_{u+1} A^i P_u.$$

Proof. ✓ Proof Choose the projection family contained in the cyclic sector decomposition. It satisfies

$$\begin{aligned} P_u^\dagger &= P_u, & P_u^2 &= P_u, \\ \sum_u P_u &= \mathbb{1}, \\ \mathcal{E}_A^\dagger(P_{u+1}) &= P_u. \end{aligned}$$

Apply Theorem 22.2.3.7. □

Theorem 22.2.3.24 (Cyclic sector decomposition of a periodic tensor). Lean ✓ Stmt *If A is periodic with period m , then the blocked tensor $A^{(m)}$ admits a cyclic sector decomposition. Thus there are projectors P_u and corner tensors $C_u = P_u A^{(m)}$ which reproduce the blocked matrix-product vectors, are left-canonical, have nonzero bond dimensions, and realize the cyclic adjoint-transfer relation of Lemma bdcf in [DICSPG17, lines 404–423]. The normality and non-repetition conclusions of Lemma bdcf are recorded separately in the component lemmas below.*

Proof. ✓ Proof This is the cyclic-sector existence part of Lemma bdcf, translated into the inverse indexing convention of Definition 22.2.3.5. □

Theorem 22.2.3.25 (Cyclic-sector decomposition with compressed corner letters). Lean ✓ Stmt *Under the hypotheses of the cyclic-sector construction for the blocked tensor $A^{(m)}$, there are sector tensors C_k , projections P_k , and linear corner isomorphisms*

$$\varphi_k: M_{\dim C_k}(\mathbb{C}) \longrightarrow P_k M_D(\mathbb{C}) P_k$$

whose sector tensors are left-canonical, reproduce the blocked matrix-product vectors, have nonzero bond dimensions, and carry the cyclic adjoint-transfer decomposition of Theorem 22.2.3.24. In addition, for every blocked physical letter i ,

$$\varphi_k(C_k^i) = P_k(A^{(m)})^i P_k.$$

This is the corner-letter identity used in [DICCSPG17, Lemma bdcf, eq. Cu].

Proof. ✓ Proof Let U diagonalize P_k , so that

$$U^\dagger P_k U = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Write $B^i = U^\dagger (A^{(m)})^i U$, and let B_{kk}^i be the corner block on the support of P_k . The sector tensor is $C_k^i = V_k^\dagger B_{kk}^i V_k$, where V_k identifies the support with $\mathbb{C}^{\dim C_k}$. The linear corner isomorphism is the inverse transport

$$X \mapsto U \begin{pmatrix} V_k X V_k^\dagger & 0 \\ 0 & 0 \end{pmatrix} U^\dagger.$$

Hence

$$\varphi_k(C_k^i) = U \begin{pmatrix} B_{kk}^i & 0 \\ 0 & 0 \end{pmatrix} U^\dagger = P_k(A^{(m)})^i P_k.$$

□

Theorem 22.2.3.26 (Periodic cyclic sectors with support isometries). Lean ✓ Stmt *If A is periodic with period m , then the cyclic-sector decomposition of $A^{(m)}$ may be chosen with rectangular support isometries V_k such that*

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ V_k V_k^\dagger &= P_k, \\ \varphi_k(X) &= V_k X V_k^\dagger. \end{aligned}$$

In particular, the same isometries realize the corner-letter identity $\varphi_k(C_k^i) = P_k(A^{(m)})^i P_k$.

Proof. ✓ Proof Apply Theorem 8.11.1 to the original tensor A with its period- m peripheral-spectrum hypotheses. The resulting decomposition is the decomposition of $A^{(m)}$, and the isometries V_k in its conclusion are the required support isometries. □

Lemma 22.2.3.27 (One-site transport of sector overlaps). Lean ✓ Stmt *Let A and B be left-canonical tensors with cyclic sector decompositions $\{A_u\}_{u \in \mathbb{Z}_m}$ and $\{B_v\}_{v \in \mathbb{Z}_m}$. For every $u, v \in \mathbb{Z}_m$ and every $N > 0$, $O_{A_{u+1}, B_{v+1}}(N) = O_{A_u, B_v}(N)$.*

Proof. ✓ Proof Choose the projection families P_u and Q_v from the two cyclic sector decompositions. The single-site relations $P_{u+1} A^i = A^i P_u$ and $Q_{v+1} B^i = B^i Q_v$ give the following trace-sum identity, where T is the one-site rotation of physical words:

$$\begin{aligned} O_{A_{u+1}, B_{v+1}}(N) &= \sum_{\tau \in [d]^{Nm}} \text{tr}(P_{u+1} A^\tau) \overline{\text{tr}(Q_{v+1} B^\tau)} \\ &= \sum_{\tau \in [d]^{Nm}} \text{tr}(P_{u+1} A^{T\tau}) \overline{\text{tr}(Q_{v+1} B^{T\tau})} \\ &= \sum_{\tau \in [d]^{Nm}} \text{tr}(P_u A^\tau) \overline{\text{tr}(Q_v B^\tau)} = O_{A_u, B_v}(N). \end{aligned} \tag{22.1}$$

The first and last equalities in (22.1) are the cyclic-sector trace formula after reindexing blocked words as physical words of length Nm . $\square$

Lemma 22.2.3.28 (Off-diagonal eigenvectors of the period transfer vanish). [Lean](#) [✓ Stmt](#) *Let A be periodic of period m . Let $P_0, \dots, P_{m-1}$ be orthogonal projections satisfying*

$$\sum_{k=0}^{m-1} P_k = \mathbb{1},$$

$$\mathcal{E}_A^\dagger(P_{k+1}) = P_k.$$

If

$$U = P_u U P_v, \quad P_u P_v = 0,$$

$$\mathcal{E}_A^m(U) = \zeta U, \quad |\zeta| = 1,$$

then $U = 0$.

Proof. [✓ Proof](#) Since the peripheral spectrum of $\mathcal{E}_A$ consists of m -th roots of unity, the corresponding lemma in the companion quantum-channel volume [LTC26, “Powering collapses peripheral eigenvalues”] gives $\text{spec}_{\text{per}}(\mathcal{E}_A^m) = \{1\}$, hence $\zeta = 1$. The cyclic trace gives $\text{tr } U = \text{tr}(P_u U P_v) = \text{tr}(P_v P_u U) = 0$.

Put $W = \sum_{t=0}^{m-1} \mathcal{E}_A^t(U)$. The equation $\mathcal{E}_A^m(U) = U$ gives $\mathcal{E}_A(W) = W$, and trace preservation gives $\text{tr } W = 0$. Irreducibility of A therefore gives $W = 0$. The shift relation $P_{k+1} A^i = A^i P_k$ moves $\mathcal{E}_A^t(U)$ into the $(u+t, v+t)$ corner, so only the $t = 0$ term contributes after multiplying by P_u and P_v . Thus $U = P_u W P_v = 0$. $\square$

Theorem 22.2.3.29 (Different periods imply asymptotic orthogonality). [Lean](#) [✓ Stmt](#) *If A and B have periods $m_A \neq m_B$, then $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$.*

Proof. [✓ Proof](#) If $D_A \neq D_B$, use Theorem 22.2.3.40. Assume $D_A = D_B$ and $A \sim_{\text{gp}} B$, say $B^i = \zeta X A^i X^{-1}$ with $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \neq 0$. The transfer maps satisfy

$$\mathcal{E}_B(Y) = \zeta \bar{\zeta} X \mathcal{E}_A(X^{-1} Y X^{-*}) X^*.$$

With $\mathcal{E}_A(\rho_A) = \rho_A$, put $\sigma = X \rho_A X^*$. Then $\mathcal{E}_B(\sigma) = |\zeta|^2 \sigma$, while $\mathcal{E}_B(\rho_B) = \rho_B$ for a nonzero positive fixed point ρ_B . Perron–Frobenius eigenvalue uniqueness [Vol12, Theorem 6.3] gives $|\zeta|^2 = 1$. Hence $\mathcal{E}_B$ is conjugate to $\mathcal{E}_A$, so $\text{spec}_{\text{per}}(\mathcal{E}_A) = \text{spec}_{\text{per}}(\mathcal{E}_B)$. Since

$$\begin{aligned} \text{spec}_{\text{per}}(\mathcal{E}_A) &= \{e^{2\pi i r/m_A} : 0 \leq r < m_A\}, \\ \text{spec}_{\text{per}}(\mathcal{E}_B) &= \{e^{2\pi i r/m_B} : 0 \leq r < m_B\}, \end{aligned} \tag{22.2}$$

equality of the two finite sets in (22.2) gives $m_A = m_B$, a contradiction. Thus $A \not\sim_{\text{gp}} B$, and the irreducible trace-preserving overlap dichotomy gives $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$. $\square$

Lemma 22.2.3.30 (Sector-block normality for periodic tensors). [Lean](#) [✓ Stmt](#) *For a periodic tensor equipped with a cyclic sector decomposition, each nonzero blocked sector tensor is normal.*

Theorem 22.2.3.31 (No sector match implies asymptotic orthogonality). [Lean](#) [✓ Stmt](#) Suppose A and B have the same total bond dimension and the same period m . Let A_u and B_v be period-blocked sector tensors of virtual dimensions d_u and e_v , indexed by $u, v \in \mathbb{Z}_m$, such that

$$\begin{aligned}\bar{A} &\sim \bigoplus_{u \in \mathbb{Z}_m} A_u, \\ \bar{B} &\sim \bigoplus_{v \in \mathbb{Z}_m} B_v,\end{aligned}$$

with unit weights, and the two decompositions are cyclic sector decompositions. Assume each sector is left-canonical:

$$\begin{aligned}\sum_{\alpha} (A_u^{\alpha})^{\dagger} A_u^{\alpha} &= I_{d_u}, \\ \sum_{\alpha} (B_v^{\alpha})^{\dagger} B_v^{\alpha} &= I_{e_v}.\end{aligned}$$

Assume $d_u \neq 0$ for every u and $e_v \neq 0$ for every v , and whenever $d_u = e_v$, the corresponding sector tensors are not gauge-phase equivalent after identifying their virtual spaces:

$$d_u = e_v \implies A_u \not\sim_{\text{gp}} B_v.$$

Then $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$.

Proof. [✓ Proof](#) Write $\bar{A} \sim \bigoplus_{u \in \mathbb{Z}_m} A_u$ and $\bar{B} \sim \bigoplus_{v \in \mathbb{Z}_m} B_v$. For every N ,

$$O_{\bar{A}\bar{B}}(N) = \sum_{u \in \mathbb{Z}_m} \sum_{v \in \mathbb{Z}_m} O_{A_u B_v}(N). \quad (22.3)$$

For each sector pair (u, v) ,

$$\begin{aligned}d_u = e_v, \quad A_u \sim_{\text{gp}} B_v &\implies \lim_{N \rightarrow \infty} O_{A_u B_v}(N) = 0, \\ d_u \neq e_v &\implies \lim_{N \rightarrow \infty} O_{A_u B_v}(N) = 0.\end{aligned}$$

The finite double sum (22.3) therefore gives

$$\lim_{N \rightarrow \infty} O_{\bar{A}\bar{B}}(N) = \sum_{u \in \mathbb{Z}_m} \sum_{v \in \mathbb{Z}_m} \lim_{N \rightarrow \infty} O_{A_u B_v}(N) = 0.$$

If $A \sim_{\text{gp}} B$, then $\bar{A} \sim_{\text{gp}} \bar{B}$, so for some $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \neq 0$, $(\bar{B})^{\alpha} = \zeta X (\bar{A})^{\alpha} X^{-1}$. Thus $V^{(N)}(\bar{B})_{\sigma} = \zeta^N V^{(N)}(\bar{A})_{\sigma}$, and

$$\begin{aligned}O_{\bar{A}\bar{B}}(N) &= \bar{\zeta}^N O_{\bar{A}\bar{A}}(N), \\ O_{\bar{B}\bar{B}}(N) &= |\zeta|^{2N} O_{\bar{A}\bar{A}}(N).\end{aligned}$$

Since $\lim_{N \rightarrow \infty} O_{\bar{A}\bar{A}}(N) = \lim_{N \rightarrow \infty} O_{\bar{B}\bar{B}}(N) = m > 0$,

$$1 = \lim_{N \rightarrow \infty} \frac{O_{\bar{B}\bar{B}}(N)}{O_{\bar{A}\bar{A}}(N)} = \lim_{N \rightarrow \infty} |\zeta|^{2N},$$

hence $|\zeta| = 1$. Therefore

$$\lim_{N \rightarrow \infty} |O_{\bar{A}\bar{B}}(N)| = \lim_{N \rightarrow \infty} O_{\bar{A}\bar{A}}(N) = m \neq 0,$$

contradicting $\lim_{N \rightarrow \infty} O_{\bar{A}\bar{B}}(N) = 0$. Thus $A \not\sim_{\text{gp}} B$, and the irreducible trace-preserving overlap dichotomy gives $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$. $\square$

Lemma 22.2.3.32 (Sector-match propagation under translation). [Lean](#) [✓ Stmt](#) Let A and B be left-canonical tensors with period m , and let $\bar{A} \sim \bigoplus_u A_u$ and $\bar{B} \sim \bigoplus_v B_v$ be unit-weight cyclic sector decompositions whose sectors are left-canonical. Suppose $d_{u_0} = e_{v_0}$, $d_{u_0} \neq 0$, and, after identifying the virtual spaces, $A_{u_0} \sim_{\text{gp}} B_{v_0}$. Then for every $l \in \mathbb{Z}_m$ there is an equality $d_{u_0+l} = e_{v_0+l}$ and, after this identification, $A_{u_0+l} \sim_{\text{gp}} B_{v_0+l}$.

Lemma 22.2.3.33 (Finite-cycle coboundary). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)

Let G be a commutative group, and let $\kappa_0, \dots, \kappa_n \in G$. If $\prod_{k=0}^n \kappa_k = 1$, then there are $\varphi_0, \dots, \varphi_n \in G$ such that $\kappa_k = \varphi_k \varphi_{k+1}^{-1}$ for every $k < n$, and $\kappa_n = \varphi_n \varphi_0^{-1}$. Equivalently, on a nonempty finite cyclic index set, a product-one family satisfies $\kappa_k = \varphi_k \varphi_{k+1}^{-1}$ for every cyclic edge. In particular, a product-one family of complex unit phases admits such a choice with all φ_k again of unit modulus.

Proof. [✓ Proof](#)

Set $\varphi_k = (\kappa_0 \cdots \kappa_{k-1})^{-1}$, with the empty product equal to 1. For $k < n$,

$$\varphi_k \varphi_{k+1}^{-1} = (\kappa_0 \cdots \kappa_{k-1})^{-1} (\kappa_0 \cdots \kappa_k) = \kappa_k.$$

Separating the final factor in the product-one hypothesis gives $(\kappa_0 \cdots \kappa_{n-1}) \kappa_n = 1$, which gives

$$\varphi_n \varphi_0^{-1} = (\kappa_0 \cdots \kappa_{n-1})^{-1} = \kappa_n.$$

□

Definition 22.2.3.34 (Global gauge). [Lean](#) [✓ Stmt](#) Given a positive period m , cyclic projectors P_u and Q_v , a cyclic offset q , and corner operators U_v , the global gauge is

$$U = \sum_u P_u U_{u+q} Q_{u+q}.$$

Lemma 22.2.3.35 (Global gauge as a sector sum). [Lean](#) [✓ Stmt](#) Let $m \geq 1$. If each corner operator satisfies $U_v = P_{v-q} U_v Q_v$, then $P_u U_{u+q} Q_{u+q} = U_{u+q}$, so the global gauge is the plain sector sum $U = \sum_v U_v$.

Proof. [✓ Proof](#) The corner relation $U_v = P_{v-q} U_v Q_v$ gives $P_u U_{u+q} Q_{u+q} = U_{u+q}$; reindexing the sum by $v = u + q$ turns $\sum_u P_u U_{u+q} Q_{u+q}$ into $\sum_v U_v$. □

Lemma 22.2.3.36 (Unit-phase gauge relation gives repeated blocks). [Lean](#) [✓ Stmt](#) Suppose $B^i = \zeta X A^i X^{-1}$ and $|\zeta| = 1$, with X invertible. Then A and B satisfy the repeated-block relation $A^i = \zeta^{-1} X^{-1} B^i X$.

Proof. [✓ Proof](#) Since $|\zeta| = 1$, the scalar is nonzero. Multiplication by X^{-1} on the left and by X on the right gives $X^{-1} B^i X = \zeta A^i$; multiplication by ζ^{-1} proves the claim. □

Lemma 22.2.3.37 (Global-gauge construction). [Lean](#) [✓ Stmt](#) Let $m \geq 1$, and let A and B carry the cyclic off-diagonal structure

$$\begin{aligned} A^i &= \sum_u P_u A^i P_{u+1}, \\ B^i &= \sum_v Q_v B^i Q_{v+1}, \end{aligned}$$

with P_u and Q_v families of orthogonal projections summing to the identity. Let q be the cyclic offset between the matched sector labels, and let each U_v be a corner partial isometry from the Q_v corner onto the P_{v-q} corner, that is,

$$\begin{aligned} U_v &= P_{v-q} U_v Q_v, \\ U_v^\dagger U_v &= Q_v, \\ U_v U_v^\dagger &= P_{v-q}. \end{aligned}$$

Assume the per-site proportionality of [DICCSPG17, eq. result],

$$P_u A^i P_{u+1} = \zeta U_{u+q} Q_{u+q} B^i Q_{u+q+1} U_{u+q+1}^\dagger, \quad |\zeta| = 1.$$

Then the global gauge $U = \sum_u P_u U_{u+q} Q_{u+q}$ is unitary and satisfies $A^i = \zeta U B^i U^\dagger$ for every i , so A and B are repeated blocks.

Proof. ✓ Proof The corner relation $U_v = P_{v-q} U_v Q_v$ gives $P_u U_{u+q} Q_{u+q} = U_{u+q}$, so $U = \sum_v U_v$ after reindexing. Distinct projections are orthogonal, $Q_v Q_w = 0$ and $P_{v-q} P_{w-q} = 0$ for $v \neq w$, which kills the cross terms in $UU^\dagger$ and $U^\dagger U$; the diagonal terms are $U_v U_v^\dagger = P_{v-q}$ and $U_v^\dagger U_v = Q_v$, so

$$\begin{aligned} UU^\dagger &= \sum_u P_u = \mathbb{1}, \\ U^\dagger U &= \sum_v Q_v = \mathbb{1}. \end{aligned}$$

The off-diagonal structure of B makes $Q_v B^i Q_w$ vanish unless $w = v+1$, so $U B^i U^\dagger = \sum_v U_v Q_v B^i Q_{v+1} U_{v+1}^\dagger$. Summing the per-site proportionality over u and reindexing by $v = u + q$ gives

$$\zeta U B^i U^\dagger = \sum_v \zeta U_v Q_v B^i Q_{v+1} U_{v+1}^\dagger = \sum_u P_u A^i P_{u+1} = A^i.$$

The last equality is the off-diagonal structure of A . The unit-modulus scalar and the unitary U exhibit A and B as repeated blocks. $\square$

Lemma 22.2.3.38 (Cyclic Ω_u -contraction for blocked sector gauges). Lean ✓ Stmt Let A and B be left-canonical tensors with period m , and let $\tilde{A} \sim \bigoplus_u A_u$ and $\tilde{B} \sim \bigoplus_v B_v$ be unit-weight cyclic sector decompositions whose sectors are left-canonical. Let the specified cyclic projectors be P_u and Q_v , and let the specified corner identifications realize the blocked sector tensors by $A_u^\alpha = P_u \tilde{A}^\alpha P_u$ and $B_v^\alpha = Q_v \tilde{B}^\alpha Q_v$. Assume $d_u \neq 0$ and A_u is normal for every u . Fix $q \in \mathbb{Z}_m$ and assume that for every u there are $d_u = e_{u+q}$, $\zeta_u \neq 0$, and $X_u \in \text{GL}_{d_u}(\mathbb{C})$ such that $B_{u+q}^\alpha = \zeta_u X_u A_u^\alpha X_u^{-1}$ for every blocked physical index α . The source obtains a phase-only product identity after absorbing these sector gauges into the B -sectors: with Q_v the cyclic sector projectors of B , it writes $\tilde{B}_v^i = U_v Q_v B^i Q_{v+1} U_{v+1}^\dagger$. The contraction input is then, for every physical block $(i_1, \dots, i_m)$,

$$A_u^{i_1} A_{u+1}^{i_2} \dots A_{u+m-1}^{i_m} = e^{i\lambda_{u+q}} \tilde{B}_{u+q}^{i_1} \tilde{B}_{u+q+1}^{i_2} \dots \tilde{B}_{u+q+m-1}^{i_m}.$$

Choose N_0 so that the N_0 -fold repetition of each cyclic m -site sector product is injective. For $\mathbf{i} = (i_1, \dots, i_{mN_0})$, form the repeated product

$$F_u^{\mathbf{i}} = (A_u^{i_1} \dots A_{u+m-1}^{i_m}) \dots (A_u^{i_{m(N_0-1)+1}} \dots A_{u+m-1}^{i_{mN_0}}),$$

and choose right inverses Ω_u satisfying

$$\sum_{\mathbf{i}} (\Omega_u^{\mathbf{i}})_{\alpha, \beta} (F_u^{\mathbf{i}})_{\alpha', \beta'} = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}.$$

Let $\widetilde{F}_{u+q+r}^{\mathbf{i}}$ denote the same cyclic repeated product formed from $\widetilde{B}_{u+q+r}, \dots, \widetilde{B}_{u+q+r+m-1}$. For every $r \in \mathbb{Z}_m$, the block identity gives

$$\begin{aligned} F_{u+r}^{\mathbf{i}} &= e^{iN_0 \lambda_{u+q+r}} \widetilde{F}_{u+q+r}^{\mathbf{i}}, \\ \widetilde{F}_{u+q+r}^{\mathbf{i}} &= e^{-iN_0 \lambda_{u+q+r}} F_{u+r}^{\mathbf{i}}. \end{aligned}$$

Let $\mathbf{i}_1, \dots, \mathbf{i}_m$ be independent words of length mN_0 , one for each inserted repeated product. Sum each $\mathbf{i}_r$ in the contraction by the right-inverse relation for Ω_{u+r} :

$$\begin{aligned} & A_u^{i_1} F_{u+1}^{\mathbf{i}_1} A_{u+1}^{i_2} F_{u+2}^{\mathbf{i}_2} \dots A_{u+m-1}^{i_m} F_u^{\mathbf{i}_m} \\ & \xrightarrow{\Omega_{u+1} \otimes \dots \otimes \Omega_{u+m-1} \otimes \Omega_u} A_u^{i_1} \otimes A_{u+1}^{i_2} \otimes \dots \otimes A_{u+m-1}^{i_m}. \end{aligned}$$

Expand the words $\mathbf{i}_1, \dots, \mathbf{i}_m$, and write $j_{s,t}$ for the physical index in the t th sector of the s th consecutive m -site block of the expanded word, where $0 \leq s \leq mN_0$ and $1 \leq t \leq m$. Since each inserted product $F_{u+r}^{\mathbf{i}_r}$ has length mN_0 , insertion shifts the cyclic sector position by $mN_0 \equiv 0 \pmod{m}$. Hence every consecutive m -site block begins at the sector u , and the interleaved word is

$$\begin{aligned} & A_u^{i_1} F_{u+1}^{\mathbf{i}_1} A_{u+1}^{i_2} F_{u+2}^{\mathbf{i}_2} \dots A_{u+m-1}^{i_m} F_u^{\mathbf{i}_m} \\ &= (A_u^{j_{0,1}} \dots A_{u+m-1}^{j_{0,m}}) \dots (A_u^{j_{mN_0,1}} \dots A_{u+m-1}^{j_{mN_0,m}}). \end{aligned}$$

Applying the block identity at the sector u to each of these m -site blocks gives

$$\begin{aligned} & A_u^{i_1} F_{u+1}^{\mathbf{i}_1} A_{u+1}^{i_2} F_{u+2}^{\mathbf{i}_2} \dots A_{u+m-1}^{i_m} F_u^{\mathbf{i}_m} \\ &= e^{i\lambda_{u+q}(mN_0+1)} \widetilde{B}_{u+q}^{i_1} \widetilde{F}_{u+q+1}^{\mathbf{i}_1} \widetilde{B}_{u+q+1}^{i_2} \widetilde{F}_{u+q+2}^{\mathbf{i}_2} \dots \widetilde{B}_{u+q+m-1}^{i_m} \widetilde{F}_{u+q}^{\mathbf{i}_m}. \end{aligned} \quad (22.4)$$

For each $r \in \mathbb{Z}_m$, substituting $\mathbf{i} = \mathbf{i}_r$ in (22.4) gives $\widetilde{F}_{u+q+r}^{\mathbf{i}_r} = e^{-iN_0 \lambda_{u+q+r}} F_{u+r}^{\mathbf{i}_r}$. Therefore the right-inverse relation gives the per-sector contraction identity

$$\sum_{\mathbf{i}_r} (\Omega_{u+r}^{\mathbf{i}_r})_{\alpha, \beta} (\widetilde{F}_{u+q+r}^{\mathbf{i}_r})_{\alpha', \beta'} = e^{-iN_0 \lambda_{u+q+r}} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}.$$

Since $r \mapsto u+q+r$ is a permutation of $\mathbb{Z}_m$, $\sum_{r \in \mathbb{Z}_m} \lambda_{u+q+r} = \sum_{r \in \mathbb{Z}_m} \lambda_r$. Applying $\Omega_{u+1} \otimes \dots \otimes \Omega_{u+m-1} \otimes \Omega_u$ to both sides of (22.4) and collecting the sector phases gives

$$\begin{aligned} & A_u^{i_1} \otimes A_{u+1}^{i_2} \otimes \dots \otimes A_{u+m-1}^{i_m} = e^{i\lambda_{u+q}(mN_0+1) - iN_0 \sum_{r \in \mathbb{Z}_m} \lambda_r} \widetilde{B}_{u+q}^{i_1} \otimes \widetilde{B}_{u+q+1}^{i_2} \otimes \dots \otimes \widetilde{B}_{u+q+m-1}^{i_m} \\ &= e^{i\eta_{u+q}} \widetilde{B}_{u+q}^{i_1} \otimes \widetilde{B}_{u+q+1}^{i_2} \otimes \dots \otimes \widetilde{B}_{u+q+m-1}^{i_m}. \end{aligned}$$

Here $\eta_{u+q} = \lambda_{u+q}(mN_0+1) - N_0 \sum_{r \in \mathbb{Z}_m} \lambda_r$. Then there are a scalar ξ with $|\xi| = 1$ and a unitary matrix U such that $A^i = \xi U B^i U^\dagger$ for every physical index i .

