

Fundamental Theorem of Matrix Product States

A Formalization Blueprint

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Chapter 1

Introduction

The central result is the Fundamental Theorem for translation-invariant matrix product vectors. The basic object is a family of matrices $A = \{A^i\}_{i=0}^{d-1}$, which assigns to each chain length N the vector $|V^{(N)}(A)\rangle$ with coefficients indexed by $\sigma = (i_1, \dots, i_N)$,

$$V^{(N)}(A)_\sigma = \text{tr}(A^{i_1} \dots A^{i_N}). \quad (1.1)$$

The main argument, for translation-invariant tensors, follows the canonical-form and basis-of-normal-tensors argument of [CPGSV16, CPGSV21], with the single-block injective case anchored in the earlier representation theorem of [PGVWC07].

Notation

$\mathbb{C}, \mathbb{N}, \mathbb{Z}$	complex numbers, natural numbers, integers
$M_D(\mathbb{C}), \text{GL}_D(\mathbb{C})$	complex matrices and invertible complex matrices
$\text{tr}(X), \text{span}_{\mathbb{C}}(S)$	trace and complex linear span
$A = \{A^i\}_{i=0}^{d-1}$	a translation-invariant MPS tensor
$w = (i_1, \dots, i_L), A^w$	a word and the product $A^{i_1} \dots A^{i_L}$
$ V^{(N)}(A)\rangle$	the length- N matrix product vector
$V^{(N)}(A)_\sigma$	the coefficient of $ V^{(N)}(A)\rangle$ at σ
$O_{AB}(N)$	the bilinear overlap $\sum_\sigma V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}$
$\mathcal{E}_A(X)$	the transfer map $\sum_i A^i X (A^i)^\dagger$
$A^{[L]}$	the length- L blocked tensor

In canonical form one writes

$$A^i = \bigoplus_{j=1}^g \mu_j A_j^i, \quad (1.2)$$

$$B^i = \bigoplus_{k=1}^h \nu_k B_k^i. \quad (1.3)$$

The blocks in equations 1.2 and 1.3 are normal tensors. If $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every positive chain length N , then the bases of normal tensors have the same size. After relabelling the blocks of B , each matched pair satisfies

$$B_j^i = \zeta_j X_j A_j^i X_j^{-1}, \quad (1.4)$$

for some invertible matrix X_j and nonzero scalar ζ_j , and the equal case compares the coefficient power sums attached to the relabelled bases. The block gauges in equation 1.4 and the weight comparison then combine into a single gauge conjugacy of the two weighted block-diagonal tensors in equations 1.2 and 1.3. For a single injective block this reduces to the familiar conclusion $B^i = X A^i X^{-1}$ for all physical indices i , the single-block specialization of equation 1.4.

Chapters 2–10 carry out this argument. Chapter 2 fixes the matrix product vector notation, word evaluations, transfer maps, gauge equivalence, injectivity, blocking, and site-periodic tensor families. Chapter 3 proves the single-block theorem by trace pairings, linear extension, simplicity of matrix algebras, and Skolem–Noether. Chapters 4, 5, and 6 develop the transfer-map, Perron–Frobenius, and overlap-decay tools used to control normal blocks. The general theory of Schwarz inequalities and multiplicative domains is developed in the companion quantum-channel volume [LTC26, Chapter “Schwarz Inequalities and Multiplicative Domains”]. Chapters 7 and 8 supply the blocking, primitivity, irreducibility, and canonical-form reductions. Chapter 8 includes the invariant-projection splitting. Chapter 9 supplies bases of normal tensors and the power-sum comparison.

Chapter 11 uses the Fundamental Theorem to obtain virtual symmetry gauges, projective bond representations, and string-order criteria; its Kraus-mixing step uses the rectangular Kraus-freedom theorem from the companion quantum-channel volume [LTC26, “Rectangular Kraus freedom”], not any additional input for the Fundamental Theorem.

Beyond Chapter 11, the later chapters develop parent-Hamiltonian consequences, correlation estimates, tensor-network examples, matrix product operators and density operators, renormalization fixed points, and periodic tensor families. They also retain the matrix-product-specific normal-form and asymptotic results needed for these applications. General quantum-channel representations, positive-map examples, dynamical semigroups, operator convexity, and entropy are developed in the companion quantum-channel volume [LTC26, Chapters “Quantum Channels and Positive Maps,” “Channel Representations and Normal Forms,” “Positive but Not Completely Positive Maps,” “Operator Convexity and Jensen Inequalities,” “Quantum Dynamical Semigroups,” and “Quantum Entropy”].

Chapter 2

Matrix Product Vectors

Fix a physical dimension d and a bond dimension D . This chapter studies the tensors that generate matrix product vector (MPV) families. All tensors here are translation-invariant with periodic boundary conditions (PBC).

2.1 Basic definitions

Definition 2.1.1 (MPS tensor). [Lean](#) [✓ Stmt](#) A (translation-invariant, PBC) *MPS tensor* with physical dimension d and bond dimension D is a collection of matrices $\{A^i\}_{i=0}^{d-1}$, where $A^i \in M_D(\mathbb{C})$, indexed by a physical index $i \in \{0, \dots, d-1\}$. Such a tensor defines an MPV family. Diagrammatically,

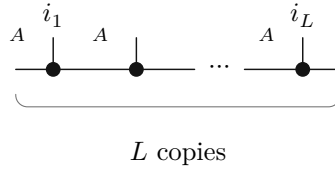


The black node denotes the tensor A , the horizontal legs are virtual, and the upper leg is the physical index i .

Definition 2.1.2 (Word evaluation). [Lean](#) [✓ Stmt](#) Given a word $w = (i_1, \dots, i_L) \in \{0, \dots, d-1\}^L$, the *word evaluation* is the matrix product

$$A^w := A^{i_1} A^{i_2} \dots A^{i_L} \in M_D(\mathbb{C}). \quad (2.1)$$

The empty word evaluates to the identity, $A^\emptyset = \mathbb{1}_D$. In tensor-network notation,



in which each black node denotes the same local tensor A , the virtual legs remain open, and the physical legs are labelled by the word $(i_1, \dots, i_L)$.

Lemma 2.1.3 (Word evaluation along a listed configuration). [Lean](#) [✓ Stmt](#) For a configuration $\sigma = (\sigma_0, \dots, \sigma_{N-1})$, let $w_\sigma = [\sigma_0, \dots, \sigma_{N-1}]$. Then $A^{w_\sigma} = A^{\sigma_0} A^{\sigma_1} \dots A^{\sigma_{N-1}}$.

Proof. ✓ Proof This follows by induction on N from the recursive definition of word evaluation. $\square$

Lemma 2.1.4 (Multiplicativity of word evaluation). Lean ✓ Stmt For any words w_1, w_2 , $A^{w_1 \cdot w_2} = A^{w_1} A^{w_2}$.

Proof. ✓ Proof The claim follows from the definition of A^w . $\square$

Lemma 2.1.5 (Centre of $M_D(\mathbb{C})$). Lean ✓ Stmt If a matrix $Z \in M_D(\mathbb{C})$ commutes with every element of $M_D(\mathbb{C})$, then there exists $\lambda \in \mathbb{C}$ such that $Z = \lambda \mathbb{1}_D$.

Proof. ✓ Proof Since Z commutes with every matrix unit, all off-diagonal entries of Z vanish and all diagonal entries are equal. $\square$

Lemma 2.1.6 (Rectangular amplified commutant). Lean ✓ Stmt Let $(A_i)_{i \in I}$ span $M_D(\mathbb{C})$, and let $C : \mathbb{C}^{m_2} \otimes \mathbb{C}^D \rightarrow \mathbb{C}^{m_1} \otimes \mathbb{C}^D$ satisfy, for every $i \in I$,

$$(\mathbb{1}_{m_1} \otimes A_i)C = C(\mathbb{1}_{m_2} \otimes A_i).$$

Then there is a matrix $F : \mathbb{C}^{m_2} \rightarrow \mathbb{C}^{m_1}$ such that $C = F \otimes \mathbb{1}_D$.

Proof. ✓ Proof Each (a, b) multiplicity block of C commutes with every A_i . Since the A_i span the full matrix algebra, Lemma 2.1.5 makes this block a scalar multiple of the identity. These scalars are the entries of F . $\square$

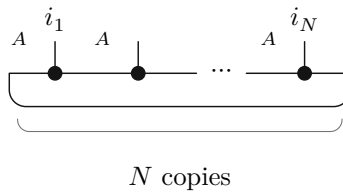
Definition 2.1.7 (Matrix product vector). Lean ✓ Stmt The *matrix product vector* (MPV) of a tensor A at system size N is the vector

$$|V^{(N)}(A)\rangle = \sum_{i_1, \dots, i_N=0}^{d-1} \text{tr}(A^{i_1} A^{i_2} \dots A^{i_N}) |i_1, \dots, i_N\rangle \in (\mathbb{C}^d)^{\otimes N}. \quad (2.2)$$

Equivalently, for a configuration $\sigma = (i_1, \dots, i_N) \in \{0, \dots, d-1\}^N$, we write

$$V^{(N)}(A)_\sigma := \text{tr}(A^{i_1} A^{i_2} \dots A^{i_N}). \quad (2.3)$$

The coefficient function $\sigma \mapsto V^{(N)}(A)_\sigma$ gives the components of the vector in (2.2); the displayed ket is the corresponding vector in $(\mathbb{C}^d)^{\otimes N}$. For a general word w , we also write $c_w(A) := \text{tr}(A^w)$. The *MPV family* generated by A is the collection $\mathcal{V}(A) = \{|V^{(N)}(A)\rangle\}_{N \geq 1}$. The coefficient $V^{(N)}(A)_\sigma$ is the periodic contraction

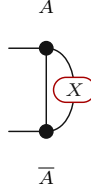


of N copies of the local tensor A , with the outer virtual legs closed by the trace.

Definition 2.1.8 (Transfer map). Lean ✓ Stmt The *transfer map* associated to a tensor A is the linear map $\mathcal{E}_A : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ defined by

$$\mathcal{E}_A(X) = \sum_{i=0}^{d-1} A^i X (A^i)^\dagger. \quad (2.4)$$

Diagrammatically, the transfer map is the double-layer contraction



in which the upper node denotes A , the lower node denotes $A^\dagger$, and the physical index is summed over between them.

Remark 2.1.9. ✓ Stmt The transfer map is automatically positive: if $X \geq 0$ then $\mathcal{E}_A(X) \geq 0$, because each summand $A^i X (A^i)^\dagger$ is positive semidefinite. The abstract framework is developed in the companion quantum-channel volume [LTC26, “Positive map”].

2.2 Site-periodic tensor families

Definition 2.2.1 (Site-periodic tensor family). Lean ✓ Stmt For a fixed period m , a *site-periodic tensor family* is a family of local tensors indexed by $\{0, \dots, m-1\}$, with local physical dimension d and bond dimension D .

Definition 2.2.2 (Periodic same-state and gauge-equivalence relations). Lean Lean ✓ Stmt For site-periodic tensors A, B at fixed period m :

- equality of the induced site-periodic states;
- cyclic gauge equivalence.

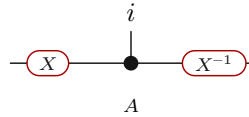
Lemma 2.2.3 (Equivalence structures for periodic relations). Lean Lean ✓ Stmt Both periodic relations are equivalence relations (reflexive, symmetric, transitive).

2.3 Gauge equivalence and same MPV

Definition 2.3.1 (Gauge equivalence). Lean ✓ Stmt Two tensors A and B of the same bond dimension D are *gauge equivalent* if there exists an invertible matrix $X \in \text{GL}_D(\mathbb{C})$ such that, for every $i \in \{0, \dots, d-1\}$,

$$B^i = X A^i X^{-1}. \quad (2.5)$$

In tensor-network notation, (2.5) is represented by



where the black node denotes the tensor A , the upper leg is the physical index i , and the two red side nodes denote the gauge matrices acting on the virtual legs.

Definition 2.3.2 (Same MPV). Lean ✓ Stmt Two tensors A and B (of the same bond dimension) generate the same MPV family if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every system size N .

Definition 2.3.3 (Same MPV - different bond dimensions). [Lean](#) [✓ Stmt](#) For tensors A and B of possibly different bond dimensions D_1 and D_2 , we write $\mathcal{V}(A) = \mathcal{V}(B)$ if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every word length N , including the empty word. This algebraic relation is used for exact direct-sum identities.

Definition 2.3.4 (Positive-length same MPV - different bond dimensions). [Lean](#) [✓ Stmt](#) For tensors A and B of possibly different bond dimensions, we write $\mathcal{V}^+(A) = \mathcal{V}^+(B)$ if $|V^{(N)}(A)\rangle = |V^{(N)}(B)\rangle$ for every $N > 0$.

Remark 2.3.5. [✓ Stmt](#) Definition 2.3.2 is the same-bond-dimension specialization of Definition 2.3.3. We keep both because later results use both the same-dimension and different-dimension forms.

Definition 2.3.6 (Nonzero proportional MPV). [Lean](#) [✓ Stmt](#) Two tensors A and B (of possibly different bond dimensions) generate nonzero proportional MPV families if for every $N > 0$ there is a nonzero scalar $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. This is the projective reading of the proportionality hypothesis in [CPGSV16, Theorem 2.10].

Definition 2.3.7 (Eventually nonzero proportional MPV). [Lean](#) [✓ Stmt](#) Two tensors A and B are eventually nonzero proportional if, for all sufficiently large N , there is a nonzero scalar $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. This auxiliary form is used with the eventual linear-independence statement Lemma E.1.10.

Lemma 2.3.8 (Elementary nonzero proportionality rules). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Nonzero proportionality implies eventual nonzero proportionality, is symmetric, and equality of MPV families is the special case with scalar 1, at every positive length.*

Proof. [✓ Proof](#) The eventual form follows by viewing an all-length statement as an eventual one. Symmetry follows by inverting the scalar c_N at each length, and equality is the case $c_N = 1$. $\square$

Lemma 2.3.9 (Coefficient isolation after complementary matching). [Lean](#) [✓ Stmt](#) *Let R be a ring, let V be an R -module, and let I and K be finite index sets. Let $T \subseteq I$, and let $\{u_i\}_{i \in I}$ and $\{v_k\}_{k \in K}$ be vectors in V . Suppose that, for every $i \notin T$, there are a scalar α_i and an index $\phi(i) \in K$ such that $u_i = \alpha_i v_{\phi(i)}$. If*

$$\sum_{i \in I} a_i u_i = \sum_{k \in K} b_k v_k$$

and the coproduct-indexed family $w: T \amalg K \rightarrow V$, defined by $w(\text{inl}(i)) = u_i$ and $w(\text{inr}(k)) = v_k$, is linearly independent, then $a_i = 0$ for every $i \in T$.

Proof. [✓ Proof](#) Substitute $u_i = \alpha_i v_{\phi(i)}$ for $i \notin T$ and collect the resulting coefficients over the fibres of ϕ . The given equality is then a linear relation in the combined family. Linear independence makes every coefficient vanish, in particular each a_i with $i \in T$. $\square$

Lemma 2.3.10 (Scaling of word evaluation). [Lean](#) [✓ Stmt](#) *Let ζA denote the tensor with matrices $\{\zeta A^i\}_{i=0}^{d-1}$. Then, for any word w , $(\zeta A)^w = \zeta^{|w|} A^w$, where $|w|$ is the length of w .*

Proof. [✓ Proof](#) Induction on w : the base case gives $\zeta^0 = 1$, and each step picks up one factor of ζ . $\square$

Lemma 2.3.11 (Scaling of MPV components). [Lean](#) [✓ Stmt](#) *Let ζA denote the tensor with matrices $\{\zeta A^i\}_{i=0}^{d-1}$. Then, for every N and every configuration $\sigma = (i_1, \dots, i_N)$, $V^{(N)}(\zeta A)_\sigma = \zeta^N V^{(N)}(A)_\sigma$.*

Proof. ✓ Proof Apply Lemma 2.3.10 to the word $w = (i_1, \dots, i_N)$ of length N , then take the trace; the scalar ζ^N pulls out by linearity. □

Definition 2.3.12 (Gauge-phase equivalence). Lean Lean ✓ Stmt Two tensors A and B are *gauge-phase equivalent* if there exist $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \in \mathbb{C} \setminus \{0\}$ such that, for every physical index i ,

$$B^i = \zeta X A^i X^{-1}. \quad (2.6)$$

Proof. ✓ Proof Gauge equivalence is the special case $\zeta = 1$. □

Definition 2.3.13 (Unit gauge-phase equivalence). Lean ✓ Stmt Two tensors are *unit gauge-phase equivalent* when the gauge-phase equation holds with $|\zeta| = 1$.

Lemma 2.3.14 (Unit gauge phase implies gauge phase). Lean ✓ Stmt *Unit gauge-phase equivalence implies gauge-phase equivalence.*

Proof. ✓ Proof A unit-modulus scalar is nonzero. □

Lemma 2.3.15 (Gauge-phase equivalence across equal bond dimensions). Lean Lean ✓ Stmt *Gauge-phase equivalence is symmetric after identifying equal bond dimensions. Moreover, suppose that A , B , and C have identified bond dimensions, that X_1 and X_2 are invertible, that $\zeta_1, \zeta_2 \neq 0$, and that, for every physical index i ,*

$$\begin{aligned} B^i &= \zeta_1 X_1 A^i X_1^{-1}, \\ C^i &= \zeta_2 X_2 A^i X_2^{-1}. \end{aligned}$$

Then

$$C^i = (\zeta_2 \zeta_1^{-1}) (X_2 X_1^{-1}) B^i (X_1 X_2^{-1}), \quad (2.7)$$

so B and C are gauge-phase equivalent after the corresponding identification of their bond dimensions.

Proof. ✓ Proof Symmetry follows by replacing (X, ζ) with (X^{-1}, ζ^{-1}) . Formula (2.7) follows by substituting the first gauge formula into the second. □

Lemma 2.3.16 (Gauge covariance of word evaluation). Lean ✓ Stmt *If $B^i = X A^i X^{-1}$ for all i , then $B^w = X A^w X^{-1}$ for every word w .*

Proof. ✓ Proof Induction on the word length, using $XX^{-1} = \mathbb{1}$. □

Theorem 2.3.17 (Gauge invariance of MPVs). Lean ✓ Stmt *If A and B are gauge equivalent, then they generate the same MPV family.*

Proof. ✓ Proof Let $B^i = X A^i X^{-1}$. By Lemma 2.3.16, $B^w = X A^w X^{-1}$ for every word w . Then $\text{tr}(B^w) = \text{tr}(X A^w X^{-1}) = \text{tr}(A^w)$ by cyclicity of the trace. □

Theorem 2.3.18 (Gauge-phase scaling of MPVs). Lean ✓ Stmt *If $B^i = \zeta X A^i X^{-1}$ for every physical index i , then for every system size N and configuration σ one has $V^{(N)}(B)_\sigma = \zeta^N V^{(N)}(A)_\sigma$.*

Proof. ✓ Proof Combine gauge covariance of word evaluation with scalar scaling, then take traces. □

2.4 Injectivity and normality

Definition 2.4.1 (Injectivity). [Lean](#) [✓ Stmt](#) A tensor A is *injective* if its matrices span the full matrix algebra: $\text{span}_{\mathbb{C}}\{A^i : i = 0, \dots, d-1\} = M_D(\mathbb{C})$.

The inverse-tensor characterization of injectivity says that A is injective exactly when there is a tensor A^{-1} such that

$$\sum_i (A^i)_{\alpha,\beta} (A^{-1,i})_{\alpha',\beta'} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \quad (2.8)$$

Thus (2.8) identifies the two pairs of virtual indices when A^{-1} is contracted with A through their common physical index. In shorthand, $A^{-1}A = \mathbb{1}$. This is the identity used in [CPGSV16, Section 2.3, lines 324–331].

Definition 2.4.2 (L -block injectivity). [Lean](#) [✓ Stmt](#) A tensor A is *L -block injective* if

$$\text{span}_{\mathbb{C}}\{A^{i_1} \dots A^{i_L} : (i_1, \dots, i_L) \in \{0, \dots, d-1\}^L\} = M_D(\mathbb{C}). \quad (2.9)$$

An injective tensor is 1-block injective.

Lemma 2.4.3 (One-site injectivity gives one-block injectivity). [Lean](#) [✓ Stmt](#) If the one-site matrices $\{A^i\}_{i=0}^{d-1}$ span $M_D(\mathbb{C})$, then the length-one word products span $M_D(\mathbb{C})$, so A is 1-block injective.

Proof. [✓ Proof](#) The length-one words are exactly the letters $i \in \{0, \dots, d-1\}$, and evaluating a length-one word gives the matrix A^i . Thus the length-one word span is the same span as in one-site injectivity. $\square$

Definition 2.4.4 (Normal tensor). [Lean](#) [✓ Stmt](#) A tensor A is *normal* [CPGSV21] if there exists a positive integer L_0 such that A is L_0 -block injective.

Lemma 2.4.5 (Gauge invariance of block injectivity). [Lean](#) [✓ Stmt](#) If A and B are gauge equivalent and A is L -block injective, then B is L -block injective.

Proof. [✓ Proof](#) Conjugation by an invertible matrix maps the span of the length- L words of A onto the span of the length- L words of B . Since conjugation is an automorphism of the full matrix algebra, fullness of the first span implies fullness of the second. $\square$

Theorem 2.4.6 (Gauge transport of normality). [Lean](#) [✓ Stmt](#) If A and B are gauge equivalent and A is normal, then B is normal.

Proof. [✓ Proof](#) Apply Lemma 2.4.5 at a blocking length for which A is block-injective. $\square$

Remark 2.4.7. [✓ Stmt](#) The algebraic characterization of normality in Definition 2.4.4 — eventual full matrix span, equivalently eventual block injectivity — is equivalent to the spectral characterization used in [CPGSV21, Definition IV.1]: after TP normalization, the transfer map $\mathcal{E}_A$ is a primitive channel, equivalently irreducible with trivial peripheral spectrum. This equivalence is proved in [SPGWC10, Proposition 3]. In what follows, the directions of the equivalence are supplied by Lemma 7.7.2.1, Theorem 7.7.4.2 for primitive normalized tensors with positive-definite fixed point, the Burnside route through Theorem 7.8.2 for the aperiodic algebraic criterion, and Theorem 8.9.2.1. In particular, every subsequent use of “normal” here refers to this single notion; Definition 8.9.2 employs the spectral formulation, but it describes the same class of tensors.

2.5 Canonical form

Definition 2.5.1 (Block-injective canonical form). [Lean](#) [✓ Stmt](#) Here, a *block-injective canonical form* for a tensor consists of:

1. a number of blocks $r \geq 1$,
2. bond dimensions $D_1, \dots, D_r$,
3. injective tensors $\{A_k^i\}_{i=0}^{d-1}$ with $A_k^i \in M_{D_k}(\mathbb{C})$ for each block $k \in \{1, \dots, r\}$,
4. scaling factors $\mu_1, \dots, \mu_r \in \mathbb{C}$.

The associated tensor is the block-diagonal tensor

$$A^i = \bigoplus_{k=1}^r \mu_k A_k^i. \quad (2.10)$$

This block-injective formulation of the canonical form is used in [\[PGVWC07\]](#); see also [\[CPGSV21\]](#) for the normal canonical form.

Remark 2.5.2. [✓ Stmt](#) Here we use the block-injective canonical form: each block is injective in the sense of Definition 2.4.1. This is stronger than the NT-block canonical form of [\[CPGSV21\]](#), where the blocks are only normal in the spectral sense. Chapter 8 treats the normal canonical form needed for the blocking-based multi-block theory, while the periodic irreducible-form extension is kept separate. This section isolates the stronger block-injective form used in the injective and block-injective arguments.

Definition 2.5.3 (Block projection). [Lean](#) [✓ Stmt](#) For a finite direct sum $\bigoplus_k \mathbb{C}^{D_k}$, the block projection P_j is the matrix that is the identity on the j -th summand and zero on all other summands.

Definition 2.5.4 (Block-diagonal matrix). [Lean](#) [✓ Stmt](#) A matrix on $\bigoplus_k \mathbb{C}^{D_k}$ is block diagonal if it is a direct sum of square matrices, one on each summand.

Lemma 2.5.5 (Off-block characterization). [Lean](#) [✓ Stmt](#) A matrix on $\bigoplus_k \mathbb{C}^{D_k}$ is block diagonal if and only if every entry from one summand to a distinct summand is zero.

Proof. [✓ Proof](#) The forward direction follows from the definition of a direct sum of matrices. Conversely, if all off-block entries vanish, the diagonal blocks reconstruct the original matrix. $\square$

Lemma 2.5.6 (Commutation with block projections). [Lean](#) [✓ Stmt](#) If a matrix on $\bigoplus_k \mathbb{C}^{D_k}$ commutes with every block projection P_j , then it is block diagonal.

Proof. [✓ Proof](#) Compare the (i, j) block of the identity $XP_j = P_jX$ for $i \neq j$. The right multiplication by P_j keeps that block, while the left multiplication by P_j kills it. Hence every off-block entry is zero, and Lemma 2.5.5 applies. $\square$

Definition 2.5.7 (Block-diagonal tensor from blocks). [Lean](#) [✓ Stmt](#) Given r blocks with bond dimensions $(D_k)_{k=1}^r$, per-block tensors $\{A_k^i\}$, and scaling factors $(\mu_k)_{k=1}^r$, the associated *block-diagonal tensor* is the tensor of total bond dimension $D = \sum_{k=1}^r D_k$ defined by

$$A^i = \bigoplus_{k=1}^r \mu_k A_k^i.$$

Lemma 2.5.8 (Word evaluation of a block-diagonal tensor). [Lean](#) [✓ Stmt](#) For a word w , the word evaluation of a block-diagonal tensor remains block diagonal:

$$A^w = \bigoplus_k \mu_k^{|w|} A_k^w. \quad (2.11)$$

Proof. [✓ Proof](#) Induct on the word and multiply block-diagonal matrices block by block; every letter contributes one scalar factor μ_k on the k -th block. $\square$

Theorem 2.5.9 (MPV decomposition). [Lean](#) [✓ Stmt](#) The MPV of a block-diagonal tensor decomposes as

$$|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(A_k)\rangle. \quad (2.12)$$

Equivalently, for each configuration σ one has

$$V^{(N)}(A)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(A_k)_\sigma.$$

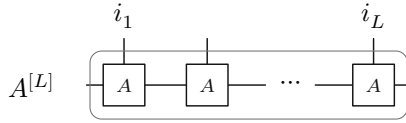
Proof. [✓ Proof](#) By Lemma 2.5.8, the full word evaluation is block diagonal with blocks $\mu_k^N A_k^w$ for a word w of length N . Taking the trace of a block-diagonal matrix gives the sum of the block traces: $\text{tr}(A^w) = \sum_k \mu_k^N \text{tr}(A_k^w)$. $\square$

2.6 Blocking

Definition 2.6.1 (Blocked tensor). [Lean](#) [✓ Stmt](#) Given a tensor A and a blocking length L , the L -blocked tensor is the tensor with physical index set $\{0, \dots, d-1\}^L$ (hence physical dimension d^L) and the same bond dimension D , whose matrices are indexed by words $(i_1, \dots, i_L) \in \{0, \dots, d-1\}^L$:

$$(A^{[L]})^{(i_1, \dots, i_L)} = A^{i_1} A^{i_2} \dots A^{i_L}. \quad (2.13)$$

Blocking coarse-grains L neighbouring sites into one tensor: the inner virtual bonds are contracted, the L physical legs merge into a single composite index, and the two outer bonds remain as the bond of $A^{[L]}$.



Lemma 2.6.2 (Blocked word evaluation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) If $w = (J_1, \dots, J_m)$ is a word in the blocked alphabet, where each J_k is an L -tuple in $\{0, \dots, d-1\}^L$, then concatenating the block words gives a word $\tilde{w}$ of length mL in the original alphabet and $(A^{[L]})^w = A^{\tilde{w}}$.

Proof. [✓ Proof](#) Induct on the blocked word w and split off the first block word. Lemma 2.1.4 turns concatenation of blocks into multiplication of the corresponding evaluations. $\square$

Lemma 2.6.3 (Blocking preserves MPV coefficients). [Lean](#) [✓ Stmt](#) For each basis state $\sigma = (\sigma_1, \dots, \sigma_N)$ of the blocked system, where each σ_k is an L -tuple in $\{0, \dots, d-1\}^L$, there exists a basis state $\tilde{\sigma} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_{NL})$ of the original system such that $V^{(N)}(A^{[L]})_\sigma = V^{(NL)}(A)_{\tilde{\sigma}}$.

Proof. ✓ Proof Concatenate the block words and apply the blocked word evaluation lemma (Lemma 2.6.2). $\square$

Theorem 2.6.4 (Blocking preserves same MPV). Lean ✓ Stmt *If A and B generate the same MPV family, then for any blocking length L , the blocked tensors $A^{[L]}$ and $B^{[L]}$ also generate the same MPV family.*

Proof. ✓ Proof By Lemma 2.6.3, each MPV coefficient of the blocked tensor $A^{[L]}$ equals a coefficient of A at the corresponding concatenated configuration. The same holds for $B^{[L]}$ (with the same concatenated configuration). Since A and B agree on all configurations, so do $A^{[L]}$ and $B^{[L]}$. $\square$

2.7 MPV overlap

Definition 2.7.1 (MPV as Hilbert-space vector). Lean ✓ Stmt We regard $|V^{(N)}(A)\rangle$ as an element of $\mathbb{C}^{d^N}$ indexed by configurations $\sigma \in \{0, \dots, d-1\}^N$, with σ -component $V^{(N)}(A)_\sigma$ as in Definition 2.1.7. This allows us to use the standard Euclidean inner product on $\mathbb{C}^{d^N}$.

Definition 2.7.2 (MPV inner product). Lean ✓ Stmt The *MPV inner product* between two tensors A and B at system size N is

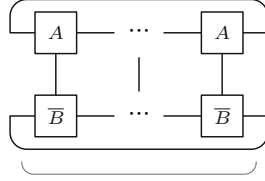
$$\langle V^{(N)}(A) | V^{(N)}(B) \rangle := \sum_{\sigma \in \{0, \dots, d-1\}^N} \overline{V^{(N)}(A)_\sigma} V^{(N)}(B)_\sigma, \quad (2.14)$$

i.e. the Euclidean inner product on $\mathbb{C}^{d^N}$ (conjugate-linear in the first argument).

Definition 2.7.3 (MPV overlap). Lean ✓ Stmt The *MPV overlap* between two tensors A (of bond dimension D_1) and B (of bond dimension D_2) at system size N is

$$O_{AB}(N) := \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}. \quad (2.15)$$

Diagrammatically, the overlap contracts the MPV ring of A against the conjugate ring of B along their shared physical indices, both virtual rings closed by the trace:



N sites

This pairing is linear in the first MPV coefficient and conjugate-linear in the second. By Lemma 2.7.4, it is the complex conjugate of the Hilbert-space inner product from Definition 2.7.2.