Theorem 22.2.3.39 (Sector match yields repeated-block equivalence). [Lean](#) [Stmt](#) Let A and B be periodic tensors with the same period m . Let $\bar{A} \sim \bigoplus_u A_u$ and $\bar{B} \sim \bigoplus_v B_v$ be unit-weight cyclic sector decompositions whose sectors are left-canonical, with specified cyclic projectors P_u, Q_v and corner identifications satisfying $A_u^\alpha = P_u \bar{A}^\alpha P_u$ and $B_v^\alpha = Q_v \bar{B}^\alpha Q_v$. Assume every sector A_u has nonzero virtual dimension. If there are sectors u_0, v_0 with $d_{u_0} = e_{v_0}$ and $A_{u_0} \sim_{\text{gp}} B_{v_0}$ after identifying their virtual spaces, then A and B are repeated blocks.

Theorem 22.2.3.40 (Different bond dimensions imply asymptotic orthogonality). [Lean](#) [✓ Stmt](#) If two periodic tensors have different bond dimensions, then $\langle V^{(N)}(A) | V^{(N)}(B) \rangle \rightarrow 0$ as $N \rightarrow \infty$.

Theorem 22.2.3.41 (Periodic overlap dichotomy). [Lean](#) [✓ Stmt](#) Let A and B be periodic tensors. Then at least one of the following alternatives holds:

1. $\langle V^{(N)}(A) | V^{(N)}(B) \rangle \rightarrow 0$ as $N \rightarrow \infty$;
2. A and B are related by the repeated-block relation, i.e. they differ by a unit-modulus phase and an invertible gauge.

Proof. [✓ Proof](#) If $m_A \neq m_B$, then $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$ by Theorem 22.2.3.29. Assume $m_A = m_B = m$. If $D_A \neq D_B$, then $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$ by Theorem 22.2.3.40. Assume $D_A = D_B$.

Choose period-blocked sector decompositions

$$\begin{aligned}\bar{A} &\sim \bigoplus_{u \in \mathbb{Z}_m} A_u, \\ \bar{B} &\sim \bigoplus_{v \in \mathbb{Z}_m} B_v.\end{aligned}$$

If no pair (u, v) satisfies $d_u = e_v$ and $A_u \sim_{\text{gp}} B_v$, then Theorem 22.2.3.31 gives $\lim_{N \rightarrow \infty} O_{AB}(N) = 0$.

Otherwise there are $u_0, v_0 \in \mathbb{Z}_m$ such that $d_{u_0} = e_{v_0}$ and $A_{u_0} \sim_{\text{gp}} B_{v_0}$. Theorem 22.2.3.39 then gives repeated-block equivalence between A and B . $\square$

Theorem 22.2.3.42 (Gram criterion for a periodic family). [Lean](#) [✓ Stmt](#) Let $\{A_j\}_{j \in J}$ be a finite family of periodic tensors whose periods divide a common positive integer p . If

$$\langle V^{(N)}(A_i) | V^{(N)}(A_j) \rangle \rightarrow 0 \quad (i \neq j),$$

then the vectors $\{|V^{(pN)}(A_j)\rangle\}_{j \in J}$ are linearly independent for every sufficiently large N .

Proof. [✓ Proof](#) The Gram matrix converges to the diagonal matrix whose j -th diagonal entry is the positive period of A_j . Its determinant therefore has a nonzero limit, so the Gram matrix is invertible for every sufficiently large N . $\square$

Theorem 22.2.3.43 (Eventual linear independence of periodic vectors). [Lean](#) [✓ Stmt](#) For a finite family $\{A_j\}_{j \in J}$ of pairwise non-repeated periodic tensors, in the sense that no two are related by a gauge transformation and a unit-modulus scalar, whose periods divide a common positive integer p , there is N_0 such that for every $N \geq N_0$, the map

$$\begin{aligned}\Phi_N: \mathbb{C}^J &\rightarrow (\mathbb{C}^d)^{\otimes pN}, \\ (c_j)_{j \in J} &\mapsto \sum_{j \in J} c_j |V^{(pN)}(A_j)\rangle\end{aligned}$$

satisfies $\ker \Phi_N = \{0\}$.

Scope restriction. The source consequence following the periodic-overlap dichotomy [DICCSPG17, lines 589–609] states eventual independence at every sufficiently large length and also states that, for an ambient tensor A in irreducible form having $\{A_j\}_{j \in J}$ as its basis of periodic tensors, the non-zero vectors $|V^{(N)}(A_j)\rangle$ span the vector $|V^{(N)}(A)\rangle$; this spanning assertion is what the paper cites as justification for the phrase “basis of periodic vectors”. The theorem recorded here proves independence only along the common-multiple subsequence pN and omits the spanning assertion. The corresponding paper-gap note, [con26s], Section “Scope restriction: independence without spanning”, records this restriction and its elimination plan.

Proof. ✓ Proof Periodicity makes every bond dimension positive. For distinct i, j , the repeated-block alternative in Theorem 22.2.3.41 contradicts pairwise non-repetition, so the cross overlap tends to zero. Theorem 22.2.3.42 now gives the claimed independence along the common-period subsequence. $\square$

Remark 22.2.3.44 (Comparison with the blocking approach). Not ready Theorem 22.2.2.1 and the equal-MPV corollary of Section 10.4 both characterize the gauge freedom between two tensors with equal MPV families, but they do so at different levels:

1. *Blocking approach* (Section 10.4, following [CPGSV16]): for a common multiple p of the periods,

$$\begin{aligned} A^{[p]} &\sim \bigoplus_a C_a, \\ B^{[p]} &\sim \bigoplus_b D_b, \end{aligned}$$

with all C_a, D_b normal. The Fundamental Theorem gives, after a permutation of the normal summands,

$$D_{\pi(a)}^\alpha = X_a C_a^\alpha X_a^{-1}.$$

This is a gauge statement for $A^{[p]}$ and $B^{[p]}$; the cyclic phases have been absorbed into the period- p words.

2. *Periodic approach* (Theorem 22.2.2.1, following [DICCSPG17]): keep the cyclic sectors

$$\begin{aligned} A^i &= \sum_{u \in \mathbb{Z}_m} P_u A^i P_{u+1}, \\ B^i &= \sum_{v \in \mathbb{Z}_m} Q_v B^i Q_{v+1}. \end{aligned}$$

Sector matching gives phases ϕ_v , a shift q , and unitaries U_v such that

$$P_u A^i P_{u+1} = e^{i\xi} e^{i\phi_{u+q}} U_{u+q} Q_{u+q} B^i Q_{u+q+1} U_{u+q+1}^\dagger e^{-i\phi_{u+q+1}}.$$

In the equal-MPV theorem the proportional phase is absorbed, so one obtains $ZA^i = UB^iU^{-1}$ with $Z^m = \mathbb{1}$; the trace only sees powers of the corresponding m -th roots of unity.

Chapters 8 and 10 follow only the blocking argument: remove periodicity by blocking, then apply the Fundamental Theorem. The periodic extension treats periodic blocks directly.

Theorem 22.2.3.45 (Peripheral proportional case from positive-length MPV equality). Lean ✓ Stmt *Let A and B be periodic tensors with the same matrix product vectors at every positive length. Then their bond dimensions agree, and after identifying the bond spaces, A and B are repeated blocks.*

Proof. ✓ Proof By Theorem 22.2.3.41, either the overlap $\langle V^{(N)}(A) | V^{(N)}(B) \rangle$ tends to 0 or the bond dimensions agree and A and B are repeated blocks. In the decay case, equality of the MPV families gives, for every $N > 0$,

$$\langle V^{(N)}(A) | V^{(N)}(B) \rangle = \langle V^{(N)}(A) | V^{(N)}(A) \rangle. \quad (22.5)$$

Along multiples of the period m_a , (22.5) would imply

$$0 = \lim_{N \rightarrow \infty} \langle V^{(m_a N)}(A) | V^{(m_a N)}(B) \rangle = \lim_{N \rightarrow \infty} \langle V^{(m_a N)}(A) | V^{(m_a N)}(A) \rangle = m_a,$$

contradicting $m_a > 0$. $\square$

Lemma 22.2.3.46 (A pure similarity is a repeated-block relation). Lean ✓ Stmt *If $B^i = X A^i X^{-1}$ for an invertible matrix X , then A and B are repeated blocks with equal bond dimension and unit phase.*

Proof. ✓ Proof Invert the pure gauge and take the repeated-block phase to be one. $\square$

Lemma 22.2.3.47 (Pure similarities preserve periodic overlap hypotheses). Lean ✓ Stmt *Let $\{A_j\}_{j \in J}$ and $\{B_k\}_{k \in K}$ be finite tensor families, and suppose that A'_j is purely similar to A_j and B'_k is purely similar to B_k for every j, k . Assume that every tensor in each primed family has a non-decaying mixed-overlap partner in the other primed family, and that every non-decaying primed pair is a repeated pair. Then the same three assertions hold for the two original families.*

Proof. ✓ Proof Cyclicity of the trace makes every closed-chain coefficient, and hence every mixed overlap, invariant under the two similarities. Thus decay and non-decay are unchanged. For a non-decaying pair, compose its primed repeated-block relation with the two pure similarities. $\square$

Theorem 22.2.3.48 (Non-decaying mixed overlaps for normalized bases of periodic tensors).

Lean ✓ Stmt *Let*

$$A^i = \bigoplus_{j \in J} (R_j \otimes A_j^i),$$

$$B^i = \bigoplus_{k \in K} (S_k \otimes B_k^i),$$

where $R_j = \text{diag}(\mu_{j,1}, \dots, \mu_{j,r_j})$ and $S_k = \text{diag}(\nu_{k,1}, \dots, \nu_{k,s_k})$ have nonzero complex diagonal entries. Suppose that the finite families $\{A_j\}_{j \in J}$ and $\{B_k\}_{k \in K}$ are pairwise non-repeated families of left-canonical periodic tensors. If, for every positive integer N , there is a scalar $c_N \neq 0$ such that

$$|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle.$$

Then every A_j has an index $k \in K$ for which

$$\langle V^{(N)}(A_j) | V^{(N)}(B_k) \rangle \not\rightarrow 0,$$

and, conversely, every B_k has such a partner in the first family. Moreover, whenever the overlap between A_j and B_k does not tend to zero, the two blocks have equal bond dimension and there are an invertible matrix Y and a scalar ζ with $|\zeta| = 1$ such that, for every physical index i ,

$$A_j^i = \zeta Y B_k^i Y^{-1}.$$

Normalization boundary. The source initially assumes that the transfer maps of the basis tensors are irreducible and have spectral radius one, and then obtains the trace-preserving normalization by a block-diagonal similarity [DICCSPG17, lines 313–332]. The present theorem starts with the normalized basis tensors. The multiplicity matrices are otherwise arbitrary diagonal matrices with nonzero complex entries, the displayed direct sums are literal, and proportionality is assumed only at positive lengths. Theorem 22.1.5 constructs each normalizing similarity, while Theorem 22.1.8 assembles them without changing the multiplicity matrices. Theorem 22.2.3.49 below applies the present result and returns its conclusion to the unnormalized blocks.

Proof. ✓ Proof Fix $j_0 \in J$ and suppose that its overlap with every B_k tends to zero. Let $T \subseteq J$ be the set of all indices with this property. For every $j \notin T$, choose $k(j)$ for which the mixed overlap does not decay. The periodic overlap dichotomy identifies A_j with $B_{k(j)}$ up to a gauge transformation and a phase; consequently their length- N vectors are scalar multiples of one another.

Choose a common positive multiple p of all periods. Along the lengths pN , define the coproduct-indexed family $w_N: T \amalg K \rightarrow (\mathbb{C}^d)^{\otimes pN}$ by $w_N(\text{inl}(j)) = |V^{(pN)}(A_j)\rangle$ and $w_N(\text{inr}(k)) = |V^{(pN)}(B_k)\rangle$. This family is eventually linearly independent. Indeed, distinct vectors within either summand have decaying overlaps by non-repetition and the overlap dichotomy, while the mixed overlaps vanish by the definition of T .

Expand the proportionality relation at length pN using the grouped coefficients

$$\text{tr}(R_j^{pN}) = \sum_q (\mu_{j,q}^p)^N, \quad \text{tr}(S_k^{pN}) = \sum_r (\nu_{k,r}^p)^N.$$

Replace each term indexed by $j \notin T$ by its scalar multiple of the corresponding $B_{k(j)}$ term, and collect terms with the same value of $k(j)$. Linear independence forces $\text{tr}(R_{j_0}^{pN}) = 0$ for every sufficiently large N . Since each $\mu_{j_0,q}^p$ is nonzero, geometric extrapolation gives the same identity at exponent zero. This says $r_{j_0} = 0$, contradicting the positive size of R_{j_0} . Hence j_0 has a non-decaying partner. Interchanging A and B , inverting the nonzero scalars c_N , and using conjugate symmetry of the mixed overlap gives the converse assertion. For any non-decaying pair, the periodic overlap dichotomy excludes its decay alternative and supplies the repeated-block relation. $\square$

Theorem 22.2.3.49 (Non-decaying mixed overlaps for spectrally periodic bases). Lean ✓ Stmt
Let

$$A^i = \bigoplus_{j \in J} (R_j \otimes A_j^i), \quad B^i = \bigoplus_{k \in K} (S_k \otimes B_k^i),$$

where the diagonal entries of every R_j and S_k are nonzero. Suppose that the two representative families are pairwise non-repeated. Assign positive integers m_j and n_k to their representatives, and suppose that every transfer map is irreducible, has spectral radius one, and has peripheral spectrum respectively $\{z \in \mathbb{C} : z^{m_j} = 1\}$ or $\{z \in \mathbb{C} : z^{n_k} = 1\}$. If, for every positive N , there is a scalar $c_N \neq 0$ such that

$$|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle,$$

then every representative in either family has a representative in the other family whose mixed overlap does not tend to zero. Every such pair has equal bond dimension and differs by an invertible similarity and a unit-modulus scalar.

Proof. ✓ Proof Normalize all representatives by their pure Perron gauges and assemble the gauges block diagonally. The multiplicity matrices and their grouped coefficients remain unchanged. Pure similarity preserves the total positive-length proportionality and pairwise non-repetition, so Theorem 22.2.3.48 applies to the normalized families. Finally apply Lemma 22.2.3.47. $\square$

Theorem 22.2.3.50 (Scalar-weight specialization of the mixed-overlap theorem). Lean ✓ Stmt
Let A and B each have the same length- N matrix-product vector, for every $N \geq 0$, as a scalar-weight direct sum formed from a finite, pairwise non-repeated family of left-canonical periodic

tensors $\{A_j\}_{j \in J}$ and $\{B_k\}_{k \in K}$, with a positive real scalar weight attached to each block. Suppose that, for every positive integer N , there is a scalar $c_N \neq 0$ such that

$$|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle.$$

Then every block in either family has a block in the other family whose mixed overlap does not tend to zero. Every such non-decaying pair has equal bond dimension and differs by an invertible similarity and a unit-modulus scalar.

Scope restriction. This formulation records one positive scalar per listed block and equality with the corresponding direct sum also at the empty word. Thus it is a specialization of the preceding multiplicity-bearing theorem and is not itself the unrestricted irreducible-form statement of [DICCSPG17, Theorem 3.4].

Proof. ✓ Proof Repeat the coefficient argument above with the singleton coefficients μ_j^N and ν_k^N . Their nonvanishing at every length replaces the geometric-extrapolation step. Equality with the two scalar-weight direct sums transports the positive-length proportionality relation to the corresponding block-diagonal tensors. $\square$

Theorem 22.2.3.51 (Proportional Fundamental Theorem for irreducible-form MPS). Not ready
Let A and B have multiplicity-bearing irreducible forms

$$A^i = \bigoplus_{j \in J} (R_j \otimes A_j^i),$$

$$B^i = \bigoplus_{k \in K} (S_k \otimes B_k^i),$$

where R_j and S_k are diagonal with nonzero diagonal entries and the two displayed block families are pairwise non-repeated bases of periodic tensors. Suppose that, for every positive N , the vectors $|V^{(N)}(A)\rangle$ and $|V^{(N)}(B)\rangle$ are proportional by a nonzero scalar. Then there is a bijection $\pi: J \rightarrow K$ such that, for every $j \in J$, the blocks A_j and $B_{\pi(j)}$ have the same period and there are a unit-modulus scalar ξ_j and an invertible matrix Y_j such that, for every physical index i ,

$$A_j^i = \xi_j Y_j B_{\pi(j)}^i Y_j^{-1}.$$

Each B_k occurs for exactly one j . This is [DICCSPG17, Theorem 3.4, lines 613–632].

The theorem relates the two bases of periodic tensors but makes no assertion relating their multiplicity matrices.

22.3 Physical symmetries on periodic MPS

Theorem 22.3.1 (Conditional group-action Z -gauge reformulation of the symmetry corollary).

Lean ✓ Stmt Let A be an MPS tensor in irreducible form II and let $U: G \rightarrow \text{GL}_d(\mathbb{C})$ be a representation of a group G on the physical leg, acting unitarily ($U(g)U(g)^\dagger = \mathbb{1}$). Assume moreover the periodic equal-case Fundamental Theorem of MPS (Theorem 22.2.2.1) as an explicit hypothesis on the pair (d, D) , expressed by the hypothesis Lean $d \mid D$. If A is on-site symmetric under U — that is, the twisted tensor $\tilde{A}_g$ has the same MPV family as A for every $g \in G$ — then for each $g \in G$ there exists a positive period m_g and a $\mathbb{Z}_{m_g}$ -gauge equivalence between A and $\tilde{A}_g$.

Concretely there are matrices Z_g and Y_g , with $Z_g^{m_g} = \mathbb{1}$ and $[A^i, Z_g] = 0$, satisfying $Z_g A^i = Y_g \tilde{A}_g^i Y_g^{-1}$ for every physical index i .

Proof. ✓ Proof The twisted tensor $\tilde{A}_g$ coincides with the physical-index rotation $A_{U(g)}$ of Definition G.1.1.1. By Theorem 22.1.19 the rotated tensor remains in irreducible form II, and the on-site symmetry hypothesis supplies the equality $\mathcal{V}(A) = \mathcal{V}(A_{U(g)})$. Apply the periodic equal-case Fundamental Theorem (Theorem 22.2.2.1) to obtain the asserted $\mathbb{Z}_{m_g}$ -gauge equivalence. $\square$

Definition 22.3.2 (Conditional periodic projective-rigidity hypothesis). Lean ✓ Stmt Let A be an MPS tensor with physical dimension d and bond dimension D , and assume it is on-site symmetric under a group action $U : G \rightarrow \mathrm{GL}_d(\mathbb{C})$. The tensor A satisfies the *periodic projective-rigidity hypothesis* if one may choose a family of virtual gauges $Y : G \rightarrow \mathrm{GL}_D(\mathbb{C})$ and a scalar 2-cochain $u : G \times G \rightarrow \mathbb{C}^\times$ such that

1. every Y_g realizes the pointwise Z -gauge intertwining modeled on the symmetry corollary with the twisted tensor $\tilde{A}_g$ for some accompanying Z_g (with $Z_g^{m_g} = \mathbb{1}$ and $[A^i, Z_g] = 0$); and
2. the family obeys the projective multiplication law on the bond space: $Y_h Y_g = u(g, h) Y_{gh}$.

The source supplies the corresponding intertwining for a single local unitary. The group-indexed coherent choice and the projective multiplication law are additional assumptions: after proving the symmetry corollary, the source paper says that its consequences will be explored elsewhere. The injective case reduces this coherence to scalar uniqueness of gauges (cf. §11.1); in the genuinely periodic setting there is a non-trivial $\mathbb{Z}_m$ -gauge ambiguity, so coherence of the factor system u is an independent analytic assumption.

Theorem 22.3.3 (Conditional projective representation from symmetry gauges). Lean ✓ Stmt Assume the periodic projective-rigidity hypothesis (Definition 22.3.2) for the tensor A and the physical action U . Then the virtual gauge family supplied by that hypothesis defines a projective representation of G on the bond space: there exist a scalar 2-cochain $\omega : G \times G \rightarrow \mathbb{C}^\times$ and virtual gauges $X : G \rightarrow \mathrm{GL}_D(\mathbb{C})$ satisfying $X(g)X(h) = \omega(g, h)X(gh)$, while for each g the matrix $X(g^{-1})$ still realizes the pointwise $\mathbb{Z}_{m_g}$ -intertwining, encoded in the rigidity hypothesis, between A and the twisted tensor $\tilde{A}_g$.

Proof. ✓ Proof Unpack the rigidity hypothesis to extract the family (Y, u) together with the pointwise Z -gauge intertwiners. Set $X(g) := Y(g^{-1})$ and $\omega(g, h) := u(h^{-1}, g^{-1})$. The projective multiplication law $X(g)X(h) = \omega(g, h)X(gh)$ is then exactly the (h^{-1}, g^{-1}) -instance of the factor-system identity, using $(gh)^{-1} = h^{-1}g^{-1}$. The intertwining relation between A and $\tilde{A}_g$ transports unchanged. $\square$

Theorem 22.3.4 (Cocycle identity for the upgraded factor system). Lean ✓ Stmt If the bond dimension D is positive, the factor system ω produced by Theorem 22.3.3 satisfies the multiplicative 2-cocycle identity $\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k)$.

Proof. ✓ Proof Associativity of matrix multiplication applied to $X(g)X(h)X(k)$, combined with the projective multiplication law, gives

$$\omega(g, h)\omega(gh, k)X((gh)k) = \omega(g, hk)\omega(h, k)X(g(hk)). \quad (22.6)$$

Since $(gh)k = g(hk)$, both sides of (22.6) are scalar multiples of the same matrix $X((gh)k)$. The matrix lies in $\mathrm{GL}_D(\mathbb{C})$ and is therefore invertible, but the scalar cancellation leading from an equality of scalar multiples of a matrix to equality of the scalar coefficients additionally requires $D > 0$ so that the underlying index set is inhabited. Cancelling then yields $\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k)$ in $\mathbb{C}^\times$, hence in $\mathrm{Units}(\mathbb{C})$. $\square$

Remark 22.3.5 (External cohomological interpretation). In the one-dimensional SPT literature, a $U(1)$ -valued factor system of a symmetric MPS gives a cohomology class in $H^2(G, U(1))$ (cf. Chen–Gu–Wen and Pérez-García et al.). This interpretation is external to [DICCSPG17]: Section 4.2 proves the Z -gauge symmetry corollary and leaves its consequences to later work. The result established here is only the 2-cocycle identity of Theorem 22.3.4, under the conditional rigidity hypothesis of Definition 22.3.2.

Theorem 22.3.6 (Conditional projective representation and cocycle from symmetry). [Lean](#) [✓ Stmt](#) Theorems 22.3.3 and 22.3.4 combine into a single statement: from the periodic projective-rigidity hypothesis (Definition 22.3.2), one obtains a projective representation of G on the bond space whose factor system is a genuine 2-cocycle whenever $D > 0$.

22.4 Refinement and divisibility

The next theorem relates refinement of a matrix-product-vector family to divisibility of its transfer map, following [DICCSPG17, Theorem 4.1].

Definition 22.4.1 (p -divisible CP map). [Lean](#) [✓ Stmt](#) A linear endomorphism $\mathcal{E}$ of $M_D(\mathbb{C})$ is p -divisible if there exists a CPTP map $\mathcal{E}'$ of $M_D(\mathbb{C})$ such that $\mathcal{E} = (\mathcal{E}')^p$.

Definition 22.4.2 (p -refinable MPS tensor). [Lean](#) [✓ Stmt](#) Given a tensor B , the family $\mathcal{V}(B)$ can be p -refined if there exist another tensor A and an isometry $W : \mathbb{C}^d \rightarrow (\mathbb{C}^d)^{\otimes p}$ such that, for every N ,

$$|V^{(pN)}(A)\rangle = W^{\otimes N} |V^{(N)}(B)\rangle.$$

Equivalently, after identifying a p -letter with an element of $\{0, \dots, d^p - 1\}$, the p -blocked tensor $A^{[p]}$ satisfies

$$V^{(N)}(A^{[p]})_\tau = \sum_{\sigma \in \{0, \dots, d^p - 1\}^N} \left(\prod_k W_{\tau(k), \sigma(k)} \right) V^{(N)}(B)_\sigma,$$

for every $N \in \mathbb{N}$ and every $\tau \in \{0, \dots, d^p - 1\}^N$.

Theorem 22.4.3 (The tensor C associated to a refinement witness). [Lean](#) [✓ Stmt](#) Let B be an MPS tensor in irreducible form II. If $\mathcal{V}(B)$ can be p -refined, then there exist a tensor A , an isometry $W : \mathbb{C}^d \rightarrow (\mathbb{C}^d)^{\otimes p}$, and a tensor C defined by $C^\alpha := \sum_{i=0}^{d-1} W_{\alpha, i} B^i$ for $\alpha \in \{0, \dots, d^p - 1\}$, such that C is again in irreducible form II, $\mathcal{E}_C = \mathcal{E}_B$, and C has the same MPV family as $A^{[p]}$.

Proof. [✓ Proof](#) Choose (A, W) from the refinement data. The definition of C gives $\mathcal{E}_C = \mathcal{E}_B$ and equality of MPV families between C and $A^{[p]}$. To see that C is still in irreducible form II, transport each periodic block B_j of B through the same physical isometry W . The left-canonical normalization is preserved because $W^\dagger W = \mathbb{1}$, while irreducibility and the peripheral spectrum are preserved because the transfer map of each transported block agrees with that of B_j . Reassembling these transported blocks with the same weights gives an irreducible-form decomposition of C . $\square$

Definition 22.4.4 (Z -gauge after applying the blocked equal-case theorem). [Lean](#) [✓ Stmt](#) Fix (d, D, p) . This hypothesis says that whenever a blocked tensor C in irreducible form II has the same MPV family as the p -blocked tensor $A^{[p]}$, there exists an integer $m \geq 1$ and a Z_m -gauge equivalence between C and $A^{[p]}$.

Theorem 22.4.5 (Z-gauge equivalence from the periodic theorem). [Lean](#) [✓ Stmt](#) Assume the periodic equal-case Fundamental Theorem at the blocked physical dimension. Let C and $A^{[p]}$ be MPS tensors of the same bond dimension, both in irreducible form II, and suppose that they have the same MPV family. Then there exists an integer $m \geq 1$ and a $\mathbb{Z}_m$ -gauge equivalence between C and $A^{[p]}$.

Definition 22.4.6 (Construction of $\tilde{A}$ from the Z-gauge). [Lean](#) [✓ Stmt](#) Fix (d, D, p) . This hypothesis says that if $m \geq 1$ and a blocked tensor C in irreducible form II is $\mathbb{Z}_m$ -gauge equivalent to $A^{[p]}$, then one can construct a tensor $\tilde{A}$ with $\sum_i (\tilde{A}^i)^\dagger \tilde{A}^i = \mathbb{1}$ and $\mathcal{E}_C = \mathcal{E}_{\tilde{A}^{[p]}}$. This channel-level hypothesis is the transfer-map reading of the construction in [DICCSPG17, lines 765–810]: for each periodic block A_j , the scalars $c_{j,\alpha}$ define matrices A'_j , and then $\tilde{A} = U^\dagger A' U$. This source description is the intended irreducible-form subcase of the hypothesis; the definition above keeps the construction as a conditional input and does not assert these block scalars $c_{j,\alpha}$ and matrices A'_j for arbitrary A . The cited source moreover obtains the blocked-tensor identity $\tilde{A}^{[p]} = C$, whereas the formal hypothesis above records only its channel-level consequence $\mathcal{E}_C = \mathcal{E}_{\tilde{A}^{[p]}}$ (equality of p -blocked transfer maps). This conditional input is therefore the transfer-map weakening of the source construction, not the full blocked tensor equality.

Definition 22.4.7 (Construction of $\tilde{A}$ after the equal-case theorem). [Lean](#) [✓ Stmt](#) Fix (d, D, p) . This states the combined construction: whenever a blocked tensor C in irreducible form II has the same MPV family as a p -blocked tensor $A^{[p]}$, one can replace A by a left-canonical tensor $\tilde{A}$ with $\mathcal{E}_C = \mathcal{E}_{\tilde{A}^{[p]}}$.

Theorem 22.4.8 (Construction of $\tilde{A}$ from Z-gauge extraction). [Lean](#) [✓ Stmt](#) If the blocked equal-case Z-gauge extraction hypothesis and the construction of $\tilde{A}$ from that Z-gauge both hold, then the combined equal-case construction holds.

Proof. [✓ Proof](#) Apply the Z-gauge extraction hypothesis to obtain a blocked $\mathbb{Z}_m$ -gauge witness between C and $A^{[p]}$, and then apply the construction of $\tilde{A}$ to that witness. $\square$

Theorem 22.4.9 (Reduction to the blocked equal-case construction). [Lean](#) [✓ Stmt](#) Assume the blocked equal-case construction. Then every p -refinable tensor B in irreducible form II admits a left-canonical tensor $\tilde{A}$ with $\mathcal{E}_B = \mathcal{E}_{\tilde{A}^{[p]}}$.

Proof. [✓ Proof](#) Apply Theorem 22.4.3 to the refinement witness of B . This produces a blocked tensor C in irreducible form II with $\mathcal{E}_C = \mathcal{E}_B$ and the same MPV family as $A^{[p]}$. The blocked equal-case construction then gives a left-canonical tensor $\tilde{A}$ with $\mathcal{E}_C = \mathcal{E}_{\tilde{A}^{[p]}}$. Combining the two transfer-map identities yields $\mathcal{E}_B = \mathcal{E}_{\tilde{A}^{[p]}}$. $\square$

Theorem 22.4.10 (Forward direction from the blocked equal-case construction). [Lean](#) [✓ Stmt](#) Assume the blocked equal-case construction. Then for every tensor B in irreducible form II, p -refinability of B implies p -divisibility of $\mathcal{E}_B$.

Proof. [✓ Proof](#) Combine Theorem 22.4.9 with the witness-level forward implication: once $\mathcal{E}_B$ is identified with $\mathcal{E}_{\tilde{A}^{[p]}}$ for a left-canonical $\tilde{A}$, the blocking identity gives $\mathcal{E}_B = (\mathcal{E}_{\tilde{A}})^p$. $\square$

The following conditional statements separate the refinability–divisibility equivalence of [DICCSPG17, Theorem 4.1] from the additional constructions used below.

Theorem 22.4.11 (p -refinability iff transfer-map p -divisibility, conditional). [Lean](#) [✓ Stmt](#) Let B be an MPS tensor in irreducible form II and let $p \geq 1$. Assume the forward construction of $\tilde{A}$ for refinability $\Rightarrow$ divisibility and the reverse transfer-map construction for divisibility $\Rightarrow$ refinability. Then B is p -refinable iff its transfer map $\mathcal{E}_B$ is p -divisible.

Proof. ✓ Proof Combine the forward implication (Theorem 22.4.12) with the reverse implication (Theorem 22.4.15) into a single biconditional. The forward direction assumes the construction of $\tilde{A}$; the reverse direction assumes the transfer-map construction. Together they give the full equivalence under those hypotheses. $\square$

Theorem 22.4.12 (*p*-refinability implies transfer-map *p*-divisibility). Lean ✓ Stmt Let B be an MPS tensor in irreducible form II. Assume that every refinement witness gives a tensor $\tilde{A}$ with $\sum_i (\tilde{A}^i)^\dagger \tilde{A}^i = \mathbb{1}$ and $\mathcal{E}_B = \mathcal{E}_{\tilde{A}^{[p]}}$. Then *p*-refinability of B implies *p*-divisibility of $\mathcal{E}_B$.

Proof. ✓ Proof The construction supplies $\tilde{A}$ with $\sum_i (\tilde{A}^i)^\dagger \tilde{A}^i = \mathbb{1}$ and $\mathcal{E}_B = \mathcal{E}_{\tilde{A}^{[p]}}$. The left-canonical property makes $\mathcal{E}_{\tilde{A}}$ a CPTP channel, and the blocking identity $\mathcal{E}_{\tilde{A}^{[p]}} = (\mathcal{E}_{\tilde{A}})^p$ then identifies $\mathcal{E}_B$ with the *p*-th power of the channel $\mathcal{E}_{\tilde{A}}$. $\square$

Definition 22.4.13 (Blocked-to-root channel-root hypothesis from *Z*-gauge). Lean ✓ Stmt Fix (d, D, p) . This sharpened reconstruction hypothesis asks for a CPTP map $\mathcal{E}'$ such that $\mathcal{E}_C = (\mathcal{E}')^p$ and the Choi/Kraus rank of $\mathcal{E}'$ is at most *d*, whenever C is related to $A^{[p]}$ by a blocked $\mathbb{Z}_m$ -gauge witness.

Theorem 22.4.14 (Channel-root criterion for blocked-to-root reconstruction). Lean ✓ Stmt If the blocked-to-root channel-root hypothesis holds, then the tensor-level blocked-to-root reconstruction hypothesis holds as well.

Proof. ✓ Proof Choose a tensor A' whose matrix family realizes the root channel with at most *d* nonzero terms. Since this channel is trace-preserving, the matrices of A' satisfy $\sum_i (A'^i)^\dagger A'^i = \mathbb{1}$. The identity $\mathcal{E}_C = (\mathcal{E}_{A'})^p$ is then equivalent to $\mathcal{E}_C = \mathcal{E}_{A'^{[p]}}$ by the blocking formula. $\square$

Theorem 22.4.15 (Transfer-map *p*-divisibility implies *p*-refinability). Lean ✓ Stmt Let B be an MPS tensor in irreducible form II and let $p \geq 1$. Assume the reverse transfer-map root hypothesis: there is a tensor A with $\mathcal{E}_B = \mathcal{E}_{A^{[p]}}$ whenever $\mathcal{E}_B$ is *p*-divisible. Then *p*-divisibility of $\mathcal{E}_B$ implies *p*-refinability of B .

Proof. ✓ Proof From the reverse transfer-map root hypothesis, extract A with $\mathcal{E}_B = \mathcal{E}_{A^{[p]}}$. This matches two finite Kraus families of the same completely positive map: the blocked family $A^{[p]}$ with d^p operators and the original family B with *d* operators. The cardinality comparison $d \leq d^p$ holds whenever $p \geq 1$, so [Wol12, Theorem 2.1(4)] supplies an isometry $V \in M_{d^p \times d}(\mathbb{C})$ with $V^\dagger V = \mathbb{1}$ and $(A^{[p]})^\alpha = \sum_j V_{\alpha,j} B^j$. Expanding the coefficient of $A^{[p]}$ at the word τ by linearity of matrix multiplication and of the trace, the contributions reorganise into the *V*-weighted sum over length-*N* words of B required by Definition 22.4.2, exhibiting *V* as the isometry *W* in the statement of *p*-refinability. $\square$

Remark 22.4.16 (Continuum-limit application). Not ready The original motivation for Theorem 22.4.11 is a continuum-limit construction in the spirit of [DICCSPG17]: refinability along arbitrarily large *p* is equivalent to infinite divisibility of the transfer map, which selects out continuum limits of translationally invariant MPS. The further consequences are explored elsewhere.

Part I

Appendices

Appendix A

Perron–Frobenius Theory for Channels and Transfer Maps: Supporting Results

This appendix supplies the canonical-gauge calculations used in Chapter 5.

A.1 Canonical-gauge algebra

For the right-canonical gauge in Theorem [LTC26, “Right-canonical (unital) gauge”], each summand becomes

$$\begin{aligned}(S^{-1}A^iS)(S^{-1}A^iS)^\dagger &= S^{-1}A^iSS^\dagger(A^i)^\dagger(S^\dagger)^{-1} \\ &= S^{-1}A^i\rho(A^i)^\dagger(S^\dagger)^{-1}.\end{aligned}$$

Hence

$$\sum_i A'^i(A'^i)^\dagger = S^{-1}\mathcal{E}_A(\rho)(S^\dagger)^{-1} = S^{-1}\rho(S^\dagger)^{-1} = \mathbb{1}.$$

For the left-canonical gauge in Theorem 5.4.2,

$$\begin{aligned}(SA^iS^{-1})^\dagger(SA^iS^{-1}) &= (S^\dagger)^{-1}(A^i)^\dagger S^\dagger SA^iS^{-1} \\ &= (S^\dagger)^{-1}(A^i)^\dagger \sigma A^iS^{-1},\end{aligned}$$

and therefore

$$\sum_i (A'^i)^\dagger A'^i = (S^\dagger)^{-1} \left(\sum_i (A^i)^\dagger \sigma A^i \right) S^{-1} = (S^\dagger)^{-1} \sigma S^{-1} = \mathbb{1}.$$

Finally, for the square-root gauge (5.1), self-adjointness of $\sigma^{1/2}$ gives

$$\begin{aligned}(B^i)^\dagger B^i &= \sigma^{-1/2}(A^i)^\dagger \sigma^{1/2} \sigma^{1/2} A^i \sigma^{-1/2} \\ &= \sigma^{-1/2}(A^i)^\dagger \sigma A^i \sigma^{-1/2}.\end{aligned}$$

Summing, substituting the adjoint fixed-point equation, and cancelling the outer square-root factors yields

$$\sum_i (B^i)^\dagger B^i = \sigma^{-1/2} \left(\sum_i (A^i)^\dagger \sigma A^i \right) \sigma^{-1/2} = \sigma^{-1/2} \sigma \sigma^{-1/2} = \mathbb{1}.$$

Lemma A.1.1 (Square-root trace-preserving gauge is a gauge equivalence). [Lean](#) [✓ Stmt](#) *If σ is positive definite and B is defined by (5.1), then A and B are gauge equivalent.*

Proof. [✓ Proof](#) The gauge matrix is $\sigma^{1/2}$, which is invertible because σ is positive definite. Thus (5.1) has the form of the gauge relation (2.5). $\square$

Appendix B

Peripheral Channel Structure and Transfer-Operator Gaps: Supporting Results

This appendix supports Chapter 6. It proves the algebraic mixed-transfer identities, an auxiliary trace-positivity result, the peripheral intertwiners behind the gauge-rigidity theorems, the Banach-algebra convergence facts behind the overlap-decay theorems, the rank-one Perron projection behind the primitive overlap limit, and the cyclic-decomposition machinery that removes peripheral periodicity.

B.1 Mixed-transfer powers and overlap traces

This section proves the self-transfer identity, the word expansion of the mixed transfer operator, and the trace identities used in Theorem 6.2.1 and Theorem 6.2.2. Recall the mixed transfer operator F_{AB} of Definition 6.1.1 and its rectangular form F_{AB}^{rect} of Definition 6.1.2.

Theorem B.1.1 (Self-transfer). Lean ✓ Stmt *Setting $B = A$ recovers the ordinary transfer map: $F_{AA} = \mathcal{E}_A$.*

Proof. ✓ Proof Substituting $B = A$ into (6.2) gives $F_{AA}(X) = \sum_i A^i X (A^i)^\dagger = \mathcal{E}_A(X)$ by (2.4). □

Lemma B.1.2 (Iterated mixed transfer). Lean ✓ Stmt *For every $X \in M_D(\mathbb{C})$ and $N \geq 0$,*

$$F_{AB}^N(X) = \sum_{\sigma \in \{0, \dots, d-1\}^N} A^\sigma X (B^\sigma)^\dagger, \quad (\text{B.1})$$

where $A^\sigma = A^{\sigma_1} \dots A^{\sigma_N}$ denotes the word evaluation of Definition 2.1.2.

Proof. ✓ Proof Induct on N . The base case is $F_{AB}^0(X) = X$. For the inductive step, apply F_{AB} to each summand in (B.1) and reindex using $\{0, \dots, d-1\}^{N+1} \cong \{0, \dots, d-1\} \times \{0, \dots, d-1\}^N$. □

Theorem 6.2.1 and Theorem 6.2.2 combine Lemma B.1.2 and Lemma [LTC26, “Trace expansion over matrix units”] the same way, over the square and rectangular matrix-unit bases respectively: expanding $\text{Tr}(T)$ by the trace-expansion formula of [LTC26, “Trace expansion over

matrix units”], applying (B.1) to each $T(E_{pq})$ for $T = F_{AB}^N$ or $T = (F_{AB}^{\text{rect}})^N$, and reassembling the sums over matrix-unit indices gives

$$\text{Tr}(T) = \sum_{\sigma} \text{tr}(A^{\sigma}) \overline{\text{tr}(B^{\sigma})} = \sum_{\sigma} V^{(N)}(A)_{\sigma} \overline{V^{(N)}(B)_{\sigma}} = O_{AB}(N).$$

B.2 Auxiliary trace positivity

B.3 Rectangular spectral-radius formulation

Lemma B.3.1 (Rectangular spectral radius bound). [Lean](#) [✓ Stmt](#) For normalized tensors of bond dimensions D_1 and D_2 , $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) \leq 1$.

Proof. [✓ Proof](#) This is the MPS formulation of the corresponding theorem in the companion quantum-channel volume [LTC26, “Spectral-radius bound for a trace-preserving mixed Kraus map”]: the rectangular mixed transfer is the mixed Kraus map, and the two tensor normalization identities are its trace-preserving hypotheses. $\square$

B.4 Peripheral intertwiners and gauge rigidity

This section proves the intertwining relations behind Theorem 6.6.1 and Theorem 6.6.4: transporting a modulus-one mixed-transfer eigenvector through separate unital gauges produces Kraus-level intertwiners, and these intertwiners force equal bond dimension once they are invertible.

Lemma B.4.1 (Gauged peripheral eigenvector yields Kraus intertwining). [Lean](#) [✓ Stmt](#) Let A be a normalized tensor of bond dimension D_1 and B a normalized tensor of bond dimension D_2 . Suppose ρ_A, ρ_B are fixed points of the respective transfer maps and S_A, S_B are invertible matrices with $S_A S_A^{\dagger} = \rho_A$ and $S_B S_B^{\dagger} = \rho_B$ (for instance the positive definite Perron–Frobenius fixed points of Theorem 5.3.2 and their square roots, as in the right-canonical gauge of Theorem [LTC26, “Right-canonical (unital) gauge”]). Gauge to unital families $A'^i = S_A^{-1} A^i S_A$ and $B'^i = S_B^{-1} B^i S_B$. If $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$ satisfies $F_{AB}^{\text{rect}}(X) = \mu X$ with $|\mu| = 1$, then $X' = S_A^{-1} X (S_B^{\dagger})^{-1}$ is nonzero, the families A' and B' are unital, and, for every i ,

$$X'(B'^i)^{\dagger} = \mu(A'^i)^{\dagger} X', \quad A'^i X' = \mu X' B'^i. \quad (\text{B.2})$$

Proof. [✓ Proof](#) Embed the gauged families and transported eigenvector into the block matrices

$$C^i := \begin{pmatrix} A'^i & 0 \\ 0 & B'^i \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & X' \\ 0 & 0 \end{pmatrix}, \quad (\text{B.3})$$

on $M_{D_1+D_2}(\mathbb{C})$; the matrices $\{C^i\}$ form a unital Kraus family, and $E_C(Y) = \mu Y$. The positive definite fixed points of the adjoint maps give a faithful weighted trace for the block map. Applying the Kadison–Schwarz inequality to Y and pairing its positive semidefinite gap with this fixed point shows that the gap vanishes. The Kraus relation of the corresponding theorem in the companion quantum-channel volume [LTC26, “KS equality implies Kraus commutation”] then gives $Y(C^i)^{\dagger} = \mu(C^i)^{\dagger} Y$; applying the same argument to the complementary product gives $C^i Y = \mu Y C^i$. Expanding these block relations gives (B.2). $\square$

Lemma B.4.2 (Intertwining yields gauge-phase equivalence). [Lean](#) [✓ Stmt](#) Let A and B be tensors of the same bond dimension. Suppose invertible gauges take them to A' and B' , and suppose there are an invertible matrix X' and a scalar μ with $|\mu| = 1$ such that $A'^i X' = \mu X' B'^i$ for every physical index i . Then A and B are gauge-phase equivalent.

Proof. ✓ Proof Compose the two gauge matrices with the intertwiner to obtain one invertible matrix conjugating A to a scalar multiple of B . $\square$

Combining these two lemmas proves Theorem 6.6.1: Lemma B.4.1 supplies the intertwining relations (B.2) for the separately gauged, unital families. The Kadison–Schwarz equality used inside its own proof makes $X'X'^\dagger$ and $X'^\dagger X'$ nonzero positive semidefinite fixed points of the gauged transfer maps; under irreducibility, Theorem 5.1.1 upgrades these to positive definite matrices, so the transported intertwiner X' is invertible. Lemma B.4.2 then converts the invertible intertwiner, together with the two gauge matrices, into a single invertible matrix realizing the gauge-phase relation $B^i = \bar{\mu} X A^i X^{-1}$ via the Skolem–Noether theorem (Theorem [LTC26, “Skolem–Noether”]).

Lemma 6.4.1 gives the companion kernel-invariance argument, independent of irreducibility: if a modulus-one rectangular intertwiner (6.6) exists between injective tensors of bond dimensions D_1 and D_2 , then the first relation shows that $\ker(X)$ is invariant under every $(B^i)^\dagger$; since the $\{B^i\}$ span the full matrix algebra, so do the $\{(B^i)^\dagger\}$, so $\ker(X)$ is trivial and $D_2 \leq D_1$. Taking adjoints of the second relation and applying the same argument to $X^\dagger$ shows that $\ker(X^\dagger)$ is invariant under every $(A^i)^\dagger$ and hence trivial, giving $D_1 \leq D_2$. Combined with Lemma B.4.1, this proves $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$ whenever A and B are injective tensors of unequal bond dimension, which is exactly Theorem 6.4.2.

B.5 Spectral-radius decay and overlap limits

This section collects the Banach-algebra convergence facts underlying Theorem 6.5.4, Theorem 6.5.5, and the block-separation consequences of the transfer-operator gap.

Theorem 6.4.3 applies Lemma [LTC26, “Powers vanish below unit spectral radius”] to the rectangular mixed transfer operator and then evaluates its powers at a fixed matrix. Theorem 6.5.4 combines the gap $\rho_{\text{spec}}(F_{AB}) < 1$ of Theorem 6.3.6 with the equal-bond-dimension specialization of this rectangular decay theorem. Since Tr is continuous on the finite-dimensional matrix-endomorphism algebra, this gives $\text{Tr}(F_{AB}^N) \rightarrow 0$ term by term over the finitely many matrix-unit entries in the trace-expansion formula of [LTC26, “Trace expansion over matrix units”]; combined with the overlap–trace identity (6.4), this proves Theorem 6.5.5.

Theorem B.5.1 (Cross-correlation decay). Lean ✓ Stmt *If A and B are injective normalized tensors that are not gauge-phase equivalent, then $\text{tr}(F_{AB}^N(X)) \rightarrow 0$ as $N \rightarrow \infty$ for every $X \in M_D(\mathbb{C})$.*

Proof. ✓ Proof Theorem 6.5.4 gives $F_{AB}^N(X) \rightarrow 0$ in operator norm, and continuity of the trace gives the conclusion. $\square$

Theorem B.5.2 (Self-correlation persists). Lean ✓ Stmt *If ρ is a fixed point of $\mathcal{E}_A$, then $\text{tr}(\mathcal{E}_A^N(\rho)) = \text{tr}(\rho)$ for every $N \geq 0$. In particular, this applies to the distinguished Perron–Frobenius fixed point from Theorem 5.3.2.*

Proof. ✓ Proof Since $\mathcal{E}_A(\rho) = \rho$, one has $\mathcal{E}_A^N(\rho) = \rho$ for every N . Taking the trace proves the claim. $\square$

B.6 Rank-one Perron projection

This section proves the rank-one Perron limit (Theorem 6.10.1) underlying the primitive overlap convergence (Theorem 6.10.2). Let $E : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ be trace-preserving with a fixed point

$\rho \neq 0$, $\text{tr}(\rho) \neq 0$. Write $P := P_\rho$ for the fixed-point projection of the corresponding definition in the companion quantum-channel volume [LTC26, “Fixed-point projection”],

$$P(X) = \frac{\text{tr}(X)}{\text{tr}(\rho)} \rho,$$

and $N := E - P$ for the complementary part. The power decomposition (the corresponding theorem in the companion quantum-channel volume [LTC26, “Power decomposition”]) gives $E^n = P + N^n$ for $n \geq 1$.

Under the hypothesis $\rho_{\text{spec}}(N) < 1$ of Theorem 6.10.1, Lemma [LTC26, “Powers vanish below unit spectral radius”] gives $N^n \rightarrow 0$, so $\text{Tr}(E^n) = \text{Tr}(P) + \text{Tr}(N^n) \rightarrow \text{Tr}(P) = 1$ by Lemma [LTC26, “Trace of the fixed-point projection”]. This is exactly the mechanism recorded in Theorem 6.10.1.

When $E = \mathcal{E}_A$ is the transfer map of a normalized MPS tensor and ρ is its Perron–Frobenius fixed point (Theorem 5.3.2, Chapter 5), the overlap–trace identity (6.4) with $A = B$ gives $O_{AA}(N) = \text{Tr}(\mathcal{E}_A^N)$, so Theorem 6.10.1 yields the primitive self-overlap limit of Theorem 6.10.2. The hypothesis $\rho_{\text{spec}}(N) < 1$ is exactly the complementary transfer-map gap that the complementary spectral-radius theorem for primitive maps (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]) supplies: for a primitive E all eigenvalues bounded by 1 in modulus and no trace-zero fixed point, every eigenvalue of $N = E - P$ has modulus strictly less than 1, and since $M_D(\mathbb{C})$ is finite-dimensional this bounds the spectral radius $\rho_{\text{spec}}(N)$ itself.