Lemma 2.7.4 (Overlap equals conjugate inner product). Lean ✓ Stmt $O_{AB}(N) = \overline{\langle V^{(N)}(A) | V^{(N)}(B) \rangle}$.

Proof. ✓ Proof The overlap sums $V_\sigma \overline{W_\sigma}$ while the inner product sums $\overline{V_\sigma} W_\sigma$. Complex conjugation swaps these. $\square$

Lemma 2.7.5 (Overlap determined by positive-length MPV). Lean ✓ Stmt If $V^{(N)}(A)_\sigma = V^{(N)}(A')_\sigma$ and $V^{(N)}(B)_\sigma = V^{(N)}(B')_\sigma$ for all $N > 0$ and all σ , then $O_{AB}(N) = O_{A'B'}(N)$ for all positive N .

Proof. ✓ Proof Direct pointwise rewriting: if the MPV coefficients agree for all positive chain lengths, the overlap sums are identical. $\square$

Chapter 3

The Single-Block Fundamental Theorem

An injective MPS tensor that generates the same matrix-product vectors as a second tensor of equal bond dimension is conjugate to it [CPGSV21, Corollary IV.5]: there is an invertible X with $B^i = XA^iX^{-1}$ for every physical index i . The argument is purely algebraic. Equality of the matrix-product vectors gives $\text{tr}(A^w) = \text{tr}(B^w)$ for every word w ; the length-two and length-three traces produce a linear map T with $T(A^i) = B^i$ that is multiplicative and nonzero; a nonzero multiplicative endomorphism of the simple algebra $M_D(\mathbb{C})$ is a unital automorphism; and by Skolem–Noether every automorphism of $M_D(\mathbb{C})$ is conjugation by an invertible X . Evaluating at A^i closes the proof. The same conjugation, one normal block at a time, is the central step of the multi-block theorem in Chapter 10.

3.1 The multiplicative linear extension

Nondegeneracy of the trace pairing on $M_D(\mathbb{C})$ is a generic algebra fact, proved in the companion quantum-channel volume [LTC26, “Nondegeneracy of the trace pairing”].

Lemma 3.1.1 (Same matrix-product vectors give trace agreement). [Lean](#) [✓ Stmt](#) *If A and B generate the same matrix-product vectors, then $\text{tr}(A^w) = \text{tr}(B^w)$ for every word w .*

Proof. [✓ Proof](#) By (2.3), the configuration $\sigma = w$ gives $V^{(|w|)}(A)_w = \text{tr}(A^w)$. Equality of the matrix-product vectors gives $V^{(N)}(A) = V^{(N)}(B)$ for all N , so $\text{tr}(A^w) = \text{tr}(B^w)$. $\square$

The trace conditions constrain every product of the matrices A^i . To bring the algebra structure of $M_D(\mathbb{C})$ to bear, record how a matrix pairs against the family $\{A^i\}$ under the trace.

Definition 3.1.2 (Trace pairing map). [Lean](#) [✓ Stmt](#) For a tensor A , the trace pairing map $\Phi_A : M_D(\mathbb{C}) \rightarrow \mathbb{C}^d$ is defined by

$$\Phi_A(M)_i = \text{tr}(MA^i). \quad (3.1)$$

Theorem 3.1.3 (The trace pairing map is injective for injective tensors). [Lean](#) [Lean](#) [✓ Stmt](#) *If A is injective, then $\ker \Phi_A = \{0\}$.*

Proof. [✓ Proof](#) If $\Phi_A(M) = 0$, then (3.1) gives $\text{tr}(MA^i) = 0$ for every i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so $\text{tr}(MN) = 0$ for all N , and $M = 0$ by the corresponding theorem in the companion quantum-channel volume [LTC26, “Nondegeneracy of the trace pairing”]. $\square$

Lemma 3.1.4 (Injectivity transfers across a range inclusion). Lean ✓ Stmt Let $\Phi, \Psi : V \rightarrow W$ be $\mathbb{C}$ -linear maps with V finite-dimensional. If $\ker \Phi = \{0\}$ and $\text{range } \Phi \subseteq \text{range } \Psi$, then $\ker \Psi = \{0\}$.

Proof. ✓ Proof By rank-nullity, $\ker \Phi = \{0\}$ gives $\dim \text{range } \Phi = \dim V$, and the range inclusion gives

$$\dim V = \dim \text{range } \Phi \leq \dim \text{range } \Psi \leq \dim V.$$

so $\dim \text{range } \Psi = \dim V$ and $\ker \Psi = \{0\}$. □

Theorem 3.1.5 (The linear extension exists and is unique). Lean ✓ Stmt If A is injective and A, B generate the same matrix-product vectors, then there is a unique linear map $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ with $T(A^i) = B^i$ for all i .

Proof. ✓ Proof Length-two trace agreement gives $\Phi_A(A^i) = \Phi_B(B^i)$ for every i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so $\text{range } \Phi_A \subseteq \text{range } \Phi_B$. With $\ker \Phi_A = \{0\}$ (Theorem 3.1.3), Lemma 3.1.4 gives $\ker \Phi_B = \{0\}$. Let g be a left inverse of Φ_B and set $T = g \circ \Phi_A$; then

$$T(A^i) = g(\Phi_A(A^i)) = g(\Phi_B(B^i)) = B^i.$$

Uniqueness holds because the $\{A^i\}$ span $M_D(\mathbb{C})$. □

Theorem 3.1.6 (The linear extension is multiplicative). Lean ✓ Stmt Under the hypotheses of Theorem 3.1.5, the map T satisfies $T(MN) = T(M)T(N)$ for all $M, N \in M_D(\mathbb{C})$.

Proof. ✓ Proof Length-two trace agreement gives $\Phi_B(T(A^i)) = \Phi_B(B^i) = \Phi_A(A^i)$; since the $\{A^i\}$ span $M_D(\mathbb{C})$, this extends to $\Phi_B \circ T = \Phi_A$, so $\text{range } \Phi_A = \text{range}(\Phi_B \circ T) \subseteq \text{range } \Phi_B$. With $\ker \Phi_A = \{0\}$ (Theorem 3.1.3), Lemma 3.1.4 makes Φ_B injective. Length-three trace agreement gives, for all i, j , and k ,

$$\Phi_B(T(A^i A^j))_k = \Phi_A(A^i A^j)_k = \text{tr}(A^i A^j A^k) = \text{tr}(B^i B^j B^k) = \Phi_B(B^i B^j)_k.$$

Hence injectivity of Φ_B yields $T(A^i A^j) = B^i B^j$. Extend in two stages: for fixed j , the maps $M \mapsto T(MA^j)$ and $M \mapsto T(M)B^j$ agree on the spanning $\{A^i\}$, hence $T(MA^j) = T(M)B^j$ for all M ; then for fixed M , the maps $N \mapsto T(MN)$ and $N \mapsto T(M)T(N)$ agree on the spanning $\{A^j\}$, hence $T(MN) = T(M)T(N)$ for all N . □

3.2 Inner automorphism and the single-block theorem

The rigidity of the full matrix algebra now finishes the proof. A nonzero multiplicative map on $M_D(\mathbb{C})$ is forced to be bijective (simplicity), unit-preserving (surjectivity), and conjugation by a single invertible matrix (Skolem–Noether). The next four results establish these steps and the nonvanishing of T , which the final theorem combines into the gauge X .

Theorem 3.2.1 (A nonzero multiplicative endomorphism of $M_D(\mathbb{C})$ is bijective). Lean ✓ Stmt A nonzero multiplicative $\mathbb{C}$ -linear endomorphism of $M_D(\mathbb{C})$ is bijective.

Proof. ✓ Proof The kernel of a multiplicative linear map is a two-sided ideal. The algebra $M_D(\mathbb{C})$ is simple, so the kernel is $\{0\}$ or all of $M_D(\mathbb{C})$; the latter forces the map to be zero. Hence the map is injective, and surjective by finite-dimensionality. □

Lemma 3.2.2 (A surjective multiplicative map preserves the unit). [Lean](#) [✓ Stmt](#) *If $T : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ is a surjective multiplicative $\mathbb{C}$ -linear map, then $T(\mathbb{1}) = \mathbb{1}$, so T is a $\mathbb{C}$ -algebra homomorphism.*

Proof. [✓ Proof](#) Choose X with $T(X) = \mathbb{1}$. Then

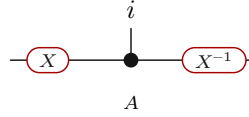
$$T(\mathbb{1}) = T(X)T(\mathbb{1}) = T(X\mathbb{1}) = T(X) = \mathbb{1}.$$

□

Lemma 3.2.3 (An injective tensor forces a nonzero partner). [Lean](#) [✓ Stmt](#) *Let $D \geq 1$. If A is injective and A, B generate the same matrix-product vectors, then the B^i are not all zero.*

Proof. [✓ Proof](#) If $B^i = 0$ for every i , then Lemma 3.1.1 on length-one words gives $\text{tr}(A^i) = 0$ for all i . The $\{A^i\}$ span $M_D(\mathbb{C})$, so the trace functional vanishes identically, contradicting $\text{tr}(\mathbb{1}) = D \neq 0$. □

Theorem 3.2.4 (Single-block Fundamental Theorem). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor and B a tensor of the same bond dimension. If A and B generate the same matrix-product vectors, then there is $X \in \text{GL}_D(\mathbb{C})$ with $B^i = XA^iX^{-1}$ for all i . In tensor-network notation the gauge conjugates the local tensor on its virtual legs, with the physical index i on the upper leg:*



Proof. [✓ Proof](#) Theorem 3.1.5 gives the unique linear T with $T(A^i) = B^i$, and Theorem 3.1.6 makes it multiplicative. It is nonzero: otherwise $B^i = 0$ for all i , against Lemma 3.2.3. By Theorem 3.2.1, T is bijective, and by Lemma 3.2.2 it fixes $\mathbb{1}$, so T is a $\mathbb{C}$ -algebra automorphism. Theorem [LTC26, “Skolem–Noether”] gives $X \in \text{GL}_D(\mathbb{C})$ with $T(M) = XM X^{-1}$; evaluating at $M = A^i$ gives $B^i = XA^iX^{-1}$. □

Chapter 4

Transfer Maps and Quantum-Channel Interfaces

This chapter records the transfer-map interfaces used by the MPS development. The underlying theory is developed in the companion quantum-channel volume [LTC26, “Positive map” and “Quantum channel”].

4.1 Irreducibility

Theorem 4.1.1 (Positive-definite transport by a coisometry). [Lean](#) [✓ Stmt](#) *Let $A > 0$ be a square complex matrix and let V be a possibly rectangular matrix satisfying $VV^\dagger = \mathbb{1}$. Then*

$$VAV^\dagger > 0. \quad (4.1)$$

Proof. [✓ Proof](#) For a row vector x , one has

$$x(VAV^\dagger)x^\dagger = (xV)A(xV)^\dagger. \quad (4.2)$$

Moreover, $xV = 0$ implies

$$x = x(VV^\dagger) = 0. \quad (4.3)$$

Thus every nonzero x gives $xV \neq 0$, and positive definiteness of A makes the displayed quadratic form strictly positive. $\square$

Theorem 4.1.2 (Injectivity implies irreducibility). [Lean](#) [✓ Stmt](#) *If A is an injective MPS tensor, then its transfer map $\mathcal{E}_A$ is irreducible.*

Proof. [✓ Proof](#) If P is an invariant projection for $\mathcal{E}_A$, then $(\mathbb{1} - P)A^iP = 0$ for all i . Since the $\{A^i\}$ span $M_D(\mathbb{C})$, $(\mathbb{1} - P)MP = 0$ for all M , so $P = 0$ or $P = \mathbb{1}$. $\square$

4.2 Transfer maps

Theorem 4.2.1 (Transfer map is CP). [Lean](#) [✓ Stmt](#) *The transfer map $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ is completely positive.*

Proof. ✓ Proof It is already written in the Kraus form (established in the companion quantum-channel volume [LTC26, “Completely positive map”]), with Kraus operators $\{A^i\}$. $\square$

Theorem 4.2.2 (Transfer matrix of an MPS transfer map). Lean ✓ Stmt *For an MPS tensor $A = \{A^i\}_{i=0}^{d-1}$, the matrix representing $\mathcal{E}_A$ under vectorization is*

$$K_A = \sum_{i=0}^{d-1} \overline{A^i} \otimes A^i.$$

Proof. ✓ Proof Apply the standard vectorization identity $\text{vec}(AXB) = (B^\top \otimes A) \text{vec}(X)$ to each summand $A^i X (A^i)^\dagger$ and sum over the physical index. $\square$

Lemma 4.2.3 (Trace adjointness of a transfer map). Lean ✓ Stmt *The trace-adjoint identity of Theorem [LTC26, “Trace adjointness of a Kraus map”] holds when the Kraus family is the family of matrices of an MPS tensor and the two Kraus maps are written as transfer maps.*

Proof. ✓ Proof

Identify the two Kraus maps with the corresponding transfer maps. $\square$

Lemma 4.2.4 (Transfer map under scalar multiplication). Lean ✓ Stmt *Let A be an MPS tensor and let $w \in \mathbb{C}$. Then, for every bond matrix X ,*

$$\mathcal{E}_{wA}(X) = w\bar{w} \mathcal{E}_A(X).$$

Proof. ✓ Proof Expand the transfer map and pull the scalar w and its conjugate out of each summand. $\square$

Lemma 4.2.5 (Left-canonical transfer maps preserve trace). Lean ✓ Stmt *If A is left canonical, then, for every bond matrix X ,*

$$\text{tr}(\mathcal{E}_A(X)) = \text{tr}(X).$$

Proof. ✓ Proof Cyclicity of trace gives

$$\text{tr}(\mathcal{E}_A(X)) = \text{tr}\left(\left(\sum_i (A^i)^\dagger A^i\right) X\right) = \text{tr}(X),$$

since left canonicity makes the sum in parentheses equal to the identity. $\square$

Theorem 4.2.6 (Transfer map is a channel). Lean ✓ Stmt *If $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, then the transfer map $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ is a quantum channel (CPTP).*

Proof. ✓ Proof Complete positivity is Theorem 4.2.1. Trace preservation follows by cyclicity and the normalization hypothesis:

$$\text{tr}(\mathcal{E}_A(X)) = \sum_i \text{tr}(A^i X (A^i)^\dagger) = \text{tr}\left(\left(\sum_i (A^i)^\dagger A^i\right) X\right) = \text{tr}(X).$$

$\square$

Chapter 5

Perron–Frobenius Theory for Channels and Transfer Maps

This chapter establishes elementary fixed-point existence for quantum channels and the Perron–Frobenius theory: existence, positive definiteness, and uniqueness of fixed points for injective and irreducible transfer maps, canonical gauge constructions, and the spectral-radius identification of Perron eigenvalues. The ergodicity of irreducible channels and the CP-map irreducibility criterion modeled on Wolf’s spectral characterization are developed in the companion quantum-channel volume [LTC26, “Cesàro convergence for irreducible channels” and “Irreducibility iff the restricted CP spectral properties hold”]. The conceptual source is Wolf’s treatment of irreducible positive maps [Wol12, Chapter 6, especially Theorems 6.2–6.5 and Corollary 6.3] and [EHK78].

5.1 Positive definiteness

Theorem 5.1.1 (Transfer-map PSD fixed point — irreducible version). [Lean](#) [✓ Stmt](#) *Let A be an MPS tensor whose transfer map $\mathcal{E}_A$ is irreducible. If $\rho \geq 0$, $\rho \neq 0$, and $\mathcal{E}_A(\rho) = \rho$, then ρ is positive definite.*

Proof. [✓ Proof](#) The transfer map is completely positive, so Theorem [LTC26, “PSD fixed point is positive definite — irreducible version”] applies. $\square$

Theorem 5.1.2 (PSD fixed point is positive definite). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor and let $\rho \geq 0$, $\rho \neq 0$, be a fixed point of $\mathcal{E}_A$. Then ρ is positive definite.*

Proof. [✓ Proof](#) Injectivity implies irreducibility of the transfer map. Apply Theorem 5.1.1. $\square$

5.2 Uniqueness

Theorem 5.2.1 (Irreducible fixed-point uniqueness). [Lean](#) [✓ Stmt](#) *Let A be an MPS tensor whose transfer map is irreducible. If $\rho, \sigma \geq 0$ are fixed points of $\mathcal{E}_A$ and $\rho \neq 0$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.*

Proof. [✓ Proof](#) The transfer map is completely positive, so Theorem [LTC26, “Irreducible fixed-point uniqueness — CP version”] applies. $\square$

Theorem 5.2.2 (Uniqueness of PSD fixed point). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor. If $\rho, \sigma \geq 0$ are nonzero fixed points of $\mathcal{E}_A$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.*

Proof. [✓ Proof](#) Injectivity implies irreducibility of the transfer map. Apply Theorem 5.2.1. $\square$

5.3 Existence and the Perron–Frobenius theorem

Theorem 5.3.1 (Existence of PSD fixed point). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, so that $\mathcal{E}_A$ is trace-preserving. Then there exists $\rho \geq 0$, $\rho \neq 0$, with $\mathcal{E}_A(\rho) = \rho$.*

Proof. [✓ Proof](#) Under the normalization $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, the transfer map is a quantum channel (Theorem 4.2.6). The Cesàro fixed-point theorem (the corresponding theorem in the companion quantum-channel volume [LTC26, “Cesàro fixed-point theorem”]) then provides the fixed point; injectivity is not needed for this step. $\square$

Theorem 5.3.2 (Quantum Perron–Frobenius). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an injective MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then the transfer map $\mathcal{E}_A$ has a unique positive semidefinite fixed point ρ up to scaling, and ρ is positive definite.*

Proof. [✓ Proof](#) Combine Theorems 5.3.1, 5.1.2, and 5.2.2. $\square$

Theorem 5.3.3 (Quantum Perron–Frobenius for irreducible transfer maps). [Lean](#) [✓ Stmt](#) *Assume $D \geq 1$. Let A be an MPS tensor such that its transfer map is irreducible and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, so that $\mathcal{E}_A$ is trace-preserving. Then $\mathcal{E}_A$ has a unique positive definite fixed point, up to scalar multiple.*

Proof. [✓ Proof](#) Existence is the channel fixed-point theorem (Theorem 5.3.1). Irreducibility upgrades the fixed point to positive definite, and Theorem 5.2.1 gives uniqueness up to scalar multiple. $\square$

5.4 Right- and left-canonical gauges

Definition 5.4.1 (Left-canonical normalization). [Lean](#) [✓ Stmt](#) *An MPS tensor $A = (A^i)_i$ is left-canonical when*

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}.$$

Equivalently, its Kraus map is trace-preserving.

Theorem 5.4.2 (Left-canonical (trace-preserving) gauge). [Lean](#) [Lean](#) [✓ Stmt](#) *Let σ be a fixed point of the adjoint transfer map $\sum_i (A^i)^\dagger \sigma A^i = \sigma$, and let S be invertible with $S^\dagger S = \sigma$. Define $A'^i = S A^i S^{-1}$. Then $\sum_i (A'^i)^\dagger A'^i = \mathbb{1}$, i.e. the gauged Kraus map is trace-preserving. This is the left-canonical normalization.*

Proof. [✓ Proof](#) Conjugating each summand gives $(S A^i S^{-1})^\dagger (S A^i S^{-1}) = (S^\dagger)^{-1} (A^i)^\dagger \sigma A^i S^{-1}$. The adjoint fixed-point substitution leaves $(S^\dagger)^{-1} \sigma S^{-1}$, and $\sigma = S^\dagger S$ cancels the outer factors. See Section A.1 for the full calculation. $\square$

Theorem 5.4.3 (Square-root trace-preserving gauge from an adjoint fixed point). [Lean](#) [✓ Stmt](#)
 Let σ be positive definite and satisfy $\sum_i (A^i)^\dagger \sigma A^i = \sigma$. Define

$$B^i = \sigma^{1/2} A^i \sigma^{-1/2}. \quad (5.1)$$

Then B is trace-preserving: $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

Proof. [✓ Proof](#)

This is the specialization $K_i = A^i$ of Theorem [\[LTC26\]](#), “Trace-preserving gauge from an adjoint fixed point, Kraus form”. Each conjugated summand is $(B^i)^\dagger B^i = \sigma^{-1/2} (A^i)^\dagger \sigma A^i \sigma^{-1/2}$. Substituting $\sum_i (A^i)^\dagger \sigma A^i = \sigma$ and cancelling the outer inverse square roots gives $\mathbb{1}$. The complete calculation appears in Section [A.1](#). $\square$

Theorem 5.4.4 (Square-root trace-preserving gauge from an adjoint eigenvector). [Lean](#) [✓ Stmt](#)
 Let σ be positive definite and satisfy $\sum_i (A^i)^\dagger \sigma A^i = r\sigma$ for $r > 0$. Define

$$B^i = \sigma^{1/2} (r^{-1/2} A^i) \sigma^{-1/2}. \quad (5.2)$$

Then B is trace-preserving: $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

Proof. [✓ Proof](#) Rescale A by $r^{-1/2}$. The adjoint transfer map is quadratic in the tensor, so the eigenvector equation becomes a fixed-point equation. Theorem [5.4.3](#) then gives the result. $\square$

Remark 5.4.5. [✓ Stmt](#) The right-canonical gauge of Theorem [\[LTC26\]](#), “Right-canonical (unital) gauge” and the left-canonical gauge of Theorem [5.4.2](#) are generally different similarity transforms: one is built from a fixed point of $\mathcal{E}_A$, the other from a fixed point of the adjoint transfer map. This section does not claim that a single gauge makes the transfer map simultaneously unital and trace-preserving.

5.5 Perron–Frobenius eigenvector existence

This section establishes the existence of a positive definite eigenvector for the adjoint transfer map of an irreducible MPS tensor, and uses it to construct a TP-gauge normalization. The PSD-eigenvector step corresponds to [\[Wol12, Theorem 6.5\]](#) (general positive maps) and [\[EHK78\]](#); the upgrade to positive definiteness uses the irreducible-map theory of [\[Wol12, Theorem 6.3\]](#). The TP-gauge application follows [\[CPGSV16, Appendix A\]](#).

Theorem 5.5.1 (Irreducibility of the transfer map of an irreducible tensor). [Lean](#) [✓ Stmt](#) If an MPS tensor A is irreducible, then its transfer map $\mathcal{E}_A$ is irreducible.

Proof. [✓ Proof](#) If P is a nontrivial invariant projection for $\mathcal{E}_A$, then

$$(\mathbb{1} - P)A^i P = 0$$

for every i , which is exactly a nontrivial invariant projection for the family $\{A^i\}$, contradicting irreducibility of A . $\square$

Theorem 5.5.2 (Positive-definite adjoint eigenvector for irreducible tensors). [Lean](#) [✓ Stmt](#) Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive definite matrix σ and a positive real $r > 0$ such that $\mathcal{E}_A^\dagger(\sigma) = r\sigma$.

This combines the eigenvector-existence part of [\[Wol12, Theorem 6.5\]](#) with the irreducible upgrade to positive definiteness ([\[Wol12, Theorem 6.3\(2\)–\(3\)\]](#)); the application to the adjoint transfer map follows [\[CPGSV16, Appendix A\]](#).

Proof. ✓ Proof By Theorem 5.5.1, $\mathcal{E}_A$ is irreducible, and since $\mathcal{K}_A = \mathcal{E}_A$ (the corresponding theorem in the companion quantum-channel volume [LTC26, “Kraus-map form of the transfer map”]), its Kraus map $\mathcal{K}_A$ is irreducible too (the corresponding theorem in the companion quantum-channel volume [LTC26, “Irreducibility in Kraus-map notation”]). Applying Theorem [LTC26, “Positive-definite adjoint eigenvector for irreducible Kraus families”] to $K_i = A^i$ gives a positive definite σ and a positive real $r > 0$ with $\sum_i (A^i)^\dagger \sigma A^i = r\sigma$, i.e. $\mathcal{E}_A^\dagger(\sigma) = r\sigma$. $\square$

Theorem 5.5.3 (TP-gauge existence for irreducible tensors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive real r , a positive definite matrix σ , and a tensor B gauge-equivalent to $r^{-1/2}A$ such that*

$$B^i = \sigma^{1/2} (r^{-1/2} A^i) \sigma^{-1/2}, \quad (5.3)$$

and $\sum_i (B^i)^\dagger B^i = \mathbb{1}$.

This is the “spectral rescaling + TP gauge” step of [CPGSV16, Appendix A]. The similarity is generally non-unitary.

Proof. ✓ Proof By Theorem 5.5.2, obtain σ positive definite and $r > 0$ with $\mathcal{E}_A^\dagger(\sigma) = r\sigma$. Set $c = r^{-1/2}$ and $A'^i = cA^i$. Then $\mathcal{E}_{A'}^\dagger(\sigma) = \sigma$. By the square-root trace-preserving eigenvector gauge construction (Theorem 5.4.4), the tensor defined by (5.3) satisfies $\sum_i (B^i)^\dagger B^i = \mathbb{1}$. Lemma A.1.1 gives the required gauge equivalence to $r^{-1/2}A$. $\square$

Theorem 5.5.4 (Unital-gauge existence for irreducible tensors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and some $A^i \neq 0$. Then there exist a positive real r , a positive definite matrix ρ , and a tensor B gauge-equivalent to $r^{-1/2}A$ such that*

$$B^i = r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}, \quad (5.4)$$

and $\sum_i B^i (B^i)^\dagger = \mathbb{1}$.

This is the Perron–Frobenius unital-gauge orientation used in [PGVWC07, Theorem 4, lines 765–770], stated for one irreducible nonzero block.

Proof. ✓ Proof Apply Theorem 5.5.2 to the conjugate-transposed Kraus family. Irreducibility transfers to that family, and the nonzero Kraus hypothesis is preserved by taking adjoints. Translating the resulting adjoint eigenvector equation back gives $\mathcal{E}_A(\rho) = r\rho$ with ρ positive definite and $r > 0$. The unital gauge theorem then gives the representative (5.4), and the same square-root similarity gives gauge equivalence to $r^{-1/2}A$. $\square$

Chapter 6

Peripheral Channel Structure and Transfer-Operator Gaps

This chapter studies two spectral phenomena for finite-dimensional quantum channels and MPS transfer maps, following [PGVWC07] and [Wol12].

The first is *peripheral* structure: for an irreducible unital Schwarz map with a faithful adjoint fixed point, the eigenvalues on the unit circle form a finite cyclic group (the corresponding theorem in the companion quantum-channel volume [LTC26, “Peripheral eigenvalues: cyclic structure for Kraus maps”]), and under trace preservation the order of this group divides the bond dimension D (Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”]). Removing this period by unitary conjugation and cyclic corner projections produces the periodicity-free cyclic-sector decomposition (Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”]) used by the canonical-form reduction of Chapter 8.

The second is the *mixed transfer operator* F_{AB} of two MPS tensors A and B , and its spectral radius $\rho_{\text{spec}}(F_{AB})$. We show that $\rho_{\text{spec}}(F_{AB}) \leq 1$ under trace-preserving normalization (Theorem 6.3.1), that equality forces A and B to be gauge-phase equivalent (Theorem 6.3.4), and that mismatched bond dimensions always force a strict gap (Theorem 6.4.2). Since the MPV overlap $O_{AB}(N)$ equals the operator trace $\text{Tr}(F_{AB}^N)$ (Theorem 6.2.1), these gaps translate directly into the overlap-decay theorems (Theorem 6.5.5 and its rectangular and irreducible-trace-preserving variants) used throughout the fundamental theorem. The complementary case — a single tensor whose transfer map has a trivial peripheral spectrum away from the fixed-point projection — gives the rank-one Perron limit $\text{Tr}(\mathcal{E}_A^n) \rightarrow 1$ (Theorem 6.10.1) and hence primitive self-overlap convergence $O_{AA}(N) \rightarrow 1$ (Theorem 6.10.2).

Throughout, $\rho_{\text{spec}}(\cdot)$ denotes the spectral radius of a linear endomorphism of a finite-dimensional matrix space: $\rho_{\text{spec}}(F_{AB}) < 1$ for a mixed transfer operator and $\rho_{\text{spec}}(\mathcal{E}_A - P) < 1$ for a transfer map after removing its fixed-point projection P . The unqualified phrase *spectral gap* is reserved for the energy gap of a many-body Hamiltonian, discussed later in the full blueprint.

The algebraic mixed-transfer identities in the first two sections do not use normalization. Starting with the spectral-radius estimates, we assume the trace-preserving normalization

$$\sum_i (A^i)^\dagger A^i = \mathbb{1}. \quad (6.1)$$

Right-canonical (unital) and left-canonical (trace-preserving) gauges are generally different similarity transforms of the same tensor, recorded in Remark 6.3.5; both are used below, depending on which fixed point — of the transfer map or of its adjoint — drives the gauge.

The proofs of the technical mixed-transfer identities, Frobenius-norm estimates, gauge-rigidity intertwiners, spectral-radius decay estimates, the rank-one Perron projection, and the cyclic-decomposition machinery removing peripheral periodicity are collected in Appendix B.

6.1 Mixed transfer operator

Definition 6.1.1 (Mixed transfer operator). Lean ✓ Stmt The *mixed transfer operator* (or *cross transfer operator*) for two MPS tensors A and B of the same bond dimension D is the linear map $F_{AB} : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ defined by

$$F_{AB}(X) := \sum_{i=0}^{d-1} A^i X(B^i)^\dagger. \quad (6.2)$$

Definition 6.1.2 (Rectangular mixed transfer operator). Lean ✓ Stmt For MPS tensors A of bond dimension D_1 and B of bond dimension D_2 , possibly with $D_1 \neq D_2$, the *rectangular mixed transfer operator* is the map $F_{AB}^{\text{rect}} : M_{D_1 \times D_2}(\mathbb{C}) \rightarrow M_{D_1 \times D_2}(\mathbb{C})$ defined by

$$F_{AB}^{\text{rect}}(X) := \sum_{i=0}^{d-1} A^i X(B^i)^\dagger. \quad (6.3)$$

The self-transfer identity $F_{AA} = \mathcal{E}_A$, the iterated word expansion of F_{AB} , and the trace-expansion lemma over matrix units are proved in Section B.1.

6.2 MPV overlap as transfer trace

Recall from (2.15) that $O_{AB}(N) = \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma}$. This section rewrites that scalar as the operator trace of the mixed transfer power.