Theorem B.6.1 (Complementary spectral-radius gap for irreducible primitive tensors). [Lean]
✓ Stmt *Let A be an irreducible normalized MPS tensor whose transfer map $\mathcal{E}_A$ is primitive. Then there is a nonzero positive semidefinite fixed point ρ of $\mathcal{E}_A$ with $\text{tr}(\rho) \neq 0$ such that, for $P := P_\rho$ and $N := \mathcal{E}_A - P$, $\rho_{\text{spec}}(N) < 1$.*

Proof. ✓ Proof The channel fixed-point existence theorem (Theorem 5.3.1) supplies a nonzero positive semidefinite fixed point ρ . Apply Theorem [LTC26, “Complementary gap at a prescribed fixed point”] to $\mathcal{E}_A$ and ρ . □

This concretely instantiates the hypothesis of Theorem 6.10.1 and, via Theorem 6.10.2, gives the primitive self-overlap limit for every irreducible peripherally primitive normalized MPS tensor.

B.7 Periodicity removal

This section proves the cyclic-decomposition machinery of Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”] and the divisibility statements of Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”] and Theorem [LTC26, “Transfer-map period divides bond dimension”]: normalizing a peripheral eigenvector to a unitary, forming its discrete Fourier (cyclic) spectral projections, and tracking how the channel shifts the resulting corners.

Combining these lemmas proves Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”]: Lemma [LTC26, “Normalized peripheral unitary”] supplies a unitary generator of exact order m for a primitive m -th peripheral root, and Lemma [LTC26, “Cyclic projections from a peripheral unitary”] builds the m Fourier projections out of its powers, using Lemma [LTC26, “Powers on a peripheral-unitary orbit”] to identify how the channel shifts them.

B.7.1 Cyclic corners and restricted primitivity

Corner preservation, the associated corner subspace, irreducibility of the restricted map, and corner rank are defined in the companion quantum-channel volume [LTC26, “Corner preservation” and the following corner definitions]. We use those definitions here in the MPS periodicity-removal argument.

B.7.2 Group structure and divisibility

Theorem [LTC26, “Peripheral eigenspaces are one-dimensional”] follows the same pattern: given a peripheral unitary eigenvector U for γ (Lemma [LTC26, “Peripheral unitary eigenvector”]), the multiplicative-domain identity gives $E(XU^\dagger) = XU^\dagger$ for any γ -eigenvector X , so the scalar fixed-point lemma for irreducible unital maps (Lemma [LTC26, “Scalar fixed points for irreducible unital maps”]) gives $XU^\dagger = c\mathbb{1}$, hence $X = cU$ and the eigenspace is one-dimensional.

The divisibility statements of Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”] and Theorem [LTC26, “Transfer-map period divides bond dimension”] follow from Lemma [LTC26, “Cyclic projections from a peripheral unitary”] and Theorem [LTC26, “Rank equals trace for an orthogonal projection”]: the m cyclic Fourier projections $P_0, \dots, P_{m-1}$ sum to $\mathbb{1}$ and are cyclically permuted by E , so trace preservation gives $\text{tr}(P_{k+1}) = \text{tr}(P_k)$ for every k , hence $D = \text{tr}(\mathbb{1}) = \sum_{k=0}^{m-1} \text{tr}(P_k) = m \text{tr}(P_0)$. Since the trace of an orthogonal projection is a natural number (Theorem [LTC26, “Rank equals trace for an orthogonal projection”]), $m \mid D$.

This periodicity-removal chain — unitary normalization, Fourier projections, cyclic Kraus shift (Lemma 6.8.1.1), the blocked cyclic corners of this section, and restricted primitivity — is exactly the machinery behind the period-removing blocking theorems of Chapter 8: [Wol12, Theorem 6.6] supplies the cyclic peripheral spectrum of the adjoint transfer map, and Theorem 8.3.1.1 (Section 8.3.1) and Theorem 8.11.1 apply it to realize the cyclic-sector decomposition of a blocked irreducible tensor with rectangular support isometries.

Appendix C

Wielandt Bound: Supporting Results

This appendix supports Chapter 7. It collects exact-word, cumulative-span, augmentation, blocking, complementary-gap, and one-step padding results used by the main quantitative argument.

C.1 Exact-word and block-injectivity support

This section proves the exact-word identities and low-dimensional facts used in Lemma 7.1.1.20, Theorem 7.1.1.21, Theorem 7.5.2, and Theorem 7.9.1.

Lemma C.1.1 (Word span products). [Lean](#) [✓ Stmt](#) *One has $S_m(A)S_n(A) \subseteq S_{m+n}(A)$: the product of word spans at lengths m and n is contained in the word span at length $m + n$.*

Proof. [✓ Proof](#) Every product $A^{w_1}A^{w_2}$ with $|w_1| = m$ and $|w_2| = n$ equals $A^{w_1w_2}$, where $|w_1w_2| = m + n$. $\square$

Lemma C.1.2 (Zero-length word span). [Lean](#) [✓ Stmt](#) *If $D \geq 2$, then $S_0(A) \neq M_D(\mathbb{C})$.*

Proof. [✓ Proof](#) The zero-length products consist only of the identity, so $S_0(A) = \text{span}_{\mathbb{C}}\{1\}$ has dimension 1. Since $\dim M_D(\mathbb{C}) = D^2 \geq 4$, this span cannot equal $M_D(\mathbb{C})$. $\square$

Corollary C.1.3 (0-block injectivity). [Lean](#) [✓ Stmt](#) *If $D \geq 1$ and A is 0-block injective, then $D = 1$.*

Proof. [✓ Proof](#) By Lemma 7.1.1.20, 0-block injectivity gives $S_0(A) = M_D(\mathbb{C})$. Lemma C.1.2 excludes $D \geq 2$, so the nonzero bond dimension must be 1. $\square$

Lemma C.1.4 (Exact words around a nonzero middle matrix). [Lean](#) [✓ Stmt](#) *If A is L -block injective and $X \neq 0$, then*

$$\text{span}_{\mathbb{C}}\{A^u X A^v : |u| = |v| = L\} = M_D(\mathbb{C}).$$

Proof. [✓ Proof](#) By L -block injectivity, the length- L word products span $M_D(\mathbb{C})$. Bilinearity of $(R, S) \mapsto RXS$ reduces the claim to Lemma [LTC26, “Two-sided span from one nonzero matrix”], with R and S chosen among those word products. $\square$

C.2 Cumulative-span and spectral linear algebra

This section supplies the finite-dimensional stabilization and spectral facts used in Theorem 7.2.2, Theorem 7.3.2, and Theorem 7.4.8.

The next two lemmas connect irreducible matrix actions with the cumulative word span used throughout Chapter 7.

Remark C.2.1 (Burnside input for cumulative spanning). The passage from irreducible tensors to full algebra spanning first passes through the irreducible action on $\mathbb{C}^D$. Burnside’s theorem then says that the resulting irreducible subalgebra of $M_D(\mathbb{C})$ is all of $M_D(\mathbb{C})$. Once the algebra span is full, finite-dimensionality gives a single cumulative span that already equals $M_D(\mathbb{C})$.

Lemma C.2.2 (Irreducible action gives an irreducible tensor). [Lean](#) [✓ Stmt](#) *If the letters of A act irreducibly on $\mathbb{C}^D$, then A has no nontrivial invariant orthogonal projection.*

Proof. [✓ Proof](#) If P were a nontrivial invariant orthogonal projection, then its range would be invariant under every letter of A . Irreducibility of the action forces this range to be either 0 or all of $\mathbb{C}^D$, so P must be 0 or $\mathbb{1}$, a contradiction. $\square$

Lemma C.2.3 (The algebra span reaches a finite cumulative span). [Lean](#) [✓ Stmt](#) *If the generated unital algebra satisfies $\text{alg}(A) = M_D(\mathbb{C})$, then there exists N such that $T_N(A) = M_D(\mathbb{C})$.*

Proof. [✓ Proof](#) Every element of $\text{alg}(A)$ lies in some cumulative span $T_n(A)$: the generators lie in $T_1(A)$, scalar multiples of the identity lie in $T_0(A)$, and the family $\{T_n(A)\}_{n \geq 0}$ is closed under addition and under multiplication with lengths adding. Since $M_D(\mathbb{C})$ is finite-dimensional, the ascending chain $T_0(A) \subseteq T_1(A) \subseteq \dots$ stabilizes, so one can choose a single N with $T_N(A) = M_D(\mathbb{C})$. $\square$

Theorem C.2.4 (Cumulative span reaches $M_D(\mathbb{C})$ — explicit bound). [Lean](#) [✓ Stmt](#) *Let $D > 0$. If A is normal, then $T_{D^2}(A) = M_D(\mathbb{C})$.*

Proof. [✓ Proof](#) By Theorem 7.1.1.19, if $T_n \neq T_{n+1}$, then $\dim T_{n+1} > \dim T_n$. Since $\dim T_n \leq D^2$ by Lemma 7.1.1.17, after at most D^2 steps either $T_n = M_D(\mathbb{C})$ or T_n has stabilised. Stabilisation before reaching $M_D(\mathbb{C})$ contradicts normality, which requires $S_{L_0} = M_D(\mathbb{C})$ for some L_0 and hence $T_{L_0} = M_D(\mathbb{C})$. $\square$

Theorem C.2.5 (Nonzero trace product). [Lean](#) [✓ Stmt](#) *Let $D > 0$. If A is normal, then there exists a word w of length at most D^2 such that $\text{tr}(A^w) \neq 0$. This is the weaker D^2 corollary of [SPGWC10, Lemma 1]. The sharp source statement is Theorem 7.2.4.*

Proof. [✓ Proof](#) By Theorem 7.2.1, $\mathbb{1} \in T_{D^2}(A)$, so $\mathbb{1}$ is a linear combination of word products of length at most D^2 . At least one has nonzero trace because $\text{tr}(\mathbb{1}) = D \neq 0$. $\square$

Theorem C.2.6 (Eigenvalue extraction). [Lean](#) [✓ Stmt](#) *Let $D > 0$. If A is normal, then there exist a word w_0 with $|w_0| \leq D^2$, a nonzero scalar $\mu \neq 0$, and a nonzero vector $\varphi \neq 0$ such that $A^{w_0}\varphi = \mu\varphi$.*

Proof. [✓ Proof](#) By Theorem C.2.5, there is a word w_0 with $\text{tr}(A^{w_0}) \neq 0$ and $|w_0| \leq D^2$. By Theorem [LTC26, “Nonzero trace implies eigenvector”], A^{w_0} has a nonzero eigenvector. $\square$

Theorem C.2.7 (Vector span stabilisation). [Lean](#) [✓ Stmt](#) *If $K_n(A, \varphi) = K_{n+1}(A, \varphi)$, then $K_m(A, \varphi) = K_n(A, \varphi)$ for every $m \geq n$.*

Proof. [✓ Proof](#) The argument is the same as for Theorem 7.1.1.18: left multiplication by A^i cannot escape the stabilised span. $\square$

Lemma C.2.8 (Vector span from matrix span). [Lean](#) [✓ Stmt](#) If $T_N(A) = M_D(\mathbb{C})$ and $\varphi \neq 0$, then $K_N(A, \varphi) = \mathbb{C}^D$.

Proof. [✓ Proof](#) Every matrix $M \in M_D(\mathbb{C})$ lies in $T_N(A)$, so in particular every rank-one matrix sending φ to any target vector v lies in $T_N(A)$. Applying such matrices to φ recovers all of $\mathbb{C}^D$. $\square$

C.3 One-step augmentation

This section proves the one-step augmentation results used in Theorem 7.4.2 and Theorem 7.4.6.

Definition C.3.1 (One-step augmentation). [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with Kraus family $\{A^i\}_{i=0}^{d-1}$, and let $X \in M_D(\mathbb{C})$. The *one-step augmentation* of A by X is the tensor $\tilde{A}$ with physical dimension $d + 1$ whose first Kraus operator is X and whose remaining Kraus operators are those of A :

$$\begin{aligned}\tilde{A}^0 &= X, \\ \tilde{A}^{i+1} &= A^i.\end{aligned}$$

Lemma C.3.2 (Kraus operators lie in the one-step span). [Lean](#) [✓ Stmt](#) Every Kraus operator A^i belongs to the one-step span $S_1(A)$.

Proof. [✓ Proof](#) This is the length-one word with letter i . $\square$

Lemma C.3.3 (Redundant one-step augmentation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $X \in S_1(A)$, and let $\tilde{A}$ be the one-step augmentation of A by X . Then $S_n(\tilde{A}) = S_n(A)$ for every $n \geq 0$.

Proof. [✓ Proof](#) The new generator X is already in $S_1(A)$, and each original generator remains in the augmented family. Hence the one-step spans agree. Multiplying word spans and arguing by induction on the word length gives equality at every length. $\square$

Lemma C.3.4 (Normality and Kraus rank under one-step augmentation). [Lean](#) [Lean](#) [✓ Stmt](#) Let $X \in S_1(A)$, and let $\tilde{A}$ be the one-step augmentation of A by X . If A is normal, then $\tilde{A}$ is normal, and $\text{kr}(\tilde{A}) = \text{kr}(A)$.

Proof. [✓ Proof](#) Since all exact word spans are unchanged by Lemma C.3.3, eventual full word span for A gives eventual full word span for $\tilde{A}$. The equality of Kraus ranks is the equality of the dimensions of the common one-step span. $\square$

Theorem C.3.5 (Wielandt case (2), invertible generator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be a normal MPS tensor with bond dimension $D > 0$, and suppose that some Kraus operator A^{i_0} is invertible. Then

$$\begin{aligned}S_{D^2 - \text{kr}(A) + 1}(A) &= M_D(\mathbb{C}), \\ \iota(A) &\leq D^2 - \text{kr}(A) + 1.\end{aligned}$$

Proof. [✓ Proof](#) The dimensions of the exact word spans cannot decrease, because left multiplication by the invertible Kraus operator is injective. If this dimension growth stopped before the full matrix algebra was reached, the same argument at every later length would contradict normality. Since $\dim S_1(A) = \text{kr}(A)$ and $\dim M_D(\mathbb{C}) = D^2$, the full algebra is reached by length $D^2 - \text{kr}(A) + 1$. The index bound follows from the definition of $\iota(A)$. $\square$

Corollary C.3.6 (Wielandt case (2), invertible Kraus operator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If some Kraus operator A^{i_0} is invertible, then

$$S_{D^2 - \text{kr}(A) + 1}(A) = M_D(\mathbb{C}),$$

$$\iota(A) \leq D^2 - \text{kr}(A) + 1.$$

Proof. [✓ Proof](#) Apply the one-step case to $X = A^{i_0}$, which belongs to $S_1(A)$ by Lemma C.3.2. $\square$

Corollary C.3.7 (Wielandt case (3), non-invertible Kraus operator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If some Kraus operator A^{i_0} is non-invertible and has an eigenvector $\varphi \neq 0$ with nonzero eigenvalue μ , then

$$S_{D^2}(A) = M_D(\mathbb{C}),$$

$$\iota(A) \leq D^2.$$

Proof. [✓ Proof](#) Apply the one-step case to $X = A^{i_0}$, which belongs to $S_1(A)$ by Lemma C.3.2. $\square$

C.4 Blocking and fixed-length spanning

This section proves the blocking and fixed-length constructions used in Theorem 7.4.8, Theorem 7.5.5, and Theorem 7.9.1.

Theorem C.4.1 (Blocking preserves normality). [Lean](#) [✓ Stmt](#) If A is normal and $L > 0$, then the blocked tensor $A^{[L]}$ is also normal.

Proof. [✓ Proof](#) Choose N with $S_N(A) = M_D(\mathbb{C})$. Then $\mathbb{1} \in S_N(A)$. For any $M \in S_N(A)$, write $\mathbb{1} = \sum_r c_r A^{w_r}$ with each $|w_r| = N$. Then

$$M = M\mathbb{1} = \sum_r c_r M A^{w_r} \in S_N(A) S_N(A) \subseteq S_{2N}(A). \quad (\text{C.1})$$

By Lemma C.1.1, iterating (C.1) gives $S_N(A) \subseteq S_{kN}(A)$ for every $k \geq 1$. Taking $k = L$ yields $M_D(\mathbb{C}) = S_N(A) \subseteq S_{NL}(A)$. By Lemma C.4.2, $S_{NL}(A) \subseteq S_N(A^{[L]})$, so $S_N(A^{[L]}) = M_D(\mathbb{C})$. $\square$

Lemma C.4.2 (Chunking: word span embeds into blocked word span). [Lean](#) [✓ Stmt](#) One has $S_{nL}(A) \subseteq S_n(A^{[L]})$ for all n, L .

Proof. [✓ Proof](#) Every length- nL word in the original alphabet can be split into n consecutive chunks of length L , each of which is a single blocked Kraus operator by (2.13). The claim follows by induction on n , using Lemma C.1.1. $\square$

Theorem C.4.3 (Word eigenvector gives blocked eigenvector). [Lean](#) [✓ Stmt](#) If $A^{w_0} \varphi = \mu \varphi$ for a word w_0 of length L , then $(A^{[L]})^{j_0} \varphi = \mu \varphi$, where j_0 is the encoded blocked index corresponding to w_0 .

Proof. [✓ Proof](#) By (2.13), $(A^{[L]})^{j_0} = A^{w_0}$. $\square$

Theorem C.4.4 (Blocked fixed-length matrix spanning). [Lean](#) [✓ Stmt](#) Suppose A has bond dimension $D > 0$, is normal, and $L > 0$, and we have:

1. a word w_0 of length L with eigenvector $A^{w_0}\varphi = \mu\varphi$ ($\mu \neq 0, \varphi \neq 0$),
2. a word τ_0 of length L with transpose eigenvector $(A^{\tau_0})^T\psi = \nu\psi$ ($\nu \neq 0, \psi \neq 0$),
3. a rank-one element $\varphi\psi^T \in S_m(A^{[L]})$ for some m .

Then there exists N such that $S_N(A) = M_D(\mathbb{C})$.

Remark. The proof gives the explicit bound $N = ((D-1) + m + (D-1))L$; the theorem states only the existential conclusion.

Proof. ✓ Proof Set $B = A^{[L]}$, which is normal by Theorem C.4.1. The word eigenvectors transfer to single-index eigenvectors of B and B^T by Theorem C.4.3. Eigenvector spreading and padding then give $H_{D-1}(B, \varphi) = H_{D-1}(B^T, \psi) = \mathbb{C}^D$.

For each standard basis vector e_j , choose $R_j \in S_{D-1}(B^T)$ with $R_j\psi = e_j$. Reversing each word after transposition shows that $R_j^T \in S_{D-1}(B)$, and therefore $(\varphi\psi^T)R_j^T = \varphi e_j^T$ belongs to $S_{m+D-1}(B)$. Theorem 7.5.2, applied with $n = D-1$ and $m' = m + D-1$, now gives $S_{(D-1)+m+(D-1)}(B) = M_D(\mathbb{C})$. Finally, Lemma 7.5.4 transfers this identity to $S_{((D-1)+m+(D-1))L}(A) = M_D(\mathbb{C})$. $\square$

Theorem C.4.5 (Rank-one extraction for blocked tensor). Lean ✓ Stmt Suppose A has bond dimension $D > 0$, is normal, and $N_0 \geq 1$ with $S_{N_0}(A) = M_D(\mathbb{C})$. Then there exist words σ_0, τ_0 of length N_0 , nonzero vectors φ, ψ , nonzero scalars μ, ν , and $m \in \mathbb{N}$ such that:

1. $A^{\sigma_0}\varphi = \mu\varphi$,
2. $(A^{\tau_0})^T\psi = \nu\psi$,
3. $\varphi\psi^T \in S_m(A^{[N_0]})$.

Proof. ✓ Proof Set $B = A^{[N_0]}$. By (2.13) and $S_{N_0}(A) = M_D(\mathbb{C})$, one has $S_1(B) = M_D(\mathbb{C})$. Hence $\mathbb{1} \in S_1(B)$, so not all blocked generators can have trace zero. Choose j_0 with $\text{tr}(B^{j_0}) \neq 0$. By Theorem [LTC26, “Nonzero trace implies eigenvector”], B^{j_0} has a nonzero eigenvector φ with eigenvalue $\mu \neq 0$. Writing j_0 as the blocked index corresponding to a word σ_0 of length N_0 gives $A^{\sigma_0}\varphi = \mu\varphi$. Applying the same argument to the transpose blocked tensor gives a word τ_0 of length N_0 , a nonzero vector ψ , and $\nu \neq 0$ with $(A^{\tau_0})^T\psi = \nu\psi$. Since A is normal and $N_0 \geq 1$, the blocked tensor B is normal by Theorem C.4.1. Therefore there exists M with $S_M(B) = M_D(\mathbb{C})$. In particular, $\varphi\psi^T \in S_M(B)$. Taking $m = M$ proves (3). $\square$

Theorem C.4.6 (Fixed-length word-span saturation via blocking). Lean ✓ Stmt If A is a normal MPS tensor with bond dimension $D > 0$, then there exists N such that $S_N(A) = M_D(\mathbb{C})$. This is a blocked fixed-length construction under the stated normality hypothesis. The quantitative result of [SPGWC10, Lemma 2(b)] is Theorem 7.4.5.

Proof. ✓ Proof By normality, some $N_0 > 0$ has $S_{N_0}(A) = M_D(\mathbb{C})$. Theorem C.4.5 produces eigenvectors and a rank-one element $\varphi\psi^T \in S_m(A^{[N_0]})$. Theorem C.4.4 then gives $S_{((D-1)+m+(D-1))N_0}(A) = M_D(\mathbb{C})$. $\square$

C.5 Complementary-gap consequences

This section proves the complementary-gap consequences used in Lemma 7.7.2.1, Lemma 7.7.4.1, and Theorem 7.7.4.2.

Theorem C.5.1 (Primitive overlap convergence). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor, then $O_{AA}(N) \rightarrow 1$.

Proof. [✓ Proof](#) The claim follows from the self-overlap convergence under a complementary transfer-map gap (Theorem 6.10.2). $\square$

Theorem C.5.2 (Peripheral primitivity and irreducibility give normalized overlap). [Lean](#) [✓ Stmt](#) Let A have positive bond dimension and satisfy $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. If A is tensor-irreducible and the transfer map is primitive, then $O_{AA}(N) \rightarrow 1$.

Proof. [✓ Proof](#) Irreducible Perron–Frobenius theory supplies a nonzero positive fixed point. Peripheral primitivity gives a strict spectral-radius bound on the complementary transfer map. Thus A is a primitive MPS tensor, and Theorem C.5.1 applies. $\square$

Lemma C.5.3 (Primitive MPS tensors have nonzero word products). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor, then for every $n \geq 0$ there exists a word w of length exactly n with $A^w \neq 0$.

Proof. [✓ Proof](#) The PSD fixed point ρ of $\mathcal{E}_A$ satisfies $\mathcal{E}_A^n(\rho) = \rho \neq 0$. Since $\mathcal{E}_A^n(\rho) = \sum_{|w|=n} A^w \rho (A^w)^\dagger$, at least one summand is nonzero, hence some $A^w \neq 0$. $\square$

Lemma C.5.4 (Iterated transfer does not vanish). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor, then $\mathcal{E}_A^n \neq 0$ for all $n \geq 0$.

Proof. [✓ Proof](#) By Lemma C.5.3, there exists a word w of length n with $A^w \neq 0$. The expansion $\mathcal{E}_A^n(\mathbb{1}) = \sum_{|u|=n} A^u (A^u)^\dagger$ contains the nonzero positive semidefinite summand $A^w (A^w)^\dagger$, so $\mathcal{E}_A^n(\mathbb{1}) \neq 0$. $\square$

Lemma C.5.5 (Fixed-point uniqueness from a complementary transfer-map gap). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor with PSD fixed point ρ , then every fixed point σ of $\mathcal{E}_A$ is proportional to ρ : $\sigma = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$.

Proof. [✓ Proof](#) Set $\sigma' = \sigma - \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$. Then $\text{tr}(\sigma') = 0$ and $(\mathcal{E}_A - P_\rho)(\sigma') = \sigma'$. If $\sigma' \neq 0$, then 1 is an eigenvalue of $\mathcal{E}_A - P_\rho$, contradicting the complementary transfer-map gap $\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1$. $\square$

Lemma C.5.6 (Complement powers tend to zero). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor with PSD fixed point ρ , then $(\mathcal{E}_A - P_\rho)^n \rightarrow 0$ in operator norm.

Proof. [✓ Proof](#) The spectral radius of $\mathcal{E}_A - P_\rho$ is strictly less than 1 by the complementary transfer-map gap hypothesis. Apply the general result that powers of an operator with spectral radius < 1 tend to zero. $\square$

Lemma C.5.7 (Primitive tensors with positive-definite fixed point have irreducible transfer map). [Lean](#) [✓ Stmt](#) If A is a primitive MPS tensor with positive-definite fixed point ρ , then the transfer map $\mathcal{E}_A$ is irreducible.

The proof uses fixed-point uniqueness (Lemma C.5.5): every PSD fixed point is proportional to ρ , so Wolf’s criterion for irreducibility (a positive-definite fixed point together with uniqueness of positive fixed points implies irreducibility) applies directly.

Proof. [✓ Proof](#) By Lemma C.5.5, if $\sigma \geq 0$ and $\mathcal{E}_A(\sigma) = \sigma$, then $\sigma = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$. Since $\rho > 0$, Wolf’s uniqueness criterion for irreducibility applies. $\square$

C.6 Identity in the one-step span

This section proves the one-step padding conversions used in Theorem 7.8.2. It also records why cumulative spanning alone does not imply normality.

Remark C.6.1 (Identity-in-the-one-step-span condition). ✓ Stmt The *one-step padding condition* is $\mathbb{1} \in S_1(A)$: the identity matrix lies in the span of the Kraus operators $\{A^i\}_{i=0}^{d-1}$. This is an additional hypothesis. The normalization $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ gives a quadratic identity and does not imply $\mathbb{1} \in S_1(A)$. Theorem 7.7.4.2 obtains normality from a positive-definite fixed point and a complementary gap without this padding condition.

Remark C.6.2 (Counterexample: cumulative spanning does not imply normality). ✓ Stmt In $M_2(\mathbb{C})$, let $A^0 = E_{12}$ and $A^1 = E_{21}$ be the off-diagonal matrix units. Then $\text{alg}(A) = M_2(\mathbb{C})$ and $T_2(A) = M_2(\mathbb{C})$, but the word spans alternate: $S_n(A)$ contains only off-diagonal matrices for odd n and only diagonal matrices for even n . Hence $S_n(A) \neq M_2(\mathbb{C})$ for every n , and A is not normal. The one-step padding condition fails because $\mathbb{1} \notin S_1(A) = \text{span}_{\mathbb{C}}\{E_{12}, E_{21}\}$.

Theorem C.6.3 (Word-span monotonicity from one-step padding). Lean ✓ Stmt If $\mathbb{1} \in S_1(A)$, then $S_n(A) \leq S_{n+1}(A)$ for every n .

Proof. ✓ Proof For any $M \in S_n(A)$, one has $M = M\mathbb{1} \in S_n(A)S_1(A) \subseteq S_{n+1}(A)$ by Lemma C.1.1. □

Theorem C.6.4 (Cumulative span equals word span under one-step padding). Lean ✓ Stmt If $\mathbb{1} \in S_1(A)$, then $T_n(A) = S_n(A)$ for every n .

Proof. ✓ Proof By Theorem C.6.3, the word spans form the chain $S_0(A) \leq S_1(A) \leq \dots \leq S_n(A)$. Therefore their sum $T_n(A) = S_0(A) + \dots + S_n(A)$ equals the largest term $S_n(A)$. □

Theorem C.6.5 (Cumulative spanning plus one-step padding implies normality). Lean ✓ Stmt If $T_N(A) = M_D(\mathbb{C})$ and $\mathbb{1} \in S_1(A)$, then A is normal.

Proof. ✓ Proof Theorem C.6.4 gives $S_N(A) = T_N(A) = M_D(\mathbb{C})$. □

Appendix D

Canonical Form Reduction: Supporting Results

This appendix supports Chapter 8. It collects the projection, gauge, blocking, compression, and reindexing arguments used by the main reduction while keeping the source-named canonical-form statements and the principal reductions in the chapter.

Throughout, $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$, $A^{[p]}$ denotes blocking by p , and $V^{(N)}(A)_\sigma = \text{tr}(A^{\sigma_1} \dots A^{\sigma_N})$. Weighted direct sums are written $\bigoplus_k \mu_k A_k$.

D.1 Similarities, strict splitting, and zero blocks

Similarity invariance and strict dimension decrease justify the induction used in Theorem 8.1.1.2; the final result removes blocks that vanish at every positive length.

Theorem D.1.1 (Unitary conjugation preserves the MPV family). [Lean](#) [✓ Stmt](#) *If U is a unitary matrix, then A and $U^\dagger A U$ generate the same MPV family.*

Proof. [✓ Proof](#) By cyclicity of the trace, $\text{tr}((U^\dagger A^{i_1} U) \dots (U^\dagger A^{i_N} U)) = \text{tr}(A^{i_1} \dots A^{i_N})$. □

Theorem D.1.2 (Nontrivial invariant projection gives strict two-block reduction). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 8.1.2, assume in addition that $P \neq 0$ and $P \neq \mathbb{1}$. Then there exist n, m with $n + m = D$, $n < D$, and $m < D$, together with block tensors A_1, A_2 , such that A and $A_1 \oplus A_2$ generate the same MPV family in the sense of Definition 2.3.3.*

Proof. [✓ Proof](#) Apply Theorem 8.1.2. In the diagonal form of P , the 1-eigenspace is nonempty because $P \neq 0$, and the 0-eigenspace is nonempty because $P \neq \mathbb{1}$. Hence both block sizes are positive. Since $n + m = D$, this implies $n < D$ and $m < D$. □

Theorem D.1.3 (Nonzero irreducible block decomposition). [Lean](#) [✓ Stmt](#) *Every MPS tensor A admits a finite family of nonzero irreducible blocks $(B_k)_{k=1}^r$, each with at least one nonzero Kraus operator and positive bond dimension. For every $N \geq 1$ and every σ ,*

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r B_k\right)_\sigma. \quad (\text{D.1})$$

Moreover, $\sum_{k=1}^r D_k \leq D$.

Proof. ✓ Proof Apply the irreducible block decomposition (Theorem 8.1.1.2) and retain the blocks having a nonzero Kraus operator. For every all-zero irreducible block C_j and every $N \geq 1$, $V^{(N)}(C_j)_\sigma = 0$, so (D.1) holds. The dimension inequality follows by restricting the structural identity $\sum_j D_j = D$ to the retained subset. $\square$

D.2 Projector closure and isometric corner decompositions

The results below turn reducing projections into support isometries, compress tensors to their corners, and reconstruct closed-chain traces from an orthogonal family of corners. They support Theorem 8.7.2 and Theorem 8.7.4.

Theorem D.2.1 (Invariant projections are reducing under projector closure). Lean ✓ Stmt Suppose that A has invariant-projector closure and that P is an orthogonal projection satisfying $(\mathbb{1} - P)A^iP = 0$ for all i . Then P reduces every tensor matrix: $PA^i = A^iP$ for all i .

Proof. ✓ Proof The lower-corner relation gives $(\mathbb{1} - P)A^i = (\mathbb{1} - P)A^i(\mathbb{1} - P)$. Applying projector closure to $\mathbb{1} - P$ gives $A^i(\mathbb{1} - P) = (\mathbb{1} - P)A^i(\mathbb{1} - P)$, hence the upper corner $PA^i(\mathbb{1} - P)$ also vanishes. Therefore $PA^i = PA^iP = A^iP$. $\square$

Theorem D.2.2 (Invariant-projector closure gives a reducing projection). Lean ✓ Stmt If A has invariant-projector closure and admits a nontrivial invariant projection, then it admits a nontrivial orthogonal projection commuting with every A^i .

Proof. ✓ Proof Apply Theorem D.2.1 to the projection supplied by Definition 8.1.3. $\square$

Lemma D.2.3 (Support isometry of an orthogonal projection). Lean ✓ Stmt

Every orthogonal projection P on $\mathbb{C}^D$ factors through an isometry: there are $n \in \mathbb{N}$ and $V \in \mathbb{C}^{D \times n}$ such that

$$\begin{aligned} V^\dagger V &= \mathbb{1}, \\ VV^\dagger &= P, \\ n &= \text{tr}(P). \end{aligned} \tag{D.2}$$

Proof. ✓ Proof Since P is Hermitian and idempotent, its eigenvalues are 0 and 1. Writing $n = \text{tr}(P)$, unitary diagonalization gives

$$P = U \text{diag}(\underbrace{1, \dots, 1}_n, 0, \dots, 0) U^\dagger.$$

The matrix $V \in \mathbb{C}^{D \times n}$ formed by the first n columns of U satisfies (D.2). $\square$

The rectangular Gram-equality criterion that turns $B^\dagger B = A^\dagger A$ into a unitary U with $B = UA$ is a generic algebra fact, proved in the companion quantum-channel volume [LTC26, “Rectangular Gram equality gives a unitary”].

Lemma D.2.4 (Corner tensors inherit projector closure). Lean ✓ Stmt Let A have invariant-projector closure, and let V be an isometry, $V^\dagger V = \mathbb{1}$, whose range projection $VV^\dagger$ commutes with every A^i . Then the corner tensor $B^i = V^\dagger A^i V$ again has invariant-projector closure.

Proof. ✓ Proof Let Q be an orthogonal projection with $QB^i = QB^iQ$ for all i . Then $VQV^\dagger$ is an orthogonal projection. Commutation of $VV^\dagger$ with each A^i gives $V^\dagger A^i = B^i V^\dagger$, so left-invariance of Q yields

$$(VQV^\dagger)A^i = VQB^iV^\dagger = VQB^iQV^\dagger = (VQV^\dagger)A^i(VQV^\dagger).$$

Projector closure for A yields $A^i(VQV^\dagger) = (VQV^\dagger)A^i(VQV^\dagger)$, and compressing both sides between $V^\dagger$ and V returns $B^iQ = QB^iQ$. $\square$

Lemma D.2.5 (Transfer map of an intertwined corner). Lean Lean ✓ Stmt Let V satisfy $A^iV = VB^i$ for all i . Then, for every X ,

$$\mathcal{E}_A(VXV^\dagger) = V\mathcal{E}_B(X)V^\dagger. \quad (\text{D.3})$$

If moreover $V^\dagger V = \mathbb{1}$, then every eigenvalue of $\mathcal{E}_B$ is an eigenvalue of $\mathcal{E}_A$.

Proof. ✓ Proof Expanding the left-hand side of (D.3) termwise, the intertwining $A^iV = VB^i$ and its adjoint $V^\dagger(A^i)^\dagger = (B^i)^\dagger V^\dagger$ move both factors past V and $V^\dagger$. For the eigenvalue statement, if $\mathcal{E}_B(X) = \mu X$ with $X \neq 0$, then $\mathcal{E}_A(VXV^\dagger) = \mu VXV^\dagger$, and $VXV^\dagger \neq 0$ because compression with the isometry gives $V^\dagger(VXV^\dagger)V = X$. $\square$

Lemma D.2.6 (Matrix product vectors split along an isometric partition). Lean ✓ Stmt Let $V_1, \dots, V_r$ be isometries, $V_k^\dagger V_k = \mathbb{1}$, with $\sum_k V_k V_k^\dagger = \mathbb{1}$, and suppose $A^i V_k = V_k B_k^i$ for some tensors $B_1, \dots, B_r$. Then $\mathcal{V}(A) = \mathcal{V}(\bigoplus_{k=1}^r B_k)$.

Proof. ✓ Proof The intertwining relation propagates from letters to words, so $A^w V_k = V_k B_k^w$ for every word w . Inserting the partition of unity $\sum_k V_k V_k^\dagger = \mathbb{1}$ into the trace and cycling gives

$$\text{tr}(A^w) = \sum_{k=1}^r \text{tr}(V_k^\dagger A^w V_k) = \sum_{k=1}^r \text{tr}(B_k^w),$$

which is the matrix product vector of the direct sum. $\square$

Lemma D.2.7 (Canonical inclusion of a dependent direct summand). Lean Lean Lean Lean ✓ Stmt Let $E_k : H_k \rightarrow \bigoplus_l H_l$ be the canonical inclusion and let $B = \bigoplus_l B_l$. For distinct k and ℓ , the corresponding inclusions have orthogonal ranges. The identities are

$$\begin{aligned} E_k^\dagger E_k &= \mathbb{1}, \\ E_k^\dagger E_\ell &= 0, \\ BE_k &= E_k B_k, \\ E_k^\dagger B &= B_k E_k^\dagger, \\ E_k^\dagger BE_k &= B_k, \\ B &= \sum_k E_k B_k E_k^\dagger. \end{aligned} \quad (\text{D.4})$$

In particular, B commutes with the range projection $E_k E_k^\dagger$.

Proof. ✓ Proof The identities in (D.4) follow directly from the defining block-diagonal entries. The relations $BE_k = E_k B_k$ and $E_k^\dagger B = B_k E_k^\dagger$ imply that B commutes with $E_k E_k^\dagger$. $\square$

Lemma D.2.8 (Canonical hypotheses descend to dependent direct summands). [Lean](#) [✓ Stmt](#) Suppose that, after a change of coordinates, A is the dependent direct sum of tensors B_k . If A has invariant-projector closure and no nontrivial periodic vectors, then every B_k has both properties. The assertion also applies to summands of dimension zero.

Proof. [✓ Proof](#) Conjugate the canonical inclusion of the k -th summand back to the original bond coordinates of A . This is an isometry which intertwines B_k with A on both sides, and its range projection commutes with every letter of A . Projector closure therefore descends by exact compression. A periodic vector of B_k would embed through the same isometry as a periodic vector of A . $\square$

Lemma D.2.9 (Closed-chain equality from an exact isometric decomposition). [Lean](#) [✓ Stmt](#) Suppose that isometries V_k satisfy

$$\begin{aligned} V_k^\dagger A^i &= (\mu_k B_k^i) V_k^\dagger, \\ A^i &= \sum_k V_k (\mu_k B_k^i) V_k^\dagger. \end{aligned} \tag{D.5}$$

Then A and $\bigoplus_k \mu_k B_k$ have the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Expand the first letter by the reconstruction formula in (D.5). In each summand, move the remaining word through $V_k^\dagger$ using the intertwining relation. Cyclicity of the trace places $V_k^\dagger V_k = \mathbb{1}$ next to the weighted block word, leaving precisely its closed-chain trace. $\square$

D.3 Perron gauges and preservation under conjugation

This section records the positive Perron eigenpair, its spectral normalization, and the behavior of irreducibility and eigenvalues under the corresponding gauge.

Lemma D.3.1 (Perron eigenpair of a nonzero irreducible tensor). [Lean](#) [✓ Stmt](#) Let B be an irreducible MPS tensor of positive bond dimension, not all of whose matrices vanish. Then there are $\rho > 0$ and $r > 0$ with $\mathcal{E}_B(\rho) = r\rho$.

Proof. [✓ Proof](#)

Irreducibility of the tensor makes the transfer map an irreducible completely positive map, and a nonzero matrix makes it nonzero, so it does not annihilate any nonzero PSD matrix. The Perron–Frobenius theorem for positive maps (Theorem [LTC26, “Positive-eigenvalue specialization”]) gives a nonzero PSD eigenvector with eigenvalue $r > 0$, and irreducibility upgrades it to a positive-definite one ([Wol12, Theorem 6.3(2)–(3)]). $\square$

Lemma D.3.2 (Spectral normalization of an irreducible block). [Lean](#) [✓ Stmt](#) Let B be an irreducible MPS tensor of positive bond dimension with $\mathcal{E}_B(\rho) = r\rho$ for some $\rho > 0$ and $r > 0$, and suppose every eigenvalue of $\mathcal{E}_B$ of modulus r equals r . Then $r^{-1/2}B$ is a normal tensor.

Proof. [✓ Proof](#) The rescaling identity $\mathcal{E}_{r^{-1/2}B}(X) = r^{-1}\mathcal{E}_B(X)$ gives $\mathcal{E}_{r^{-1/2}B}(\rho) = r^{-1}r\rho = \rho$, so ρ is a positive-definite fixed point of $\mathcal{E}_{r^{-1/2}B}$. The rescaled map remains irreducible and completely positive, so Theorem [LTC26, “Spectral radius is the Perron eigenvalue”] identifies its spectral radius with the eigenvalue one. The eigenvalues of modulus one of $\mathcal{E}_{r^{-1/2}B}$ are the eigenvalues of $\mathcal{E}_B$ of modulus r , divided by r ; by hypothesis the only such eigenvalue is one, so the peripheral eigenvalue set is $\{1\}$. Irreducibility is unchanged by a nonzero rescaling. $\square$

Lemma D.3.3 (Range factorization of the support projection). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)

The support projection π of a positive semidefinite matrix ρ factors as $\pi = \rho W$ for a spectral pseudo-inverse W , witnessing that the range of π is the range of ρ . For a rectangular matrix Y , the support projection π of $YY^\dagger$ fixes Y , $\pi Y = Y$; and whenever $AY = YG$ for some matrix G —that is, A maps the range of Y into itself—the one-sided invariance $(\mathbb{1} - \pi)A\pi = 0$ holds.

Proof. [✓ Proof](#)

In the spectral decomposition $\rho = U \text{diag}(\lambda) U^\dagger$, take $W = U \text{diag}(\lambda^+) U^\dagger$, where λ^+ inverts the positive eigenvalues and vanishes on the kernel. Then $\rho W = U \text{diag}(\mathbf{1}_{\lambda>0}) U^\dagger = \pi$. From $\pi(YY^\dagger) = YY^\dagger$, the matrix $((\mathbb{1} - \pi)Y)((\mathbb{1} - \pi)Y)^\dagger$ vanishes, hence $\pi Y = Y$. Finally,

$$(\mathbb{1} - \pi)A\pi = (\mathbb{1} - \pi)AY(Y^\dagger W) = (\mathbb{1} - \pi)Y G(Y^\dagger W) = 0.$$

□

Theorem D.3.4 (Irreducibility under rescaled conjugation). [Lean](#) [✓ Stmt](#) Let B be an irreducible tensor, let X be invertible, and let $c \neq 0$. Then the rescaled conjugated tensor with letters $cX^{-1}B^vX$ is irreducible. In particular, the rescaled unital gauge of an irreducible tensor is irreducible; this is the normalization of [CPGSV16, Section 2.3] applied to an irreducible corner.

Proof. [✓ Proof](#) An invariant orthogonal projection $P \notin \{0, \mathbb{1}\}$ of the conjugated tensor makes the columns of $Y = XP$ span a subspace invariant under every letter of B : $B^vY = YG_v$, where $G_v = X^{-1}B^vXP$. The support projection of $YY^\dagger$ is then a nontrivial invariant orthogonal projection of B by Lemma D.3.3: it is nonzero because $Y \neq 0$, and it is not the identity because any nonzero vector in the kernel of P transports through $(X^\dagger)^{-1}$ to the kernel of $YY^\dagger$. This contradicts irreducibility of B . □

Theorem D.3.5 (Unital Schwarz data for an irreducible corner). [Lean](#) [✓ Stmt](#) Let B be an irreducible tensor with a positive definite transfer eigenvector, $\mathcal{E}_B(\rho) = r\rho$, where $\rho > 0$ and $r > 0$. Then the rescaled unital gauge $K^v = r^{-1/2}\rho^{-1/2}B^v\rho^{1/2}$ is a unital Kraus family, is an irreducible tensor with an irreducible transfer map, and its adjoint map has a positive definite fixed point. These are the hypotheses of the cyclic decomposition of the peripheral spectrum, [Wol12, Theorem 6.6].

Proof. [✓ Proof](#) Unitality is Theorem 8.8.2; irreducibility is Theorem D.3.4. The adjoint of a unital family is trace-preserving, so its transfer map is a channel and has a positive semidefinite fixed point by the Cesàro argument; irreducibility of the adjoint map upgrades the fixed point to a positive definite one. □

Lemma D.3.6 (Eigenvalue transport to the unital gauge). [Lean](#) [✓ Stmt](#) In the setting of Theorem D.3.5, every transfer eigenvalue μ of B becomes the transfer eigenvalue μ/r of the rescaled unital gauge: the gauge conjugates the transfer map by the congruence with $\rho^{1/2}$ and divides it by r . In particular, the peripheral eigenvalues $e^{2\pi i q/p}$ described in [CPGSV16, Section 2.3] arise from the transfer eigenvalues of modulus r .

Proof. [✓ Proof](#) If $\mathcal{E}_B(Z) = \mu Z$, then $Z' = \rho^{-1/2}Z\rho^{-1/2}$ is nonzero and satisfies $\mathcal{E}_K(Z') = (\mu/r)Z'$ by direct computation. □

D.4 Blocking identities and primitive stability

These identities control transfer-map powers, weighted direct sums, and positive-length MPV equalities under repeated blocking. They are used in the common-length reduction of Theorem 8.3.1.4.

Blocking by a length p commutes with weighted direct sums up to the p th power of each weight: for nonzero weights (μ_k) and blocks (B_k) ,

$$\left(\bigoplus_k \mu_k B_k \right)^{[p]} \sim \bigoplus_k \mu_k^p B_k^{[p]}, \quad (\text{D.6})$$

where $\sim$ is equality of MPV families. When two block families are reblocked to a common length $p = m_k e_k$, the p -blocked alphabet is identified with the e_k -fold tensor power of the m_k -blocked alphabet, so each sector reblocked through its own period agrees with the directly p -blocked block.

Theorem D.4.1 (Common blocking period for a block family). Lean Stmt *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension MPS tensors, each admitting a blocking period p_k making its transfer map primitive. Then $p = \text{lcm}(p_1, \dots, p_r)$ is a common period: $\mathcal{E}_{B_k^{[p]}}$ is primitive for every k .*

Proof. Proof Each p_k divides p , so primitivity of $\mathcal{E}_{B_k^{[p_k]}}$ carries over to $\mathcal{E}_{B_k^{[p]}}$ via the divisibility identity for transfer-map powers. □

Theorem D.4.2 (Common primitive blocking for irreducible left-canonical blocks). Lean Stmt *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension, left-canonical, tensor-irreducible MPS tensors. Then there is a single blocking length $p > 0$ such that $\mathcal{E}_{B_k^{[p]}}$ is primitive for every k .*

Proof. Proof Apply Theorem 8.9.2.1 to each block, then apply Theorem D.4.1 to the resulting finite family of positive periods. Their least common multiple is positive and makes every blocked transfer map primitive. □

Theorem D.4.3 (Quasi-idempotence under physical blocking). Lean Stmt *Let A be an MPS tensor and let $c \in \mathbb{C}$. If*

$$\mathcal{E}_A^2 = c \mathcal{E}_A,$$

then, for every integer $L \geq 0$,

$$\mathcal{E}_{A^{[L]}}^2 = c^L \mathcal{E}_{A^{[L]}}.$$

Proof. Proof The transfer map of $A^{[L]}$ is $\mathcal{E}_A^L$. Therefore

$$(\mathcal{E}_A^L)^2 = (\mathcal{E}_A^2)^L = (c \mathcal{E}_A)^L = c^L \mathcal{E}_A^L,$$

which is the required identity. □

Lemma D.4.4 (Adjoint primitivity equivalence). Lean Lean Stmt *Let E be a linear endomorphism of a finite-dimensional complex inner product space. Then E is primitive if and only if its adjoint $E^\dagger$ is primitive. Equivalently, the eigenvalues λ of E with $|\lambda| = 1$ form the singleton $\{1\}$ if and only if the same holds for $E^\dagger$, because adjunction sends every eigenvalue to its complex conjugate.*

Lemma D.4.5 (Primitivity under blocking by a multiple). [Lean](#) [✓ Stmt](#) Let A have positive bond dimension and let $p_0, p \geq 1$ with $p_0 \mid p$. If $\mathcal{E}_{A^{[p_0]}}$ is primitive, then $\mathcal{E}_{A^{[p]}}$ is primitive.

Theorem D.4.6 (Left-canonical normalization under blocking). [Lean](#) [✓ Stmt](#) If a finite matrix family satisfies

$$\sum_i (A^i)^\dagger A^i = \mathbb{1},$$

then every blocked family satisfies the same identity.

Proof. [✓ Proof](#) Expand the blocked matrices as products indexed by words and sum one letter at a time. $\square$

Theorem D.4.7 (Irreducibility from a faithful fixed point). [Lean](#) [✓ Stmt](#)

Let E be a channel with a positive-definite fixed point. If all nonzero positive semidefinite fixed points of E are proportional, then E is irreducible.

Proof. [✓ Proof](#) A nontrivial invariant projection would produce a positive semidefinite fixed point not proportional to the faithful one. $\square$

Lemma D.4.8 (Fixed-point criteria for irreducibility after primitive blocking). [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let A be left-canonical, tensor-irreducible, and have primitive transfer map. Then A has primitive fixed-point data (ρ, P_ρ, N) with $\rho > 0$ and $N^n \rightarrow 0$. For every blocking length P , the tensor $A^{[P]}$ is left-canonical, its transfer map is $\mathcal{E}_A^P$, and ρ is fixed by this blocked map. For every $n \geq 1$,

$$\mathcal{E}_A^n = P_\rho + N^n. \tag{D.7}$$

Finally, a channel with a positive-definite fixed point whose positive semidefinite fixed points are all proportional is irreducible as a map; irreducibility of a transfer map implies tensor irreducibility.

Proof. [✓ Proof](#) Peripheral primitivity, tensor irreducibility, and left-canonical normalization first give primitive fixed-point data, and tensor irreducibility makes its positive semidefinite fixed point positive definite. Blocking preserves the left-canonical identity, replaces the transfer map by its positive power, and therefore preserves the fixed point. The defining complementary-power identity gives (D.7). The channel fixed-point criterion then gives map irreducibility from uniqueness of positive semidefinite fixed points, and the final implication converts map irreducibility to tensor irreducibility. $\square$

Theorem D.4.9 (Blocking preserves irreducibility for primitive irreducible blocks). [Lean](#) [✓ Stmt](#)

Let A be left-canonical, irreducible, primitive, of positive bond dimension. For every $P > 0$, the blocked tensor $A^{[P]}$ is irreducible: there is no orthogonal projection Q with $Q \neq 0$, $Q \neq \mathbb{1}$ such that $(\mathbb{1} - Q)(A^{[P]})^\alpha Q = 0$ for every P -blocked index α .