Lemma 6.2.1 (Overlap equals transfer trace). Lean ✓ Stmt For tensors A and B of bond dimension $D \geq 1$,

$$O_{AB}(N) = \text{Tr}(F_{AB}^N). \quad (6.4)$$

Proof. ✓ Proof Section B.1 expands the operator trace over matrix units (Lemma [LTC26, “Trace expansion over matrix units”]) and the transfer power by words (Lemma B.1.2); reassembling the sums over matrix-unit indices gives

$$\text{Tr}(F_{AB}^N) = \sum_{\sigma} \text{tr}(A^\sigma) \overline{\text{tr}(B^\sigma)} = \sum_{\sigma} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma} = O_{AB}(N).$$

The rectangular case, Lemma 6.2.2, is the same argument over the rectangular matrix-unit basis. $\square$

Lemma 6.2.2 (Rectangular overlap as transfer trace). Lean ✓ Stmt For tensors A of bond dimension $D_1 \geq 1$ and B of bond dimension $D_2 \geq 1$,

$$O_{AB}(N) = \text{Tr}((F_{AB}^{\text{rect}})^N). \quad (6.5)$$

Proof. ✓ Proof The detailed word-index calculation is the same as the proof of Theorem 6.2.1, over the rectangular matrix-unit basis; see Section B.1. $\square$

6.3 Eigenvalue bound and transfer-operator gap

The rectangular eigenvalue bound is the MPS formulation of the generic bound for trace-preserving mixed Kraus maps. Under the identification of the rectangular mixed transfer with the mixed Kraus map, the two tensor normalizations are precisely the trace-preserving conditions. For the rigidity theorem, one gauges A and B separately using positive fixed points of their transfer maps to obtain unital Kraus families, and then applies the Kadison–Schwarz equality argument ([Wol12, Theorem 5.3]) to a block embedding of a mixed-transfer eigenvector.

Theorem 6.3.1 (Eigenvalue bound). Lean ✓ Stmt *Let A and B be normalized MPS tensors. Every eigenvalue μ of F_{AB} satisfies $|\mu| \leq 1$.*

Proof. ✓ Proof This is the equal-bond-dimension case $D_1 = D_2 = D$ of Theorem 6.3.2. The argument uses only the trace-preserving normalization (6.1) and the general eigenvalue bound for trace-preserving Kraus maps; it does not appeal to Perron–Frobenius theory. $\square$

Theorem 6.3.2 (Rectangular eigenvalue bound). Lean ✓ Stmt *Let A and B be normalized MPS tensors of bond dimensions $D_1 \geq 1$ and $D_2 \geq 1$. Every eigenvalue μ of F_{AB}^{rect} satisfies $|\mu| \leq 1$.*

Proof. ✓ Proof This is the MPS formulation of the corresponding theorem in the companion quantum-channel volume [LTC26, “Eigenvalue bound for a trace-preserving mixed Kraus map”]: the map F_{AB}^{rect} is the mixed Kraus map of the two matrix families, and the normalizations of A and B are exactly its two trace-preserving hypotheses. $\square$

Lemma 6.3.3 (Spectral radius bound). Lean ✓ Stmt *For normalized tensors, $\rho_{\text{spec}}(F_{AB}) \leq 1$.*

Proof. ✓ Proof The spectral radius is the maximum of $|\mu|$ over the eigenvalues, and each satisfies $|\mu| \leq 1$ by Theorem 6.3.1. $\square$

Theorem 6.3.4 (Spectral radius ≥ 1 implies gauge-phase equivalence). Lean ✓ Stmt *Let A and B be injective normalized MPS tensors. If $\rho_{\text{spec}}(F_{AB}) \geq 1$, then A and B are gauge-phase equivalent.*

Proof. ✓ Proof Since $\rho_{\text{spec}}(F_{AB}) \leq 1$ by Lemma 6.3.3, the hypothesis gives an eigenvalue μ with $|\mu| = 1$ and a nonzero eigenvector X satisfying $F_{AB}(X) = \mu X$. The proof proceeds in five steps, elaborated in Section B.4.

Step 1 (Separate gauging). Choose positive definite fixed points ρ_A and ρ_B of $\mathcal{E}_A$ and $\mathcal{E}_B$ by Theorem 5.3.2. Gauge the tensors separately to unital Kraus families by Theorem [LTC26, “Right-canonical (unital) gauge”]: $A'^i = \rho_A^{-1/2} A^i \rho_A^{1/2}$ and similarly for B . Set $X' = \rho_A^{-1/2} X \rho_B^{-1/2}$.

Step 2 (Block Kraus family and off-diagonal embedding). Form

$$C^i := \begin{pmatrix} A'^i & 0 \\ 0 & B'^i \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & X' \\ 0 & 0 \end{pmatrix}.$$

The matrices $\{C^i\}$ form a unital Kraus family on $M_{2D}(\mathbb{C})$, and $E_C(Y) = \mu Y$.

Step 3 (Kadison–Schwarz equality and Kraus relations). The positive definite fixed points of the adjoint maps give a faithful weighted trace for the block map. Applying the Kadison–Schwarz inequality to Y and pairing its positive semidefinite gap with this fixed point shows that the gap vanishes. The Kraus relation of the corresponding theorem in the companion quantum-channel volume [LTC26, “KS equality implies Kraus commutation”] then gives $Y(C^i)^\dagger = \mu(C^i)^\dagger Y$. Applying the same argument to the complementary product gives $C^i Y = \mu Y C^i$.

Step 4 (Intertwining identity and invertibility). Expanding the block relations from Step 3 gives

$$X'(B'^i)^\dagger = \mu(A'^i)^\dagger X', \quad A'^i X' = \mu X' B'^i.$$

The corresponding multiplicative identities make $X'^\dagger X'$ and $X'X'^\dagger$ nonzero positive semidefinite fixed points of the appropriate transfer maps. Injectivity upgrades them to positive definite matrices, so X' is invertible.

Step 5 (Skolem–Noether conclusion). The intertwining identity and invertibility of X' give an algebra isomorphism between the full matrix algebras generated by the two families. By Theorem [LTC26, “Skolem–Noether”], this isomorphism is inner. Undoing the separate gauges gives $B^i = \bar{\mu} X A^i X^{-1}$ for every i , which is the gauge-phase relation (2.6).

The gauging is applied separately to $\mathcal{E}_A$ and $\mathcal{E}_B$; the mixed transfer map F_{AB} need not be simultaneously unital and trace-preserving. $\square$

Remark 6.3.5 (Two gauge conventions). ✓ Stmt The right-canonical gauge used here, $A'^i = \rho^{-1/2} A^i \rho^{1/2}$, makes $\{A'^i\}$ a unital Kraus family: $\sum_i A'^i (A'^i)^\dagger = \mathbb{1}$. The convention $\tilde{A}^i = \rho^{1/2} A^i \rho^{-1/2}$ makes the family trace-preserving: $\sum_i (\tilde{A}^i)^\dagger \tilde{A}^i = \mathbb{1}$. Both conventions appear in practice: the unital form is used in the transfer-operator gap proof above, while the trace-preserving form is used in the canonical-form reduction of Chapter 8.

Theorem 6.3.6 (Strict transfer-operator gap). Lean ✓ Stmt *Let A and B be injective normalized MPS tensors that are not gauge-phase equivalent. Then $\rho_{\text{spec}}(F_{AB}) < 1$.*

Proof. ✓ Proof Lemma 6.3.3 gives $\rho_{\text{spec}}(F_{AB}) \leq 1$. Equality would imply gauge-phase equivalence by Theorem 6.3.4, contradicting the hypothesis. $\square$

6.4 Transfer-operator gap — rectangular case

Lemma 6.4.1 (Rectangular intertwining forces equal bond dimension). Lean ✓ Stmt *Let A and B be injective tensors of bond dimensions D_1 and D_2 . If there are $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$ and $\mu \in \mathbb{C}$ such that, for every physical index i ,*

$$X(B^i)^\dagger = \mu(A^i)^\dagger X, \quad A^i X = \mu X B^i, \quad (6.6)$$

then $D_1 = D_2$.

Proof. ✓ Proof The first relation in (6.6) shows that $\ker(X)$ is invariant under every $(B^i)^\dagger$: if $Xv = 0$, then $X(B^i)^\dagger v = \mu(A^i)^\dagger Xv = 0$. Since the $\{B^i\}$ span the full matrix algebra, so do the $\{(B^i)^\dagger\}$ (conjugate transposition is a conjugate-linear bijection and therefore preserves complex linear spans), so $\ker(X)$ is invariant under every matrix; injectivity of the resulting map forces $\ker(X) = 0$, giving $D_2 \leq D_1$. Taking adjoints of the second relation gives $X^\dagger(A^i)^\dagger = \bar{\mu}(B^i)^\dagger X^\dagger$, so the same argument applied to $X^\dagger$ shows $\ker(X^\dagger)$ is invariant under every $(A^i)^\dagger$ and hence trivial, giving $D_1 \leq D_2$. $\square$

Theorem 6.4.2 (Rectangular transfer-operator gap). Lean ✓ Stmt *Let A be an injective normalized tensor of bond dimension D_1 and B an injective normalized tensor of bond dimension D_2 , with $D_1 \neq D_2$. Then $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$.*

Proof. ✓ Proof Theorem 6.3.2 gives $|\mu| \leq 1$ for every eigenvalue of F_{AB}^{rect} . If its spectral radius were 1, there would be a modulus-one eigenvector $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$.

Gauge A and B to unital families A' and B' and transport X to a nonzero rectangular intertwiner X' satisfying (B.2). Lemma 6.4.1 would then give $D_1 = D_2$, contradicting the hypothesis. $\square$

Theorem 6.4.3 (Rectangular transfer powers below unit spectral radius). Lean ✓ Stmt *Let A and B have bond dimensions D_1 and D_2 , respectively. If $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$, then, for every $X \in M_{D_1 \times D_2}(\mathbb{C})$,*

$$(F_{AB}^{\text{rect}})^n(X) \rightarrow 0.$$

This is the decay mechanism for the off-diagonal transfer operators in [CPGSV16, Appendix B, equations (B.3)–(B.4) and the paragraph following equation (B.6)].

Proof. ✓ Proof

The Gelfand formula implies convergence of $(F_{AB}^{\text{rect}})^n$ to zero in operator norm. Evaluation at a fixed rectangular matrix is continuous, and therefore preserves this limit. $\square$

Theorem 6.4.4 (Rectangular overlap decay). Lean ✓ Stmt *If $D_1 \neq D_2$ and both tensors are injective and normalized, then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. ✓ Proof The gap $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$ from Theorem 6.4.2 and Lemma [LTC26, “Powers vanish below unit spectral radius”] imply operator-norm convergence $(F_{AB}^{\text{rect}})^N \rightarrow 0$, so its operator trace tends to zero. The identity (6.5) therefore gives $O_{AB}(N) \rightarrow 0$. $\square$

6.5 MPV overlap decay

We first record two spectral-radius identities for complex normed algebras. They will also control the scalar weights of normal fixed-point blocks in Chapter ??.

Theorem 6.5.1 (Spectral radius under nonzero scaling). Lean ✓ Stmt *Let a belong to a complex Banach algebra and let $c \in \mathbb{C} \setminus \{0\}$. Then*

$$\rho_{\text{spec}}(ca) = |c| \rho_{\text{spec}}(a).$$

Proof. ✓ Proof Multiplication by c sends the spectrum of a onto the spectrum of ca . Taking the supremum of the moduli of the spectral values gives the formula. $\square$

Theorem 6.5.2 (Spectral radius of a nonzero idempotent). Lean ✓ Stmt *Let a be a nonzero element of a complex normed algebra. If $a^2 = a$, then*

$$\rho_{\text{spec}}(a) = 1.$$

Proof. ✓ Proof The identity $a^2 = a$ gives $a(a - 1) = 0$. Since $a \neq 0$, the element $a - 1$ is not invertible, so 1 belongs to the spectrum of a . Every spectral value λ satisfies $\lambda^2 = \lambda$, hence the spectrum is contained in $\{0, 1\}$. $\square$

The Banach-algebra fact that powers below unit spectral radius vanish is proved in Section B.5.

Lemma 6.5.3 (An idempotent below unit spectral radius vanishes). Lean ✓ Stmt *Let a be an element of a complex Banach algebra with $a^2 = a$ and $\rho_{\text{spec}}(a) < 1$. Then $a = 0$.*

Proof. ✓ Proof

Lemma [LTC26, “Powers vanish below unit spectral radius”] gives $a^n \rightarrow 0$. Idempotence gives $a^{n+1} = a$ for every n , so uniqueness of limits yields $a = 0$. $\square$

Theorem 6.5.4 (Transfer powers converge to zero). Lean ✓ Stmt *Let A and B be injective normalized tensors that are not gauge-phase equivalent. Then $F_{AB}^n(X) \rightarrow 0$ for every $X \in M_D(\mathbb{C})$.*

Proof. ✓ Proof The gap $\rho_{\text{spec}}(F_{AB}) < 1$ from Theorem 6.3.6 and the equal-bond-dimension case of Theorem 6.4.3 give $F_{AB}^n(X) \rightarrow 0$ for every X . $\square$

Theorem 6.5.5 (Overlap decay). Lean ✓ Stmt *Let A and B be injective normalized tensors of the same bond dimension that are not gauge-phase equivalent. Then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. ✓ Proof By (6.4) and the trace-expansion formula of [LTC26, “Trace expansion over matrix units”],

$$O_{AB}(N) = \text{Tr}(F_{AB}^N) = \sum_{p,q} (F_{AB}^N(E_{pq}))_{pq}.$$

Theorem 6.5.4 makes every summand tend to zero. Since the sum is finite, $O_{AB}(N) \rightarrow 0$. $\square$

The block-separation consequences of this gap — cross-correlation decay between distinct blocks and the persistence of self-correlation at a fixed point — used in the canonical-form separation of Chapter 8, are recorded in Section B.5.

6.6 Transfer-operator gap under irreducible-TP hypotheses

The preceding transfer-operator gap results require *injectivity* of both tensors. The following variants weaken this to *irreducibility* together with the trace-preserving normalization (6.1), following Cirac et al. [CPGSV16] and the irreducibility theory of [Wol12, Section 6.2]. The argument replaces the Kraus-commutation step with a Cauchy–Schwarz rigidity argument for the Perron–Frobenius fixed point.

Theorem 6.6.1 (Gauge-equivalence from irreducible-TP spectral radius). Lean Lean ✓ Stmt *Let A and B be irreducible trace-preserving normalized MPS tensors of bond dimension D . If $\rho_{\text{spec}}(F_{AB}) \geq 1$, then A and B are gauge-phase equivalent.*

Proof. ✓ Proof Choose a modulus-one eigenvector as in Theorem 6.3.4. The proof follows its five steps, with one change. Separate gauging, the block Kraus family, and the Kadison–Schwarz equality proceed as before. The equality makes the relevant positive products of the transported intertwiner nonzero positive semidefinite fixed points. Under irreducibility, Theorem 5.1.1 makes these fixed points positive definite, so the transported intertwiner is invertible. Lemma B.4.1 supplies (B.2), and Lemma B.4.2 then gives gauge-phase equivalence. $\square$

Theorem 6.6.2 (Strict transfer-operator gap (irreducible-TP)). Lean ✓ Stmt *Let A and B be irreducible trace-preserving normalized MPS tensors of the same bond dimension that are not gauge-phase equivalent. Then $\rho_{\text{spec}}(F_{AB}) < 1$.*

Proof. ✓ Proof Lemma 6.3.3 gives $\rho_{\text{spec}}(F_{AB}) \leq 1$, while the contrapositive of Theorem 6.6.1 excludes equality. $\square$

Theorem 6.6.3 (Overlap decay (irreducible-TP)). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving normalized tensors of the same bond dimension that are not gauge-phase equivalent. Then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) Theorem 6.6.2 gives $\rho_{\text{spec}}(F_{AB}) < 1$, so $F_{AB}^N \rightarrow 0$. The overlap–trace identity (6.4) gives $O_{AB}(N) \rightarrow 0$. $\square$

Theorem 6.6.4 (Rectangular transfer-operator gap (irreducible-TP)). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving normalized tensors of bond dimensions D_1 and D_2 , respectively, with $D_1 \neq D_2$. Then $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$.*

Proof. [✓ Proof](#) Suppose instead that $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) = 1$, and choose a modulus-one eigenvector $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$. Gauge A and B separately to unital families A' and B' and transport X to $X' \neq 0$. The matrix $X'X'^\dagger$ is a nonzero positive semidefinite fixed point of the appropriate transfer map on $M_{D_1}(\mathbb{C})$, so Theorem 5.1.1 makes it positive definite. Thus X' has full row rank and $D_1 \leq D_2$. Similarly, $X'^\dagger X'$ is positive definite, so X' has full column rank and $D_2 \leq D_1$. Hence $D_1 = D_2$, a contradiction. $\square$

Theorem 6.6.5 (Rectangular overlap decay (irreducible-TP)). [Lean](#) [✓ Stmt](#) *If $D_1 \neq D_2$ and both tensors are irreducible and trace-preserving normalized, then $O_{AB}(N) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) Theorem 6.6.4 gives $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$, so $(F_{AB}^{\text{rect}})^N \rightarrow 0$. The rectangular overlap–trace identity (6.5) gives $O_{AB}(N) \rightarrow 0$. $\square$

6.7 Overlap rigidity for normal blocks

The BNT matching argument ultimately turns a non-decaying mixed overlap into a gauge-phase match. The following two results isolate that spectral step.

Lemma 6.7.1 (Mixed-transfer spectral radius from unit-modulus overlap). [Lean](#) [✓ Stmt](#) *Let A and B have the same bond dimension. If $\|\langle V^{(N)}(A) | V^{(N)}(B) \rangle\| \rightarrow 1$, then the mixed transfer map has spectral radius at least 1.*

Proof. [✓ Proof](#) If the mixed-transfer spectral radius were less than 1, the iterated mixed transfer map would tend to zero in operator norm, hence so would its trace. By (6.4), the overlap would tend to zero, contradicting the hypothesis. $\square$

Theorem 6.7.2 (Gauge-phase equivalence from unit-modulus overlap). [Lean](#) [✓ Stmt](#) *Let A and B be irreducible MPS tensors of the same bond dimension with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ and $\sum_i (B^i)^\dagger B^i = \mathbb{1}$. If $\|\langle V^{(N)}(A) | V^{(N)}(B) \rangle\| \rightarrow 1$, then A and B are gauge-phase equivalent.*

Proof. [✓ Proof](#) By Lemma 6.7.1, the mixed-transfer spectral radius is at least 1, and Theorem 6.6.1 then gives gauge-phase equivalence. $\square$

6.8 Peripheral spectral refinements

Every result in this section concerning peripheral eigenvalues, cyclic projections, or peripheral eigenspaces is stated for a finite Kraus map and therefore assumes complete positivity. These results do not supply the abstract Schwarz-map case of Wolf Theorem 6.6; the restriction and its elimination plan are documented in the scope-restriction note [con26a].

6.8.1 Cyclic decomposition of irreducible finite Kraus maps

The construction — normalizing a peripheral eigenvalue to a unitary eigenvector, forming its discrete Fourier (cyclic) spectral projections, and the supporting corner-algebra apparatus (restriction to an invariant corner, corner rank, and the compression isometry $PM_D(\mathbb{C})P \cong M_n(\mathbb{C})$) — is given in Section B.7.

Lemma 6.8.1.1 (Cyclic shift of the Kraus operators). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let $P_0, \dots, P_{m-1}$ be orthogonal projections summing to the identity and cyclically permuted by a Kraus map, $\sum_v K_v P_{k+1} K_v^\dagger = P_k$. Then the projections are mutually orthogonal, none of them vanishes, and each Kraus operator intertwines consecutive projections:*

$$K_v P_{k+1} = P_k K_v. \quad (6.7)$$

Proof. [✓ Proof](#)

Compressing $\sum_v K_v P_{k+1} K_v^\dagger = P_k$ by $\mathbb{1} - P_k$ exhibits a vanishing sum of positive semidefinite matrices, so each summand vanishes: $(\mathbb{1} - P_k) K_v P_{k+1} = 0$. Mutual orthogonality follows by compressing the resolution of the identity by a fixed P_k , another vanishing sum of positive semidefinite matrices. Expanding $K_v P_{k+1}$ and $P_k K_v$ through the resolution of the identity, the off-diagonal compressions vanish by orthogonality, leaving (6.7). If some projection vanished, the cyclic action would propagate the vanishing around the cycle, contradicting the resolution of the identity. $\square$

Theorem 6.8.1.2 (Cyclic shift along a Kraus word). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Lemma 6.8.1.1, a product of ℓ Kraus operators moves each cyclic projection back by ℓ steps. In particular, every product of m Kraus operators commutes with every cyclic projection.*

Proof. [✓ Proof](#) Induct on the length of the word using (6.7). $\square$

The primitivity and irreducibility of the restricted sector dynamics, the invariance of each cyclic corner under the period, and the closure of the peripheral eigenvalues under products and inverses are proved in Section B.7.

6.9 Self-overlap and the period-one criterion

Theorem 6.9.1 (Self-overlap along peripheral-period multiples). [Lean](#) [✓ Stmt](#) *Let $D > 0$ and $m > 0$. Let A be a left-canonical irreducible MPS tensor with bond dimension D , and suppose that the peripheral eigenvalues of its transfer map are exactly $\{z \in \mathbb{C} : z^m = 1\}$. Then $O_{AA}(mn) \rightarrow m$.*

Proof. [✓ Proof](#) Apply Theorem 8.11.1 to the period- m blocking $A^{[m]}$. It gives m nonzero cyclic-sector tensors $C_0, \dots, C_{m-1}$, each left-canonical, whose unit-weight direct sum has the same MPV family as $A^{[m]}$. The cyclic-sector relations make every C_u primitive and irreducible. Distinct sectors are not gauge-phase equivalent. Hence the primitive self-overlap limit and the equal- and unequal-dimension overlap-decay theorems give

$$O_{C_u C_v}(n) \rightarrow \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}$$

Expanding the overlap of the unit-weight direct sum and taking the finite sum of these limits yields

$$O_{A^{[m]}A^{[m]}}(n) = \sum_{u,v \in \mathbb{Z}_m} O_{C_u C_v}(n) \rightarrow m.$$

Finally, blocking identifies $O_{A^{[m]}A^{[m]}}(n) = O_{AA}(mn)$, which proves the claim. $\square$

Theorem 6.9.2 (Primitivity and normality from normalized self-overlap). [Lean](#) [✓ Stmt](#) *Let A be an irreducible left-canonical tensor of positive bond dimension. If $O_{AA}(N) \rightarrow 1$, then the transfer map of A is primitive and A is normal.*

Proof. [✓ Proof](#) Let m be the peripheral period. Along its multiples, $O_{AA}(mn) \rightarrow m$. The assumed full limit gives $O_{AA}(mn) \rightarrow 1$, hence $m = 1$. The transfer map is therefore primitive, and irreducibility and left-canonicity give normality. $\square$

Theorem 6.9.3 (Normality from normalized self-overlap). [Lean](#) [✓ Stmt](#) *Let A be an irreducible left-canonical tensor of positive bond dimension. If $O_{AA}(N) \rightarrow 1$, then A is normal.*

Proof. [✓ Proof](#) This is the normality conclusion of Theorem 6.9.2. $\square$

6.10 Primitive overlap convergence (complementary transfer-map gap form)

The theorems in this section are stated directly in terms of the complementary map $E - P$. This is the analytic form needed for the later overlap argument, corresponding to the primitive-case specialization of [Wol12, Theorem 6.7] where T_ϕ is a rank-one projection. For primitive normalized transfer maps, the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”] supplies this complementary transfer-map input under explicit hypotheses; the rank-one Perron projection $P = P_\rho$, its trace, and the concrete irreducible instantiation of the complementary spectral-radius gap are given in Section B.6.

Theorem 6.10.1 (Complementary transfer-map gap implies trace convergence to 1). [Lean](#) [✓ Stmt](#) *Let E be trace-preserving with fixed point ρ , where $\text{tr}(\rho) \neq 0$, and let P be the fixed-point projection of the corresponding definition in the companion quantum-channel volume [LTC26, “Fixed-point projection”]. If $\rho_{\text{spec}}(E - P) < 1$, then $\text{Tr}(E^n) \rightarrow 1$ as $n \rightarrow \infty$.*

Proof. [✓ Proof](#)

By the power decomposition (established in the companion quantum-channel volume [LTC26, “Power decomposition”]), $E^n = P + (E - P)^n$ for $n \geq 1$. Since $\rho_{\text{spec}}(E - P) < 1$, the Gelfand formula gives $(E - P)^n \rightarrow 0$, so $\text{Tr}(E^n) \rightarrow \text{Tr}(P)$. Lemma [LTC26, “Trace of the fixed-point projection”] (Section B.6) gives $\text{Tr}(P) = 1$. $\square$

Theorem 6.10.2 (Self-overlap convergence from a complementary transfer-map gap). [Lean](#) [✓ Stmt](#) *Let A be a normalized MPS tensor with a nonzero PSD fixed point ρ of the transfer map. If $\rho_{\text{spec}}(\mathcal{E}_A - P) < 1$, where P is the fixed-point projection, then $O_{AA}(N) \rightarrow 1$ as $N \rightarrow \infty$.*

Proof. [✓ Proof](#) By the overlap–trace identity (6.4) with $A = B$, one has $O_{AA}(N) = \text{Tr}(\mathcal{E}_A^N)$. By Theorem 6.10.1, $\text{Tr}(\mathcal{E}_A^N) \rightarrow 1$. $\square$

Chapter 7

Wielandt Bound

This chapter proves the quantum Wielandt bounds of [SPGWC10]. For a primitive normalized tensor, it compares the uniform vector-spreading index $q(\mathcal{E}_A)$ with the first exact word length $\iota(A)$ satisfying $S_{\iota(A)}(A) = M_D(\mathbb{C})$ and proves $\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2$. The cumulative bound $T_{D^2}(A) = M_D(\mathbb{C})$ is an intermediate step in the proof of Lemma 1.

7.1 Cumulative span

The exact-word identities used in this section are proved in Section C.1. The stabilization and spectral linear algebra behind the cumulative estimates are collected in Section C.2.

Definition 7.1.1 (Word span). [Lean](#) [✓ Stmt](#) The *word span* at length n is the subspace

$$S_n(A) = \text{span}_{\mathbb{C}}\{A^w : |w| = n\} \subseteq M_D(\mathbb{C}).$$

This is [SPGWC10, Section II].

Definition 7.1.2 (Cumulative span). [Lean](#) [✓ Stmt](#) The *cumulative span* at level n is $T_n(A) = S_0(A) + S_1(A) + \dots + S_n(A)$.

Definition 7.1.3 (Fixed-length vector span of an MPS tensor). [Lean](#) [✓ Stmt](#)

For an MPS tensor A , its fixed-length vector span is the fixed-length vector span of the underlying finite matrix family.

7.1.1 Paper primitivity and indices

For the quantitative bounds, write $d' = \dim S_1(A)$ for the number of linearly independent Kraus operators. The source definitions and Proposition 3 of [SPGWC10] distinguish uniform vector spreading, eventual exact-word spanning, and spectral strong irreducibility.

Definition 7.1.1.1 (Kraus rank). [Lean](#) [✓ Stmt](#) The *Kraus rank* is $\text{kr}(A) = \dim S_1(A)$.

Definition 7.1.1.2 (Eventually full Kraus rank). [Lean](#) [✓ Stmt](#) The Kraus family has *eventually full Kraus rank* if there is a positive integer i such that $S_i(A) = M_D(\mathbb{C})$.

Definition 7.1.1.3 (Paper primitivity). [Lean](#) [✓ Stmt](#)

The transfer map is *primitive in the sense of [SPGWC10]* if there is a positive integer q such that

$$H_q(A, \varphi) = \mathbb{C}^D \quad (7.1)$$

for every nonzero $\varphi \in \mathbb{C}^D$. The same q works for all nonzero vectors.

Definition 7.1.1.4 (Full-Kraus-rank index). [Lean](#) [✓ Stmt](#) The *full-Kraus-rank index* is

$$\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}.$$

If the set is empty, its value is defined to be zero.

Definition 7.1.1.5 (Primitivity index). [Lean](#) [✓ Stmt](#) The *primitivity index* is

$$q(\mathcal{E}_A) = \min\{q \geq 1 : H_q(A, \varphi) = \mathbb{C}^D \text{ for every } \varphi \neq 0\}.$$

If the set is empty, its value is defined to be zero.

Definition 7.1.1.6 (Strengthened strong irreducibility). [Lean](#) [✓ Stmt](#) We call the tensor *strongly irreducible* here if there is a matrix $\rho > 0$ such that

$$\begin{aligned} \mathcal{E}_A(\rho) &= \rho, \\ \text{spec}_{\text{per}}(\mathcal{E}_A) &= \{1\}, \end{aligned}$$

and $\mathcal{E}_A$ is irreducible. The condition in [SPGWC10, Proposition 3(c)] requires that 1 be the only peripheral eigenvalue and that its eigenspace be one-dimensional, generated by a positive-definite fixed point. The definition above strengthens this condition by also requiring irreducibility explicitly.

The next seven results use the following notation. For nonnegative integers d and D , let $K = \{K^i\}_{i=0}^{d-1}$ be a finite family of $D \times D$ matrices, and let $\mathcal{E}_K$ be its Kraus map.

Theorem 7.1.1.7 (Powers of a transfer-map eigenmatrix). [Lean](#) [✓ Stmt](#) If $\mathcal{E}_A(X) = \mu X$, then $\mathcal{E}_A^n(X) = \mu^n X$ for every $n \geq 0$.

Proof. [✓ Proof](#) This is the corresponding theorem in the companion quantum-channel volume [LTC26, “Powers on an eigenspace”] for the eigenspace of μ . $\square$

Theorem 7.1.1.8 (Hermitian power-fixed point from a peripheral eigenvector). [Lean](#) [✓ Stmt](#) Let A be a normalized MPS tensor, and suppose $\mathcal{E}_A(X) = \mu X$ with $X \neq 0$, $\mu \neq 1$, and $\mu^p = 1$. Then there is a nonzero Hermitian matrix H such that

$$\begin{aligned} \text{tr}(H) &= 0, \\ \mathcal{E}_A^p(H) &= H. \end{aligned}$$

Moreover, H is not positive semidefinite.

Proof. [✓ Proof](#)

Apply Theorem [LTC26, “Hermitian power-fixed point from a finite-order eigenmatrix”] to the transfer channel. The resulting matrix is nonzero, Hermitian, trace zero, and fixed by $\mathcal{E}_A^p$. A nonzero positive-semidefinite matrix has positive trace, so this matrix is not positive semidefinite. $\square$

Theorem 7.1.1.9 (Uniqueness of power-fixed positive matrices). [Lean](#) [✓ Stmt](#) Suppose there is a length $q \geq 0$ such that $H_q(A, \varphi) = \mathbb{C}^D$ for every nonzero $\varphi \in \mathbb{C}^D$. Let $p > 0$. If ρ and σ are nonzero positive-semidefinite fixed points of $\mathcal{E}_A^p$, then $\sigma = c\rho$ for some $c \in \mathbb{C}$.

Proof. [✓ Proof](#)

This is Theorem [LTC26](#), “Uniqueness of power-fixed positive matrices for a Kraus map”] for the Kraus family formed by the matrices of the tensor. $\square$

Theorem 7.1.1.10 (Kraus-span equivalence). [Lean](#) [✓ Stmt](#) Under normalization, primitivity in Wolf’s sense is equivalent to eventual full Kraus rank.