Proof. [✓ Proof](#) Lemma D.4.8 gives primitive fixed-point data with $\rho > 0$. In particular, if $P_\rho(X) = \text{tr}(X)\rho / \text{tr}(\rho)$ and $N := \mathcal{E}_A - P_\rho$, then

$$\begin{aligned} \mathcal{E}_A &= P_\rho + N, \\ N^n &\rightarrow 0. \end{aligned}$$

The primitive fixed-point theorem supplies this complementary-power convergence.

The blocked transfer map is the corresponding positive power:

$$\mathcal{E}_{A^{[P]}}(X) = \mathcal{E}_A^P(X). \quad (\text{D.8})$$

Hence ρ remains a positive-definite fixed point after blocking. Let $\sigma \geq 0$ be any fixed point of $\mathcal{E}_A^P$, set

$$c = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)},$$

$$\sigma' = \sigma - c\rho.$$

Then $\text{tr}(\sigma') = 0$, so $P_\rho(\sigma') = 0$, while $\mathcal{E}_A^P(\sigma') = \sigma'$. Since the fixed-point projection and the complement annihilate one another, iteration gives, for every $k \geq 1$,

$$\sigma' = \mathcal{E}_A^{Pk}(\sigma') = N^{Pk}(\sigma').$$

But $N^{Pk}(\sigma') \rightarrow 0$ as $k \rightarrow \infty$, hence $\sigma' = 0$. Thus every positive semidefinite fixed point of the blocked map is proportional to ρ .

Blocking preserves the left-canonical identity, so Theorem 4.2.6 makes the blocked transfer map a channel. Its positive-definite fixed point and the uniqueness just proved imply map irreducibility by Lemma D.4.8. The final implication in that lemma then shows that $A^{[P]}$ is tensor-irreducible. $\square$

Theorem D.4.10 (Extra blocking of primitive irreducible sector blocks). Lean ✓ Stmt *Let A have positive bond dimension and be trace-preserving, primitive, and tensor-irreducible. For every $k > 0$,*

$$\sum_{\alpha} ((A^{[k]})^{\alpha})^{\dagger} (A^{[k]})^{\alpha} = \mathbb{1},$$

$$\sigma_{\partial}(\mathcal{E}_{A^{[k]}}) = \{1\}, \quad (\text{D.9})$$

and $A^{[k]}$ is tensor-irreducible. This k is a later common-refinement or injectivity length, distinct from the period-removal length that produced the sector block.

Proof. ✓ Proof Trace preservation follows from blocking a trace-preserving tensor, primitivity from the transfer-power identity (D.8), and irreducibility from Theorem D.4.9. $\square$

Lemma D.4.11 (Blocking the weighted block sum). Lean ✓ Stmt *For nonzero weights $(\mu_k)_{k=1}^r$, blocks $(B_k)_{k=1}^r$, and any $p \geq 1$, the blocking identity (D.6) holds, where $\sim$ denotes generation of the same MPV family at every length.*

Lemma D.4.12 (Positive-length equality survives blocking). Lean ✓ Stmt *If two tensors have the same MPV coefficients at every positive length, then so do their blocked tensors after any positive blocking length.*

Theorem D.4.13 (Eventual nonzero proportionality survives blocking). Lean ✓ Stmt *If two tensors have eventually nonzero proportional MPV coefficients, then so do their blocked tensors after any positive blocking length.*

Proof. ✓ Proof If proportionality holds from length N_0 onward, a blocked chain of length $N \geq N_0$ expands to an original chain of length $Np \geq N_0$. Use the original proportionality scalar at length Np . $\square$

Lemma D.4.14 (Positive-length blocking of a weighted block sum). [Lean](#) [✓ Stmt](#) If A and $\bigoplus_k \mu_k B_k$ have the same MPV coefficients at every positive length, then after any positive blocking length p , $A^{[p]}$ and $\bigoplus_k \mu_k^p B_k^{[p]}$ have the same MPV coefficients at every positive length.

Lemma D.4.15 (Positive-length equality of powered block sums). [Lean](#) [✓ Stmt](#) If $\bigoplus_j \mu_j^A A_j$ and $\bigoplus_k \mu_k^B B_k$ have the same MPV coefficients at every positive length, then after any positive common blocking length p , $\bigoplus_j (\mu_j^A)^p A_j^{[p]}$ and $\bigoplus_k (\mu_k^B)^p B_k^{[p]}$ have the same MPV coefficients at every positive length.

Lemma D.4.16 (Nonzero weights survive blocking). [Lean](#) [✓ Stmt](#) If $\mu \in \mathbb{C}$ is nonzero and $p \geq 1$, then $\mu^p \neq 0$; blocking preserves the nonzero-weight hypothesis of a weighted block decomposition.

Lemma D.4.17 (Iterated blocking at the transfer-map level). [Lean](#) [✓ Stmt](#) For any MPS tensor A and $m, n \geq 0$,

$$\mathcal{E}_{(A^{[m]})^{[n]}} = \mathcal{E}_{A^{[mn]}}. \quad (\text{D.10})$$

D.5 Support compression and cyclic-sector families

Support isometries first realize the cyclic corners. The remaining results place finitely many cyclic-sector families over one blocked alphabet and identify their weighted nonzero parts.

Theorem D.5.1 (Supported corner compression with a support isometry). [Lean](#) [✓ Stmt](#)

Let A be a tensor normalized on the support of an orthogonal projection P , so that $PA^iP = A^i$ and $\sum_i (A^i)^\dagger A^i = P$. Then there are a corner tensor C , a linear isomorphism $\varphi : M_{\dim C}(\mathbb{C}) \xrightarrow{\sim} PM_D(\mathbb{C})P$, and a rectangular isometry $V : \mathbb{C}^{\dim C} \rightarrow \mathbb{C}^D$ such that

$$\begin{aligned} V^\dagger V &= \mathbb{1}, \\ VV^\dagger &= P, \\ \varphi(X) &= V XV^\dagger. \end{aligned} \quad (\text{D.11})$$

Moreover, $\varphi(C^i) = A^i$ for every physical letter, and for every word $\sigma = (\sigma_1, \dots, \sigma_N)$,

$$V^{(N)}(C)_\sigma = \text{tr}(PA^{\sigma_1} \dots A^{\sigma_N}). \quad (\text{D.12})$$

Proof. [✓ Proof](#)

Apply Theorem [LTC26](#), “Compression of a supported finite Kraus family” to the family $(A^i)_i$. It gives C , φ , and V satisfying [\(D.11\)](#) and $\varphi(C^i) = A^i$. It remains to identify the coefficients of the associated words. For each word $\sigma = (\sigma_1, \dots, \sigma_N)$, cyclicity of the trace and $VV^\dagger = P$ give

$$V^{(N)}(C)_\sigma = \text{tr}(C^{\sigma_1} \dots C^{\sigma_N}) = \text{tr}(V^\dagger A^{\sigma_1} \dots A^{\sigma_N} V) = \text{tr}(PA^{\sigma_1} \dots A^{\sigma_N}),$$

which proves [\(D.12\)](#). □

Theorem D.5.2 (Commuting projections give sector isometries). [Lean](#) [✓ Stmt](#) Let a left-canonical tensor A commute with a family of orthogonal projections (P_k) summing to $\mathbb{1}$. Then

A decomposes into compressed sector tensors (C_k) , with a unit-weight direct-sum MPV decomposition, corner isomorphisms $\varphi_k : M_{\dim C_k}(\mathbb{C}) \xrightarrow{\sim} P_k M_D(\mathbb{C}) P_k$, and rectangular isometries V_k satisfying

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ V_k V_k^\dagger &= P_k, \\ \varphi_k(X) &= V_k X V_k^\dagger. \end{aligned} \tag{D.13}$$

The sector letters obey $\varphi_k(C_k^i) = P_k A^i P_k$.

Proof. ✓ Proof Apply Theorem D.5.1 to the compression of A on each sector P_k . Orthogonality and $\sum_k P_k = \mathbb{1}$ give the direct-sum decomposition. For every word $\sigma = (\sigma_1, \dots, \sigma_N)$, the partition of unity and (D.12) give

$$V^{(N)}(A)_\sigma = \text{tr} \left(\left(\sum_k P_k \right) A^{\sigma_1} \dots A^{\sigma_N} \right) = \sum_k \text{tr}(P_k A^{\sigma_1} \dots A^{\sigma_N}) = \sum_k V^{(N)}(C_k)_\sigma.$$

This is the unit-weight matrix-product-vector identity. □

Theorem D.5.3 (Adjoint-fixed projections give sector isometries). Lean ✓ Stmt Let A be left-canonical and let (P_k) be orthogonal projections summing to $\mathbb{1}$ and fixed by the adjoint transfer map. Then the same sector decomposition and support isometries of Theorem D.5.2 exist. In particular, the adjoint fixed-point condition $\mathcal{E}_A^\dagger(P_k) = P_k$ gives $P_k A^i = A^i P_k$, which gives the corner-letter identity $\varphi_k(C_k^i) = P_k A^i P_k$.

Proof. ✓ Proof The fixed-point identity $\mathcal{E}_A^\dagger(P_k) = P_k$, together with left-canonicity $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, places P_k in the multiplicative domain of $\mathcal{E}_A^\dagger$, forcing $P_k A^i = A^i P_k$ for every letter A^i . The result then follows from Theorem D.5.2. □

Theorem D.5.4 (Cyclic sectors after removing zero blocks). Lean ✓ Stmt If A and B generate the same MPV family at every positive length, then after removing the all-zero blocks and applying the trace-preserving gauge there are weighted nonzero-block tensors $\bigoplus_j \mu_j^A A_j$ and $\bigoplus_k \mu_k^B B_k$ such that, for every $N \geq 1$ and σ ,

$$V^{(N)}(A)_\sigma = V^{(N)} \left(\bigoplus_j \mu_j^A A_j \right)_\sigma, \tag{D.14}$$

$$V^{(N)}(B)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k^B B_k \right)_\sigma. \tag{D.15}$$

The two nonzero parts agree:

$$V^{(N)} \left(\bigoplus_j \mu_j^A A_j \right)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k^B B_k \right)_\sigma. \tag{D.16}$$

Each nonzero block has primitive irreducible cyclic sectors: for each j , there is $m_j \geq 1$ with a unit-weight sector decomposition $A_j^{[m_j]} \sim \bigoplus_{u=1}^{m_j} C_{j,u}^A$, and similarly for B_k .

Proof. [✓ Proof](#) Apply Theorem 8.2.1.1 separately to A and B . Its positive-length conclusion gives (D.14) and (D.15). Chaining these identities through $\mathcal{V}^+(A) = \mathcal{V}^+(B)$ gives (D.16). Finally apply Theorem 8.3.1.3 to every trace-preserving irreducible nonzero-weight block. $\square$

Lemma D.5.5 (Common reblocking of cyclic-sector families). [Lean](#) [✓ Stmt](#) *Given a finite family $(B_k)_{k=1}^r$, each with primitive irreducible cyclic sectors, there is a common blocking length p such that each $B_k^{[p]}$ admits a unit-weight direct-sum decomposition over a single common blocked alphabet $d^{[p]}$ into left-canonical blocks with primitive transfer maps, tensor irreducibility, and positive bond dimensions.*

Lemma D.5.6 (Common reblocking at a prescribed multiple). [Lean](#) [✓ Stmt](#) *With hypotheses as in Lemma D.5.5, and given any common multiple p of the per-block period-removal lengths (m_k) , the common reblocking can be performed at the prescribed length p .*

Lemma D.5.7 (Structural properties of the common reblocked sectors). [Lean](#) [✓ Stmt](#) *Each sector tensor in the common reblocked cyclic-sector family of Lemma D.5.5 is trace-preserving, $\sum_i (B^i)^\dagger B^i = \mathbb{1}$, has primitive transfer map, $\sigma_\partial(\mathcal{E}_B) = \{1\}$, is tensor-irreducible, and has positive bond dimension.*

Lemma D.5.8 (Common-alphabet flattening). [Lean](#) [✓ Stmt](#) *With $p = m_k e_k$ for each k , the iterated blocked tensor $(B_k^{[m_k]})^{[e_k]}$ agrees with $B_k^{[p]}$ in MPV family, after the canonical identification of the p -blocked alphabet $d^{[p]}$ with the e_k -fold tensor power of the m_k -blocked alphabet $d^{[m_k]}$.*

Lemma D.5.9 (Direct blocking in the common alphabet). [Lean](#) [✓ Stmt](#) *For every original block B_k , direct blocking at the common length $p = m_k e_k$ agrees, as an MPV family, with the same block transported to the common blocked alphabet. Write $\widetilde{B}_k$ for this common-alphabet image. Then*

$$V^{(N)}(B_k^{[p]})_\sigma = V^{(N)}(\widetilde{B}_k)_\sigma.$$

Lemma D.5.10 (Reindexed nonzero-part identity). [Lean](#) [✓ Stmt](#) *The powered nonzero-part decomposition $\bigoplus_k \mu_k^p B_k^{[p]}$ carries over under the common-alphabet identification of Lemma D.5.8: the common-alphabet sectors have the same nonzero-part decomposition as the direct p -blocked tensors.*

D.6 Prepared normal-form consequences

The first theorem derives the overlap condition from a prepared primitive family. The second does not construct primitive blocks from an arbitrary tensor: it only orders a family that already satisfies all primitive, irreducible, trace-preserving, nonzero-weight, and positive-dimension hypotheses.

Theorem D.6.1 (Self-overlap is derived in normal canonical form). [Lean](#) [✓ Stmt](#) *If (μ_k, A_k) satisfies the normal canonical form conditions, then $O_{A_k A_k}(N) \rightarrow 1$ for every block k .*

Proof. [✓ Proof](#) Fix a block k . Irreducibility and the TP normalization give a nonzero positive semidefinite fixed point ρ of $\mathcal{E}_{A_k}$ with fixed-point projector P ; primitivity gives the complementary gap $\rho_{\text{spec}}(\mathcal{E}_{A_k} - P) < 1$ (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]). Theorem 6.10.2 then yields $O_{A_k A_k}(N) \rightarrow 1$. $\square$

Theorem D.6.2 (Reordering prepared primitive blocks into normal canonical form). [Lean](#)
 $\checkmark$ *Stmnt* Suppose A generates the same MPV family as $\bigoplus_k \mu_k A_k$, and that each A_k is irreducible, satisfies $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$, and has primitive transfer map, all $\mu_k \neq 0$, and all bond dimensions are positive. Then there exist a blocking length $p > 0$, weights (ν_k) , and blocked tensors (B_k) such that $A^{[p]}$ generates the same MPV family as $\bigoplus_k \nu_k B_k$, and (ν_k, B_k) satisfies the normal canonical form conditions. The weight moduli need not be distinct; equal-modulus blocks are allowed.

Proof. $\checkmark$ *Proof* The given blocks are already primitive, so take $p = 1$. Reorder the blocks by non-increasing weight modulus, with ties allowed, and set $\nu_k := \mu_k$ and $B_k := A_k$ in that order. Irreducibility, TP normalization, primitivity, nonzero weights, and positive bond dimensions are preserved, so the reordered family satisfies Definition 8.9.2 and generates the same MPV family as A . $\square$

Remark D.6.3 (Scope of Theorem D.6.2). Theorem D.6.2 assumes all six block hypotheses in advance and produces the normal canonical form by reordering. It is not the arbitrary-input existence theorem of [CPGSV16, Section 2.3]; that primitive-block reduction, including its structural bond-dimension bound, is Theorem 8.11.1.2 in Section 8.11.1.

Appendix E

Basis of Normal Tensors: Supporting Results

This appendix supports Chapter 9. It proves the elementary normal-tensor consequences, Gram-matrix criteria, finite-dimensional change-of-basis facts, sector-blocking identities, direct-sum separation lemmas, and prepared-family bookkeeping used by the main BNT characterization, matching, and block-injectivity results.

E.1 Elementary normal-tensor and Gram-matrix support

Throughout this section, $V^{(N)}(A)$ denotes the length- N matrix product vector of a tensor A , and $O_{AB}(N)$ denotes the corresponding mixed overlap. We first record elementary consequences of the two notions of normality used in Chapter 9, and then prove the finite-family linear-independence criteria used in the BNT characterization and matching arguments.

Lemma E.1.1 (Positive bond dimensions in a coefficient-form BNT). [Lean](#) [✓ Stmt](#) *Every block in a coefficient-form basis of normal tensors has positive bond dimension.*

Proof. [✓ Proof](#) Eventual linear independence makes the matrix product vector of each block nonzero at some positive length. A tensor with zero bond dimension has zero matrix product vector at every length, which is a contradiction. $\square$

Lemma E.1.2 (Bond dimension one implies irreducibility). [Lean](#) [✓ Stmt](#) *Every tensor with a one-dimensional bond space is irreducible.*

Proof. [✓ Proof](#) The only orthogonal projections on a one-dimensional complex vector space are zero and the identity. $\square$

Lemma E.1.3 (Normality in bond dimension one). [Lean](#) [✓ Stmt](#) *A tensor with one-dimensional bond space and identity transfer map is a normal tensor.*

Proof. [✓ Proof](#) Irreducibility follows from the preceding lemma. The identity transfer map has spectral radius one, and its only eigenvalue is one. $\square$

Lemma E.1.4 (Perron gauges preserve primitivity). [Lean](#) [✓ Stmt](#) *Let σ be positive definite. The transfer map of the Perron gauge determined by σ is primitive if and only if the original transfer map is primitive.*

Proof. ✓ **Proof** The two transfer maps are similar through the invertible congruence $X \mapsto \sigma^{-1/2} X \sigma^{-1/2}$. Similar linear maps have the same spectrum, so the condition that 1 is the only eigenvalue on the unit circle is unchanged. $\square$

Lemma E.1.5 (Preservation of an algebraic BNT under positive-length MPV equality). Lean ✓ **Stmt** *Let A and A' have the same matrix product vectors at every positive length. If $(A_j)_{j=1}^g$ is a basis of normal tensors for A , then the same family is a basis of normal tensors for A' .*

Proof. ✓ **Proof** Substitute the positive-length MPV equality into the spanning identity for A . Normality and eventual linear independence depend only on the family $(A_j)_{j=1}^g$. $\square$

Corollary E.1.6 (Positive-length geometric proportionality phase). Lean ✓ **Stmt** *Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for every positive N , their matrix product vectors are proportional by a nonzero scalar. Then there is $a \in \mathbb{C}$, with $|a| = 1$, such that $|V^{(N)}(A)\rangle = a^N |V^{(N)}(B)\rangle$ for every positive N . This is the positive-length one-normal-block consequence of [CPGSV16, Corollary A.3 and Theorem 2.10].*

Proof. ✓ **Proof** Positive-length nonzero proportionality implies eventual nonzero proportionality. Apply Theorem 9.2.12. $\square$

Lemma E.1.7 (Eventual linear independence from finite-index overlap orthonormality). Lean ✓ **Stmt** *Let I be a finite index set and let $(A_i)_{i \in I}$ be a finite family of tensors such that $O_{A_i A_i}(N) \rightarrow 1$ for each i , and $O_{A_i A_j}(N) \rightarrow 0$ whenever $i \neq j$. Then the MPV states $\{|V^{(N)}(A_i)\rangle\}_{i \in I}$ are linearly independent for all sufficiently large N .*

Proof. ✓ **Proof** The Gram matrix of the family $\{|V^{(N)}(A_i)\rangle\}_{i \in I}$ converges entrywise to the identity matrix. Hence for all sufficiently large N it is invertible, so the MPV states are linearly independent. $\square$

The same conclusion is often more convenient with an explicit threshold rather than the eventual quantifier.

Theorem E.1.8 (Existential BNT independence from overlap limits). Lean ✓ **Stmt** *If the self-overlaps of a finite block family tend to 1 and the cross-overlaps of distinct blocks tend to 0, then the MPV states are linearly independent for all sufficiently large system sizes.*

Proof. ✓ **Proof** This is Lemma E.1.10, restated with an explicit threshold N_0 . $\square$

Lemma E.1.9 (Eventual linear independence for two families). Lean ✓ **Stmt** *Let $(A_j)_{j=1}^{g_A}$ and $(B_k)_{k=1}^{g_B}$ be finite families of tensors. Suppose the overlaps inside each family are asymptotically orthonormal, and suppose every mixed overlap $O_{A_j B_k}(N)$ tends to 0. Then the combined MPV family*

$$\{|V^{(N)}(A_j)\rangle\}_{j=1}^{g_A} \cup \{|V^{(N)}(B_k)\rangle\}_{k=1}^{g_B}$$

is linearly independent for all sufficiently large N .

Proof. ✓ **Proof** Apply the finite-index Gram-matrix criterion to the disjoint union of the two index sets. The within-family hypotheses give the diagonal and off-diagonal limits inside each block, and the mixed overlap hypothesis gives the off-diagonal limits between the two blocks. $\square$

The following two linear-algebra consequences package the eventual Gram-matrix criterion and its coefficient-extraction corollary.

Lemma E.1.10 (Eventual linear independence from overlap orthonormality). [Lean](#) [Lean](#) [✓ Stmt](#) Let $(A_j)_{j=1}^g$ be a finite family of tensors such that $O_{A_j A_j}(N) \rightarrow 1$ for each j and $O_{A_i A_j}(N) \rightarrow 0$ for $i \neq j$. Then the MPV states $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$ are linearly independent for all sufficiently large N .

Proof. [✓ Proof](#) Apply Lemma E.1.7 with $I = \{1, \dots, g\}$. □

Eventual linear independence is also what converts an equality of two finite linear combinations into equality of their coefficients; this is the mechanism behind the coefficient identities of Chapter 10.

Lemma E.1.11 (Eventual coefficient extraction from linear independence). [Lean](#) [✓ Stmt](#) If a finite family of vectors is linearly independent for all sufficiently large N , and two finite linear combinations of that family are equal for all sufficiently large N , then the corresponding coefficients are equal for all sufficiently large N .

Proof. [✓ Proof](#) Move the two linear combinations to one side. Linear independence forces each coefficient difference to vanish at every sufficiently large length. □

When two BNT families must be compared against each other, the same Gram-matrix criterion applies to their disjoint union, provided the mixed overlaps also decay. This is the form used in the exact sector matching of Section 9.8.

E.2 Change of basis, power sums, and sector bookkeeping

This section collects the finite-dimensional linear algebra behind alternative BNT comparisons, the extrapolation identities for finite sums of geometric sequences, and the bookkeeping identities for sector decompositions. A sector has copy weights $\mu_{j,q} \neq 0$, coefficient $c_{N,j} = \sum_q \mu_{j,q}^N$, representative bond dimension D_j , and total bond dimension $D_{\text{tot}} = \sum_j r_j D_j$.

Lemma E.2.1 (Invertible change of basis). [Lean](#) [✓ Stmt](#) Let $v_1, \dots, v_g$ and $w_1, \dots, w_g$ be two linearly independent families in a complex vector space with the same span. Then there exists an invertible matrix $U \in \text{GL}_g(\mathbb{C})$ such that $w_j = \sum_{i=1}^g U_{ij} v_i$ for every j .

Proof. [✓ Proof](#) Because the two families span the same subspace, each w_j has a unique expansion in the basis $\{v_i\}_{i=1}^g$; these coefficients form the columns of U . If $Ua = 0$, then the corresponding linear relation among the w_j is trivial because the w_j are linearly independent. Thus U is injective, hence invertible. □

Lemma E.2.2 (Eventual change of basis from equal spans). [Lean](#) [✓ Stmt](#) If two finite families of blocks have asymptotically orthonormal overlaps within each family, and if their MPV spans agree for all sufficiently large system sizes N , then for all sufficiently large N there exists an invertible matrix U_N expressing the MPV states of one family as linear combinations of the MPV states of the other.

Proof. [✓ Proof](#) By Lemma E.1.10, both families are linearly independent for all sufficiently large N . Hence, for all sufficiently large N , both families are linearly independent and the assumed span equality holds. Lemma E.2.1 applies at each such N . □

Lemma E.2.3 (Eventual vanishing of a finite geometric sum). [Lean](#) [✓ Stmt](#) Let $w_1, \dots, w_n$ be nonzero complex numbers. If $\sum_i c_i w_i^N = 0$ for every sufficiently large N , then the same equality holds for every non-negative exponent.

Proof. [✓ Proof](#) Induct on the number of terms. Subtracting w_1 times the equality at exponent N from the equality at exponent $N+1$ gives $\sum_{i=2}^n c_i(w_i - w_1)w_i^N = 0$. The induction hypothesis yields this equality at every exponent, so the original sum $S_N = \sum_i c_i w_i^N$ satisfies $S_{N+1} = w_1 S_N$ for every N . Since S_N eventually vanishes and $w_1 \neq 0$, it follows that $S_0 = 0$ and hence $S_N = 0$ for every N . $\square$

Lemma E.2.4 (Eventual power-sum equality holds at every exponent). [Lean](#) [✓ Stmt](#) Let $a_1, \dots, a_m$ and $b_1, \dots, b_n$ be nonzero complex numbers. If their power sums agree for every sufficiently large exponent, then $\sum_i a_i^k = \sum_j b_j^k$ for every non-negative exponent k .