Theorem 7.1.1.11 (Combined primitivity equivalence). [Lean](#) [✓ Stmt](#) Under normalization, primitivity in Wolf’s sense is equivalent to the conjunction of eventual full Kraus rank, normality, and strong irreducibility.

Remark 7.1.1.12 (Eventually full Kraus rank and normality). [Lean](#) [✓ Stmt](#) Eventual full Kraus rank is equivalent to normality. The remaining pairwise primitivity equivalences are stated in Theorem [7.1.1.13](#).

Theorem 7.1.1.13 (Strengthened form of the primitivity equivalence). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $D > 0$ and suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. The following are equivalent:

1. there is a positive q for which $H_q(A, \varphi) = \mathbb{C}^D$ for every $\varphi \neq 0$;
2. there is a positive i for which $S_i(A) = M_D(\mathbb{C})$;
3. $\mathcal{E}_A$ is strongly irreducible.

Proposition 3 of [\[SPGWC10\]](#) states the same equivalence with item (3) replaced by the literal source condition: 1 is the only peripheral eigenvalue, and its one-dimensional eigenspace is generated by a positive-definite fixed point. Here item (3) includes the additional irreducibility condition.

Proof. [✓ Proof](#)

If $S_i(A) = M_D(\mathbb{C})$ and $\varphi \neq 0$, then $S_i(A)\varphi = \mathbb{C}^D$, proving (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (3), regard the matrices of A as a finite Kraus family. Theorems [LTC26](#), “Positive-definite fixed point from vector spreading”] and [LTC26](#), “Primitivity from vector spreading”] give a positive-definite fixed point and show that 1 is the only peripheral eigenvalue. Theorem [LTC26](#), “Irreducibility from vector spreading”] gives irreducibility. Together these facts say that $\mathcal{E}_A$ is strongly irreducible.

For (3) $\Rightarrow$ (2), write $\mathcal{E}_A^m = P_\rho + (\mathcal{E}_A - P_\rho)^m$. The complementary powers tend to zero. If $S_m(A) \neq M_D(\mathbb{C})$ for every m , choose a nonzero matrix B orthogonal to $S_m(A)$. The associated trace pairing vanishes on the left, while positive definiteness gives a lower bound $c\|B\|^2$ for the P_ρ term and the complementary term is $o(\|B\|^2)$. For large m this is a contradiction. $\square$

Theorem 7.1.1.14 (Primitive-channel specialization of the index bound). [Lean](#) [✓ Stmt](#) Under the hypotheses of Theorem [7.1.1.13](#), paper primitivity implies

$$q(\mathcal{E}_A) \leq \iota(A).$$

Proposition 1 of [\[SPGWC10\]](#) states this inequality for every quantum channel; the present theorem records its specialization under normalization and paper primitivity.

Proof. [✓ Proof](#) By Theorem [7.1.1.13](#), $\iota(A)$ is a positive length with $S_{\iota(A)}(A) = M_D(\mathbb{C})$. Therefore $H_{\iota(A)}(A, \varphi) = \mathbb{C}^D$ for every $\varphi \neq 0$, so the defining minimum for $q(\mathcal{E}_A)$ is at most $\iota(A)$. $\square$

Theorem 7.1.1.15 (Unique fixed points of full rank). [Lean](#) [✓ Stmt](#) *Let $D > 0$ and suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. If there is a positive n for which $S_n(A) = M_D(\mathbb{C})$, then there is a unique positive definite density matrix $\rho \in M_D(\mathbb{C})$ such that every fixed point of $\mathcal{E}_A$ is a scalar multiple of ρ .*

This is [Wol12, Theorem 6.15].

Proof. [✓ Proof](#) By Theorem 7.1.1.13, the span hypothesis makes $\mathcal{E}_A$ strongly irreducible, giving a positive-definite fixed point ρ_0 together with peripheral primitivity (unique peripheral eigenvalue 1) and irreducibility of $\mathcal{E}_A$. Peripheral primitivity and irreducibility of $\mathcal{E}_A$ together give the complementary transfer-map gap at ρ_0 (Lemma C.5.5): the restriction of $\mathcal{E}_A$ off the fixed-point projection at ρ_0 has spectral radius strictly less than one. This gap forces every fixed point of $\mathcal{E}_A$ to be a scalar multiple of ρ_0 ; rescaling ρ_0 by the inverse of its (positive) trace gives the unique density-matrix representative ρ .

The source proves this instead by passing to $\mathcal{E}_A^n$, showing every $\mathcal{E}_A^n(|\psi\rangle\langle\psi|)$ has full rank from the span hypothesis, and ruling out a fixed-point space of dimension larger than one via Corollary 6.5 of [Wol12]. Both routes use only Wolf’s own Chapter 6 machinery; the present proof reuses the already-formalized Proposition 3 equivalence. $\square$

Lemma 7.1.1.16 (Monotonicity of cumulative span). [Lean](#) [✓ Stmt](#) *For every n , one has $T_n(A) \leq T_{n+1}(A)$.*

Proof. [✓ Proof](#) The span T_{n+1} includes all words of length at most $n+1$ and therefore contains all words of length at most n . $\square$

Lemma 7.1.1.17 (Dimension bound). [Lean](#) [✓ Stmt](#) *For every n , one has $\dim T_n(A) \leq D^2$.*

Proof. [✓ Proof](#) Since $T_n(A) \subseteq M_D(\mathbb{C})$ and $\dim M_D(\mathbb{C}) = D^2$, the result follows. $\square$

Theorem 7.1.1.18 (Cumulative span stabilisation). [Lean](#) [✓ Stmt](#) *If $T_n(A) = T_{n+1}(A)$, then $T_m(A) = T_n(A)$ for every $m \geq n$.*

Proof. [✓ Proof](#) If $T_n = T_{n+1}$, then $S_{n+1} \subseteq T_n$. Left-multiplying any element of S_{n+1} by A^i gives an element of S_{n+2} , so $S_{n+2} \subseteq T_{n+1} = T_n$. By induction, $S_m \subseteq T_n$ for every $m > n$, hence $T_m = T_n$. $\square$

Theorem 7.1.1.19 (Dimension growth). [Lean](#) [✓ Stmt](#) *If $T_n(A) \neq T_{n+1}(A)$, then $\dim T_{n+1}(A) > \dim T_n(A)$.*

Proof. [✓ Proof](#) The inclusion $T_n \leq T_{n+1}$ from Lemma 7.1.1.16, together with $T_n \neq T_{n+1}$, gives strict subspace inclusion and hence strict dimension growth. $\square$

Lemma 7.1.1.20 (Word span characterises block injectivity). [Lean](#) [✓ Stmt](#) *One has $S_L(A) = M_D(\mathbb{C})$ if and only if A is L -block injective.*

Proof. [✓ Proof](#) Both conditions assert that the span of all products of length L equals $M_D(\mathbb{C})$. $\square$

Theorem 7.1.1.21 (Positive-length propagation of block injectivity). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *For every n ,*

$$S_{n+1}(A) = \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_n(A).$$

Consequently, if $L > 0$ and A is L -block injective, then A is m -block injective for every $m \geq L$. This auxiliary length-shift statement combines the block-injectivity discussion in [CPGSV16, Section II] with [CPGSV16, Appendix C.3, Lemma L]; it is not stated there as a separate lemma.

Proof. ✓ Proof Every word of length $n + 1$ factors into its first letter and a word of length n , which proves the displayed equality by bilinearity. Write $L = n + 1$. Since $S_L(A) = M_D(\mathbb{C})$, one has $S_n(A) \leq S_L(A)$. Hence

$$M_D(\mathbb{C}) = S_L(A) = \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_n(A) \leq \text{span}_{\mathbb{C}}\{A^i\}_{i=0}^{d-1} S_L(A) = S_{L+1}(A).$$

Iterating this implication gives $S_m(A) = M_D(\mathbb{C})$ for every $m \geq L$. □

7.2 Nonzero trace product

Theorem 7.2.1 (Cumulative span reaches $M_D(\mathbb{C})$). Lean ✓ Stmt *Let $D > 0$. If A is normal, then $T_{D^2}(A) = M_D(\mathbb{C})$. This cumulative estimate supports the proof of [SPGWC10, Lemma 1]; it is not a conclusion of source Theorem 1.*

Proof. ✓ Proof This is Theorem C.2.4. □

Theorem 7.2.2 (Cumulative span reaches $M_D(\mathbb{C})$ — sharp bound). Lean ✓ Stmt *Let $D > 0$. If A is normal, then $T_{D^2-d'+1}(A) = M_D(\mathbb{C})$, where $d' = \dim S_1(A) = \text{kr}(A)$. This is the cumulative-span estimate used to prove the sharp trace conclusion in [SPGWC10, Lemma 1].*

Proof. ✓ Proof If $T_n(A) \neq T_{n+1}(A)$, then the dimension strictly increases. Since $S_1(A) \subseteq T_1(A)$ and $\dim S_1(A) = d'$, one has $\dim T_1(A) \geq d'$. After at most $D^2 - d'$ additional strict-growth steps, the dimension reaches $D^2 = \dim M_D(\mathbb{C})$. Stabilisation before reaching $M_D(\mathbb{C})$ contradicts normality. Hence $T_{D^2-d'+1}(A) = M_D(\mathbb{C})$. □

Theorem 7.2.3 (Eigenvector extraction for an MPS tensor). Lean ✓ Stmt *The conclusion of Theorem [LTC26, “Eigenvector extraction from a full cumulative span”] holds for the matrix family of an MPS tensor whenever its cumulative span at level N is the full matrix algebra.*

Proof. ✓ Proof

Apply the finite-family theorem to the matrices of the tensor. □

Theorem 7.2.4 (Sharp nonzero trace product for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. Then there is a word w with*

$$1 \leq |w| \leq D^2 - \text{kr}(A) + 1, \\ \text{tr}(A^w) \neq 0.$$

This is the sharp trace conclusion of [SPGWC10, Lemma 1].

Proof. ✓ Proof Theorem 7.1.1.13 gives eventual full Kraus rank, hence normality. Put $b = D^2 - \text{kr}(A) + 1$ and let V_n be the span of A^w with $1 \leq |w| \leq n$. Then $S_1(A) \subseteq V_1$, so $\dim V_1 \geq \text{kr}(A)$. If $V_n = V_{n+1}$ for some $n \geq 1$, then V_n is closed under left multiplication by every A^i . It therefore contains every positive-length word product; normality forces $V_n = M_D(\mathbb{C})$. Thus, until the full matrix algebra is reached, the dimensions increase strictly, and the bound on $\dim M_D(\mathbb{C})$ gives $V_b = M_D(\mathbb{C})$. Express $\mathbb{1}$ as a linear combination of the positive-length words defining V_b . Since $\text{tr}(\mathbb{1}) = D \neq 0$, at least one has nonzero trace. □

Theorem 7.2.5 (Sharp nonzero trace product). Lean ✓ Stmt *Let $D > 0$. If A is normal, then there exists a word w of length at most $D^2 - \text{kr}(A) + 1$ such that $\text{tr}(A^w) \neq 0$. This normality variant has the same sharp length bound as [SPGWC10, Lemma 1]; the source hypotheses are stated in Theorem 7.2.4.*

Proof. ✓ Proof By Theorem 7.2.2, the identity $\mathbb{1}$ lies in $T_{D^2-d'+1}(A)$ and is therefore a linear combination of word products of length at most $D^2 - d' + 1$. At least one such product has nonzero trace. $\square$

Theorem 7.2.6 (Sharp nonzero trace product — positive length). Lean ✓ Stmt *If A is normal and $D \geq 1$, then there exists a word w of positive length $1 \leq |w| \leq D^2 - \text{kr}(A) + 1$ such that $\text{tr}(A^w) \neq 0$. This strengthens Theorem 7.2.5 by requiring $|w| \geq 1$, a feature needed for the blocking argument in the general Wielandt bound below.*

Proof. ✓ Proof Define the *positive-level* cumulative span $V_n = \text{span}_{\mathbb{C}}\{A^w : 1 \leq |w| \leq n\}$. The same stabilization-or-strict-growth dichotomy shows $V_{D^2-d'+1} = M_D(\mathbb{C})$: if V_n stabilises and is not $M_D(\mathbb{C})$, then V_n is closed under left multiplication by each A^i , so $S_N(A) \subseteq V_n \neq M_D(\mathbb{C})$ for every $N \geq 1$, contradicting normality. Since $V_{D^2-d'+1} = M_D(\mathbb{C})$, the identity $\mathbb{1}$ lies in $V_{D^2-d'+1}$ and is therefore a linear combination of word products of positive length. At least one such product has nonzero trace because $\text{tr}(\mathbb{1}) = D \neq 0$ for $D \geq 1$. $\square$

7.3 Eigenvector spreading

Definition 7.3.1 (Cumulative vector span). Lean ✓ Stmt Given an MPS tensor A and a vector $\varphi \in \mathbb{C}^D$, the *cumulative vector span* is

$$K_n(A, \varphi) = \text{span}_{\mathbb{C}}\{A^w \varphi : |w| \leq n\} \subseteq \mathbb{C}^D.$$

This is [SPGWC10, proof of Lemma 2(a)].

Theorem 7.3.2 (Eigenvector spreading). Lean ✓ Stmt *Let A be a normal MPS tensor and let $A^{i_0} \varphi = \mu \varphi$, where $\mu \neq 0$ and $\varphi \neq 0$. Then $K_{D-1}(A, \varphi) = \mathbb{C}^D$. This cumulative form supplies the dimension-growth step in [SPGWC10, Lemma 2(a)].*

Proof. ✓ Proof The vector span K_n is monotone and $\dim K_n \leq D$. If $K_n = K_{n+1}$ for some $n < D - 1$, the span stabilises by Theorem C.2.7. But $T_{D^2}(A) = M_D(\mathbb{C})$ by Theorem 7.2.1, so Lemma C.2.8 gives $K_{D^2}(A, \varphi) = \mathbb{C}^D$, contradicting early stabilisation. Hence $\dim K_n$ grows by at least 1 at each step, reaching D by step $D - 1$. $\square$

Theorem 7.3.3 (Fixed-length eigenvector spreading for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. If $\varphi \neq 0$, $\mu \neq 0$, and $A^{i_0} \varphi = \mu \varphi$, then $H_{D-1}(A, \varphi) = \mathbb{C}^D$. This is [SPGWC10, Lemma 2(a)].*

Proof. ✓ Proof Proposition 3, in the form of Theorem 7.1.1.13, gives eventual full Kraus rank and hence normality. The cumulative spaces $K_n(A, \varphi)$ cannot stabilise below $\mathbb{C}^D$, because every matrix eventually lies in an exact word span. Their dimensions therefore reach D by $n = D - 1$, giving $K_{D-1}(A, \varphi) = \mathbb{C}^D$. Finally, the eigenvector relation pads every shorter word to length $D - 1$ by a nonzero power of μ , so $K_{D-1}(A, \varphi) = H_{D-1}(A, \varphi)$. $\square$

7.4 Quantum Wielandt bounds

Section C.3 proves the one-step augmentation facts used in cases (2) and (3). The blocking identities used in the general case are proved in Section C.4.

Remark 7.4.1 (Explicit blocking-length estimates). ✓ Stmt Theorem 7.2.1 gives the D^2 cumulative-span bound. The full-rank index bound $\iota(A) \leq (D^2 - d' + 1)D^2$ ([SPGWC10, Theorem 1, case (1)]) is proved in Theorem 7.4.8 below. The left-canonical normal blocked-injectivity estimate D^4 is proved below in Theorem 7.5.5. A sharper $3D^5$ canonical-form block-separation estimate, developed in the full blueprint, combines this D^4 estimate with the direct-sum argument of [PGVWC07].

Theorem 7.4.2 (Wielandt case (2), one-step invertible element). Lean ✓ Stmt *Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If $S_1(A)$ contains an invertible matrix X , then*

$$S_{D^2 - \text{kr}(A) + 1}(A) = M_D(\mathbb{C}). \quad (7.2)$$

In the terminology of [SPGWC10], this is the case where the first application space $S_1(A)$ contains an invertible operator. This is the exact word-span form of [SPGWC10, Theorem 1, case (2)].

Proof. ✓ Proof Adjoin X as an extra first Kraus operator, obtaining $\tilde{A}$. Theorem 7.1.1.13 gives normality of A , hence normality of $\tilde{A}$; the Kraus rank and all exact word spans are unchanged. The first Kraus operator of $\tilde{A}$ is invertible, so Theorem C.3.5 gives $S_{D^2 - \text{kr}(\tilde{A}) + 1}(\tilde{A}) = M_D(\mathbb{C})$. Transfer this equality back to A . □

Corollary 7.4.3 (Wielandt case (2), index bound). Lean ✓ Stmt *Under the hypotheses of Theorem 7.4.2, one has $\iota(A) \leq D^2 - \text{kr}(A) + 1$, where $\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}$.*

Proof. ✓ Proof The word-span equality (7.2) gives an admissible value in the defining minimum for $\iota(A)$. □

Lemma 7.4.4 (Rank-one extraction from a non-invertible eigenvector). Lean Lean ✓ Stmt *Let A be a normal MPS tensor with bond dimension $D \geq 1$. Suppose that a Kraus operator A^{i_0} is non-invertible and that there exist a nonzero vector $\varphi \in \mathbb{C}^D$ and a nonzero scalar μ such that $A^{i_0}\varphi = \mu\varphi$. Then, for every $\psi \in \mathbb{C}^D$, $|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A)$.*

Proof. ✓ Proof Let $\mathcal{L}(M) = A^{i_0}M$, and choose r so that $\mathcal{L}^r$ annihilates the generalized zero-eigenspace of $\mathcal{L}$; equivalently, $\text{range}(\mathcal{L}^r)$ is its stabilized nonzero-spectral part. The dimension-growth argument gives a level n_0 for which

$$(A^{i_0})^r S_{n_0}(A) = \{(A^{i_0})^r M : M \in M_D(\mathbb{C})\}.$$

Since $A^{i_0}\varphi = \mu\varphi$ and $\mu \neq 0$, iterating gives $(A^{i_0})^r\varphi = \mu^r\varphi$ and hence $\varphi = \mu^{-r}(A^{i_0})^r\varphi \in \text{range}((A^{i_0})^r)$. Therefore, for every $\psi \in \mathbb{C}^D$, $|\varphi\rangle\langle\psi| \in S_{r+n_0}(A)$. The corresponding Fitting-index estimate gives $r + n_0 \leq D^2 - D + 1$. Since

$$A^{i_0}|\varphi\rangle\langle\psi| = |A^{i_0}\varphi\rangle\langle\psi| = \mu|\varphi\rangle\langle\psi|,$$

one has $|\varphi\rangle\langle\psi| = \mu^{-1}A^{i_0}|\varphi\rangle\langle\psi|$. Thus $|\varphi\rangle\langle\psi| \in S_k(A)$ implies $|\varphi\rangle\langle\psi| \in S_{k+1}(A)$, because left multiplication by A^{i_0} maps $S_k(A)$ into $S_{k+1}(A)$. Iterating from $k = r + n_0$ and using $r + n_0 \leq D^2 - D + 1$ gives $|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A)$. □

Theorem 7.4.5 (Sharp rank-one extraction for a primitive channel). Lean ✓ Stmt *Let $D > 0$, suppose $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and assume that A is primitive in the sense of Definition 7.1.1.3. Suppose A^{i_0} is not invertible and $A^{i_0}\varphi = \mu\varphi$ with $\varphi \neq 0$ and $\mu \neq 0$. Then, for every $\psi \in \mathbb{C}^D$,*

$$|\varphi\rangle\langle\psi| \in S_{D^2 - D + 1}(A).$$

This is the quantitative rank-one conclusion of [SPGWC10, Lemma 2(b)].

Proof. ✓ Proof Theorem 7.1.1.13 first gives normality. Apply the Fitting decomposition to left multiplication $\mathcal{L}(M) = A^{i_0}M$. Choose r so that $\mathcal{L}^r$ kills its generalized zero-eigenspace. Dimension growth then gives n_0 with

$$(A^{i_0})^r S_{n_0}(A) = \{(A^{i_0})^r M : M \in M_D(\mathbb{C})\},$$

$$r + n_0 \leq D^2 - D + 1.$$

Since $(A^{i_0})^r \varphi = \mu^r \varphi$ and $\mu \neq 0$, the vector φ lies in the range of $(A^{i_0})^r$. Hence $|\varphi\rangle\langle\psi| \in S_{r+n_0}(A)$ for every ψ . The identity $|\varphi\rangle\langle\psi| = \mu^{-1} A^{i_0} |\varphi\rangle\langle\psi|$ pads this element to the exact length $D^2 - D + 1$. $\square$

Theorem 7.4.6 (Wielandt case (3), one-step non-invertible element). Lean ✓ Stmt *Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If $S_1(A)$ contains a non-invertible matrix X with a nonzero eigenvalue, then $S_{D^2}(A) = M_D(\mathbb{C})$. More explicitly, it is enough to have $X\varphi = \mu\varphi$ with $\varphi \neq 0$ and $\mu \neq 0$. In the terminology of [SPGWC10], this is the case where the first application space $S_1(A)$ contains a non-invertible operator with a nonzero eigenvalue. This is the exact word-span form of [SPGWC10, Theorem 1, case (3)].*

Proof. ✓ Proof Theorem 7.1.1.13 gives normality. Adjoin X as an extra first Kraus operator. The augmented tensor remains normal and has the same exact word spans as A . The eigenvector relation for X is now a one-generator eigenvector relation. The eigenvector-spreading argument gives full fixed-length vector span by length $D-1$, and Lemma 7.4.4 gives the rank-one operators $|\varphi\rangle\langle\psi|$ at length $D^2 - D + 1$. Multiplying these two fixed-length spans gives S_{D^2} for the augmented tensor, hence for A . $\square$

Corollary 7.4.7 (Wielandt case (3), index bound). Lean ✓ Stmt *Under the hypotheses of Theorem 7.4.6, one has $\iota(A) \leq D^2$.*

Proof. ✓ Proof The equality $S_{D^2}(A) = M_D(\mathbb{C})$ gives an admissible value in the defining minimum for $\iota(A)$. $\square$

Theorem 7.4.8 (Wielandt's inequality — general bound). Lean ✓ Stmt *Let A be a normalized MPS tensor with $D > 0$, $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, that is primitive in the sense of Definition 7.1.1.3. Then*

$$\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2,$$

where $\iota(A) = \min\{n \geq 1 : S_n(A) = M_D(\mathbb{C})\}$ is the full-Kraus-rank index. This is [SPGWC10, Theorem 1, case (1)].

Proof. ✓ Proof Theorem 7.1.1.13 gives normality. By Theorem 7.2.4, there exists a *positive-length* word w with $|w| \leq D^2 - d' + 1$ and $\text{tr}(A^w) \neq 0$, where $d' = \text{kr}(A)$. The matrix A^w has nonzero trace, hence a nonzero eigenvalue μ and eigenvector φ by Theorem [LTC26, “Nonzero trace implies eigenvector”]. Set $n = |w|$ and let $B = A^{[n]}$ be the blocked tensor. Then B is normal by Theorem C.4.1, and the encoding of w as a single block index gives $B^{i_0} = A^w$.

1. If A^w is invertible, then $S_{D^2}(B) = M_D(\mathbb{C})$, whence $S_{D^2n}(A) = M_D(\mathbb{C})$.
2. If A^w is non-invertible, then the eigenvector φ spans a proper subspace, and the rank-one extraction argument gives $S_{D^2}(B) = M_D(\mathbb{C})$, whence $S_{D^2n}(A) = M_D(\mathbb{C})$.

In the invertible case, the invertible Kraus operator B^{i_0} allows direct dimension growth, giving $S_{D^2-k_B+1}(B) = M_D(\mathbb{C})$, where $k_B = \text{kr}(B)$. In the non-invertible case, the bound $S_{D^2}(B) = M_D(\mathbb{C})$ follows from tracking indices through the rank-one extraction argument of [SPGWC10, Lemma 2(b)]; the separate blocked support construction in Theorem C.4.6 does not supply this explicit bound. In both cases, $\iota(A) \leq D^2 n \leq D^2(D^2 - d' + 1)$. $\square$

Remark 7.4.9 (Quantitative and qualitative Wielandt bounds). ✓ Stmt The quantitative bound $\iota(A) \leq (D^2 - d' + 1)D^2$ is cited from [SPGWC10, Theorem 1, case (1)]. The later Fundamental Theorem needs only the qualitative conclusion $S_N(A) = M_D(\mathbb{C})$ for some N . The blocked construction in Theorem C.4.6 supplies this conclusion directly from normality; the explicit index bound is not needed there.

7.5 Fixed-length matrix spanning

The preceding sections establish the cumulative-spanning part of the Lemma 2(a) argument from [SPGWC10]: eigenvector spreading gives a cumulative vector span $K_{D-1}(A, \varphi) = \mathbb{C}^D$. The next stage is to pass to a span at one fixed length, namely to find n_0 such that $S_{n_0}(A) = M_D(\mathbb{C})$ (i.e. word products of *exactly* one length span all matrices). The product-span facts used here are proved in Section C.1; the blocked fixed-length construction is in Section C.4.

Lemma 7.5.1 (Eigenvector padding). Lean ✓ Stmt Suppose $A^{i_0}\varphi = \mu\varphi$ with $\mu \neq 0$. Then $K_n(A, \varphi) = H_n(A, \varphi)$ for every n . In particular, if $K_n(A, \varphi) = \mathbb{C}^D$, then $H_n(A, \varphi) = \mathbb{C}^D$.

Proof. ✓ Proof Any word w with $|w| \leq n$ can be padded to exact length n by appending $k = n - |w|$ copies of the index i_0 . The eigenvector relation gives $A^{w i_0^k}\varphi = \mu^k A^w\varphi$, so the padded vector is a nonzero scalar multiple of the original. Hence every generator of K_n lies in H_n . The reverse inclusion $H_n \leq K_n$ is immediate. $\square$

Theorem 7.5.2 (Fixed-length vector spanning implies matrix spanning). Lean ✓ Stmt Assume:

1. $H_n(A, \varphi) = \mathbb{C}^D$ (length- n word products applied to φ span all of $\mathbb{C}^D$), and
2. for each standard basis vector e_j , the rank-one operator $|\varphi\rangle\langle e_j|$ lies in $S_m(A)$.

Then $S_{n+m}(A) = M_D(\mathbb{C})$.

Proof. ✓ Proof Fix a matrix unit E_{ij} . By hypothesis (1), there exists $M_i \in S_n(A)$ with $M_i\varphi = e_i$. By hypothesis (2), $|\varphi\rangle\langle e_j| \in S_m(A)$. Their product satisfies

$$M_i|\varphi\rangle\langle e_j| = |M_i\varphi\rangle\langle e_j| = |e_i\rangle\langle e_j| = E_{ij}.$$

Lemma C.1.1 therefore gives $E_{ij} \in S_{n+m}(A)$. Since the matrix units span $M_D(\mathbb{C})$, $S_{n+m}(A) = M_D(\mathbb{C})$. $\square$

Lemma 7.5.3 (Block injectivity and blocked-tensor injectivity). Lean ✓ Stmt A tensor A is L -block injective if and only if its L -blocked tensor $A^{[L]}$ is injective.

Proof. ✓ Proof The blocked physical alphabet is a reindexing of words of length L . Decoding blocked indices therefore identifies the two spanning families, and hence the two injectivity conditions are equivalent. $\square$

Lemma 7.5.4 (Blocking transfer). Lean ✓ Stmt If the blocked tensor $A^{[L]}$ satisfies $S_n(A^{[L]}) = M_D(\mathbb{C})$, then $S_{nL}(A) = M_D(\mathbb{C})$. In particular, A is L -block injective if and only if $A^{[L]}$ is one-site injective.

Proof. ✓ Proof By the blocked-tensor definition (2.13), each length- n word in the blocked alphabet $\{0, \dots, d^L - 1\}$ decodes to a length- nL word in the original alphabet $\{0, \dots, d - 1\}$, so $S_n(A^{[L]}) \subseteq S_{nL}(A)$. For $n = 1$, the d^L blocked letters are indexed by the length- L words of A , giving $S_1(A^{[L]}) = S_L(A)$, which proves the stated equivalence. $\square$

Theorem 7.5.5 (Bounded injective blocking for left-canonical normal tensors). Lean ✓ Stmt *Let A be a left-canonical normal MPS tensor with bond dimension $D > 0$. Then there is a positive blocking length $L \leq D^4$ such that the blocked tensor $A^{[L]}$ is one-site injective.*

This is the blocked-injectivity form of the quantum Wielandt input used in the Fundamental Theorem for translation-invariant tensors. It corresponds to the statement in [CPGSV16, Section II] that every normal tensor becomes injective after at most D^4 blockings, using the index estimate of [SPGWC10, Theorem 1]. The left-canonical/trace-preserving hypothesis is included because this is the canonical-form context in which the blocked-injectivity input is used.

Proof. ✓ Proof Normality is eventual full Kraus rank, so Theorem 7.1.1.13 gives paper primitivity. Theorem 7.4.8 then yields

$$\iota(A) \leq (D^2 - \text{kr}(A) + 1)D^2.$$

Trace preservation gives a nonzero Kraus operator, hence $\text{kr}(A) \geq 1$ and the right-hand side is at most D^4 . Set $L = \iota(A)$. By definition, $L > 0$ and $S_L(A) = M_D(\mathbb{C})$. Lemma 7.5.4 identifies the latter equality with one-site injectivity of $A^{[L]}$. $\square$

Theorem 7.5.6 (Exact quantum-Wielandt blocking length). Lean ✓ Stmt *Let A be a left-canonical normal MPS tensor with bond dimension $D > 0$. Then $S_{D^4}(A) = M_D(\mathbb{C})$; equivalently, A is D^4 -block injective.*

Proof. ✓ Proof Choose L with $0 < L \leq D^4$ and $S_L(A) = M_D(\mathbb{C})$, and write $D^4 = L + k$. Induction on k , using positive-length propagation at each step, gives $S_{L+k}(A) = M_D(\mathbb{C})$. $\square$

Theorem 7.5.7 (Common exact block-injectivity length). Lean ✓ Stmt *Let $(A_j)_{j=1}^g$ be a finite family of positive-bond-dimension, left-canonical normal tensors. There is a positive integer L such that every A_j is L -block injective.*

Proof. ✓ Proof Apply Theorem 7.5.5 to each block. The product of the finitely many positive lengths is a common positive multiple, and exact block injectivity persists under positive multiples. $\square$

Corollary 7.5.8 (Uniform injective blocking for separated normal-canonical families). Lean ✓ Stmt *Let $(A_x)_{x \in X}$ be a finite separated normal-canonical family whose blocks have positive virtual bond dimension. Then there is a positive integer L such that every blocked tensor $A_x^{[L]}$ is one-site injective. The same conclusion holds simultaneously for two finite separated normal-canonical families whose blocks have positive virtual bond dimension.*

Here the family is a basis of normal-canonical representatives in the sense of Definition 8.9.1.2; the blocks are distinct and the weight moduli are non-increasing but need not be strictly ordered. The representative family is assumed to have been selected; this statement does not reconstruct the full CPSV sector decomposition with repeated copies inside one sector. This scope restriction is recorded in [con26b].

Proof. ✓ Proof Apply Theorem 7.5.5 to each block. Taking the product of the finitely many resulting positive blocking lengths gives a common positive multiple, and injectivity persists under positive further blocking. $\square$

Remark 7.5.9 (Quantitative and blocked rank-one arguments). ✓ Stmt The quantitative rank-one statement of [SPGWC10, Lemma 2(b)] is Theorem 7.4.5. Theorem 7.5.2 turns that rank-one conclusion and fixed-length vector spreading into matrix spanning. Section C.4 also proves a separate blocked construction, Theorem C.4.6, which first extracts a rank-one operator in a blocked tensor and then applies the fixed-length matrix-spanning argument; its fixed-length conclusion is a consequence of its normality hypothesis, not the quantitative statement of source Lemma 2(b).

Remark 7.5.10 (Use of the quantum Wielandt theorem in the Fundamental Theorem). The cumulative estimate $T_{D^2}(A) = M_D(\mathbb{C})$ in Theorem 7.2.1 supports the proof of Lemma 1; it is not a conclusion of source Theorem 1. The source theorem instead bounds the first positive N for which $S_N(A) = M_D(\mathbb{C})$. This exact-length statement matches injectivity of Γ_N and passes through blocking by Lemma 7.5.4.

The blocked normality and rank-one constructions used below are proved in Section C.4.

7.6 Primitive MPS tensors

Definition 7.6.1 (Primitive MPS tensor). Lean ✓ Stmt Assume $D > 0$. A tensor A together with a matrix $\rho \in M_D(\mathbb{C})$ is *primitive* if

$$\begin{aligned} \sum_i (A^i)^\dagger A^i &= \mathbb{1}, \\ \rho &\geq 0, \quad \rho \neq 0, \quad \mathcal{E}_A(\rho) = \rho, \\ \rho_{\text{spec}}(\mathcal{E}_A - P) &< 1, \end{aligned}$$

where P is the fixed-point projection associated to ρ . When the choice of ρ is irrelevant, we simply say that A is a primitive MPS tensor. This condition combines a complementary spectral gap with a nonzero positive semidefinite fixed point. It is not the paper definition in Definition 7.1.1.3, which is the uniform spreading condition (7.1). With the additional hypothesis $\rho > 0$, the complementary-gap condition implies strong irreducibility and hence paper primitivity by Theorem 7.1.1.13.

7.7 Primitivity and normality

Throughout this section, primitivity means the condition of Definition 7.6.1: the tensor is in TP gauge, it has a nonzero PSD fixed point, and the complementary map has spectral radius < 1 .

7.7.1 From primitivity to normality

The primitivity condition of Definition 7.6.1 combines TP normalization, a nonzero PSD fixed point, and a complementary transfer-map gap. It is connected to normality through the following results.

Remark 7.7.1.1 (Complementary-gap and paper primitivity). ✓ Stmt Definition 7.6.1 assumes TP normalization, a nonzero positive semidefinite fixed point ρ , and $\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1$. Without $\rho > 0$, these hypotheses do not imply irreducibility or normality; for example, a fixed point supported on $\text{diag}(1, 0)$ need not see the complementary invariant subspace.

If $\rho > 0$, the transfer map is irreducible and the complementary gap gives peripheral spectrum $\{1\}$. Thus the tensor is strongly irreducible in the sense of Definition 7.1.1.6. Proposition 3 then gives uniform spreading and eventual full Kraus rank. This implication compares the two notions; it does not identify Definition 7.6.1 with the paper definition.

7.7.2 Primitive implies irreducible

Lemma 7.7.2.1 (Primitive tensors with positive-definite fixed point are irreducible). [Lean](#)

[✓ Stmt](#) If A is a primitive MPS tensor with PSD fixed point ρ and ρ is positive definite, then A is an irreducible tensor. This is part of [SPGWC10, Proposition 3] (see also [CPGSV16, Section 2.3]).

Proof. [✓ Proof](#) By contrapositive. Assume A has an invariant projection P ($P \neq 0$, $P \neq 1$, $(1 - P)A^i P = 0$ for all i). Set $\sigma = P\rho P$.

1. *Invariance of σ under $\mathcal{E}_A^n$:* the projection relation $(1 - P)A^i P = 0$ implies $(1 - P)\mathcal{E}_A^n(\sigma) = 0$ for all n (by induction on n).
2. *Convergence:* By Lemma C.5.6, $\mathcal{E}_A^n(\sigma) \rightarrow \frac{\text{tr}(\sigma)}{\text{tr}(\rho)}\rho$. Hence $(1 - P)\frac{\text{tr}(\sigma)}{\text{tr}(\rho)}\rho = 0$.
3. *Nonzero scalar:* Since $P \neq 0$ and ρ is positive definite, $\text{tr}(P\rho P) > 0$, so $\frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \neq 0$. Therefore $(1 - P)\rho = 0$.
4. *Contradiction:* Since ρ is positive definite, it is a unit in $M_D(\mathbb{C})$. From $(1 - P)\rho = 0$ we get $1 - P = 0$, i.e. $P = 1$, contradicting $P \neq 1$.

□

7.7.3 From irreducibility to full spanning via Burnside's theorem

The irreducibility condition (Definition 8.1.1.1) is formulated in terms of invariant *projections*. For Burnside-style arguments it is more natural to work with invariant *subspaces* of $\mathbb{C}^D$ under a finite matrix family.

Definition 7.7.3.1 (Algebra span). [Lean](#) [✓ Stmt](#) For a finite matrix family A , let $\text{alg}(A)$ be the unital $\mathbb{C}$ -subalgebra of $M_D(\mathbb{C})$ generated by the matrices $\{A^i\}_{i=0}^{d-1}$: $\text{alg}(A) = \mathbb{C}\langle A^i : i = 0, \dots, d-1 \rangle \subseteq M_D(\mathbb{C})$.

Definition 7.7.3.2 (Invariant submodule). [Lean](#) [✓ Stmt](#) A subspace $W \leq \mathbb{C}^D$ is A -invariant if $A^i W \subseteq W$ for every i .

Definition 7.7.3.3 (Irreducible action). [Lean](#) [✓ Stmt](#) The matrices in A act *irreducibly* on $\mathbb{C}^D$ if every A -invariant subspace is trivial, i.e. equal to 0 or to $\mathbb{C}^D$.

Theorem 7.7.3.4 (Irreducible tensor implies irreducible action). [Lean](#) [✓ Stmt](#) If A is irreducible as a tensor (Definition 8.1.1.1), then the Kraus operators act irreducibly on $\mathbb{C}^D$ (Definition 7.7.3.3).

Proof. [✓ Proof](#) Let $W \leq \mathbb{C}^D$ be an A -invariant subspace. In finite dimension the orthogonal projection P onto W exists. Invariance of W implies $(1 - P)A^i P = 0$ for all i . Thus P is a nontrivial invariant projection unless $W = 0$ or $W = \mathbb{C}^D$, contradicting irreducibility of the tensor. □

Theorem 7.7.3.5 (Burnside's theorem for matrix algebras). [Lean](#) [✓ Stmt](#) Assume $D > 0$. If A acts irreducibly on $\mathbb{C}^D$, then the generated unital subalgebra is the full matrix algebra: $\text{alg}(A) = M_D(\mathbb{C})$. This is the complex finite-dimensional case of Burnside's theorem (equivalently, Jacobson's density theorem); see [Jac09].

Proof. ✓ Proof Identify $\text{alg}(A)$ with its image under the algebra equivalence $M_D(\mathbb{C}) \simeq_{\mathbb{C}} \text{End}_{\mathbb{C}}(\mathbb{C}^D)$. The irreducible-action hypothesis makes $\mathbb{C}^D$ a simple module for this algebra. Over the algebraically closed field $\mathbb{C}$, Schur's lemma identifies the endomorphisms commuting with the action with scalars, and Jacobson density then forces the action map to be surjective onto all of $\text{End}_{\mathbb{C}}(\mathbb{C}^D)$. $\square$

Remark 7.7.3.6. ✓ Stmt This Burnside/Jacobson-density step is a substantial external algebraic ingredient in the chapter. We use it here as the finite-dimensional complex Burnside theorem, namely the statement that an irreducible matrix algebra action already generates the full endomorphism algebra.

7.7.4 From primitivity to strong irreducibility and normality

The results in this subsection remove the one-step padding assumption from the normality conclusions. They connect primitivity in Definition 7.6.1, formulated via a complementary transfer-map gap, directly to normality (eventually full Kraus rank), using the convergence of the transfer-map iterates to the projection onto the fixed-point subspace.

Lemma 7.7.4.1 (Primitive tensors with positive-definite fixed point are strongly irreducible).

Lean ✓ Stmt *If A is a primitive MPS tensor with positive-definite fixed point ρ , then it satisfies the strengthened condition in Definition 7.1.1.6: $\mathcal{E}_A$ is irreducible, the fixed point ρ is positive definite, and the peripheral spectrum of $\mathcal{E}_A$ is $\{1\}$. The source clause is [SPGWC10, Proposition 3(c)]; the definition used here records irreducibility separately.*

Proof. ✓ Proof Lemma C.5.7 gives irreducibility of $\mathcal{E}_A$. The complementary-gap hypothesis gives peripheral primitivity directly: the only eigenvalue of modulus one is 1. Together with the positive-definite fixed point ρ , these are exactly the three conditions in Definition 7.1.1.6. The complementary-gap consequences are proved in Section C.5. $\square$

Theorem 7.7.4.2 (Primitive MPS with positive-definite fixed point is normal). Lean ✓ Stmt *If A is a primitive MPS tensor with positive-definite fixed point ρ , then A is normal (has eventually full Kraus rank). No identity-in-the-one-step-span condition is needed.*

This connects the complementary-gap condition in Definition 7.6.1 directly to normality.

Proof. ✓ Proof By Lemma 7.7.4.1, A is strongly irreducible. Theorem 7.1.1.13 identifies strong irreducibility with eventual full Kraus rank, hence with normality. $\square$

7.8 One-step padding and exact word spans

Cumulative spanning, $T_N(A) = M_D(\mathbb{C})$, is weaker than exact-length spanning $S_L(A) = M_D(\mathbb{C})$. If $\mathbb{1} \in S_1(A)$, shorter words can be padded to a common length. Section C.6 proves this conversion and records a counterexample without the one-step padding condition.

Theorem 7.8.1 (Homogeneous word-span propagation under normalization). Lean Lean Lean

✓ Stmt *Suppose*

$$\sum_i A^i (A^i)^\dagger = \mathbb{1}, \tag{7.3}$$

and $S_L(A) = M_D(\mathbb{C})$. Then $S_m(A) = M_D(\mathbb{C})$ for every $m \geq L$. Equivalently, L -block injectivity propagates to every larger homogeneous length.

Proof. ✓ **Proof** If $S_n(A) = M_D(\mathbb{C})$, then for every $X \in M_D(\mathbb{C})$, substituting (7.3) gives

$$X = \left(\sum_i A^i (A^i)^\dagger \right) X = \sum_i A^i ((A^i)^\dagger X) \in S_1(A) S_n(A) \subseteq S_{n+1}(A).$$

Indeed, $(A^i)^\dagger X \in S_n(A)$ because $S_n(A) = M_D(\mathbb{C})$. Thus $S_{n+1}(A) = M_D(\mathbb{C})$. Iterating this implication gives the result for every $m \geq L$. □

Theorem 7.8.2 (Algebra span plus one-step padding implies normality). Lean ✓ Stmt *If $\text{alg}(A) = M_D(\mathbb{C})$ and $\mathbb{1} \in S_1(A)$, then A is normal.*

Proof. ✓ **Proof** Since $\text{alg}(A) = M_D(\mathbb{C})$, the ascending chain condition for finite-dimensional subspaces gives an N such that $T_N(A) = M_D(\mathbb{C})$. Apply Theorem C.6.5. □

Remark 7.8.3 (Normal canonical form avoids one-step padding). The identity-in-the-one-step-span results in Section C.6 are needed only when cumulative spanning is upgraded to exact-length spanning. The normal-canonical theory used later in Chapters 8–9 does not pass through this step: it works directly with irreducible TP-normalized blocks whose blocked transfer maps are primitive and, once the corresponding weighted block data are available, places them in normal canonical form in the sense of [CPGSV16]. There are analogous cumulative-span lemmas for blocked irreducible tensors and eigenvector-spreading arguments, but they are not needed for the implication stated below.

7.9 Block injectivity from TP/primitive/irreducible blocks

Theorem 7.9.1 (TP-primitive irreducible tensor is injective after blocking). Lean ✓ Stmt *Let A be a left-canonical MPS tensor with $D > 0$. Suppose that A is tensor-irreducible and that its transfer map is primitive. Then there exists a positive blocking length L such that $A^{[L]}$ is one-site injective.*

Proof. ✓ **Proof** If $D = 1$, trace preservation gives a nonzero one-site Kraus matrix, and every nonzero scalar matrix spans $M_1(\mathbb{C})$. Hence A is one-site injective, so Lemma 2.4.3 gives the positive witness $L = 1$. Suppose now that $D \geq 2$. Theorem 8.11.1.1 gives a block-injectivity witness L . If $L = 0$, Corollary C.1.3 would force $D = 1$, a contradiction. Thus $L > 0$, and the blocked-chain equivalence identifies this L -block injectivity with one-site injectivity of the blocked tensor $A^{[L]}$. □

Remark 7.9.2. Blocking preserves normality by Theorem C.4.1, proved in Section C.4.

Chapter 8

Canonical Form Reduction

This is the first of three chapters that together prove the Fundamental Theorem of matrix product states. Here an arbitrary translation-invariant MPS tensor is reduced to a weighted family of primitive blocks; Chapter 9 constructs the basis of normal tensors carried by those blocks and compares their coefficients, and Chapter 10 proves the theorem itself. The reduction follows [PGVWC07], [CPGSV16], [CPGSV21], and [Wol12, Chapter 6]. Its goal is to expose the primitive blocks inside the tensor. The transfer operator of an arbitrary tensor can be reducible and periodic. The reduction first splits invariant subspaces. For each irreducible trace-preserving block, it then powers the transfer map, extracts the adjoint-fixed cyclic projections, compresses to their corners, and proves that each corner is primitive and tensor-irreducible. The blocks are returned in a fixed gauge. The periodic decomposition theorem of [PGVWC07, Theorem 5] gives p translated p -periodic state summands when $p \mid N$ and a zero state otherwise. We state this fixed-length decomposition below before using its blocked cyclic-sector counterpart in Theorem 8.3.1.1.

Definition 8.1 (Periodic projector-component chain). [Lean] [Lean] [Lean] [✓ Stmt] Let $A = (A^i)_{i=0}^{d-1}$ have bond dimension D , let $(Q_u)_{u \in \mathbb{Z}/p\mathbb{Z}}$ be cyclic projections, and fix a starting sector u . For each $0 \leq j < N$, the corresponding chain of length N has local matrices

$$B_{u,j}^i = Q_{u+j} A^i Q_{u+j+1}, \quad (8.1)$$

where sector indices are read modulo p . It has the same bond dimension D as A and repeats with period p .

Theorem 8.2 (Boundary traces and component-chain coefficients). [Lean] [Lean] [✓ Stmt] Suppose that the orthogonal projections satisfy $\sum_u Q_u = \mathbb{1}$ and $Q_u A^i = A^i Q_{u+1}$. For a word $\sigma = (i_0, \dots, i_{N-1})$, define the boundary product

$$F_u(\sigma) = Q_u A^{i_0} Q_{u+1} A^{i_1} \dots Q_{u+N-1} A^{i_{N-1}} Q_{u+N}. \quad (8.2)$$

At every length $N \geq 0$,

$$V^{(N)}(A)_\sigma = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \text{tr} F_u(\sigma). \quad (8.3)$$

If $N > 0$, then $\text{tr} F_u(\sigma)$ is the coefficient of the explicit component chain from Definition 8.1. At $N = 0$, the boundary product is $F_u(\emptyset) = Q_u$, so

$$\text{tr}(\mathbb{1}) = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \text{tr}(Q_u). \quad (8.4)$$

This is not the sum of the ordinary empty component-chain coefficients: an empty chain evaluates to $\mathbb{1}$ and does not retain its starting projection.

Proof. ✓ **Proof** The cyclic relation and orthogonality give $F_u(\sigma) = Q_u A^{i_0} \dots A^{i_{N-1}}$. Hence

$$\mathrm{tr}(A^{i_0} \dots A^{i_{N-1}}) = \mathrm{tr} \left(\left(\sum_u Q_u \right) A^{i_0} \dots A^{i_{N-1}} \right) = \sum_u \mathrm{tr} F_u(\sigma). \quad (8.5)$$

For $N > 0$, multiplying the local matrices in (8.1) gives $F_u(\sigma)$, proving the coefficient identity. For $N = 0$, the same trace calculation reduces to (8.4). $\square$

Theorem 8.3 (Normalized periodic state-vector decomposition). Lean Lean Lean ✓ **Stmt** This is a normalized form of the state-vector conclusion in [PGVWC07, Theorem 5]. Let A be an irreducible left-canonical tensor whose peripheral transfer spectrum is the set of p -th roots of unity. If $p \mid N$, there are orthogonal cyclic projections $(Q_u)_{u \in \mathbb{Z}/p\mathbb{Z}}$ such that $\sum_u Q_u = \mathbb{1}$, $Q_u A^i = A^i Q_{u+1}$, and $Q_{u+N} = Q_u$. For every ring length $N \geq 0$, its coefficients satisfy the boundary-aware decomposition

$$V^{(N)}(A)_\sigma = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \mathrm{tr} F_u(\sigma). \quad (8.6)$$

For $N > 0$, this becomes the source's decomposition into the p explicit component chains:

$$|V^{(N)}(A)\rangle = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} |V^{(N)}(B_u)\rangle. \quad (8.7)$$

Each B_u is a p -periodic MPS of bond dimension D with local matrices given by (8.1). At $N = 0$, (8.6) instead uses the boundary traces $\mathrm{tr}(Q_u)$ from (8.4); it does not identify them with ordinary empty-chain coefficients. If $p \nmid N$, then $|V^{(N)}(A)\rangle = 0$.

Proof. ✓ **Proof** Choose the cyclic projections supplied by periodicity. If $p \mid N$, then $Q_{u+N} = Q_u$, and Theorem 8.2 gives (8.6) at every length. For $N > 0$, the explicit component-chain decomposition follows by inserting $\sum_u Q_u = \mathbb{1}$ and multiplying the local corner matrices; this proves (8.7). The separate coefficient identity in Theorem 8.2 identifies its summands with the boundary traces. If $p \nmid N$, then $Q_{u+N} \neq Q_u$; cyclicity of the trace and orthogonality give $\mathrm{tr} F_u(\sigma) = 0$ for every u and σ . $\square$

Two normalizations appear. The unital orientation $\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}$ is used in [PGVWC07, Theorem 4]; the trace-preserving (left-canonical) orientation $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$ is used in the TP-gauge and after-blocking reductions below. They exchange under the conjugate-transposed Kraus family $K_i := (A^i)^\dagger$, whose transfer map is the Frobenius adjoint of $\mathcal{E}_A$.

8.1 Invariant-subspace decomposition and irreducible blocks

The block decomposition rests on one principle: a positive semidefinite fixed point of $\mathcal{E}_A$ that is not full rank has a proper invariant support, which block-diagonalizes the Kraus operators without changing the MPV family. Iterating the resulting splitting decomposes any tensor into irreducible blocks.

Lemma 8.1.1 (Lower-triangular Kraus condition implies corner invariance). Lean ✓ **Stmt** Let A be an MPS tensor and let P be an orthogonal projection. If $(\mathbb{1} - P)A^i P = 0$ for every physical index i , then $P\mathcal{E}_A(PXP)P = \mathcal{E}_A(PXP)$ for every $X \in M_D(\mathbb{C})$.

Proof. ✓ Proof Expand the transfer map using (2.4) and use $(\mathbb{1} - P)A^iP = 0$ to reduce each Kraus term to the P -corner. $\square$

Theorem 8.1.2 (Invariant projection gives a two-block decomposition). Lean ✓ Stmt Suppose P is an orthogonal projection satisfying $(\mathbb{1} - P)A^iP = 0$ for every physical index i . Then there exist n, m with $n + m = D$ and MPS tensors A_1, A_2 of bond dimensions n and m such that A and the two-block tensor $A_1 \oplus A_2$ generate the same MPV family in the sense of Definition 2.3.3.

Proof. ✓ Proof Diagonalize P by a unitary U . Since $P^2 = P$, the diagonal form of P has only 0- and 1-eigenvalues, so the basis splits as $n + m = D$. Conjugating A by U preserves the MPV family by Theorem D.1.1. In this basis the relation $(\mathbb{1} - P)A^iP = 0$ makes each letter upper block triangular. Write the conjugated letters as

$$B^i = \begin{pmatrix} A_1^i & * \\ 0 & A_2^i \end{pmatrix}.$$

Since $B^{i_1} \dots B^{i_N}$ is again upper block triangular, its trace satisfies

$$\text{tr}(B^{i_1} \dots B^{i_N}) = \text{tr}(A_1^{i_1} \dots A_1^{i_N}) + \text{tr}(A_2^{i_1} \dots A_2^{i_N}). \quad (8.8)$$

Hence the block-diagonal tensor $A_1 \oplus A_2$ has the same MPV coefficients as the conjugated tensor. $\square$

Definition 8.1.3 (Invariant projection condition). Lean ✓ Stmt An MPS tensor A has an *invariant projection* if there exists an orthogonal projection P with $P \neq 0$, $P \neq \mathbb{1}$, and $(\mathbb{1} - P)A^iP = 0$ for all i . A tensor is *irreducible* if it has no nontrivial invariant projection (Definition 8.1.1.1).

Theorem 8.1.4 (PSD fixed point gives invariant projection). Lean ✓ Stmt Let $\rho \geq 0$ be a fixed point of $\mathcal{E}_A$, and let P be its support projection. Then $(\mathbb{1} - P)A^iP = 0$ for all i : the range of P is invariant under every A^i .

Proof. ✓ Proof From $\mathcal{E}_A(\rho) = \rho$ and positivity of each summand $A^i\rho(A^i)^\dagger$, one gets $(\mathbb{1} - P)A^i\rho(A^i)^\dagger(\mathbb{1} - P) = 0$ for each i . Since ρ is supported on the range of P , this forces $(\mathbb{1} - P)A^iP = 0$.

This is the finite-dimensional case of [Wol12, Proposition 6.10]: the support projection of any stationary state is sub-harmonic. The equivalence between irreducibility and absence of nontrivial invariant projections is [Wol12, Theorem 6.2]; see also [CPGSV21, Section IV.A.1]. $\square$

Theorem 8.1.5 (Two-block decomposition). Lean ✓ Stmt If $\mathcal{E}_A$ has a nonzero PSD fixed point ρ that is not positive definite, then there exist MPS tensors A_1, A_2 of smaller bond dimensions with $\mathcal{V}(A) = \mathcal{V}(A_1 \oplus A_2)$.

Proof. ✓ Proof The support projection P of ρ is neither 0 nor $\mathbb{1}$, since $\rho \neq 0$ and ρ is not positive definite. Theorem 8.1.4 gives $(\mathbb{1} - P)A^iP = 0$, so a unitary change of basis diagonalizing P puts A^i in upper-triangular block form with diagonal blocks B_{11}^i, B_{22}^i . Theorem D.1.1 preserves the MPV under the conjugation, and the trace identity (8.8) shows that deleting the off-diagonal blocks preserves all MPV coefficients. $\square$

8.1.1 Iterated reduction to irreducible blocks

Iterating the two-block splitting on bond dimension decomposes any tensor into irreducible blocks; the remaining lemmas record the spectral facts about irreducible unital blocks and the algebra-spanning input used later.

Definition 8.1.1.1 (Irreducible tensor). [Lean](#) [✓ Stmt](#) An MPS tensor A is *irreducible* if it has no nontrivial invariant projection.

Theorem 8.1.1.2 (Irreducible block decomposition). [Lean](#) [✓ Stmt](#) Every MPS tensor A admits a block-diagonal tensor $\bigoplus_{k=1}^r A_k$ with each A_k irreducible, $\mathcal{V}(A) = \mathcal{V}(\bigoplus_k A_k)$, and $\sum_{k=1}^r D_k = D$.

Proof. [✓ Proof](#) By strong induction on D . If A is irreducible, take $r = 1$. Otherwise a nontrivial invariant projection P gives the upper-triangular splitting of Theorem 8.1.5; deleting the off-diagonal part preserves the MPV family and yields two blocks of strictly smaller bond dimension. Apply the induction hypothesis to each block.

Both [PGVWC07, Theorem 4] and [CPGSV21, Section IV.A.1] obtain the block decomposition by the same invariant-projection splitting. $\square$

8.2 Nonzero blocks and the trace-preserving gauge

In an irreducible block decomposition some blocks may have all-zero Kraus operators. Such a block contributes nothing to a matrix product vector at any positive length and is therefore discarded. The structural direct-sum identity implies that the retained block dimensions satisfy $\sum_k D_k \leq D$ without using an empty-word coefficient. Each retained block is then placed in the trace-preserving gauge.

Theorem 8.2.1 (Blockwise TP-gauge normalization of irreducible blocks). [Lean](#) [✓ Stmt](#) Suppose A generates the same MPV family as $\bigoplus_{k=1}^r B_k$, where each B_k is irreducible with at least one nonzero Kraus operator. Then there are nonzero weights $(\mu_k)_{k=1}^r$ and blocks $(C_k)_{k=1}^r$ such that, for every N and every σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k C_k\right)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(C_k)_\sigma, \quad (8.9)$$

each C_k is irreducible and left-canonical, $\sum_i (C_k^i)^\dagger C_k^i = \mathbb{1}$, with positive bond dimension. This is the blockwise form used in the arbitrary-tensor trace-preserving gauge below.

Proof. [✓ Proof](#) The Perron–Frobenius TP-gauge construction is applied to each irreducible block using the gauge (5.3). Gauge equivalence preserves irreducibility and the MPV family, and the positive scaling factors are absorbed into the weights μ_k . $\square$

8.2.1 Trace-preserving gauge for arbitrary tensors

Theorem 8.2.1.1 (TP-gauge reduction for arbitrary tensors). [Lean](#) [✓ Stmt](#) For any MPS tensor A , there exist nontrivial blocks $(B_k)_{k=1}^r$ with nonzero weights $(\mu_k)_{k=1}^r$, each block irreducible, left-canonical ($\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$), and of positive bond dimension, such that, for every $N \geq 1$ and every σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma, \quad (8.10)$$

and $\sum_{k=1}^r D_k \leq D$.

Proof. ✓ Proof Theorem D.1.3 supplies the nonzero irreducible blocks $(C_k)_{k=1}^r$, with $V^{(N)}(A)_\sigma = V^{(N)}(\bigoplus_{k=1}^r C_k)_\sigma$ for every $N \geq 1$ and every σ . Applying the Perron–Frobenius TP gauge (5.3) to each C_k gives a left-canonical block B_k with weight μ_k , and

$$V^{(N)}\left(\bigoplus_{k=1}^r C_k\right)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma.$$

The gauge preserves every block dimension, so the structural bound $\sum_k D_k \leq D$ is unchanged. $\square$

8.3 Blocking, period removal, and primitive blocks

Blocking by an integer p carries MPV equalities, weights, transfer maps, and identifications of the blocked physical alphabet. For an irreducible trace-preserving block, powering first collapses the peripheral eigenvalue set to $\{1\}$; the powered map can nevertheless split into several cyclic corners. The reduction compresses to those corners and proves primitivity there. A later least-common-multiple step places a finite family of the resulting sectors over one blocked alphabet.

8.3.1 Period removal by cyclic sectors

The period-removing blocking of [CPGSV16, Section 2.3] has four distinct steps. Power the transfer map by its period, obtain the adjoint-fixed cyclic projections, compress to their corners, and prove that each corner restriction is irreducible. Only then does the peripheral set $\{1\}$ imply primitivity of the sector maps. The cyclic projections come from [Wol12, Theorem 6.6].

Theorem 8.3.1.1 (Period removal by blocking). Lean ✓ Stmt *Let A be a left-canonical irreducible tensor. Then there is a period-removing length $m \geq 1$ such that $A^{[m]}$ admits:*

1. *left-canonical sector tensors $(C_k)_{k=1}^m$ of nonzero bond dimension,*
2. *a unit-weight direct-sum MPV decomposition $A^{[m]} \sim \bigoplus_{k=1}^m C_k$,*
3. *orthogonal projections $(P_k)_{k=1}^m$ summing to $\mathbb{1}$ with $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$,*
4. *commutation relations $P_k(A^{[m]})^i = (A^{[m]})^i P_k$ for every blocked letter i ,*
5. *the sector trace formula $f_{C_k}(\sigma) = \text{tr}(P_k(A^{[m]})^{\sigma_1} \dots (A^{[m]})^{\sigma_N})$,*
6. *linear isomorphisms onto the compression corners $\varphi_k : M_{\dim C_k}(\mathbb{C}) \xrightarrow{\sim} P_k M_D(\mathbb{C}) P_k$,*
7. *transfer-intertwining identities $\varphi_k(\mathcal{E}_{C_k^\dagger}(X)) = \mathcal{E}_{(P_k A^{[m]})^\dagger}(\varphi_k(X))$,*
8. *multiplicativity and adjoint compatibility $\varphi_k(XY) = \varphi_k(X)\varphi_k(Y)$ and $\varphi_k(X^\dagger) = \varphi_k(X)^\dagger$.*

This is the cyclic-sector part of the period-removal step of [CPGSV16, Section 2.3]; the projections P_k come from the peripheral spectrum of the adjoint transfer map ([Wol12, Theorem 6.6]).