Proof. [✓ Proof](#) Set $w_l = a_l$ for $l \leq m$ and $w_{m+l} = b_l$ for $l \leq n$, with coefficients $c_l = 1$ for $l \leq m$ and $c_l = -1$ for $l > m$. Then $\sum_{l=1}^{m+n} c_l w_l^k = \sum_i a_i^k - \sum_j b_j^k$. The hypothesis makes this expression zero for every sufficiently large k . Lemma E.2.3 makes it zero for every k , which is the required power-sum equality. $\square$

Theorem E.2.5 (Equal power sums imply equal multisets). [Lean](#) [✓ Stmt](#) Let $\alpha, \beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$. If $\sum_i \alpha_i^k = \sum_i \beta_i^k$ for all $k \geq 1$, then the multisets $\{\alpha_i\}$ and $\{\beta_i\}$ are equal.

Proof. [✓ Proof](#) Construct diagonal matrices $\text{diag}(\alpha)$ and $\text{diag}(\beta)$. The power-sum hypothesis gives $\text{tr}(\text{diag}(\alpha)^k) = \text{tr}(\text{diag}(\beta)^k)$. By the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”], the characteristic polynomials agree: $\prod_i (X - \alpha_i) = \prod_i (X - \beta_i)$. Extracting roots gives multiset equality. $\square$

Theorem E.2.6 (Representative dimensions are bounded by the total bond dimension). [Lean](#) [Lean](#) [✓ Stmt](#) Let the representative A_j have bond dimension D_j , and let D be the bond dimension of the flattened sector tensor. Then $D_j \leq D$. If every D_j is positive and there are g representatives, then also $g \leq D$.

Proof. [✓ Proof](#) Since $r_j > 0$, choose one copy of A_j . Its contribution D_j is one non-negative summand in the total dimension of the flattened copy family, so $D_j \leq D$.

For the second assertion, every representative has multiplicity $r_j \geq 1$, and each copy has positive dimension $D_j > 0$. Therefore

$$g = \sum_j 1 \leq \sum_j r_j \leq \sum_j r_j D_j = D.$$

$\square$

Theorem E.2.7 (Blocked sector coefficients and assembled tensor). [Lean](#) [Lean](#) [✓ Stmt](#) The coefficient of the blocked decomposition satisfies

$$c_{P^{[P]},j}^{(N)} = \sum_q (\mu_{j,q}^p)^N = c_{P,j}^{(Np)}. \quad (\text{E.1})$$

Moreover, blocking the tensor assembled from P gives the same MPV family as the tensor assembled from the blocked sector decomposition.

Proof. [✓ Proof](#) The coefficient identity in (E.1) is $(\mu^p)^N = \mu^{pN}$. The tensor identity is Theorem D.4.11 applied to the flattened copy family. $\square$

Lemma E.2.8 (Existence of an MPV phase class). [Lean](#) [✓ Stmt](#) A nonempty finite family of blocks has at least one MPV phase class.

Proof. ✓ Proof Choose one original block index. Its occurrence in the phase-class enumeration supplies a phase class containing it. $\square$

Lemma E.2.9 (Regrouping a matrix sum by MPV phase classes). Lean ✓ Stmt *Let F_k be a finite family of square matrices indexed by the original blocks. If $k = (j, q)$ enumerates the copies in the MPV phase classes, then $\sum_j \sum_q F_{j,q} = \sum_k F_k$.*

Proof. ✓ Proof Evaluate both sides at a pair of matrix indices and apply the bijective enumeration of the finite phase-class quotient. $\square$

Definition E.2.10 (Single-copy sector decomposition). Lean ✓ Stmt Given nonzero weights $(\mu_k)_{k=1}^r$ and blocks $(B_k)_{k=1}^r$, the *single-copy sector decomposition* has r sectors, one basis block B_k per sector, multiplicity 1, and sector weight μ_k . As a finite sum,

$$\sum_{k=1}^r \mu_k^N V^{(N)}(B_k)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k B_k \right)_\sigma. \quad (\text{E.2})$$

Lemma E.2.11 (Single-copy sector decomposition preserves the represented family). Lean ✓ Stmt *The single-copy sector decomposition of $(\mu_k, B_k)_k$ represents the same MPS family as $\bigoplus_k \mu_k B_k$.*

Proof. ✓ Proof Expanding the sector decomposition gives the left-hand side of (E.2), which is the finite block-sum expression for $\bigoplus_k \mu_k B_k$. $\square$

Lemma E.2.12 (Sector-decomposition coefficient is not eventually zero). Lean Lean ✓ Stmt *For any sector decomposition P and any sector j , the coefficient $c_{P,j}^{(N)} = \sum_q \mu_{j,q}^N$ is not eventually zero in N . More generally, for every positive integer p , the subsequence $c_{P,j}^{(pN)}$ is not eventually zero.*

Proof. ✓ Proof If the coefficient vanished for every $N \geq M$, the finite geometric-sum extrapolation would make it vanish at every non-negative exponent. However, at exponent zero, $c_{P,j}^{(0)} = \sum_q \mu_{j,q}^0 = r_j > 0$, which is impossible. For the subsequence, apply the same argument to the nonzero bases $\mu_{j,q}^p$. $\square$

E.2.1 Proportional matching and finite bijection extraction

The equal-MPV matching theorems in Chapter 9 are specializations of an eventual proportionality argument. We record that general mechanism and the finite-set construction used to package matches in both directions.

Theorem E.2.1.1 (Exact sector matching under eventual proportionality). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total MPV families are eventually nonzero and proportional. For every sector j of P , there is a sector k of Q with equal bond dimension, gauge-phase-equivalent basis tensor after the dimension identification, and non-decaying cross-overlap.*

Proof. ✓ Proof Suppose the chosen sector decays against every sector of Q , and separate all such P -sectors from the remaining ones. For each remaining sector, choose a Q -sector with non-decaying overlap. Both blocks are normal by the canonical-form hypotheses. Theorem 9.2.9 rules out its decay alternative and supplies equal bond dimensions and a gauge-phase relation with a unit phase. Gauge-phase covariance therefore expresses that P -sector MPV as a geometric scalar multiple of the chosen Q -sector MPV.

Substitute these relations into the eventual proportionality identity. The combined-family Gram criterion makes the decaying residue together with the Q family eventually linearly independent. Exact coefficient extraction then makes the chosen sector coefficient eventually zero, contradicting its power-sum nonvanishing. Thus a non-decaying partner exists. A second application of Theorem 9.2.9 rules out decay and gives the required dimension equality and gauge-phase equivalence; no modulus-one overlap limit is assumed. $\square$

Theorem E.2.1.2 (Full-basis matching under eventual proportionality). Lean Stmt *Under the same hypotheses, every sector of Q has an equal-dimensional, gauge-phase-equivalent partner in P with non-decaying cross-overlap.*

Proof. Proof Reverse the eventual proportionality and apply Theorem E.2.1.1 to each sector of Q ; symmetry of the overlap converts the resulting witness to the stated orientation. $\square$

Lemma E.2.1.3 (Bijection from matches in both directions). Lean Stmt *Suppose every sector of Q has an equal-dimensional, gauge-phase-equivalent partner in P with non-decaying overlap, and the converse holds from P to Q . Then the forward partners may be chosen as a bijection between the two finite sector sets, preserving all three properties.*

Proof. Proof Basis separation makes each partner map injective: two sectors with the same partner would be gauge-phase equivalent to one another. The injections in both directions force equal finite cardinalities, so the forward map is bijective. $\square$

E.3 Direct-sum separation and prepared-family bookkeeping

We now prove the supporting separation and regrouping results used by the source-facing block-injectivity and supplier theorems in Chapter 9. The direct-sum argument follows the ground-space uniqueness mechanism of [PGVWC07]: disjoint local MPS spaces give pair-trace selectors, products of those selectors isolate one BNT block, and an injective prefix supplies an arbitrary matrix on that block. The remaining results identify dimensions while phase-equivalent prepared blocks are grouped into sectors and normalize the resulting finite weight family.

Lemma E.3.1 (Three-block separation of distinct BNT representatives). Lean Lean Lean Stmt *Let $A^1, \dots, A^r$ be irreducible left-canonical representatives with normalized self-overlaps, and suppose that distinct same-dimensional representatives are not gauge-phase equivalent. Fix $L_0 > 0$ such that every block is injective at lengths $L_0, L_0 + 1$, and $3(L_0 + 1)$. Then each ordered pair $k \neq j$ is trace-separated at the common length $S = 3(L_0 + 1)$: the map*

$$(c_\omega)_{|\omega|=S} \mapsto \left(\sum_{|\omega|=S} c_\omega A_\omega^k, \sum_{|\omega|=S} c_\omega A_\omega^j \right)$$

is onto $M_{D_k}(\mathbb{C}) \oplus M_{D_j}(\mathbb{C})$. In particular, one may prescribe an arbitrary matrix on either block while making the same word combination vanish on the other block.

Proof. Proof Put $L = L_0 + 1$ and $S = 3L$. For a block A , write

$$\Gamma_n^A(X) = \sum_{|w|=n} \text{tr}(XA_w)|w\rangle, \quad \mathcal{G}_n(A) = \text{im } \Gamma_n^A.$$

Block injectivity at L and S makes both boundary maps injective; in particular,

$$\dim \mathcal{G}_L(A^q) = D_q^2 \quad (1 \leq q \leq r). \quad (\text{E.3})$$

Fix $k \neq j$ and first order the pair so that $D_j \leq D_k$. Suppose that a nonzero vector belongs to $\mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j)$. There are boundary matrices X and Y such that

$$0 \neq \psi = \Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(Y). \quad (\text{E.4})$$

Injectivity at S gives $X \neq 0$ and $Y \neq 0$. Split every length- S word as uvw with $|u| = |v| = |w| = L$. Equality E.4 gives

$$\text{tr}(XA_u^k A_v^k A_w^k) = \text{tr}(YA_u^j A_v^j A_w^j). \quad (\text{E.5})$$

Because the length- L words of A^k span $M_{D_k}(\mathbb{C})$ and $X \neq 0$, the two-sided ideal generated by X is all of $M_{D_k}(\mathbb{C})$:

$$\text{span}\{A_u^k X A_v^k : |u| = |v| = L\} = M_{D_k}(\mathbb{C}). \quad (\text{E.6})$$

Indeed, expand arbitrary left and right matrix units in length- L words; a nonzero entry of X then produces every matrix unit. Thus, for each $Z \in M_{D_k}(\mathbb{C})$, choose coefficients a_{uv} with $Z = \sum_{u,v} a_{uv} A_u^k X A_v^k$ and set $R(Z) = \sum_{u,v} a_{uv} A_u^j Y A_v^j$. Cyclicity of the trace and E.5, applied to the word vwu , yield

$$\begin{aligned} \text{tr}(Z A_w^k) &= \sum_{u,v} a_{uv} \text{tr}(X A_v^k A_w^k A_u^k) \\ &= \sum_{u,v} a_{uv} \text{tr}(Y A_v^j A_w^j A_u^j) = \text{tr}(R(Z) A_w^j). \end{aligned}$$

Hence $\Gamma_L^{A^k}(Z) = \Gamma_L^{A^j}(R(Z))$ for every Z , and therefore

$$\mathcal{G}_L(A^k) \subseteq \mathcal{G}_L(A^j). \quad (\text{E.7})$$

Equations E.3 and E.7 give

$$D_k^2 = \dim \mathcal{G}_L(A^k) \leq \dim \mathcal{G}_L(A^j) = D_j^2.$$

Together with $D_j \leq D_k$, this forces $D_j = D_k$ and equality in E.7. This proves the unequal-dimension branch immediately: if $D_j < D_k$, a nonzero intersection is impossible. Reversing the initial ordering treats $D_k < D_j$ in the same way.

It remains to exclude the intersection when $D_j = D_k$. Gauge-phase separation and Theorem 9.4.3 give, beyond any prescribed threshold, a chain length $N > L$ for which the two periodic MPVs are not proportional. On the other hand, equality $\mathcal{G}_L(A^k) = \mathcal{G}_L(A^j)$ gives identical translated length- L constraints on the N -site ring. The parent-Hamiltonian uniqueness mechanism identifies the corresponding chain ground spaces with the one-dimensional MPV spans:

$$\mathcal{G}_{N,L}(A^k) = \mathbb{C} V^{(N)}(A^k), \quad \mathcal{G}_{N,L}(A^j) = \mathbb{C} V^{(N)}(A^j).$$

Equality of the local spaces therefore forces these two MPV lines to be equal, contradicting the choice of N . Consequently, for every distinct pair, including the unequal-dimensional pairs,

$$\mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j) = \{0\}. \quad (\text{E.8})$$

Finally, let Φ_{kj} be the displayed length- S word-evaluation map from coefficient families to $M_{D_k}(\mathbb{C}) \oplus M_{D_j}(\mathbb{C})$. Pair the target by

$$\langle (X, Y), (M, N) \rangle = \text{tr}(XM) + \text{tr}(YN).$$

A pair (X, Y) annihilates $\text{im}(\Phi_{kj})$ precisely when

$$\text{tr}(XA_w^k) + \text{tr}(YA_w^j) = 0 \quad (|w| = S),$$

or equivalently $\Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(-Y)$. There is therefore a linear map

$$\Theta : \text{im}(\Phi_{kj})^\perp \longrightarrow \mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j), \quad \Theta(X, Y) = \Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(-Y).$$

It is injective because both boundary maps are injective. Conversely, if ψ lies in the intersection, there are unique X and Z with $\Gamma_S^{A^k}(X) = \psi = \Gamma_S^{A^j}(Z)$; the pair $(X, -Z)$ annihilates $\text{im}(\Phi_{kj})$ and maps to ψ . This construction is the inverse of Θ , so the two spaces are isomorphic:

$$\text{im}(\Phi_{kj})^\perp \cong \mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j). \quad (\text{E.9})$$

The right-hand side vanishes by E.8, and hence so does the annihilator. The trace pairing is nondegenerate and all spaces are finite-dimensional, so $\text{im}(\Phi_{kj})^\perp = \{0\}$ implies that Φ_{kj} is surjective. Surjectivity first gives an arbitrary value on one block and zero on an anchor block; the selector construction then chooses $(1, 0)$ for the remaining pairs, and products of these homogeneous selectors give the simultaneous word-tuple span in Lemma 9.3.3. $\square$

Theorem E.3.2 (Common injective blocking for primitive irreducible families). [Lean](#) [✓ Stmt](#) *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension MPS tensors, each trace-preserving, primitive, and tensor-irreducible. Then there is a single $L \geq 1$ such that each $B_k^{[L]}$ is one-site injective, and this blocking preserves trace preservation, primitivity, and irreducibility for every block.*

Proof. [✓ Proof](#) For each block, Theorem 7.9.1 gives a positive length at which it becomes one-site injective; the product of these lengths is a common length. Injectivity persists under positive further blocking, and Theorem D.4.10 preserves the other hypotheses. $\square$

Lemma E.3.3 (Total dimension after grouping by BNT representatives). [Lean](#) [✓ Stmt](#) *Suppose CF summands are grouped by BNT representative, every original copy weight is nonzero, and every summand assigned to a representative has that representative's bond dimension. Then the total bond dimension of the resulting BNT sector decomposition is the sum of the original block dimensions.*

Proof. [✓ Proof](#) The collapsed decomposition counts one representative dimension for each assigned copy. Let $e(j, q)$ denote the original CF summand assigned to the q th copy of representative P_j . Since $(j, q) \mapsto e(j, q)$ reindexes the original summands and each assigned summand has the representative's bond dimension,

$$\sum_{j=1}^g \sum_{q=1}^{r_j} \dim(P_j) = \sum_{j=1}^g \sum_{q=1}^{r_j} D_{e(j,q)} = \sum_{k=1}^r D_k.$$

This is the equality between the collapsed total dimension and the original direct-sum dimension. $\square$

Lemma E.3.4 (Prepared phase-equivalent NTs have the same bond dimension). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimension. If their MPV families differ by a nonzero scalar power, then X and Y have the same bond dimension.*

Proof. [✓ Proof](#) Choose $\zeta \neq 0$ with $|V^{(N)}(Y)\rangle = \zeta^N |V^{(N)}(X)\rangle$ for every positive length N . The normalized self-overlaps give $|O_{XX}(N)| \rightarrow 1$, and the phase relation gives

$$\begin{aligned} O_{YX}(N) &= \zeta^N O_{XX}(N), \\ |O_{YX}(N)| &= |\zeta|^N |O_{XX}(N)|. \end{aligned}$$

Since the same normalization forces $|\zeta| = 1$, this gives $|O_{YX}(N)| \rightarrow 1$. If the two bond dimensions differed, the rectangular overlap-decay theorem would force $O_{YX}(N) \rightarrow 0$, contradicting the preceding limit. $\square$

Lemma E.3.5 (Prepared BNT representative classes preserve bond dimension). [Lean](#) [✓ Stmt](#) *In a prepared family of positive-bond-dimension trace-preserving primitive irreducible NTs, every CF summand assigned to a BNT representative has that representative's bond dimension.*

Proof. [✓ Proof](#) Apply Lemma E.3.4 to each representative and each assigned summand. $\square$

Lemma E.3.6 (Prepared collapsed BNT total dimension). [Lean](#) [✓ Stmt](#) *For prepared trace-preserving primitive irreducible NTs of positive bond dimension with all copy weights nonzero, grouping CF summands by BNT representative preserves the total bond dimension: the collapsed sector decomposition has total bond dimension equal to the sum of the original block dimensions.*

Proof. [✓ Proof](#) Every assigned summand shares the bond dimension of its representative, by the equal-dimension fact above. The displayed regrouping identity in Lemma E.3.3 then identifies the collapsed sum over pairs (j, q) with $\sum_{k=1}^r D_k$, the original direct-sum dimension. $\square$

Theorem E.3.7 (Projective weight normalization). [Lean](#) [✓ Stmt](#) *For nonzero weights $\mu_1, \dots, \mu_r$ with $r \geq 1$ there is a positive real m such that every normalized weight satisfies*

$$\begin{aligned} \left| \frac{\mu_k}{m} \right| &\leq 1, \\ \frac{\mu_k}{m} &\neq 0, \end{aligned}$$

and at least one normalized weight has unit modulus, $|\mu_{k_0}/m| = 1$. This is the normalization that [CPGSV16, line 246] chooses for the canonical-form weights.

Proof. [✓ Proof](#) Take $m = \max_k |\mu_k|$; the maximum over a nonempty finite family is attained, positive because every weight is nonzero, an upper bound for every modulus, and equal to the modulus of the attaining weight. $\square$

Lemma E.3.8 (Common scalar rescaling of block weights). [Lean](#) [✓ Stmt](#) *Multiplying every block weight in a block-diagonal MPS tensor by a scalar c multiplies every length- N MPV coefficient by c^N .*

Proof. [✓ Proof](#) Expand the block-diagonal MPV coefficient as a finite sum over blocks and factor c^N out of the sum. $\square$

Appendix F

Proof of the Fundamental Theorem: Supporting Results

This appendix supports Chapter 10. It proves phase normalization, coefficient extraction from matched gauges, direct-sum conjugation, the exact consequences of length-dependent proportionality, coordinate reindexing, and unitary refinements. The first part concerns coefficients and block gauges; the second identifies matched coordinates with the literal total bond space.

F.1 Matched phases and coefficient extraction

This section isolates two elementary steps used by the main proof. First, the sector partners obtained in both directions determine a finite bijection. Second, normalization of the matched normal tensors forces every gauge scalar to be a phase.

Theorem F.1.1 (Bijective matching under eventual proportionality). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total MPV families are eventually nonzero and proportional. There is a bijection $\beta : J(Q) \simeq J(P)$ such that every Q_k has an equal-dimensional, gauge-phase-equivalent partner $P_{\beta(k)}$ satisfying*

$$O_{P_{\beta(k)} Q_k}(N) \not\rightarrow 0.$$

Proof. ✓ Proof Apply the full-basis matching theorem first from Q to P and then to the reverse proportionality from P to Q . If P_j and $P_{j'}$ had the same partner Q_k , then symmetry and transitivity would give

$$P_j \sim_{\text{gp}} Q_k \sim_{\text{gp}} P_{j'} \implies P_j \sim_{\text{gp}} P_{j'},$$

contradicting BNT basis separation. Both partner maps are therefore injective. The two finite sector sets have equal cardinality, so the forward map is bijective and retains the displayed overlap limit together with the dimension and gauge data. $\square$

Lemma F.1.2 (Unit phase from matched normalized BNT blocks). Lean ✓ Stmt *Let P and Q be BNT canonical forms. If, for every positive length N and word σ , a sector P_j and a sector Q_k have MPV families related by*

$$V^{(N)}(Q_k)_\sigma = \zeta^N V^{(N)}(P_j)_\sigma, \tag{F.1}$$

then $|\zeta| = 1$.

Proof. ✓ Proof The BNT normalization gives self-overlap limits of modulus 1 for both sectors. The MPV identity (F.1) gives, for every positive N ,

$$O_{Q_k Q_k}(N) = |\zeta|^{2N} O_{P_j P_j}(N).$$

Since $O_{Q_k Q_k}(N) \rightarrow 1$ and $O_{P_j P_j}(N) \rightarrow 1$, taking limits forces $|\zeta|^{2N} \rightarrow 1$ and hence $|\zeta| = 1$. $\square$

F.2 Coefficient identities from gauge-phase equalities

The main coefficient theorem takes the matched phases as input. The next result shows how those phases are selected from gauge-phase equivalences before the linear-independence argument is applied.

Lemma F.2.1 (Full-basis coefficient identity from matched sectors). Lean ✓ Stmt *Let P and Q be BNT canonical forms with the same MPV family at every positive length, matched by a bijection $\beta : J(Q) \simeq J(P)$ with bond-dimension equalities and a gauge-phase equivalence between Q_k and $P_{\beta(k)}$ for every k . Then each sector carries a unit-modulus phase ζ_k , and eventually*

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q). \quad (\text{F.2})$$

This is the equal-MPV coefficient comparison of [CPGSV16, Appendix MPV proof, lines 1187–1188].

Proof. ✓ Proof For each matched sector, the gauge-phase equivalence gives $V^{(N)}(Q_k)_\sigma = \zeta_k^N V^{(N)}(P_{\beta(k)})_\sigma$. Lemma F.1.2 gives $|\zeta_k| = 1$. Applying Lemma 10.2.2 to the same phases gives $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ for all sufficiently large N . $\square$

F.3 The scalar-threaded proportional identity

For proportional total MPVs, coefficient extraction retains the one scalar that relates the two total vectors at each length. This identity is exact, but it does not imply the scalar-free identity needed to recover copy weights; the counterexample in Remark 10.3.6 shows that no such implication is possible.

Lemma F.3.1 (Proportional matched-phase coefficient identity, scalar-threaded). Lean ✓ Stmt *Let P be a BNT canonical form and Q a sector decomposition whose total tensors generate eventually nonzero proportional MPV families. Suppose a bijection $\beta : J(Q) \simeq J(P)$ and scalars ζ_k satisfy the matched MPV phase identities (10.2) for every sector k . Then there is a single eventually-nonzero scalar sequence c_N and a threshold N_0 such that, for all $N > N_0$ and every $k \in J(Q)$,*

$$c_N^{(\beta(k))}(P) = c_N \zeta_k^N c_N^{(k)}(Q). \quad (\text{F.3})$$

The scalar c_N is the same for every sector k : it is the single global proportionality scalar of [CPGSV16, Theorem thm1, lines 1167–1170] at length N .

Proof. ✓ Proof For each N on the proportionality tail, expand both total MPVs in their sector bases and use the global proportionality scalar c_N :

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = c_N \sum_{k \in J(Q)} c_N^{(k)}(Q) |V^{(N)}(Q_k)\rangle.$$

The matched phase identities (10.2) give $|V^{(N)}(Q_k)\rangle = \zeta_k^N |V^{(N)}(P_{\beta(k)})\rangle$; because every sector is matched, reindexing the Q -sum by β rewrites the right-hand side entirely in the P -basis. The BNT eventual linear independence of the P -basis and Lemma E.1.11 then extract the coefficient identities (F.3) for all sufficiently large N . $\square$

F.4 Direct-sum conjugation

We now prove the general block-matrix identities used by the global gauge in Section 10.3. These statements do not choose or match sectors; they only combine conjugacies that have already been obtained.

Definition F.4.1 (Invertible matrix associated to a unitary). Lean Stmt A unitary matrix U determines an invertible matrix whose inverse is $U^\dagger$.

Lemma F.4.2 (Unitary block-diagonal gauges). Lean Stmt If every block gauge U_k is unitary, then the flattened block-diagonal gauge $U = \bigoplus_k U_k$ is unitary.

Proof. Proof The products are computed blockwise:

$$UU^\dagger = \bigoplus_k U_k U_k^\dagger = \bigoplus_k \mathbb{1} = \mathbb{1}.$$

$\square$

Lemma F.4.3 (Direct-sum conjugation by the flattened gauge). Lean Lean Stmt If, for every block k and physical index i , $B_k^i = X_k A_k^i X_k^{-1}$, then the weighted direct sums satisfy

$$\bigoplus_k \mu_k B_k^i = X \left(\bigoplus_k \mu_k A_k^i \right) X^{-1},$$

where X is the flattened block-diagonal gauge from Definition 10.3.1. Hence the two assembled tensors are gauge equivalent.