Proof. ✓ Proof The peripheral cyclic-structure theorem identifies the period and its primitive root. The cyclic decomposition of the adjoint transfer map gives projections (P_k) with $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$. Iterating the cyclic relation m times gives $(\mathcal{E}_{A^\dagger})^m(P_k) = P_k$, and the blocked-transfer identity $\mathcal{E}_{(A^{[m]})^\dagger} = (\mathcal{E}_{A^\dagger})^m$ identifies these with the adjoint transfer map of $A^{[m]}$. Apply the block decomposition from adjoint-fixed projections. $\square$

Theorem 8.3.1.2 (Unconditional primitive blocked sectors). Lean ✓ Stmt Let A be a left-canonical irreducible tensor whose adjoint transfer map has peripheral spectrum $\{\gamma^j : 0 \leq j < m\}$ for a primitive m th root of unity γ , with blocked cyclic-sector decomposition $(C_u)_{u=1}^m, (P_u)_{u=1}^m$, and linear isomorphisms $\varphi_u : M_{\dim C_u}(\mathbb{C}) \xrightarrow{\sim} P_u M_D(\mathbb{C}) P_u$. Assume explicitly that the P_u are orthogonal projections summing to $\mathbb{1}$ and satisfying $\mathcal{E}_{A^\dagger}(P_{u+1}) = P_u$, that each sector has nonzero bond dimension, and that the compression maps satisfy

$$\begin{aligned}\varphi_u(\mathcal{E}_{C_u^\dagger}(X)) &= \mathcal{E}_{(P_u A^{[m]})^\dagger}(\varphi_u(X)), \\ \varphi_u(XY) &= \varphi_u(X)\varphi_u(Y), \\ \varphi_u(X^\dagger) &= \varphi_u(X)^\dagger.\end{aligned}\tag{8.11}$$

Here $P_u A^{[m]}$ denotes the blocked tensor with letters $P_u(A^{[m]})^\alpha$. Then every sector tensor satisfies $\sigma_\partial(\mathcal{E}_{C_u}) = \{1\}$ and is tensor-irreducible.

Proof. ✓ Proof Theorem 8.4.3 gives irreducibility of $(\mathcal{E}_A^\dagger)^m$ on each corner P_u . The compression equivalences in (8.11) intertwine the adjoint transfer map of C_u with the corresponding corner restriction of $(\mathcal{E}_A^\dagger)^m$. The peripheral spectrum of the blocked adjoint map is

$$\sigma_\partial((\mathcal{E}_A^\dagger)^m) = \{(\gamma^j)^m : 0 \leq j < m\} = \{1\},\tag{8.12}$$

because γ is a primitive m th root of unity. Hence each nonzero irreducible corner restriction is primitive. Primitivity and irreducibility are preserved across φ_u ; then use Theorem D.4.4 to pass from the adjoint blocked map to the primal transfer map of C_u . $\square$

Theorem 8.3.1.3 (Period removal with primitive irreducible sectors). Lean ✓ Stmt Let A be a left-canonical irreducible tensor. Then there is a period $m \geq 1$ and a unit-weight decomposition of $A^{[m]}$ into sector blocks $(C_k)_{k=1}^m$, each of which is left-canonical, has primitive transfer map, is tensor-irreducible, and has nonzero bond dimension. Equivalently, the cyclic-sector decomposition of $A^{[m]}$ already satisfies the trace-preservation, primitivity, and irreducibility hypotheses needed for the common-blocking step (Lemma D.5.5).

Proof. ✓ Proof Left-canonicity makes $K_i := (A^i)^\dagger$ unital, irreducibility passes to the adjoint map, and the quantum Perron–Frobenius theorem provides a positive-definite fixed point; the cyclic peripheral-spectrum theorem and Theorem 8.3.1.1 then produce the cyclic-sector decomposition. Keep the projections and compression equivalences, and apply Theorem 8.3.1.2 to the sector blocks. $\square$

Theorem 8.3.1.4 (Common blocking length for cyclic sectors). Lean ✓ Stmt If two tensors generate the same MPV family at every positive length, then the cyclic-sector decompositions on the two sides can be expressed at one common positive blocking length p . For every $N \geq 1$ and every blocked σ , each blocked tensor equals its powered nonzero part:

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_j (\mu_j^A)^p (A_j)^{[p]}\right)_\sigma,\tag{8.13}$$

$$V^{(N)}(B^{[p]})_\sigma = V^{(N)}\left(\bigoplus_k (\mu_k^B)^p (B_k)^{[p]}\right)_\sigma.\tag{8.14}$$

The two powered nonzero parts agree:

$$V^{(N)}\left(\bigoplus_j (\mu_j^A)^p (A_j)^{[p]}\right)_\sigma = V^{(N)}\left(\bigoplus_k (\mu_k^B)^p (B_k)^{[p]}\right)_\sigma.\tag{8.15}$$

Proof. ✓ Proof Start from Theorem D.5.4. With p_A, p_B the least common multiples of the period-removal lengths on the two sides, use $p = \text{lcm}(p_A, p_B)$. The prescribed-common-multiple lemma constructs both one-sided sector families at this length, and the positive-length blocking lemmas carry the one-sided tensor/nonzero-part equalities over to the common alphabet, giving (8.13) and (8.14). The two-sided blocking lemma applied to (D.16) gives (8.15). $\square$

8.4 Cyclic-sector irreducibility

The powered map is analyzed corner by corner before the resulting sector families are placed over a common blocked alphabet. The compression and common-alphabet identities are proved in Section D.5.

Theorem 8.4.1 (Fixed-point upgrade on an invariant corner). Lean ✓ Stmt *Let A have positive bond dimension, suppose that $\mathcal{E}_A$ is irreducible, and assume $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Let Q be an orthogonal projection. If $(\mathcal{E}_{A^\dagger})^m$ preserves the corner $QM_D(\mathbb{C})Q$, then*

$$(\mathcal{E}_{A^\dagger})^m(Q) = Q.$$

Proof. ✓ Proof Choose a positive-definite fixed point ρ of $\mathcal{E}_A$. The weighted trace $X \mapsto \text{tr}(\rho X)$ is invariant under $\mathcal{E}_{A^\dagger}$ and its powers. Corner preservation gives

$$Q - (\mathcal{E}_{A^\dagger})^m(Q) = Q(\mathcal{E}_{A^\dagger})^m(\mathbb{1} - Q)Q \geq 0.$$

The invariant weighted trace of this positive semidefinite difference is zero. Since $\rho > 0$, the difference vanishes. $\square$

Theorem 8.4.2 (Fixed-point algebra rigidity on cyclic sectors). Lean ✓ Stmt *Let A be an irreducible MPS tensor with positive bond dimension and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Let $(P_k)_{k \in \mathbb{Z}/m\mathbb{Z}}$ be orthogonal projections such that*

$$\sum_k P_k = \mathbb{1}, \quad \mathcal{E}_{A^\dagger}(P_{k+1}) = P_k.$$

If $X, Y \in P_k M_D(\mathbb{C}) P_k$ are fixed by $(\mathcal{E}_{A^\dagger})^m$, then

$$\mathcal{E}_{A^\dagger}(XY) = \mathcal{E}_{A^\dagger}(X)\mathcal{E}_{A^\dagger}(Y).$$

Proof. ✓ Proof Irreducibility passes to the adjoint transfer map. The cyclic relation gives equality in both Kadison–Schwarz inequalities:

$$\begin{aligned} \mathcal{E}_{A^\dagger}(P_k^\dagger P_k) &= \mathcal{E}_{A^\dagger}(P_k)^\dagger \mathcal{E}_{A^\dagger}(P_k), \\ \mathcal{E}_{A^\dagger}(P_k P_k^\dagger) &= \mathcal{E}_{A^\dagger}(P_k) \mathcal{E}_{A^\dagger}(P_k)^\dagger. \end{aligned}$$

Hence every P_k lies in the multiplicative domain, so for every Z ,

$$\begin{aligned} \mathcal{E}_{A^\dagger}(P_k Z) &= \mathcal{E}_{A^\dagger}(P_k) \mathcal{E}_{A^\dagger}(Z), \\ \mathcal{E}_{A^\dagger}(Z P_k) &= \mathcal{E}_{A^\dagger}(Z) \mathcal{E}_{A^\dagger}(P_k). \end{aligned}$$

If $X \in P_k M_D(\mathbb{C}) P_k$ is fixed by $(\mathcal{E}_{A^\dagger})^m$, then its orbit sum

$$R_X = \sum_{\ell=0}^{m-1} (\mathcal{E}_{A^\dagger})^\ell(X)$$

is fixed by $\mathcal{E}_{A^\dagger}$. By Theorem 8.8.1, $R_X = c\mathbb{1}$. Orthogonality of the cyclic sectors and the multiplicative-domain identities give $X = P_k R_X P_k = c P_k$. Similarly, $Y = c' P_k$. Since $\mathcal{E}_{A^\dagger}(P_k) = P_{k-1}$, both sides of the required identity equal $cc' P_{k-1}$. $\square$

Lemma 8.4.3 (Sector irreducibility for the adjoint blocked map). [Lean](#) [✓ Stmt](#) Let A be a left-canonical irreducible MPS tensor with $D > 0$, let m be the period of its cyclic-sector decomposition, and let $(P_u)_{u=1}^m$ be the cyclic projections of the adjoint transfer map with $\mathcal{E}_{A^\dagger}(P_{u+1}) = P_u$ and $\sum_u P_u = \mathbb{1}$. Then the restriction of $(\mathcal{E}_{A^\dagger})^m$ to each corner $P_u M_D(\mathbb{C}) P_u$ is irreducible.

Definition 8.4.4 (Primitive irreducible cyclic-sector decomposition). [Lean](#) [✓ Stmt](#) A has primitive irreducible cyclic sectors of period $m \geq 1$ if there is a family $(C_k)_{k=1}^m$ with $A^{[m]} \sim \bigoplus_{k=1}^m C_k$ with unit weights, each C_k left-canonical, with primitive transfer map, tensor-irreducible, and of positive bond dimension.

8.5 Translation-invariant canonical form

Definition 8.5.1 (Unnormalized positive-length PGVWC07 witness). [Lean](#) [✓ Stmt](#) An unnormalized positive-length PGVWC07 witness for a tensor A consists of finitely many blocks $(A_k)_{k=1}^r$ of positive bond dimensions and positive real weights $(a_k)_{k=1}^r$. Each block is unital, has only scalar transfer-map fixed points, and has a diagonal positive-definite fixed point Λ_k for the adjoint transfer map. Moreover,

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r a_k A_k\right)_\sigma \quad (N > 0),$$

and the retained bond dimensions satisfy $\sum_k \dim(A_k) \leq D$.

Theorem 8.5.2 (Construction of the positive-length PGVWC07 witness). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Every MPS tensor has an unnormalized positive-length PGVWC07 witness. More precisely, the construction has four stages. First, a nonzero irreducible block remains irreducible under nonzero scalar rescaling and can be put in a unital gauge with scalar fixed-point space and a diagonal positive-definite dual fixed point. Second, this construction is applied to each member of a finite nonzero irreducible block family. Third, the nonzero irreducible decomposition of an arbitrary tensor supplies that family and the total bond-dimension bound. Finally, these blocks, weights, and fixed points form a witness in the sense of Definition 8.5.1.

Proof. [✓ Proof](#) Follow [PGVWC07, Theorem 4]. The nonzero irreducible decomposition is obtained by strict dimension induction. On a nonzero current block, Perron–Frobenius gives a positive semidefinite eigenvector at a positive spectral radius. If it is singular, positivity makes its support invariant: $(\mathbb{1} - P)A^i P = 0$. After diagonalizing P , every letter is upper block triangular and the finite-ring trace splits as

$$\mathrm{tr}(A^{i_1} \dots A^{i_N}) = \mathrm{tr}(A_1^{i_1} \dots A_1^{i_N}) + \mathrm{tr}(A_2^{i_1} \dots A_2^{i_N}).$$

Discard all-zero blocks and repeat on the two strictly smaller diagonal blocks. This gives the finite nonzero irreducible family of Theorem D.1.3.

For each retained irreducible block, divide by the square root of its positive spectral radius and conjugate by the square root of its positive-definite Perron eigenvector. Theorem 5.5.4 gives the unital identity

$$\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}.$$

Nonzero scalar rescaling preserves irreducibility. The proof passes from tensor irreducibility to irreducibility of the matrix action, carries invariant subspaces through the gauge similarity, and

returns to tensor irreducibility using Theorem 7.7.3.4 and Lemma C.2.2. Hence Theorem 8.8.1 makes every fixed point scalar. Theorem 8.10.2 then diagonalizes a positive-definite dual fixed point by a unitary without changing unitality.

Apply this single-block construction to every member of the finite family. The square roots of the block spectral radii become positive weights, gauge invariance preserves every positive-length MPV coefficient, and the block dimensions are unchanged. Combining this blockwise result with the initial decomposition yields the arbitrary-input unnormalized form and its bound $\sum_k \dim(A_k) \leq D$. Together these conclusions give the structured witness. $\square$

Theorem 8.5.3 (Finite normalization of a nonempty PGVWC07 witness). [Lean](#) [Lean](#) [✓ Stmt](#) *Let W be an unnormalized positive-length witness with positive weights a_k . If some positive-length MPV coefficient of the original tensor is nonzero, then W has at least one block. For any nonempty witness, define*

$$s := \max_{1 \leq k \leq r} a_k,$$

$$\nu_k := \frac{a_k}{s}.$$

Then $s > 0$, every ν_k is positive, $|\nu_k| \leq 1$, and $|\nu_k| = 1$ for at least one k . For every $N > 0$,

$$V^{(N)}(A)_\sigma = s^N V^{(N)}\left(\bigoplus_k \nu_k A_k\right)_\sigma. \quad (8.16)$$

Proof. [✓ Proof](#) If the witness had no blocks, its weighted block tensor would have zero MPV coefficients at every positive length, contrary to the assumed nonzero coefficient. Thus $r > 0$. Since the finite family (a_k) is nonempty and every $a_k > 0$, its maximum s is positive. The definition $\nu_k = a_k/s$ gives $0 < \nu_k \leq 1$, and any maximizing index has $\nu_k = 1$. Finally, $a_k = s\nu_k$ for every k , so homogeneity of a length- N matrix product vector gives (8.16). $\square$

Theorem 8.5.4 (Normalized exact form for a nonzero MPV family). [Lean](#) [✓ Stmt](#) *Suppose that $V^{(N)}(A)_\sigma \neq 0$ for some $N > 0$ and some word σ . Then there are a scalar $s > 0$ and a nonempty finite family of positive-bond-dimension blocks $(A_k)_{k=1}^r$ with positive weights $(\mu_k)_{k=1}^r$ such that $s^{-1}A$ and $\bigoplus_k \mu_k A_k$ have identical MPV coefficients at every positive length. The weights satisfy $|\mu_k| \leq 1$, with equality for at least one block, and*

$$\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1},$$

$$\sum_i (A_k^i)^\dagger \Lambda_k A_k^i = \Lambda_k \quad (8.17)$$

for a diagonal positive-definite matrix Λ_k . Every fixed point of $\mathcal{E}_{A_k}$ is a scalar multiple of $\mathbb{1}$, and $\sum_k \dim(A_k) \leq D$.

Proof. [✓ Proof](#) Choose the structured witness from Theorem 8.5.2. The assumed nonzero coefficient makes its block family nonempty, so Theorem 8.5.3 supplies $s > 0$ and the normalized positive weights. Equation (8.16) and homogeneity give

$$V^{(N)}(s^{-1}A)_\sigma = s^{-N} V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_k \mu_k A_k\right)_\sigma.$$

The witness satisfies all required block conditions and the bond-dimension bound. $\square$

Theorem 8.5.5 (Zero/nonzero alternative for the normalized exact form). Lean ✓ Stmt For every MPS tensor A , one of the following alternatives holds:

1. $V^{(N)}(A)_\sigma = 0$ for every $N > 0$ and every word σ ;
2. there are a scalar $s > 0$ and a nonempty normalized block family satisfying all conclusions of Theorem 8.5.4.

Proof. ✓ Proof Split according to whether there exist $N > 0$ and σ with $V^{(N)}(A)_\sigma \neq 0$. In that case apply Theorem 8.5.4. If no such pair exists, every positive-length coefficient vanishes. $\square$

Theorem 8.5.6 (Translation-invariant canonical form). Lean ✓ Stmt Let A be a translation-invariant MPS representation of bond dimension D . There is a positive scalar s and a finite family of blocks A_k , possibly empty, such that the globally rescaled tensor $s^{-1}A$ has the same MPV coefficients on every positive-length ring as $\bigoplus_k \mu_k A_k^i$. The weights are positive and satisfy $1 \geq |\mu_k|$; if the family is nonempty, then some weight has norm 1. For each block,

$$\begin{aligned} \sum_i A_k^i (A_k^i)^\dagger &= \mathbb{1}, \\ \sum_i (A_k^i)^\dagger \Lambda_k A_k^i &= \Lambda_k. \end{aligned} \tag{8.18}$$

Here Λ_k is diagonal, positive, and full-rank, and each block has positive bond dimension. Moreover, the only fixed points of $\mathcal{E}_{A_k}(X) = \sum_i A_k^i X (A_k^i)^\dagger$ are the scalar multiples of $\mathbb{1}$: for every X , the identity $\mathcal{E}_{A_k}(X) = X$ implies that there exists $c \in \mathbb{C}$ such that $X = c\mathbb{1}$. The total bond dimension of the decomposition is at most D .

This is the positive-length form of [PGVWC07, Theorem 4], with the global spectral-radius normalization step made explicit.

Proof. ✓ Proof Apply Theorem 8.5.5. If every positive-length coefficient vanishes, take $s = 1$ and the empty retained family; the block conditions and bond-dimension bound are vacuous, and the MPV identity is the zero identity. In the nonzero branch, use the nonempty normalized family supplied by the second alternative. $\square$

Remark 8.5.7 (CPSV blocked canonical form (source statement)). The blocked canonical-form reduction of [CPGSV16, Section 2.3] and [CPGSV21, Section IV.A] states that an arbitrary translation-invariant MPS tensor becomes, after blocking, a weighted family of normal tensors, up to a summand that vanishes on every positive-length ring. Four statements must be distinguished. Theorem 8.5.6 gives the unconditional PGVWC07 normalized exact form before period removal. Theorem 8.11.1.2 gives the prepared trace-preserving primitive-block reduction. For a family with a nonzero positive-length coefficient, Theorem E.3.7 normalizes the weights and Theorem 9.6.3.14 gives the normalized BNT form, including its per-site scalar. If every positive-length coefficient vanishes, there is no nonempty family from which to choose a unit-modulus weight; this exceptional zero family is represented by the empty retained family instead.

8.6 Normal tensors and canonical forms

This section records the CPSV definitions of *normal tensor* and *canonical form* from [CPGSV16]. The basis of normal tensors is stated in Chapter 9. Write

$$\sigma_\partial(\mathcal{E}) = \{\lambda \in \sigma(\mathcal{E}) : |\lambda| = r(\mathcal{E})\} \tag{8.19}$$

for the peripheral spectrum of a transfer map $\mathcal{E}$, where $r(\mathcal{E})$ is its spectral radius. The stronger conditions used later (Definition 8.9.2 and the canonical-form conditions of Chapter 9) add left-canonical normalization and spectral primitivity. The weight moduli are taken non-increasing, not strictly decreasing; equal-modulus and repeated-copy sectors are retained and compared by the coefficient argument of Chapter 9.

Definition 8.6.1 (CPSV normal tensor (NT)). [Lean](#) [✓ Stmt](#) A tensor A of physical dimension d and bond dimension D is a *normal tensor* if, after the spectral-radius normalization of [CPGSV16], (i) A admits no nontrivial invariant orthogonal projection and (ii) the associated CPM $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$ has spectral radius $r(\mathcal{E}_A) = 1$ and peripheral spectrum $\sigma_\partial(\mathcal{E}_A) = \{1\}$.

Remark 8.6.2. Condition (i) cannot be dropped. A reducible tensor may have peripheral spectrum $\{1\}$ while still admitting a nontrivial invariant projection, so condition (ii) alone does not characterize normality.

Theorem 8.6.3 (Algebraically normal left-canonical tensors are CPSV normal). [Lean](#) [✓ Stmt](#) Let $D > 0$. If A is algebraically normal and left-canonical, then

$$A \text{ is a CPSV normal tensor.} \quad (8.20)$$

Proof. [✓ Proof](#) Algebraic normality together with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ makes the transfer map strongly irreducible. It consequently has a positive-definite fixed point, no nontrivial invariant projection, and peripheral spectrum $\{1\}$. The fixed-point eigenvalue and left-canonical normalization give $r(\mathcal{E}_A) = 1$. Thus no scalar rescaling remains, and the two conditions in Definition 8.6.1 hold for A itself. $\square$

Theorem 8.6.4 (Normality and left-canonical normalization under entrywise conjugation). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) For a tensor or matrix, an overline denotes entrywise conjugation. If A and B are gauge equivalent through X , then $\overline{A}$ and $\overline{B}$ are gauge equivalent through $\overline{X}$. If A is left-canonical, then $\overline{A}$ is left-canonical. If A is algebraically normal, then $\overline{A}$ is algebraically normal. Independently, if A is a CPSV normal tensor, then $\overline{A}$ is a CPSV normal tensor.

Proof. [✓ Proof](#) Entrywise conjugating $B^i = X A^i X^{-1}$ gives $\overline{B^i} = \overline{X} \overline{A^i} (\overline{X})^{-1}$. Conjugating the left-canonical identity gives

$$\sum_i (\overline{A^i})^\dagger \overline{A^i} = \overline{\sum_i (A^i)^\dagger A^i} = \overline{\mathbb{1}} = \mathbb{1}. \quad (8.21)$$

Entrywise conjugation maps the span of the length- N products of A onto the span of those of $\overline{A}$, so it preserves positive-length block injectivity and hence algebraic normality. For the normal-tensor assertion, the transfer map of $\overline{A}$ is the anti-linear conjugate $\sigma \circ \mathcal{E}_A \circ \sigma$ by (8.22). By the corresponding theorem in the companion quantum-channel volume [LTC26, “Entrywise conjugation transports peripheral eigenvalues”] this conjugation preserves the spectral radius and primitivity, and by Theorem 8.6.6 entrywise conjugation preserves irreducibility. Hence both clauses of Definition 8.6.1 hold for $\overline{A}$. $\square$

Theorem 8.6.5 (Transfer maps and positive diagonal matrices under entrywise conjugation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let σ denote entrywise conjugation. For every tensor A , as operators on the matrix space,

$$\mathcal{E}_{\overline{A}} = \sigma \circ \mathcal{E}_A \circ \sigma; \quad (8.22)$$

evaluating at $\overline{X}$ gives, for every matrix X ,

$$\mathcal{E}_{\overline{A}}(\overline{X}) = \overline{\mathcal{E}_A(X)}. \quad (8.23)$$

If Λ is positive definite and diagonal, then

$$\overline{\Lambda} = \Lambda. \quad (8.24)$$

Proof. ✓ **Proof** Entrywise conjugation commutes with finite sums, matrix products, and conjugate transposition, which gives (8.22) on every matrix, and (8.23) follows since σ is an involution. The diagonal entries of a positive-definite complex matrix are real, while all off-diagonal entries of a diagonal matrix vanish. This proves (8.24). $\square$

Theorem 8.6.6 (Irreducibility under entrywise conjugation). Lean ✓ **Stmt** If A is irreducible, then $\overline{A}$ is irreducible.

Proof. ✓ **Proof** Suppose P is a nontrivial invariant orthogonal projection for $\overline{A}$, so $(1-P)\overline{A}^i P = 0$ for every i . Conjugating entrywise gives $(1-\overline{P})A^i \overline{P} = 0$, and $\overline{P}$ is again a Hermitian idempotent distinct from 0 and 1. Hence $\overline{P}$ is a nontrivial invariant orthogonal projection for A , contradicting the irreducibility of A . $\square$

Definition 8.6.7 (CPSV canonical form). Lean Lean Lean Lean Lean ✓ **Stmt** Let A have bond dimension D . It is in the canonical form of [CPGSV16, Section 2.3] if there are positive integers $D_1, \dots, D_r$, normal tensors A_k of bond dimension D_k , nonzero weights $\mu_k \in \mathbb{C}$, and, writing $S = \sum_k D_k$, a matrix $U \in \mathbb{C}^{S \times D}$ such that $S \leq D$ and, for every physical index i ,

$$\begin{aligned} UU^\dagger &= \mathbb{1}_S, \\ A^i &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U. \end{aligned} \quad (\text{CF})$$

Thus the complement of the retained direct sum consists literally of zero bond-space coordinates. In particular, $r = 0$ is possible only when $A = 0$.

The convention in line 246 of [CPGSV16] is a separate normalization of these data, not part of canonical-form membership:

$$\begin{aligned} |\mu_k| &\leq 1 \quad \text{for every } k, \\ A \neq 0 &\implies \exists k, |\mu_k| = 1. \end{aligned} \quad (8.25)$$

Every weight μ_k is nonzero. This is the reading of [CPGSV16, Section 2.3] adopted here: the remark in line 219 there places the degenerate case in the block dimensions ($D_k = 0$), and lines 224–225 choose each μ_k so that the transfer map of $\mu_k A_k$ has spectral radius one, which a vanishing weight cannot satisfy. The reading loses nothing, since if $\mu_{k_0} = 0$ then $\mu_{k_0} A_{k_0}^i = 0$ is the zero matrix of size D_{k_0} and

$$\bigoplus_{k=1}^r \mu_k A_k^i = \bigoplus_{k \neq k_0} \mu_k A_k^i \oplus 0_{D_{k_0}}, \quad (8.26)$$

so A^i is the same coisometric image of the direct sum over the remaining indices, with the D_{k_0} coordinates of the omitted summand moved to the zero complement. No ordering or separation condition is imposed.

The basis-of-normal-tensors refinement is Definition 9.1.1; it is not part of canonical form itself.

Theorem 8.6.8 (Physical relabeling preserves CPSV normal and canonical form). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $e : \{0, \dots, d' - 1\} \rightarrow \{0, \dots, d - 1\}$ be a bijection and set

$$(e^*A)^j = A^{e(j)}. \quad (8.27)$$

The transfer maps of A and e^*A agree. In particular, physical relabeling preserves left-canonical normalization and CPSV normality. If A has canonical-form data

$$A^i = U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U,$$

then e^*A has the same weights and coisometry and the relabeled normal blocks:

$$(e^*A)^j = U^\dagger \left(\bigoplus_{k=1}^r \mu_k (e^*A_k)^j \right) U. \quad (8.28)$$

Proof. [✓ Proof](#) A bijection of the physical alphabet only permutes the summands in $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$. Hence the transfer map, its spectral radius, its peripheral spectrum, and its invariant orthogonal projections are unchanged. Formula (8.28) is obtained from Formula (CF) by replacing i with $e(j)$. $\square$

Theorem 8.6.9 (Transfer maps under a bond-space gauge). [Lean](#) [✓ Stmt](#) If $B^i = X A^i X^{-1}$ for every physical index i , then there is an invertible matrix C such that

$$\mathcal{E}_B(Y) = C^{-1} \mathcal{E}_A(CY C^\dagger) (C^\dagger)^{-1}.$$

Proof. [✓ Proof](#) Expand both transfer maps and take $C = X^{-1}$. $\square$

Theorem 8.6.10 (Gauge invariance of transfer quasi-idempotence). [Lean](#) [✓ Stmt](#) Let A and B be gauge-equivalent MPS tensors and let $c \in \mathbb{C}$. Then

$$\mathcal{E}_B^2 = c \mathcal{E}_B \iff \mathcal{E}_A^2 = c \mathcal{E}_A.$$

Proof. [✓ Proof](#) Theorem 8.6.9 identifies the two transfer maps by an algebra automorphism. Such an automorphism preserves composition and scalar multiplication, and therefore preserves the displayed equation. $\square$

Theorem 8.6.11 (Spectral radius under congruence similarity). [Lean](#) [✓ Stmt](#) For an invertible matrix C , the linear maps

$$E \quad \text{and} \quad Y \mapsto C^{-1} E (CY C^\dagger) (C^\dagger)^{-1}$$

have the same spectral radius.

Proof. [✓ Proof](#) The two maps are conjugate by the invertible linear transformation $Y \mapsto CY C^\dagger$, and therefore have the same spectrum. $\square$

Theorem 8.6.12 (Primitivity under congruence similarity). [Lean](#) [✓ Stmt](#) For an invertible matrix C , a linear map E is primitive if and only if

$$Y \mapsto C^{-1} E (CY C^\dagger) (C^\dagger)^{-1}$$

is primitive.

Proof. [✓ Proof](#) Congruence similarity is an algebra isomorphism on the endomorphism algebra and hence preserves the peripheral spectrum. $\square$

Theorem 8.6.13 (Gauge transport of CPSV normality). [Lean](#) [✓ Stmt](#) *Let X be invertible and suppose*

$$B^i = XA^iX^{-1}$$

for every physical index i . If B is a CPSV normal tensor, then A is a CPSV normal tensor.

Proof. [✓ Proof](#) With $S_X(Y) = XYX^\dagger$, the transfer maps satisfy

$$\mathcal{E}_B = S_X \circ \mathcal{E}_A \circ S_X^{-1}. \quad (8.29)$$

Similarity preserves the spectral radius and the peripheral spectrum. For irreducibility, start with an invariant orthogonal projection Q for one transfer map and form the support projection of the corresponding positive congruence $XQX^\dagger$ (using X^{-1} in the reverse direction). The intertwining in Formula (8.29) makes this support projection invariant for the other transfer map. Irreducibility forces its support to be zero or full, and invertibility of X then forces $Q = 0$ or $Q = 1$. Thus both conditions in Definition 8.6.1 are transported across the gauge. $\square$

Theorem 8.6.14 (Positive blocking preserves CPSV normal tensors). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be a CPSV normal tensor and let $L \geq 1$. Then the blocked tensor $A^{[L]}$ is a CPSV normal tensor.*

Proof. [✓ Proof](#) Choose a trace-preserving gauge $B^i = XA^iX^{-1}$. Positive blocking gives

$$(B^{[L]})^I = X(A^{[L]})^IX^{-1}. \quad (8.30)$$

The trace-preserving primitive and irreducible structure of B is preserved by positive blocking, so Theorem 8.6.3 makes $B^{[L]}$ CPSV normal. Gauge transport (Theorem 8.6.13) then gives CPSV normality of $A^{[L]}$. $\square$

Theorem 8.6.15 (Literal CPSV canonical form is preserved by positive blocking). [Lean](#) [Lean](#) [✓ Stmt](#) *Let A have canonical-form data*

$$A^i = U^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) U, \quad UU^\dagger = \mathbb{1}.$$

For every $L \geq 1$, $A^{[L]}$ has canonical-form data with the same coisometry U , the blocked normal tensors $A_k^{[L]}$, and the weights μ_k^L :

$$(A^{[L]})^I = U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L (A_k^{[L]})^I \right) U. \quad (8.31)$$

Proof. [✓ Proof](#) Write $I = (i_1, \dots, i_L)$. Since $L \geq 1$, the word I is nonempty. Repeatedly inserting $UU^\dagger = \mathbb{1}$ into $A^{i_1} \dots A^{i_L}$ gives

$$\begin{aligned} A^{i_1} \dots A^{i_L} &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L A_k^{i_1} \dots A_k^{i_L} \right) U \\ &= U^\dagger \left(\bigoplus_{k=1}^r \mu_k^L (A_k^{[L]})^I \right) U. \end{aligned}$$

The preceding theorem gives CPSV normality of every blocked block $A_k^{[L]}$, and the displayed identity is exactly the retained direct-sum reconstruction. $\square$

Theorem 8.6.16 (CPSV canonical-form representative after blocking). Lean ✓ Stmt For every tensor A , there are a positive integer p , a bond dimension D' , and a tensor B of physical dimension d^p and bond dimension D' such that $B \in \text{CF}$ and, for every $N \geq 1$,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}(B)_\sigma. \quad (8.32)$$

Here B is another tensor generating the same positive-length MPV family. The assertion $B \in \text{CF}$ is literal in the bond coordinates of B ; no assertion is made that $A^{[p]}$ itself is literally in canonical form in its original bond coordinates.

Proof. ✓ Proof Apply Theorem 8.11.1.3 to A and itself. It gives $p \geq 1$, weights μ_k , and left-canonical, primitive, tensor-irreducible blocks C_k such that, for every $N \geq 1$,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_k \mu_k C_k\right)_\sigma. \quad (8.33)$$

First, Theorem 8.11.1.1 makes each C_k algebraically normal. Second, Theorem 8.6.3 combines this conclusion with the left-canonical identity to make each C_k a CPSV normal tensor. Hence

$$B := \bigoplus_k \mu_k C_k \quad (8.34)$$

satisfies the literal block-diagonal equation (CF), so $B \in \text{CF}$. Equation (8.33) gives the second part of (8.32). $\square$

Theorem 8.6.17 (Closed-chain expansion of CPSV canonical form). Lean ✓ Stmt For canonical-form data as in (CF), every positive length N and every configuration σ satisfy

$$V^{(N)}(A)_\sigma = \sum_{k=1}^r \mu_k^N V^{(N)}(A_k)_\sigma. \quad (8.35)$$

Equivalently, $|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(A_k)\rangle$. This is the identity immediately following the canonical-form definition in [CPGSV16, Section 2.3].

Proof. ✓ Proof Induction on a nonempty physical word w of length N , using (CF) at each letter and $UU^\dagger = \mathbb{1}_S$ between adjacent letters, gives

$$A^w = U^\dagger \left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right) U. \quad (8.36)$$

Applying cyclicity of the trace and $UU^\dagger = \mathbb{1}_S$ to (8.36) gives

$$\text{tr}(A^w) = \text{tr} \left(\left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right) UU^\dagger \right) = \text{tr} \left(\bigoplus_{k=1}^r \mu_k^N A_k^w \right). \quad (8.37)$$

Applying Theorem 2.5.9 to the right-hand side of (8.37) yields $\text{tr}(A^w) = \sum_{k=1}^r \mu_k^N \text{tr}(A_k^w)$, which is (8.35). $\square$

Theorem 8.6.18 (BNT characterization for CPSV canonical form). [Lean](#) [✓ Stmt](#) Let A have canonical-form data as in (CF), and let $(B_j)_{j=1}^g$ have positive bond dimensions. The family $(B_j)_{j=1}^g$ is a basis of normal tensors for A if and only if

$$\begin{aligned} & (\forall j, B_j \text{ is normal}) \\ & \wedge (\forall k, \exists j, X_k, \phi_k, \forall i, A_k^i = e^{i\phi_k} X_k B_j^i X_k^{-1}) \\ & \wedge (\forall j \neq j', \nexists X, \phi, \forall i, B_j^i = e^{i\phi} X B_{j'}^i X^{-1}). \end{aligned} \quad (8.38)$$

Each displayed conjugacy also requires the two bond dimensions to agree, and every scalar $e^{i\phi}$ has modulus one. This is [CPGSV16, Proposition 2.7 and Appendix A], specialized to the literal canonical-form data.

Proof. [✓ Proof](#) The canonical-form data supply normality of every A_k and the positive-length identity $|V^{(N)}(A)\rangle = \sum_k \mu_k^N |V^{(N)}(A_k)\rangle$. Substitution of these two facts into Theorem 9.1.4 gives (8.38). $\square$

The projector-closure result in Theorem 8.7.4 gives normal blocks and orthogonal ambient embeddings under the source hypotheses. Combining these embeddings into one coisometry gives Definition 8.6.7.

This source definition is distinct from the stronger normal-canonical-form conditions below. The renormalization fixed-point analysis uses a separate block-injective canonical-form criterion; it is not the CPSV canonical-form definition.

Remark 8.6.19 (Relation to the stronger conditions). A tensor A which is irreducible (Definition 8.1.1.1), has transfer map of spectral radius one, and has primitive transfer map is normal in the sense of Definition 8.6.1. The reverse implications from the stronger conditions of Definition 8.9.2 and of Chapter 9 require the fundamental-theorem step or the algebraic-to-spectral normality equivalence, established later.

8.6.1 Normalization conventions

Remark 8.6.1.1 (Scalar normalization conventions). The reductions use four scalar facts: scaling preserves injectivity; the transfer map under a scalar ζ rescales by $|\zeta|^2$; unit-modulus phase rescaling preserves $\sum_i (A^i)^\dagger A^i = \mathbb{1}$; and the MPV of a weighted block-diagonal tensor factorizes as $\mu_k^N V^{(N)}(A_k)_\sigma = \eta_k^N |\mu_k|^N V^{(N)}(A_k)_\sigma$, with $\eta_k := \mu_k / |\mu_k|$.

Left-canonical normalization is required before the blocks can be compared. Without it, the implication $\mathcal{V}(\bigoplus_k \mu_k A_k) = \mathcal{V}(\bigoplus_k \mu_k B_k) \Rightarrow \forall k, \mathcal{V}(A_k) = \mathcal{V}(B_k)$ already fails for two scalar blocks: for $\mu = (1, 2)$, $A = (1, 3/2)$, and $B = (3, 1/2)$, the two direct sums generate identical MPV families but the first blocks differ at $N = 1$. Left-canonical normalization removes this particular ambiguity; the sectors are then matched, up to a permutation, by the exact fixed-length linear independence of Chapter 9, which requires no ordering of the weight moduli.

8.7 Projector criterion for canonical decomposition

Definition 8.7.1 (Invariant-projector closure). [Lean](#) [✓ Stmt](#) An MPS tensor A has *invariant-projector closure* if every orthogonal projection P satisfying $PA^i = PA^i P$ for all i also satisfies $A^i P = PA^i P$ for all i . This is the projector hypothesis in the sufficient condition for canonical form of [CPGSV16, Section 2].

Theorem 8.7.2 (Decomposition into irreducible corners under projector closure). Lean ✓ Stmt Suppose A has invariant-projector closure. Then there are isometries $V_1, \dots, V_r$ with $V_k^\dagger V_k = \mathbb{1}$, pairwise orthogonal ranges, and $\sum_k V_k V_k^\dagger = \mathbb{1}$, such that $A^i V_k = V_k A_k^i$ for the corner tensors $A_k^i = V_k^\dagger A^i V_k$, every corner tensor A_k is irreducible, and $\mathcal{V}(A) = \mathcal{V}(\bigoplus_{k=1}^r A_k)$.

Proof. ✓ Proof Proceed by strong induction on the bond dimension. If A is irreducible, the single corner $V = \mathbb{1}$ suffices. Otherwise, A has a nontrivial invariant projection P , which commutes with every A^i by Theorem D.2.1. Factoring $P = V_1 V_1^\dagger$ and $\mathbb{1} - P = V_2 V_2^\dagger$ through support isometries by Lemma D.2.3 splits A into two corner tensors of strictly smaller bond dimension. For $j = 1, 2$, commutation gives $A^i V_j = V_j (V_j^\dagger A^i V_j)$, and each corner inherits invariant-projector closure by Lemma D.2.4. The two recursively obtained families are combined. Ranges from different corners are orthogonal because

$$V_1^\dagger V_2 = V_1^\dagger P(\mathbb{1} - P)V_2 = 0.$$

Equality of matrix product vectors follows from Lemma D.2.6. □

Definition 8.7.3 (No nontrivial periodic vectors). Lean ✓ Stmt An MPS tensor A has *no nontrivial p -periodic vectors* if, for every isometry V with $V^\dagger V = \mathbb{1}$ and $A^i V = V B^i$ for all i , with B irreducible (equivalently, for every restriction B of A to a minimal invariant subspace), and for every $\rho > 0$ and $r > 0$ with $\mathcal{E}_B(\rho) = r\rho$, every eigenvalue of $\mathcal{E}_B$ of modulus r equals r . The eigenvalue r is the spectral radius of $\mathcal{E}_B$ (Theorem [LTC26, “Spectral radius is the Perron eigenvalue”]), so the condition states that no block of the invariant-subspace decomposition of [CPGSV16, Section 2], rescaled to spectral radius one, has a peripheral eigenvalue $e^{2\pi i q/p} \neq 1$. Eigenvectors with such eigenvalues are the p -periodic vectors of [CPGSV16, Section 2].

Theorem 8.7.4 (Sufficient condition for canonical form). Lean Lean ✓ Stmt Suppose A has invariant-projector closure and no nontrivial p -periodic vectors. Then there are normal tensors $A_1, \dots, A_r$ of positive bond dimensions $D_1, \dots, D_r$, with $\sum_k D_k \leq D$, and nonzero weights $\mu_1, \dots, \mu_r$ such that

$$\mathcal{V}^+(A) = \mathcal{V}^+\left(\bigoplus_{k=1}^r \mu_k A_k\right). \quad (8.39)$$

Moreover, there are isometries $V_k : \mathbb{C}^{D_k} \rightarrow \mathbb{C}^D$ with pairwise orthogonal ranges such that

$$\begin{aligned} A^i V_k &= V_k (\mu_k A_k^i), \\ A^i &= \sum_k V_k (\mu_k A_k^i) V_k^\dagger. \end{aligned} \quad (8.40)$$

This is the sufficient condition for canonical form of [CPGSV16, Section 2], with that paper’s convention $\sum_k D_k \leq D$ allowing zero blocks: a corner of A carrying the zero tensor cannot be rescaled to spectral radius one and is omitted because it vanishes at every positive length.

Proof. ✓ Proof Decompose A into irreducible corners B_k along isometries V_k by Theorem 8.7.2; every matrix product vector coefficient of A is the sum of those of the corners. A corner whose matrices all vanish contributes nothing at positive length and is dropped. Every remaining corner has a Perron eigenpair $\mathcal{E}_{B_k}(\rho_k) = r_k \rho_k$ by Lemma D.3.1. The absence of p -periodic vectors, applied to the corner B_k , gives that every eigenvalue of $\mathcal{E}_{B_k}$ of modulus r_k equals r_k , so

$A_k := r_k^{-1/2} B_k$ is normal by Lemma D.3.2. Setting $\mu_k := r_k^{1/2}$ gives $\mu_k A_k = B_k$ at the level of matrices, so for every word w of positive length N the weight cancels:

$$\mu_k^N \operatorname{tr}(A_k^w) = (r_k^{1/2})^N (r_k^{-1/2})^N \operatorname{tr}(B_k^w) = \operatorname{tr}(B_k^w).$$

Summing over k gives (8.39), while the corner reconstruction gives (8.40). The retained dimensions satisfy $\sum_k D_k \leq \sum_k \operatorname{tr}(V_k V_k^\dagger) = \operatorname{tr}(\mathbb{1}) = D$. $\square$

Theorem 8.7.5 (Non-scalar fixed point gives a further split). [Lean](#) [Lean](#) [✓ Stmt](#) Suppose A is unital, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If X is a Hermitian fixed point of $\mathcal{E}_A$ which is not a scalar multiple of $\mathbb{1}$, then there is a nonzero PSD fixed point ρ which is not positive definite, and consequently A admits a two-block MPV-equivalent decomposition with both block dimensions strictly smaller than D . This is the fixed-point shift and further decomposition step in [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) Let $\lambda_{\max}$ be the largest eigenvalue of X . Then $\rho := \lambda_{\max} \mathbb{1} - X$ is positive semidefinite, singular, and nonzero because X is not scalar. Unitality and the fixed-point equation for X give $\mathcal{E}_A(\rho) = \rho$. Apply Theorem 8.1.5. The non-scalar hypothesis is needed because scalar multiples of $\mathbb{1}$ are fixed by every unital map and yield no nontrivial support projection. $\square$

8.8 Fixed points and canonical gauges of irreducible blocks

Theorem 8.8.1 (Scalar fixed points in an irreducible unital block). [Lean](#) [✓ Stmt](#) If A is irreducible and unital, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and $\mathcal{E}_A(X) = X$, then X is a scalar multiple of $\mathbb{1}$. This is the final fixed-point assertion in the proof of [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) Tensor irreducibility gives irreducibility of the completely positive transfer map. By Wolf's Theorem 6.6 for irreducible unital Kraus maps, every fixed point is scalar. $\square$

Theorem 8.8.2 (Unital gauge from a full-rank Perron–Frobenius eigenvector). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let ρ be positive definite with $\mathcal{E}_A(\rho) = r\rho$ for some $r > 0$. For $B^i := r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}$,

$$\sum_i B^i (B^i)^\dagger = \mathbb{1}. \quad (8.41)$$

If $r = 1$, then the unscaled gauge $B^i := \rho^{-1/2} A^i \rho^{1/2}$ is unital and generates the same MPV family as A . This is the spectral-radius normalization and full-rank fixed-point gauge of [PGVWC07, Theorem 4].

Proof. [✓ Proof](#) The positive square root of ρ is invertible. Substituting $B^i = r^{-1/2} \rho^{-1/2} A^i \rho^{1/2}$ and using $\mathcal{E}_A(\rho) = r\rho$ gives

$$\sum_i B^i (B^i)^\dagger = r^{-1} \rho^{-1/2} \mathcal{E}_A(\rho) \rho^{-1/2} = \mathbb{1},$$

which is (8.41). For $r = 1$, the change from A to B is a similarity, so cyclicity of the trace gives equality of all MPV coefficients. $\square$

Definition 8.8.3 (CPSV canonical form II). [Lean](#) [Lean](#) [✓ Stmt](#) In canonical form II of [CPGSV16, Appendix A], there are positive retained dimensions D_k , normal tensors A_k , weights μ_k , and, for $S = \sum_k D_k \leq D$, a matrix $U \in \mathbb{C}^{S \times D}$ such that, for every physical index i ,

$$UU^\dagger = \mathbb{1}_S, \quad A^i = U^\dagger \left(\bigoplus_k \mu_k A_k^i \right) U. \quad (8.42)$$

Thus omitted bond-space coordinates are literally zero. Every retained block additionally satisfies

$$\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}_{D_k},$$

$$\mathcal{E}_{A_k}(\Lambda_k) = \Lambda_k,$$

where Λ_k is diagonal and positive definite. The two displayed equations are the only additional blockwise conditions; no ordering, injectivity, gauges, or basis-of-normal-tensors assumptions are included. The line-246 convention (8.25) remains a separate condition on the underlying canonical-form data.

Theorem 8.8.4 (Canonical form admits a canonical-form-II gauge). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)

Let A be a tensor in CPSV canonical form, with ambient bond dimension D . Then there is a tensor B of the same bond dimension and an $G \in \text{GL}(D, \mathbb{C})$ such that, for every physical index i ,

$$B^i = GA^iG^{-1}. \quad (8.43)$$

Moreover, B is in canonical form II and the total retained block dimension is unchanged. This is the nonsingular gauge assertion of [CPGSV16, Appendix A].

Proof. [✓ Proof](#) Write the retained part of the canonical-form decomposition as $C^i = \bigoplus_k \mu_k A_k^i$, and let $U \in \mathbb{C}^{S \times D}$ satisfy $UU^\dagger = \mathbb{1}_S$ and $A^i = U^\dagger C^i U$. For each normal block, first choose its pure trace-preserving Perron gauge and then diagonalize its positive definite fixed point by a unitary. This gives $X_k \in \text{GL}(D_k, \mathbb{C})$ and blocks B_k satisfying the two canonical-form-II identities. Put $X = \bigoplus_k X_k$ and extend it to the omitted zero coordinates by

$$G = U^\dagger XU + (\mathbb{1}_D - U^\dagger U). \quad (8.44)$$

Its inverse is

$$G^{-1} = U^\dagger X^{-1}U + (\mathbb{1}_D - U^\dagger U). \quad (8.45)$$

The identity $UU^\dagger = \mathbb{1}_S$ gives $GG^{-1} = G^{-1}G = \mathbb{1}_D$. It also gives

$$GU^\dagger = U^\dagger X, \quad (8.46)$$

and

$$UG^{-1} = X^{-1}U. \quad (8.47)$$

Therefore

$$GA^iG^{-1} = U^\dagger XC^iX^{-1}U = U^\dagger \left(\bigoplus_k \mu_k B_k^i \right) U, \quad (8.48)$$

which has the same ambient reconstruction and satisfies canonical form II. $\square$

Theorem 8.8.5 (Gauge equivalence under coisometric reconstruction). [Lean](#) [✓ Stmt](#) Let A and B have common bond dimension D , and let C and E have common retained bond dimension n . Suppose that $U_A, U_B \in \mathbb{C}^{n \times D}$ satisfy $U_A U_A^\dagger = U_B U_B^\dagger = \mathbb{1}_n$ and, for every physical index i ,

$$A^i = U_A^\dagger C^i U_A. \quad (8.49)$$

$$B^i = U_B^\dagger E^i U_B. \quad (8.50)$$

If C and E are gauge equivalent, then A and B are gauge equivalent.

Proof. ✓ Proof Choose $X \in \text{GL}(n, \mathbb{C})$ such that, for every physical index i , $E^i = XC^iX^{-1}$. Equality of the Gram matrices $U_A U_A^\dagger$ and $U_B U_B^\dagger$ gives a unitary $W \in \mathbb{C}^{D \times D}$ with

$$U_B^\dagger = W U_A^\dagger. \quad (8.51)$$

$$U_B = U_A W^\dagger. \quad (8.52)$$

Extend the retained gauge through U_A by

$$G = U_A^\dagger X U_A + (\mathbb{1}_D - U_A^\dagger U_A). \quad (8.53)$$

Its inverse is

$$G^{-1} = U_A^\dagger X^{-1} U_A + (\mathbb{1}_D - U_A^\dagger U_A). \quad (8.54)$$

The coisometry identity gives $GU_A^\dagger = U_A^\dagger X$ and $U_A G^{-1} = X^{-1} U_A$. Hence, with $Y = WG$,

$$B^i = W U_A^\dagger E^i U_A W^\dagger = W G A^i G^{-1} W^\dagger = Y A^i Y^{-1}. \quad (8.55)$$

□

Theorem 8.8.6 (Diagonal Perron normalization for an irreducible TP tensor). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ and $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that, for $B^i := U^\dagger A^i U$,*

$$\sum_i (B^i)^\dagger B^i = \mathbb{1},$$

$$\mathcal{E}_B(\Lambda) = \Lambda.$$

This proves the diagonal Perron normalization required in CFII. It does not prove that A is in the CPSV canonical form or that its block is normal; those require the corresponding primitivity and canonical-form hypotheses.

Proof. ✓ Proof The TP hypothesis makes $\mathcal{E}_A$ a channel with a nonzero PSD fixed point. Irreducibility makes $\mathcal{E}_A$ an irreducible CP map, and Theorem 5.1.1 upgrades the fixed point to positive definite. The spectral theorem diagonalizes it by a unitary conjugation, which preserves the TP normalization. □

8.9 Normal canonical form

This section defines the spectral *normal* canonical form used in the reduction. A separate block-injective canonical-form criterion is used in the renormalization fixed-point analysis. The supporting overlap consequences are collected in Section D.6 and Section 6.7.

Recall from Definition 2.7.3 and Lemma 6.2.1 that the bilinear overlap is

$$O_{AB}(N) = \sum_{\sigma \in \{0, \dots, d-1\}^N} V^{(N)}(A)_\sigma \overline{V^{(N)}(B)_\sigma} = \text{Tr}(F_{AB}^N). \quad (8.56)$$

This notation enters the self-overlap clause of the canonical-form conditions below.

Remark 8.9.1 (Normalization convention). [PGVWC07, Theorem 4] uses the unital gauge $\sum_i A_k^i (A_k^i)^\dagger = \mathbb{1}$; here the blocks are in the dual TP gauge $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$. A positive-definite fixed point of the adjoint transfer map provides the change of gauge between the two (Theorem 5.4.2).

Definition 8.9.2 (Normal canonical form conditions). [Lean](#) [✓ Stmt](#) Scaling factors $(\mu_k)_{k=1}^r$ and block tensors $(A_k)_{k=1}^r$ satisfy the *normal canonical form conditions* if:

1. each A_k is irreducible,
2. each A_k satisfies $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$,
3. each $\mathcal{E}_{A_k}$ is primitive (equivalently, $\sigma_\partial(\mathcal{E}_{A_k}) = \{1\}$),
4. $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_r|$,
5. all $\mu_k \neq 0$,
6. all bond dimensions are positive.

Here “normal” is the spectral sense of [CPGSV16], which by [SPGWC10, Proposition 3] is equivalent to the eventual block-injectivity formulation of Definition 2.4.4.

8.9.1 Separated one-copy normal canonical forms

The following specialization attaches BNT separation to the one-copy normal canonical form. Repeated representatives and their power-sum coefficients are instead recorded by the sector decomposition of Chapter 9.

Definition 8.9.1.1 (No gauge-phase-equivalent distinct blocks). [Lean](#) [✓ Stmt](#) A finite block family has no gauge-phase-equivalent distinct blocks when, for any two distinct indices with the same bond dimension, the corresponding blocks are not gauge-phase equivalent.

Definition 8.9.1.2 (Normal canonical form with BNT separation). [Lean](#) [✓ Stmt](#) This is a one-copy canonical-form condition: each listed block is one representative and repeated copies are not encoded as separate blocks. A family is in *normal canonical form with BNT separation* if it satisfies Definition 8.9.2 and distinct blocks of the same bond dimension are not gauge-phase equivalent. The weight moduli are non-increasing but need not be strictly decreasing, so equal-modulus blocks are allowed; the grouping into a BNT is governed by gauge-phase equivalence, not by distinctness of moduli. Repeated equal-modulus copies of one basis tensor are not separate blocks here; they appear as the weights $\mu_{j,q}$ in $c_N^{(j)} = \sum_q \mu_{j,q}^N$ in the sector decomposition below.

The off-diagonal decay needed for the BNT criterion follows from the separation condition together with the overlap-decay theorems of the preceding chapter.

Theorem 8.9.1.3 (Cross-overlap decay from explicit BNT hypotheses). [Lean](#) [✓ Stmt](#) Let $(A_j)_{j=1}^r$ be a family in which each A_j is injective and left-canonical, $\sum_i (A_j^i)^\dagger A_j^i = \mathbb{1}$, and distinct equal-dimension blocks are not gauge-phase equivalent. Then $O_{A_j A_k}(N) \rightarrow 0$ for every pair $j \neq k$.

Proof. [✓ Proof](#) If the bond dimensions differ, Theorem 6.4.4 gives the decay. If the bond dimensions agree, the non-equivalence hypothesis excludes gauge-phase equivalence, so Theorem 6.5.5 gives the decay. $\square$

Lemma 8.9.1.4 (BNT from explicit blockwise hypotheses). Lean ✓ Stmt *Let $(\mu_k, A_k)_{k=1}^r$ be a weighted block family. Assume each A_k is injective and left-canonical, $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$, that $O_{A_k A_k}(N) \rightarrow 1$ for each k , and that distinct equal-dimension blocks are not gauge-phase equivalent. Then the blocks form a basis of normal tensors for the block-diagonal tensor $\bigoplus_k \mu_k A_k$.*

Proof. ✓ Proof Injective blocks are normal. The block-diagonal decomposition theorem gives the MPV expansion of $\bigoplus_k \mu_k A_k$ in terms of the block MPVs. The self-overlap limits are part of the hypotheses, and Theorem 8.9.1.3 supplies the off-diagonal decay. Therefore Lemma E.1.10 gives the eventual linear independence required in Definition 9.1.1. $\square$

The normal-canonical formulation encodes normality spectrally (primitive transfer map), the BNT definition algebraically (eventual block injectivity).

Remark 8.9.1.5 (Comparing the two normality conditions). The spectral normality of Definition 8.9.2 and the algebraic normality of Definition 2.4.4 agree: the Wielandt theory of Chapter 7 supplies the implication from primitivity to eventual block injectivity used here.

Lemma 8.9.1.6 (Restricted normal-CF-BNT hypotheses give a BNT). Lean ✓ Stmt *On this one-copy surface, if a family is in normal canonical form with BNT separation and each block is also normal in the algebraic sense, then the blocks form a basis of normal tensors for the associated block-diagonal tensor. The blocks here are distinct representatives.*

Proof. ✓ Proof The normal-canonical hypotheses already give the block-diagonal MPV expansion, the self-overlap normalization, and the eventual linear independence coming from irreducible trace-preserving overlap decay. Supplying algebraic normality for each block adds the remaining clause in Definition 9.1.1. $\square$

Remark 8.9.1.7. See Theorem 6.6.1 for the modulus-one eigenvalue rigidity statement for irreducible trace-preserving blocks.

8.9.2 Canonical form from primitivity

A normal block needs a primitive transfer map; an irreducible TP block may still have a cyclic peripheral spectrum. Raising the map to its period sends every peripheral eigenvalue to 1, but eigenvalue 1 can then have several cyclic corners and need not be simple on the full powered space. Period removal therefore proceeds in this order: power the map, obtain the adjoint-fixed cyclic projections, compress to their corners, prove each corner irreducible, and only then infer primitivity of the sector maps. This is the blocking step of [CPGSV16, Section 2.3].

Theorem 8.9.2.1 (Blocking yields a peripheral-spectrum primitive transfer map). Lean ✓ Stmt *Let A be an irreducible MPS tensor with $D > 0$ and $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. Then there exists a blocking length $p > 0$ such that $\sigma_\partial(\mathcal{E}_{A^{[p]}}) = \{1\}$.*

Proof. ✓ Proof Pass to the conjugate-transposed Kraus family $K_i := (A^i)^\dagger$, which is unital with irreducible Kraus map. The TP fixed-point theorem for $\mathcal{E}_A$ provides a positive definite fixed point for the adjoint of K , so the corresponding theorem in the companion quantum-channel volume [LTC26, “Roots of unity via adjoint fixed point”] shows that every peripheral eigenvalue is a root of unity. Since $\sigma_\partial(\mathcal{E}_A)$ is finite (the corresponding theorem in the companion quantum-channel volume [LTC26, “Finiteness of peripheral eigenvalues”]), the corresponding lemma in the companion quantum-channel volume [LTC26, “Common exponent for a finite set of roots of unity”] gives a common exponent p with $\mu^p = 1$ for every $\mu \in \sigma_\partial(\mathcal{E}_A)$, and the corresponding lemma in the companion quantum-channel volume [LTC26, “Powering collapses peripheral eigenvalues”] gives $\sigma_\partial(\mathcal{E}_A^p) = \{1\}$. Blocking by p takes the p th power of the transfer

map, so $\mathcal{E}_{A^{[p]}}$ is primitive in the peripheral-spectrum sense of the corresponding definition in the companion quantum-channel volume [LTC26, “Primitive map”]. The blocked map need not remain irreducible: eigenvalue 1 may have several cyclic corners. The adjoint-fixed projections are therefore compressed before irreducibility and primitivity are asserted sector by sector in Theorem 8.3.1.3. This is the periodicity-removal step of [CPGSV16, Section 2.3] for an irreducible block already in TP gauge. $\square$

8.10 Left-canonical and dual diagonalization of irreducible blocks

Each irreducible block in trace-preserving gauge admits a unitary normalization with diagonal positive Perron data. This is the normalization component of CFII (Definition 8.8.3); the full CFII assertion also requires the CPSV canonical-form and normal-block hypotheses. The full PGVWC07 canonical form is Theorem 8.5.6.

Theorem 8.10.1 (Diagonal Perron data for an irreducible TP block). Lean ✓ Stmt *Let A be an irreducible MPS tensor in TP gauge, $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, with $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that:*

1. $B^i := U^\dagger A^i U$ is TP,
2. $\mathcal{E}_B(\Lambda) = \Lambda$,
3. Λ is diagonal and positive definite,
4. A and B generate the same MPV family.

These are the two normalization identities in the CFII definition of [CPGSV16, Appendix A]. The theorem does not by itself prove that A is in canonical form or that its block is normal. The unitary conjugation is performed inside TP gauge after TP normalization.

Proof. ✓ Proof Theorem 8.8.6 provides the unitary and the diagonal positive-definite fixed point, and unitary conjugation preserves trace preservation and the MPV family (Theorem D.1.1). $\square$

Theorem 8.10.2 (Dual diagonalization for irreducible unital blocks). Lean ✓ Stmt *Let A be an irreducible MPS tensor in unital gauge $\sum_i A^i (A^i)^\dagger = \mathbb{1}$, with $D > 0$. Then there exist a unitary U and a diagonal positive definite Λ such that, for $B^i := U^\dagger A^i U$,*

$$\begin{aligned} \sum_i B^i (B^i)^\dagger &= \mathbb{1}, \\ \sum_i (B^i)^\dagger \Lambda B^i &= \Lambda. \end{aligned} \tag{8.57}$$

Moreover, A and B generate the same MPV family. This is the dual-map fixed-point and unitary-diagonalization step of [PGVWC07, Theorem 4].

Proof. ✓ Proof Apply Theorem 8.10.1 to the conjugate-transposed family $i \mapsto (A^i)^\dagger$: unitality of A is trace preservation for this family, whose transfer map is the dual of $\mathcal{E}_A$. Translating back gives (8.57). $\square$

8.11 Cyclic-sector isometries and primitive reduction

Theorem 8.11.1 (Spectral cyclic sectors with support isometries). [Lean] [✓ Stmt] *Let A be a left-canonical irreducible tensor, and suppose the adjoint transfer map $\mathcal{E}_{A^\dagger}$ has a positive-definite fixed point, is irreducible as a positive map, and has peripheral spectrum*

$$\sigma_\partial(\mathcal{E}_{A^\dagger}) = \{\gamma^j : j \in \mathbb{Z}/m\mathbb{Z}\}, \quad (8.58)$$

where γ is a primitive m th root of unity and $m \geq 1$. Then the spectral cyclic-sector decomposition of $A^{[m]}$ has compression maps which may be chosen with rectangular support isometries V_k satisfying

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ V_k V_k^\dagger &= P_k, \\ \varphi_k(X) &= V_k X V_k^\dagger. \end{aligned} \quad (8.59)$$

These isometries also realize the blocked corner letters $\varphi_k(C_k^\alpha) = P_k(A^{[m]})^\alpha P_k$.

Proof. [✓ Proof] Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”], applied to $\mathcal{E}_{A^\dagger}$, provides projections P_k satisfying $\mathcal{E}_{A^\dagger}(P_{k+1}) = P_k$. Iterating m times gives $(\mathcal{E}_{A^\dagger})^m(P_k) = P_k$, and $(\mathcal{E}_{A^\dagger})^m = \mathcal{E}_{(A^{[m]})^\dagger}$ shows that each P_k is fixed by the adjoint transfer map of $A^{[m]}$. Applying Theorem D.5.3 to that blocked tensor gives the compressed sector tensors and the support isometries in (8.59). $\square$

8.11.1 Reduction to primitive blocks

This subsection completes the reduction after period removal. A TP-primitive-irreducible block is normal, and an arbitrary tensor becomes, after a positive blocking length, a weighted family of left-canonical primitive blocks whose total bond dimension is bounded by the original bond dimension. For two tensors with the same MPV family, the cyclic sectors can be expressed over one common blocked alphabet, which is the form used by the Fundamental Theorem.