Proof. Proof Multiplication of block-diagonal matrices is computed blockwise. The scalar weight μ_k commutes with the conjugation on the k th block, and the reindexing from dependent block coordinates to the flattened bond index preserves products and inverses. $\square$

Theorem F.4.4 (Global gauge from a copy-weight matching). Lean Stmt Suppose that the sectors of P and Q are matched, with bond-dimension equalities, nonzero phases, and block gauges satisfying $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. If the same phases also give a copy-weight matching, then the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{\tau_k} \otimes X_k)$. This is only a reformulation of Theorem 10.3.2.

Proof. Proof The copy-weight matching supplies $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Together with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$, these are exactly the two hypotheses (10.4) of Theorem 10.3.2. $\square$

F.5 Additional proportional consequences

The following implications are useful when copy weights satisfy extra relations. They are not conclusions of CPSV16 Theorem 2.10. In particular, their additional hypotheses fail for the proportional BNT forms in Remark 10.3.6.

Theorem F.5.1 (Additional proportional global gauge under copy-weight matching). [Lean](#) [✓ Stmt](#) Let P and Q be BNT canonical forms whose total tensors generate eventually nonzero proportional MPV families. Then the proportional sector-matching theorem gives β , the bond-dimension equalities, the unit phases ζ_k , and the block gauges X_k . If one additionally assumes a copy-weight matching for these same phases, then the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$.

Proof. [✓ Proof](#) The proportional sector-matching theorem supplies β , X_k , and unit phases ζ_k satisfying $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. Since $|\zeta_k| = 1$, each ζ_k is nonzero. A copy-weight matching gives $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$, so Theorem F.4.4 gives the direct-sum gauge. $\square$

Theorem F.5.2 (Additional proportional global gauge under coefficient identities). [Lean](#) [✓ Stmt](#) Let P and Q be BNT canonical forms whose total tensors generate eventually nonzero proportional MPV families. Then there are a bijection $\beta : J(Q) \simeq J(P)$, bond-dimension equalities, unit-modulus phases ζ_k , and block gauges X_k such that $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$ for every sector k and physical index i . If one additionally assumes that these same phases satisfy the matched-sector coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ for all sufficiently large N , then they determine a copy-weight matching and hence the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$.

Proof. [✓ Proof](#) The sector-matching theorem gives β , X_k , and $|\zeta_k| = 1$ with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. Since $\zeta_k \neq 0$, the coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ produce $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$ by Definition 10.2.4. Theorem F.5.1 then applies to this copy-weight matching. $\square$

F.6 Matched flattened coordinates

Let $\beta : J(Q) \simeq J(P)$ match the basis sectors and let τ_k match the copies in sector k . Flatten the pair (k, q) to the copy index of Q . The matched P -data on that index are the weight and tensor

$$\tilde{\mu}_{k,q} = \mu_{\beta(k), \tau_k(q)}, \quad \tilde{P}_{k,q} = P_{\beta(k)},$$

where the second expression is read after the matched bond-dimension identification. The block gauge repeats X_k on every copy q .

Definition F.6.1 (Matched weights, tensors, and repeated gauges). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The matched flattened data are the functions $\tilde{\mu}$, $\tilde{P}$, and $(k, q) \mapsto X_k$ defined above on the flattened copy coordinates of Q .

Lemma F.6.2 (Equality of the matched total bond dimensions). [Lean](#) [✓ Stmt](#) If matched sectors have equal bond dimensions and matched copy sets are bijective, then P and Q have the same total bond dimension.

Proof. [✓ Proof](#) Reindex first by the sector bijection and then by each copy bijection. Every summand $\dim P_{\beta(k)}$ becomes $\dim Q_k$, so

$$\sum_{j \in J(P)} r_j(P) \dim P_j = \sum_{k \in J(Q)} r_k(Q) \dim Q_k.$$

$\square$

Definition F.6.3 (Sector and bond-coordinate equivalences). [Lean](#) [Lean](#) [✓ Stmt](#) The sector bijection, copy bijections, and bond-dimension equalities induce an equivalence of the dependent triples (j, q, a) and (k, q', a') . After flattening the dependent bond coordinate, this gives an equivalence of the standard total bond-index sets.

Proof. [✓ Proof](#) Send (k, q, a) to $(\beta(k), \tau_k(q), a)$, using the bond-dimension equality to identify the last coordinate. The inverse uses β^{-1} and τ_k^{-1} . Both composites are the identity componentwise. Conjugating by the standard equivalence between dependent triples and a finite interval gives the flattened bond-coordinate equivalence. $\square$

Theorem F.6.4 (Equal-MPV gauge with all matched witnesses). [Lean](#) [✓ Stmt](#) *Under the hypotheses of the equal-MPV theorem, one may choose simultaneously the sector bijection, bond-dimension equalities, copy bijections, phases, block gauges, total-dimension equality, and the flattened global gauge. They satisfy the matched block and weight identities of Theorem 10.3.4, together with the conjugation of Q by the matched-coordinate P -tensor.*

Proof. [✓ Proof](#) Theorem 10.3.4 supplies all witnesses except the displayed equality of total dimensions. Apply Lemma F.6.2 to its sector and copy bijections. The resulting equality identifies the total bond spaces, while the global conjugation equation is unchanged. $\square$

F.7 Permutation gauges and literal coordinates

The matched P -tensor has the correct blocks but lists them in Q -order. A permutation matrix restores the literal order of P . We record its invertible and unitary forms before proving the coordinate conversion used in Theorem 10.3.5.

Definition F.7.1 (Invertible permutation gauge). [Lean](#) [✓ Stmt](#) For a permutation ρ of a finite bond-index set, let Π_ρ be its permutation matrix. The corresponding element of the general linear group is denoted G_ρ .

Lemma F.7.2 (Matrix and inverse identities for a permutation gauge). [Lean](#) [Lean](#) [✓ Stmt](#) *The matrix of G_ρ is Π_ρ , and the matrix of its inverse is $\Pi_{\rho^{-1}}$.*

Proof. [✓ Proof](#) The first identity is the definition of the permutation gauge. Since the inverse permutation reverses the coordinate relabelling, $G_\rho^{-1} = G_{\rho^{-1}}$, which gives the second identity. $\square$

Every row and every column of Π_ρ contains exactly one nonzero entry, equal to 1. Hence

$$\Pi_\rho \Pi_{\rho^{-1}} = \mathbb{1}, \quad \Pi_\rho^\dagger = \Pi_{\rho^{-1}}.$$

Together with Lemma F.7.2, these identities show that $G_\rho^\dagger = G_\rho^{-1}$, so the permutation gauge is unitary.

Lemma F.7.3 (Permutation conjugation reindexes both coordinates). [Lean](#) [✓ Stmt](#) *For a permutation ρ and a square matrix M , conjugation by the permutation matrix reindexes both coordinates by ρ :*

$$\Pi_\rho M \Pi_{\rho^{-1}} = (M_{\rho(i), \rho(j)})_{i,j}. \quad (\text{F.4})$$

Equivalently, the right-hand side is the submatrix of M along $\rho \times \rho$.

Proof. [✓ Proof](#) Left multiplication by Π_ρ selects row $\rho(i)$ in row i , and right multiplication by $\Pi_{\rho^{-1}}$ selects column $\rho(j)$ in column j . The two selections give the displayed submatrix. $\square$

The coordinate equivalence of Definition F.6.3 induces the required permutation ρ after the total-dimension identification. A convention with inverse indices merely replaces the chosen permutation by ρ^{-1} ; we use the convention in (F.4) to match the formal coordinate identity. Applying that identity to P_{tot}^i gives the matched-coordinate tensor. Composing G_ρ with the block-diagonal gauge therefore gives the literal gauge of Theorem 10.3.5. This also explains why the total-dimension cast and the sector permutation must occur in that order.

F.8 Unitary witness refinements

The main text proves that a specified gauge between left-canonical irreducible sectors can be replaced by a unitary without changing its phase. We record the existential form and then repeat the construction over every matched copy.

Lemma F.8.1 (Unitary gauge for left-canonical irreducible tensors). Lean ✓ Stmt *Let A and B be left-canonical irreducible tensors of common positive bond dimension that are gauge-phase equivalent. Then the gauge can be taken unitary: there are a unitary matrix U and a scalar ζ with $|\zeta| = 1$ such that, for every physical index i ,*

$$B^i = \zeta U A^i U^\dagger.$$

Proof. ✓ Proof Unpack the gauge-phase equivalence as $B^i = \zeta X A^i X^{-1}$ with X invertible and $\zeta \neq 0$. Lemma 10.4.1.2 replaces X by a unitary U while retaining this same phase and proves $|\zeta| = 1$. □

Lemma F.8.2 (Unitary gauge in matched sector coordinates). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total tensors generate the same MPV family at every positive length. There are a sector bijection β , copy bijections τ_k , phases ζ_k , sector unitaries U_k , and a unitary block-diagonal matrix X such that*

$$\begin{aligned} Q_k^i &= \zeta_k U_k P_{\beta(k)}^i U_k^\dagger, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}, \\ |\zeta_k| &= 1. \end{aligned}$$

The matched sectors and copies have equal bond dimensions and multiplicities. If P_{match} denotes the corresponding reordered block assembly, then $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$.

Proof. ✓ Proof Apply Lemma 10.4.1.2 to each matched sector, retaining the phase in the copy-weight identity. Repeating U_k on every copy gives $X = \bigoplus_{k,q} U_k$, which is unitary by Lemma F.4.2. For every matched copy (k, q) ,

$$\nu_{k,q} Q_k^i = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)} \zeta_k U_k P_{\beta(k)}^i U_k^\dagger = \mu_{\beta(k), \tau_k(q)} U_k P_{\beta(k)}^i U_k^\dagger.$$

These are precisely the blocks of $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$. □

Appendix G

Symmetries and Virtual-Boundary Nondecay: Supporting Results

This appendix supports Chapter 11. It collects permutation and gauge bookkeeping, factor-system identities, stationary-boundary twist calculations, common trace-preserving gauges, and the finite-dimensional limit arguments used in the virtual-boundary nondecay results. The notation is introduced locally in each part, and every reference to the main development is given by its stable label.

The physical index set is $I = \{0, \dots, d-1\}$, the bond algebra is $M_D(\mathbb{C})$, and G denotes a group. All sums below are finite. The first section treats permutation actions on I ; the second isolates scalar uniqueness for two gauges of the same injective tensor; the third records the elementary factor-system bookkeeping used in the cohomology discussion of Chapter 11.

G.1 Permutation reindexing and identity bookkeeping

The linear-combination twist $\tilde{A}_g^i = \sum_j U(g)_{ij} A^j$ is the standard form used in the virtual representation theorem. The following alternative formulation replaces the matrix action by a permutation of the physical index; it is useful for symmetries that act by reordering basis states rather than by a general linear map.

Definition G.1.1 (Permutation symmetry datum). [Lean](#) [✓ Stmt](#) An *on-site symmetry datum* for a group G on a physical space of dimension d consists of a matrix-valued map $u : G \rightarrow M_d(\mathbb{C})$ and a physical-index action $\sigma : G \rightarrow \text{End}(\{0, \dots, d-1\})$.

Definition G.1.2 (Permutation-twisted tensor). [Lean](#) [✓ Stmt](#) Given an on-site symmetry datum (u, σ) , the *permutation-twisted tensor* at group element g is

$$(A^{\sigma_g})^i := A^{\sigma(g)(i)}. \tag{G.1}$$

In tensor-network notation the local tensor is unchanged and only the physical leg is relabelled:

$$\begin{array}{ccc}
\begin{array}{c} i \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_g} & \begin{array}{c} \sigma(g)(i) \\ | \\ \bullet \\ | \\ A \end{array} .
\end{array}$$

Lemma G.1.3 (Identity permutation twist). [Lean](#) [✓ Stmt](#) If $\sigma(1) = \text{id}$, then $(A^{\sigma_1})^i = A^i$ for all i .

Proof. [✓ Proof](#) The claim follows from $\sigma(1)(i) = i$ for all i . $\square$

Composition order. The linear-combination twist (Lemma 11.1.3) composes as $\widetilde{(\widetilde{A_h})_g} = \widetilde{A_{gh}}$, applying h first then g . The permutation twist below uses the same left-action convention $\sigma(gh) = \sigma(g) \circ \sigma(h)$, so $(A^{\sigma_g})^{\sigma_h} = A^{\sigma_{gh}}$; the inner permutation $\sigma(h)$ acts first on the index.

Lemma G.1.4 (Composition of permutation twists). [Lean](#) [✓ Stmt](#) If $\sigma(gh) = \sigma(g) \circ \sigma(h)$ for all $g, h \in G$, then

$$A^{\sigma_{gh}} = (A^{\sigma_g})^{\sigma_h}. \quad (\text{G.2})$$

Proof. [✓ Proof](#) Expanding both sides gives $(A^{\sigma_{gh}})^i = A^{\sigma(g)(\sigma(h)(i))} = ((A^{\sigma_g})^{\sigma_h})^i$. Diagrammatically this is just two consecutive relabellings of the same local tensor:

$$\begin{array}{ccccc}
\begin{array}{c} i \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_h} & \begin{array}{c} \sigma(h)(i) \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_g} & \begin{array}{c} \sigma(g)(\sigma(h)(i)) \\ | \\ \bullet \\ | \\ A \end{array} .
\end{array}$$

$\square$

G.1.1 Linear physical-index rotations

The same bookkeeping applies to a general matrix acting on the physical index.

Definition G.1.1.1 (Physical-index rotation). [Lean](#) [✓ Stmt](#) Given a matrix $M \in M_d(\mathbb{C})$ and an MPS tensor A , the *physical-index rotation* is the tensor $(A_M)^i := \sum_j M_{ij} A^j$. When the matrix is a unitary u , the rotation is written A_u .

Lemma G.1.1.2 (Matrix product vector under physical rotation). [Lean](#) [✓ Stmt](#) For a length- N physical configuration $s = (s_0, \dots, s_{N-1})$, physical rotation satisfies

$$\text{tr}((A_M)^{s_0} \dots (A_M)^{s_{N-1}}) = \sum_{t \in \{0, \dots, d-1\}^N} \left(\prod_{n=0}^{N-1} M_{s_n, t_n} \right) \text{tr}(A^{t_0} \dots A^{t_{N-1}}). \quad (\text{G.3})$$

Proof. [✓ Proof](#) Substitute $(A_M)^{s_n} = \sum_{t_n} M_{s_n, t_n} A^{t_n}$ at each site, distribute the finite sums through the ordered matrix product, and use linearity of the trace to obtain (G.3). $\square$

Lemma G.1.1.3 (Word expansion for a twisted tensor). [Lean](#) [✓ Stmt](#) For a word $b : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$,

$$A_g^{b_0} \dots A_g^{b_{L-1}} = \sum_{v : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}} \left(\prod_{k=0}^{L-1} U(g)_{b_k v_k} \right) A^{v_0} \dots A^{v_{L-1}}. \quad (\text{G.4})$$

Proof. [✓ Proof](#) Substitute $A_g^{b_k} = \sum_{v_k} U(g)_{b_k v_k} A^{v_k}$ into the ordered product. Repeated distributivity gives one summand for each function v , scalar coefficients commute past the matrices and multiply, and the remaining ordered product is $A^{v_0} \dots A^{v_{L-1}}$. $\square$

G.3 Cocycle equivalence and coboundaries

Let $\omega_1, \omega_2 : G \times G \rightarrow \mathbb{C}^\times$ be arbitrary scalar 2-cochains. The definitions of coboundary and cohomology are Definitions 11.2.1 and 11.2.2.

Theorem G.3.1 (Cohomology relation is an equivalence relation). [Lean](#) [✓ Stmt](#) *The relation “cohomologous to” is reflexive, symmetric, and transitive on the set of $\mathbb{C}^\times$ -valued scalar 2-cochains, and therefore also on the subset of cocycles.*

Proof. [✓ Proof](#) Reflexivity: take $\varphi = 1$. Symmetry: replace φ by φ^{-1} . Transitivity: compose the witnesses φ and ψ pointwise in (11.10). $\square$

Lemma G.3.2 (Coboundary iff cohomologous to the trivial cocycle). [Lean](#) [✓ Stmt](#) *A scalar 2-cochain ω is a coboundary if and only if $\omega \sim 1$, where $1(g, h) = 1$ for all g, h .*

Proof. [✓ Proof](#) Both directions follow by absorbing or introducing the factor $1(g, h) = 1$ via $x \cdot 1 = x$ in (11.10). $\square$

G.4 Stationary-boundary twist support

Fix an MPS tensor $A = (A^i)_{i=0}^{d-1}$, a physical matrix u , and a stationary boundary matrix $\Lambda \in M_D(\mathbb{C})$. The twisted transfer map $\mathcal{E}_u$ is Definition 11.3.1. The next constructions are virtual-boundary functionals; they should not be confused with the physical endpoint correlator of Definition 11.3.4.

Definition G.4.1 (Twisted companion tensor). [Lean](#) [✓ Stmt](#) For an MPS tensor A and a physical-index matrix u , define the companion tensor B_u by

$$B_u^n := \sum_{n'=0}^{d-1} \overline{u_{n'n}} A^{n'}. \quad (\text{G.5})$$

Lemma G.4.2 (Twisted transfer as a mixed transfer operator). [Lean](#) [✓ Stmt](#) *The twisted transfer map of A is the mixed transfer operator of the pair (A, B_u) :*

$$\mathcal{E}_u = F_{A, B_u}. \quad (\text{G.6})$$

Proof. [✓ Proof](#) Expand both definitions and rearrange the double sum. $\square$

Lemma G.4.3 (The companion tensor has the same transfer map). [Lean](#) [✓ Stmt](#) *If $uu^\dagger = \mathbb{1}_d$, then for every $X \in M_D(\mathbb{C})$ the companion family B_u satisfies*

$$\mathcal{E}_{B_u}(X) = \mathcal{E}_A(X).$$

Proof. [✓ Proof](#) the corresponding theorem in the companion quantum-channel volume [LTC26, “Unitary freedom in Kraus operators”] applies to the unitary combination defining B_u . Equivalently, expand $B_u^n = \sum_{n'} \overline{u_{n'n}} A^{n'}$. The coefficient of $A^j X (A^k)^\dagger$ in $\mathcal{E}_{B_u}(X)$ is

$$\sum_n \overline{u_{jn}} u_{kn} = (uu^\dagger)_{kj} = \delta_{kj}.$$

All off-diagonal terms vanish and the diagonal terms sum to $\mathcal{E}_A(X)$. $\square$

Definition G.4.4 (Virtual-boundary twist functional). [Lean](#) [✓ Stmt](#) For a boundary state Λ and boundary matrices $X, Y \in M_D(\mathbb{C})$, the *virtual-boundary twist functional* is

$$R_L(u; X, Y) := \text{tr}(\Lambda X \mathcal{E}_u^L(Y)). \quad (\text{G.7})$$

The special case $X = Y = \mathbb{1}$ is the stationary-boundary block-twist functional of Definition 11.3.2. The matrices X, Y act on the virtual level. In the physical string correlator of [PGWS⁺08] the string of twists is instead flanked by local operators x, y acting on physical sites; expanding such operators in the transfer picture produces particular virtual matrices, and [PGWS⁺08] argues that for injective tensors, physical operators on sufficiently many sites produce every virtual matrix. The results below are stated directly in this virtual-boundary form.

Lemma G.4.5 (Virtual-boundary nondecay prevents convergence to zero). [Lean](#) [✓ Stmt](#) If virtual-boundary nondecay holds for A, u, Λ , then for some boundary matrices X, Y the sequence $R_L(u; X, Y)$ does not tend to 0 as $L \rightarrow \infty$.

Proof. [✓ Proof](#) A uniform lower bound $c \leq \|R_L(u; X, Y)\|$ is incompatible with convergence to 0. $\square$

G.5 Kraus mixing, transfer expansions, and a common TP gauge

The spectral estimates in the main chapter compare A with B_u . They are placed in one trace-preserving gauge obtained from their common transfer map.

Lemma G.5.1 (Unitary Kraus mixing). [Lean](#) [✓ Stmt](#) If u is unitary, then replacing Kraus operators $\{A_i\}$ by their unitary mixtures $\{\sum_j u_{ij} A_j\}$ preserves the channel:

$$\sum_i \left(\sum_j u_{ij} A_j \right) Y \left(\sum_j u_{ij} A_j \right)^\dagger = \sum_i A_i Y A_i^\dagger. \quad (\text{G.8})$$

Proof. [✓ Proof](#) This is exactly the usual invariance of a Kraus decomposition under a unitary change of Kraus operators; see the corresponding theorem in the companion quantum-channel volume [LTC26, “Unitary freedom in Kraus operators”]. $\square$

Definition G.5.2 (Common trace-preserving gauge for a twisted pair). [Lean](#) [✓ Stmt](#) A *common trace-preserving gauge setup* for (A, u) consists of a tensor B equal to the companion B_u and satisfying $\mathcal{E}_B(X) = \mathcal{E}_A(X)$ for every $X \in M_D(\mathbb{C})$. It also includes a positive definite matrix σ fixed by the adjoint transfer map of B , its positive square root S , and gauged tensors A', B' satisfying

$$\begin{aligned} B &= B_u, & \mathcal{E}_B(X) &= \mathcal{E}_A(X), & \mathcal{E}_{B^\dagger}(\sigma) &= \sigma, & \sigma &> 0, \\ S &= \sigma^{1/2}, & S^\dagger &= S, & A'^i &= S A^i S^{-1}, & B'^i &= S B^i S^{-1}. \end{aligned} \quad (\text{G.9})$$

In addition,

$$\begin{aligned} S S^{-1} &= S^{-1} S = \mathbb{1}, & S^\dagger (S^\dagger)^{-1} &= \mathbb{1}, & (S^{-1})^\dagger &= S^{-1}, \\ \sum_i (A'^i)^\dagger A'^i &= \mathbb{1}, & \sum_i (B'^i)^\dagger B'^i &= \mathbb{1}, \end{aligned} \quad (\text{G.10})$$

together with irreducibility of both A' and B' . These identities complete the common trace-preserving gauge data.

Definition G.5.3 (Construction of the common trace-preserving gauge). [Lean](#) [✓ Stmt](#) Let $D > 0$. If A is injective, $uu^\dagger = \mathbb{1}_d$, and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, define the common trace-preserving gauge setup for (A, u) by choosing the unique positive definite stationary state of the adjoint channel and using the data in Definition G.5.2.

Proof. [✓ Proof](#) Injectivity makes $\mathcal{E}_A$ irreducible. The adjoint channel therefore has a positive definite stationary state σ . By Lemma G.4.3, A and B_u have the same transfer map, hence the same adjoint fixed point. Put $S = \sigma^{1/2}$. Positivity makes S invertible, and direct substitution gives

$$\begin{aligned} \sum_i (SA^i S^{-1})^\dagger (SA^i S^{-1}) &= \mathbb{1}, \\ \sum_i (SB_u^i S^{-1})^\dagger (SB_u^i S^{-1}) &= \mathbb{1}. \end{aligned} \tag{G.11}$$

Similarity preserves irreducibility, so the two gauged families have all the asserted properties. $\square$

Theorem G.5.4 (Twisted eigenvalues survive the common gauge). [Lean](#) [✓ Stmt](#) If $V \neq 0$ and $\mathcal{E}_u(V) = \lambda V$, then λ is an eigenvalue of the mixed transfer map $F_{A', B'}$ in any common trace-preserving gauge setup.

Proof. [✓ Proof](#) By Lemma G.4.2, the original equation is $F_{A, B_u}(V) = \lambda V$. Multiplying on the left and right by S and using $S^\dagger = S$ gives

$$F_{A', B'}(SVS) = \lambda SVS.$$

Since S is invertible, $SVS \neq 0$. $\square$

G.6 Boundary and limit support

Lemma G.6.1 (Spectral radius < 1 forces virtual-boundary decay). [Lean](#) [✓ Stmt](#) If $\rho(\mathcal{E}_u) < 1$, then, as $L \rightarrow \infty$, for all boundary matrices $X, Y, \Lambda \in M_D(\mathbb{C})$ one has $R_L(u; X, Y) \rightarrow 0$.

Proof. [✓ Proof](#) In finite dimension, $\rho(\mathcal{E}_u) < 1$ implies $\mathcal{E}_u^L \rightarrow 0$ in operator norm. Composing with the fixed trace pairing $Z \mapsto \text{tr}(\Lambda X Z)$ gives the stated scalar convergence. $\square$

G.7 Routine universality corollary

The following consequence uses only the virtual MPV-family symmetry predicate and the canonical fixed-point hypotheses from the main chapter.

Theorem G.7.1 (Virtual-boundary nondecay agrees for two injective MPV-symmetric tensors). [Lean](#) [✓ Stmt](#) If A and B are injective, both on-site symmetric under a unitary representation U , and if $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, $\mathcal{E}_B(\mathbb{1}) = \mathbb{1}$, Λ_A and Λ_B are positive definite, $\text{tr}(\Lambda_A) = 1$, $\text{tr}(\Lambda_B) = 1$, $\mathcal{E}_A^\dagger(\Lambda_A) = \Lambda_A$, and $\mathcal{E}_B^\dagger(\Lambda_B) = \Lambda_B$, then for every $g \in G$, virtual-boundary nondecay holds for A with twist $U(g)$ if and only if it holds for B . In particular, since Definition 11.4.1 implies on-site symmetry for both tensors, this theorem applies to tensors in the same SPT phase; it applies equally to symmetric tensors in different SPT phases.

Proof. [✓ Proof](#) Both directions follow from Theorem 11.4.4, which shows that virtual-boundary nondecay holds universally for injective symmetric tensors with canonical normalisations. $\square$

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