Theorem 8.11.1.1 (TP-primitive irreducible tensor is normal). [Lean] [✓ Stmt] *Let A be a left-canonical MPS tensor with $D > 0$, tensor-irreducible, and with peripheral spectrum $\{1\}$. Tensor irreducibility implies irreducibility of the completely positive transfer map, so A is normal.*

Proof. [✓ Proof] The primitive peripheral spectrum, tensor irreducibility, and left-canonical normalization give primitive fixed-point data. The fixed point is positive definite by Lemma D.4.8. Theorem 7.7.4.2 then gives normality. $\square$

An arbitrary translation-invariant tensor reduces, after blocking, to a weighted direct sum of peripheral-spectrum primitive blocks. These globally blocked pieces need not remain irreducible; the cyclic-sector decomposition below refines them into primitive irreducible blocks.

Theorem 8.11.1.2 (Arbitrary tensor to primitive block decomposition). [Lean] [✓ Stmt] *For any MPS tensor A , there exist $p \geq 1$, nonzero weights $(\mu_k)_{k=1}^r$, and blocks $(B_k)_{k=1}^r$ such that, for every blocked σ of positive length N ,*

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r \mu_k B_k\right)_\sigma. \quad (8.60)$$

The retained block dimensions satisfy $\sum_{k=1}^r D_k \leq D$. Each block is left-canonical and primitive:

$$\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1},$$

$$\sigma_\partial(\mathcal{E}_{B_k}) = \{1\}.$$

Each bond dimension is positive.

Proof. ✓ **Proof** Theorem 8.2.1.1 supplies the left-canonical nontrivial blocks, the positive-length equality, and the bound $\sum_k D_k \leq D$. Apply Theorem D.4.2 directly to this finite irreducible family. Its proof first applies Theorem 8.9.2.1 to each block and then takes the least common multiple of the resulting periods through Theorem D.4.1. This gives one positive length p at which every blocked transfer map is primitive. Lemma D.4.16 keeps the blocked weights nonzero, and Lemma D.4.14 carries the positive-length equality through blocking, giving (8.60). $\square$

Theorem 8.11.1.3 (Unconditional common primitive block decompositions). Lean ✓ **Stmt** Let A and B have the same MPV family at every positive length. Then there is a positive blocking length at which both blocked tensors have the same positive-length MPV family as weighted direct sums of trace-preserving, primitive, tensor-irreducible blocks with positive bond dimensions and nonzero weights. The two weighted nonzero-sector families have the same positive-length MPV family.

Proof. ✓ **Proof** Apply Theorem 8.3.1.4 to choose the common blocking length and the two common cyclic-sector families. For either family, write G_k for its cyclic-sector blocks and (ν_x, C_x) for the weights and blocks of the flattened common-sector family from Lemma D.5.10. The common-alphabet nonzero-part identity gives, for every length N and blocked word σ ,

$$V^{(N)}\left(\bigoplus_k \mu_k^p G_k^{[p]}\right)_\sigma = V^{(N)}\left(\bigoplus_x \nu_x C_x\right)_\sigma.$$

Applying this identity on the two sides of the comparison yields a positive length p and weighted primitive block families such that, for every $N \geq 1$ and blocked word σ ,

$$V^{(N)}(A^{[p]})_\sigma = V^{(N)}\left(\bigoplus_x \mu_x^A A_x\right)_\sigma,$$

$$V^{(N)}(B^{[p]})_\sigma = V^{(N)}\left(\bigoplus_y \mu_y^B B_y\right)_\sigma.$$

The two weighted nonzero parts are equal at every positive length. The structural properties and the nonvanishing of the weights come from the common-sector families. $\square$

Chapter 9

Basis of Normal Tensors

A *basis of normal tensors* (BNT) for an MPS tensor A is a finite family of normal blocks $\{A_j\}$ whose MPV states $\{|V^{(N)}(A_j)\rangle\}$ span $|V^{(N)}(A)\rangle$ at every positive length and are eventually linearly independent. Every coefficient of a canonical form is nonzero, so equal MPV families force a bijective gauge-phase matching $B_k = e^{i\phi_k} X_k A_{j_k} X_k^{-1}$, and the sector coefficients $c_N^{(j)} = \sum_q \mu_{j,q}^N$ recover the per-copy weight multisets by Newton–Girard. The references are [PGVWC07], [CPGSV16], and [CPGSV21].

9.1 Bases of normal tensors

Definition 9.1.1 (Basis of Normal Tensors (BNT)). Lean ✓ Stmt A *basis of normal tensors* (BNT) for a total tensor A_{tot} consists of g block tensors $(A_1, \dots, A_g)$ with bond dimensions $(D_1, \dots, D_g)$ such that each A_j is normal (Definition 2.4.4); at each system size $N \geq 1$ the MPV of A_{tot} lies in the span of the block MPVs $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$; and for sufficiently large N those block MPVs are linearly independent. Here normality means eventual block injectivity, as in Definition 2.4.4. Thus this is the algebraic form of [CPGSV21, Definition IV.2], rather than its primitive-channel formulation. In the canonical-form characterization of [CPGSV16, Section 2.3], a BNT is a minimal family of representatives for $\tilde{A}_k^i = e^{i\phi_k} X_k A_j^i X_k^{-1}$ among the normal blocks.

Definition 9.1.2 (Basis of normal tensors, coefficient form). Lean ✓ Stmt The coefficient form of a basis of normal tensors: each block A_j is a CPSV normal tensor (Definition 8.6.1), the MPV of A is given at every positive length by explicit coefficients $c_j^{(N)} \in \mathbb{C}$ with $|V^{(N)}(A)\rangle = \sum_{j=1}^g c_j^{(N)} |V^{(N)}(A_j)\rangle$, and the block MPVs $\{|V^{(N)}(A_j)\rangle\}$ are eventually linearly independent. This is the literal coefficient-form definition of [CPGSV16, Section 2.3]. It differs from Definition 9.1.1 only in the chosen normality predicate. Theorem 9.2.5 proves the implication between the two forms.

Theorem 9.1.3 (Distinct BNT members are not gauge-phase equivalent). Lean ✓ Stmt *Distinct members of a basis of normal tensors are not gauge-phase equivalent after identifying equal bond dimensions.*

Proof. ✓ Proof At every sufficiently large length the matrix product vectors of the basis members are linearly independent. A gauge-phase equivalence between two distinct members would give $|V^{(N)}(A_k)\rangle = \zeta^N |V^{(N)}(A_j)\rangle$ for a nonzero scalar ζ . Thus one vector would be a scalar multiple of the other, contradicting linear independence. $\square$

Theorem 9.1.4 (Characterization by canonical-form representatives). [Lean](#) [Lean](#) [Stmt](#) Let A have a canonical-form MPV expansion

$$|V^{(N)}(A)\rangle = \sum_{k=1}^r \mu_k^N |V^{(N)}(\tilde{A}_k)\rangle$$

at every positive length, where the tensors $\tilde{A}_k$ are normal tensors of positive bond dimension and every coefficient μ_k is nonzero. Let $(A_j)_{j=1}^g$ be tensors of positive bond dimension. Then $(A_j)_{j=1}^g$ is a basis of normal tensors for A if and only if:

1. every A_j is normal;
2. for every k there are an index j , an invertible matrix X_k , and a phase $e^{i\phi_k}$ such that $\tilde{A}_k^i = e^{i\phi_k} X_k A_j^i X_k^{-1}$ for every physical index i ;
3. no two distinct tensors A_j and $A_{j'}$ are related in this way.

This is [CPGSV16, Proposition 2.7], read with the nonzero-coefficient convention of Definition 8.6.7.

Proof. [Proof](#) Group the summands into the MPV phase classes of Definition 9.6.1.7, and choose one representative from each class. By Theorem 9.1.3, eventual linear independence of a BNT excludes gauge-phase equivalence between two distinct A_j .

Suppose that some phase-class representative is not gauge-phase equivalent to any A_j . Partition all phase classes into the unmatched classes T and the remaining classes. Every remaining representative is a nonzero scalar multiple of some $|V^{(N)}(A_j)\rangle$ at each positive length. Distinct unmatched representatives have vanishing mutual overlaps, and their overlaps with every A_j also vanish. Explicitly, for $t \neq t'$ in T and every j ,

$$\begin{aligned} O_{\tilde{A}_t \tilde{A}_{t'}}(N) &\longrightarrow 0, \\ O_{\tilde{A}_t A_j}(N) &\longrightarrow 0. \end{aligned}$$

By Lemma E.1.7, the combined family

$$\{|V^{(N)}(\tilde{A}_t)\rangle : t \in T\} \cup \{|V^{(N)}(A_j)\rangle : 1 \leq j \leq g\}$$

is linearly independent for all sufficiently large N . Comparing the canonical-form expansion with the BNT expansion gives a relation

$$\sum_{t \in T} c_{N,t} |V^{(N)}(\tilde{A}_t)\rangle + \sum_{j=1}^g \alpha_{N,j} |V^{(N)}(A_j)\rangle = 0.$$

Linear independence forces every $c_{N,t}$ to vanish. Hence the coefficient of every unmatched phase class vanishes for all sufficiently large N . For an unmatched class t , let q range over its canonical-form summands, write μ_q for the corresponding canonical-form weight, and write ζ_q for the phase relating that summand to the chosen representative. Then $c_{N,t} = \sum_{q \in t} (\zeta_q \mu_q)^N$. Every base $\zeta_q \mu_q$ is nonzero because every coefficient is nonzero. Eventual vanishing of this finite geometric sum would imply vanishing at $N = 0$, where its value is the positive number of summands in the class. This contradiction proves coverage. The normal-tensor phase theorem turns the resulting MPV phase relation into the stated gauge-phase relation.

Conversely, coverage gives, for each canonical-form summand, $|V^{(N)}(\tilde{A}_k)\rangle = \zeta_k^N |V^{(N)}(A_{j(k)})\rangle$. Hence

$$|V^{(N)}(A)\rangle = \sum_j \left(\sum_{k:j(k)=j} (\mu_k \zeta_k)^N \right) |V^{(N)}(A_j)\rangle.$$

This is the required BNT span at every positive length. Pairwise gauge-phase distinction and normality give eventual linear independence, completing the converse. $\square$

9.2 Normal blocks and overlap rigidity

This section relates spectral normality to eventual block injectivity and records the overlap dichotomy that detects bond-dimension and gauge-phase matching between normal blocks.

Theorem 9.2.1 (Normality forces positive bond dimension). [Lean](#) [✓ Stmt](#) *Every normal tensor has positive bond dimension.*

Proof. [✓ Proof](#) If the bond dimension were zero, the bond-matrix space and its endomorphism algebra would each contain only zero. The transfer operator would therefore have spectral radius zero, contradicting the spectral-radius-one clause in the definition of a normal tensor. $\square$

Theorem 9.2.2 (Pure Perron normalization at spectral radius one). [Lean](#) [✓ Stmt](#) *Let A be irreducible and suppose that the spectral radius of $\mathcal{E}_A$ is one. There is a positive definite matrix σ satisfying*

$$\sum_i (A^i)^\dagger \sigma A^i = \sigma. \quad (9.1)$$

The tensor

$$B^i = \sigma^{1/2} A^i \sigma^{-1/2} \quad (9.2)$$

is left-canonical and irreducible, and A and B are related by a pure invertible similarity.

Proof. [✓ Proof](#) Positive-definite right and left Perron eigenvectors have the same positive eigenvalue by the trace pairing. The spectral-radius identity makes this eigenvalue equal to one. Thus the left Perron eigenvector is an adjoint fixed point, so its square-root gauge is trace-preserving. Similarity preserves irreducibility. $\square$

Theorem 9.2.3 (Spectral normality gives a pure trace-preserving gauge). [Lean](#) [✓ Stmt](#) *Let A be a normal tensor in the sense of [CPGSV16, Section 2.2]. Its bond dimension is necessarily positive, and there is a positive definite matrix σ such that the gauge obtained from σ is trace-preserving, primitive, and irreducible. No scalar rescaling is needed.*

Proof. [✓ Proof](#) Pure Perron normalization gives a trace-preserving gauge by a pure similarity. Irreducibility and primitivity are invariant under this similarity. $\square$

Theorem 9.2.4 (Spectral normality gives eventual block injectivity). [Lean](#) [✓ Stmt](#) *Every normal tensor in the sense of [CPGSV16, Section 2.2] is injective after blocking by some finite length; its positive bond dimension follows from normality.*

Proof. [✓ Proof](#) Apply Theorem 9.2.3. The trace-preserving, primitive, irreducible representative is eventually block-injective, and gauge invariance transports this conclusion back to the original tensor. $\square$

Theorem 9.2.5 (A coefficient-form BNT is an algebraic BNT). [Lean](#) [✓ Stmt](#) If $(A_j)_{j=1}^g$ is a coefficient-form basis of normal tensors for A , then $(A_j)_{j=1}^g$ is an algebraic basis of normal tensors for A . Only this implication from spectral normality to algebraic normality is asserted.

Proof. [✓ Proof](#) Every block has positive bond dimension, so spectral normality implies eventual block injectivity for each block. The spanning identities and eventual linear independence are unchanged. $\square$

Theorem 9.2.6 (Non-decaying rectangular overlap determines the bond dimension). [Lean](#) [✓ Stmt](#) Let A and B be irreducible trace-preserving tensors of positive, possibly different, bond dimensions. If $|O_{AB}(N)| \rightarrow 1$, then their bond dimensions are equal.

Proof. [✓ Proof](#) If the bond dimensions differed, the rectangular overlap-decay theorem would give $O_{AB}(N) \rightarrow 0$, and hence $|O_{AB}(N)| \rightarrow 0$, contradicting the assumed limit. $\square$

Theorem 9.2.7 (A normal tensor has a nonzero letter). [Lean](#) [✓ Stmt](#) Let A be a CPSV normal tensor of positive bond dimension. Then $A^i \neq 0$ for some physical letter i .

Proof. [✓ Proof](#) If every letter vanished, then the transfer map would vanish and its spectral radius would be zero, contradicting the spectral-radius-one normalization of a CPSV normal tensor. $\square$

Theorem 9.2.8 (Phase-equivalent normal tensors are gauge-phase equivalent). [Lean](#) [Lean](#) [✓ Stmt](#) Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for some nonzero $\zeta \in \mathbb{C}$, $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ for $N \geq 0$. Then the bond dimensions are equal and there is an invertible matrix X and a scalar η of unit modulus such that $B^i = \eta X A^i X^{-1}$ for every physical letter i . This is the gauge recovery used in [CPGSV16, Appendix A].

Proof. [✓ Proof](#) Apply Theorem 9.2.3 to both tensors. Gauge invariance of matrix product vectors preserves the phase relation, whose self-overlap identities imply unit modulus for the phase. The resulting rectangular cross-overlap has norm tending to one. If the bond dimensions differed, Theorem 9.2.6 would give a contradiction. Thus the bond dimensions agree, and the unit-overlap rigidity theorem gives a gauge-phase equivalence between the trace-preserving representatives. Conjugating by the two Perron gauges gives the asserted relation for A and B . $\square$

Theorem 9.2.9 (Normal-tensor overlap dichotomy). [Lean](#) [✓ Stmt](#) Let A and B be normal tensors of positive, possibly different, bond dimensions. Their self-overlaps satisfy

$$\begin{aligned} O_{AA}(N) &\rightarrow 1, \\ O_{BB}(N) &\rightarrow 1. \end{aligned}$$

Moreover, either $O_{AB}(N) \rightarrow 0$, or their bond dimensions are equal; call the common value D . Then there are $X \in \text{GL}_D(\mathbb{C})$ and $\zeta \in \mathbb{C}$, with $|\zeta| = 1$, such that $B^i = \zeta X A^i X^{-1}$ for every physical letter i . In the latter case, $|O_{AB}(N)| \rightarrow 1$. This is [CPGSV16, Lemma A.2].

Proof. [✓ Proof](#) Put both tensors in their pure trace-preserving gauges. If their bond dimensions differ, rectangular overlap decay gives $O_{AB}(N) \rightarrow 0$. If their dimensions agree but they are not gauge-phase equivalent, the equal-dimension overlap-decay theorem gives the same conclusion. In the remaining case, $B^i = \zeta X A^i X^{-1}$. By Theorem 9.2.8, $|\zeta| = 1$. Hence

$$\begin{aligned} O_{AB}(N) &= \bar{\zeta}^N O_{AA}(N), \\ |O_{AB}(N)| &= |O_{AA}(N)| \rightarrow 1. \end{aligned}$$

$\square$

Theorem 9.2.10 (Exact phase alternative for normal tensors). [Lean](#) [✓ Stmt](#) *Let A and B be normal tensors of positive, possibly different, bond dimensions. Then either $O_{AB}(N) \rightarrow 0$, or there is a phase $\zeta \in \mathbb{C}$, with $|\zeta| = 1$, such that $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ for every chain length N . This is [CPGSV16, Corollary A.3].*

Proof. [✓ Proof](#) Apply Theorem 9.2.9. In its second alternative, gauge-phase covariance gives $V^{(N)}(B)_\sigma = \zeta^N V^{(N)}(A)_\sigma$ for every configuration σ , and hence the asserted equality of vectors. $\square$

Theorem 9.2.11 (Unit overlap from eventual proportionality). [Lean](#) [✓ Stmt](#) *Let A and B be tensors, possibly with different bond dimensions, whose self-overlaps tend to one. Suppose that, for all sufficiently large N , there is $c_N \in \mathbb{C}$ such that $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. Then $|O_{AB}(N)| \rightarrow 1$.*

Proof. [✓ Proof](#) The proportionality relation gives

$$\begin{aligned} O_{AB}(N) &= c_N O_{BB}(N), \\ O_{AA}(N) &= |c_N|^2 O_{BB}(N). \end{aligned}$$

Hence $|c_N|^2 \rightarrow 1$ and therefore $|c_N| \rightarrow 1$. The first identity now gives $|O_{AB}(N)| \rightarrow 1$. $\square$

Theorem 9.2.12 (Geometric proportionality phase for normal tensors). [Lean](#) [✓ Stmt](#) *Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for all sufficiently large N , their matrix product vectors are proportional by a nonzero scalar. Then there is $a \in \mathbb{C}$, with $|a| = 1$, such that $|V^{(N)}(A)\rangle = a^N |V^{(N)}(B)\rangle$ for every positive N . This is the one-normal-block consequence of [CPGSV16, Corollary A.3 and the proof of Theorem 2.10].*

Proof. [✓ Proof](#) Normality gives $O_{AA}(N) \rightarrow 1$ and $O_{BB}(N) \rightarrow 1$, so Theorem 9.2.11 gives $|O_{AB}(N)| \rightarrow 1$. The first alternative in Theorem 9.2.10 gives $O_{AB}(N) \rightarrow 0$, hence $|O_{AB}(N)| \rightarrow 0$, a contradiction. In the remaining alternative, $|V^{(N)}(B)\rangle = \zeta^N |V^{(N)}(A)\rangle$ with $|\zeta| = 1$; take $a = \zeta^{-1}$. $\square$

Theorem 9.2.13 (Separated normal tensors are eventually independent). [Lean](#) [✓ Stmt](#) *Let $(A_j)_{j=1}^g$ be a finite family of positive-bond-dimension normal tensors, no two of which are gauge-phase equivalent after identifying equal bond dimensions. Then the matrix product vectors $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$ are linearly independent for all sufficiently large N , as in [CPGSV16, Appendix A].*

Proof. [✓ Proof](#) Put every tensor in the pure trace-preserving gauge of Theorem 9.2.3. Distinct gauge-phase classes have overlaps tending to zero, while each self-overlap tends to one. The finite Gram matrix therefore tends to the identity and is invertible for all sufficiently large lengths. Gauge invariance of matrix product vectors gives the conclusion for the original tensors. $\square$

9.3 Block-injective consequences for BNT families

The overlap separation of normal blocks yields a simultaneous product-algebra span after blocking. We first state the fixed-length span condition and then apply it to coefficient-form and sector BNTs.

Definition 9.3.1 (Fixed-length block-injective canonical-form condition). [Lean](#) [Lean](#) [✓ Stmt](#) For a fixed blocked length m , the fixed-length form of the block-injective canonical-form condition

is

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = m\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Equivalently,

$$\forall X = (X_j)_j \in \bigoplus_{j=1}^g M_{D_j}(\mathbb{C}), \quad \exists (c_\omega(X))_{|\omega|=m}, \quad (X_j)_{j=1}^g = \sum_{|\omega|=m} c_\omega(X) (A_1^\omega, \dots, A_g^\omega).$$

This is the fixed-length form of the block-injective canonical-form condition of [CPGSV21, lines 2113–2117].

Theorem 9.3.2 (Gauge invariance of simultaneous fixed-length product spans). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) Let $(A_j)_{j=1}^g$ and $(B_j)_{j=1}^g$ be tensor families with the same physical dimension and the same bond dimension D_j in each labelled block. Suppose that for every j there is an arbitrary invertible matrix $X_j \in \text{GL}_{D_j}(\mathbb{C})$ such that, for every physical index i ,

$$B_j^i = X_j A_j^i X_j^{-1}.$$

Then, at every fixed length L ,

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}) \iff \text{span}\{(B_1^\omega, \dots, B_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Both directions hold for general invertible gauges; no unitarity condition is imposed. In particular, a separation certificate proved after passing to left-canonical representatives can be transported back to a source unital gauge carrying a positive dual fixed point. This supplies only the simultaneous-span transport; the subsequent source-gauge boundary and parent-Hamiltonian arguments are separate.

Proof. [Stmt](#) [Proof](#) Given a target tuple $(M_j)_j$, conjugate it backwards to $(X_j^{-1} M_j X_j)_j$. If the word tuples of $(A_j)_j$ span the product algebra, one common coefficient family $(c_\omega)_\omega$ expresses this conjugated tuple. Gauge covariance of word evaluation then gives

$$\sum_{|\omega|=L} c_\omega B_j^\omega = X_j \left(\sum_{|\omega|=L} c_\omega A_j^\omega \right) X_j^{-1} = M_j \quad (9.3)$$

for every j . Thus the $(B_j)_j$ word tuples span the product algebra. Replacing every X_j by X_j^{-1} proves the reverse implication. $\square$

Lemma 9.3.3 (Direct-sum block injectivity at the source length). [Lean](#) [Lean](#) [Lean](#) [Stmt](#) Let $(A_j)_{j=1}^g$ be irreducible left-canonical tensors with normalized self-overlaps and with no gauge-phase-equivalent distinct blocks. Suppose $g \geq 2$, $L_0 > 0$, and every A_j is block-injective at lengths L_0 , $L_0 + 1$, and $3(L_0 + 1)$. Then

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = 3(g-1)(L_0 + 1)\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the direct-sum induction length in [PGVWC07, lines 1346–1408].

Proof. ✓ Proof Lemma E.3.1 gives homogeneous pair-trace separation at $S = 3(L_0 + 1)$. For a chosen anchor block, pair separation prescribes an arbitrary matrix there and zero on one other block. Multiplying $g - 1$ such homogeneous combinations removes the other blocks one at a time. Induction on the number of removed blocks therefore realizes every tuple supported on the anchor at length $(g - 1)S$. Summing over the anchors gives the full product algebra at $3(g - 1)(L_0 + 1)$. $\square$

Theorem 9.3.4 (Simultaneous one-letter span after blocking a BNT). Lean Lean Lean Lean ✓ Stmt Let $(A_j)_{j=1}^g$ be a basis of normal tensors, with bond dimensions D_j . There is a positive integer L such that the one-letter evaluations of the blocked tensors span the full product algebra:

$$\text{span}_{\mathbb{C}} \{((A_1^{[L]})^i, \dots, (A_g^{[L]})^i) : 0 \leq i < d^L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Moreover, there is L_0 such that, for every $L \geq L_0$, the unblocked word tuples have the simultaneous span

$$\text{span}_{\mathbb{C}} \{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the existence form of simultaneous block injectivity associated with [CPGSV16, lines 317–345]; no numerical bound on L or L_0 is asserted here.

Proof. ✓ Proof Put each BNT representative in its trace-preserving Perron gauge. The gauged family is irreducible and left-canonical, has normalized self-overlaps, and remains gauge-phase separated. Choose a common positive length p at which every gauged block is block-injective. Propagation of homogeneous word spans gives block injectivity at $p + 1$ and $3(p + 1)$.

Lemma 9.3.3 gives pairwise separation at length $S = 3(p + 1)$. Write $g \dot{-} 1 = \max\{g - 1, 0\}$. Multiplying one separator for each of the other blocks gives a selector of length $(g \dot{-} 1)S$ for every block. Transport these selectors back from the Perron gauges. Gauge invariance also gives block injectivity of every original block at length p . Concatenating the injective prefix with the selectors yields the full product-algebra span at

$$L = p + (g \dot{-} 1)S = p + 3(g \dot{-} 1)(p + 1) > 0.$$

The truncated factor makes the same construction valid for empty and singleton families. The same span provides selectors and block injectivity at length L . For every sufficiently large n , concatenate an $(n - L)$ -letter injective prefix with these selectors to obtain the full product-algebra span at length n . Finally, length- L words of A_j are the one-letter evaluations of $A_j^{[L]}$, which proves the blocked statement. $\square$

Theorem 9.3.5 (Sharp simultaneous span under a total-dimension bound). Lean ✓ Stmt Let $(A_j)_{j=1}^g$ be a basis of normal tensors with bond dimensions D_j . If $K > 0$ and $\sum_j D_j \leq K$, then there is a positive integer $L \leq 3K^5$ such that

$$\text{span}_{\mathbb{C}} \{(A_1^\omega, \dots, A_g^\omega) : |\omega| = L\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Proof. ✓ Proof Put every representative in its trace-preserving Perron gauge. Since $D_j \leq \sum_k D_k \leq K$, quantum Wielandt and homogeneous-span propagation make every representative block-injective at K^4 , $K^4 + 1$, and $3(K^4 + 1)$. Conjugation preserves the simultaneous word span.

For $g = 0$, take $L = 1$; the product algebra has only its zero element. For $g = 1$, take $L = K^4$. If $g \geq 2$, the three-block separation theorem gives

$$L = 3(g-1)(K^4 + 1).$$

Positivity of the D_j gives $g = \sum_j 1 \leq \sum_j D_j \leq K$, and hence $L \leq 3(K-1)(K^4 + 1) \leq 3K^5$. $\square$

Theorem 9.3.6 (Total dimension of the BNT representatives). [Lean](#) [Stmt](#) *Let literal canonical-form data of ambient bond dimension D have phase classes j and chosen representatives of dimensions D_j . Then*

$$\sum_j D_j \leq D.$$

Proof. [Proof](#) In every phase class, choose its first copy. The class-copy equivalence sends these choices to distinct displayed block indices, and the first copy is the chosen representative. Thus the sum of the representative dimensions is bounded by the sum over all displayed blocks, which is at most D by the canonical-form dimension bound. $\square$

Theorem 9.3.7 (Sharp block-injective form for literal CPSV canonical form). [Lean](#) [Stmt](#) *Let A be in literal CPSV canonical form with positive ambient bond dimension D . There is a basis of normal tensors $(A_j)_{j=1}^g$ and a positive integer $L \leq 3D^5$ such that*

$$\text{span}_{\mathbb{C}} \left\{ ((A_1^{[L]})^i, \dots, (A_g^{[L]})^i) : 0 \leq i < d^L \right\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

Hence A is in block-injective canonical form after blocking at most $3D^5$ sites. This is the proposition at [CPGSV16, lines 342–345].

Proof. [Proof](#) Choose one representative from every matrix-product-vector phase class. These representatives form a basis of normal tensors, and their total bond dimension is at most D by Theorem 9.3.6. Apply Theorem 9.3.5 with $K = D$. A word of length L in A_j is a one-letter evaluation of $A_j^{[L]}$, so the resulting length- L span is exactly the asserted one-letter span. $\square$

Let A be a tensor in canonical form, and let $A_1, \dots, A_g$ be a basis of normal tensors of A . For suitable coefficients $\mu_{j,q}$ and invertible matrices $X_{j,q}$, write $M_j = \text{diag}(\mu_{j,1}, \dots, \mu_{j,r_j})$ and $X = \bigoplus_{j=1}^g \bigoplus_{q=1}^{r_j} X_{j,q}$. Then

$$A^i = X \left[\bigoplus_{j=1}^g (M_j \otimes A_j^i) \right] X^{-1} = \bigoplus_{j=1}^g \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} A_j^i X_{j,q}^{-1}. \quad (9.4)$$

This is [CPGSV16, Section 2.3, Equations (20a)–(21)]. The decomposition (9.4) is used below to separate the normal blocks.

9.4 Permutation rigidity for bases of normal tensors

When two bases of normal tensors generate the same MPV subspace for all sufficiently large positive lengths, the blocks must agree up to a permutation and per-block gauge phases. The argument turns the eventual equality of spans into an invertible change-of-basis matrix, then analyses its asymptotics through the overlaps.

The change-of-basis matrix is now used to match the blocks. We first treat equal block counts, then remove that restriction.

Theorem 9.4.1 (Permutation rigidity for equal block counts). Lean ✓ Stmt Let $(A_j)_{j=1}^g$ and $(B_j)_{j=1}^g$ be injective families of positive bond dimension satisfying trace-preserving normalization, with self-overlaps tending to 1 and off-diagonal overlaps tending to 0 within each family. If the spans of the MPV states agree for all sufficiently large system sizes N , then there is a permutation π of $\{1, \dots, g\}$ such that A_j and $B_{\pi(j)}$ have the same bond dimension and are gauge-phase equivalent.

Proof. ✓ Proof For large N , Lemma E.2.2 gives an invertible matrix U_N relating the two MPV families. Fix j and write $|V^{(N)}(B_j)\rangle = \sum_i u_{ij}^{(N)} |V^{(N)}(A_i)\rangle$. With $G_A(N) = (O_{A_i A_\ell}(N))_{i,\ell}$ and $b_j(N) = (O_{A_i B_j}(N))_i$, this gives

$$G_A(N) \overline{u_j(N)} = b_j(N).$$

Moreover,

$$O_{B_j B_j}(N) = u_j(N)^\top G_A(N) \overline{u_j(N)}.$$

If all mixed overlaps $O_{A_i B_j}(N)$ tended to 0, then $b_j(N) \rightarrow 0$; since $G_A(N) \rightarrow \mathbb{1}$, one obtains $\overline{u_j(N)} \rightarrow 0$ and hence $u_j(N) \rightarrow 0$. Thus $O_{B_j B_j}(N) \rightarrow 0$, contradicting $O_{B_j B_j}(N) \rightarrow 1$. Thus each B_j has a matched block $A_{f(j)}$ with non-decaying mixed overlap. Theorems 6.4.4 and 6.5.5 then force equal bond dimension and gauge-phase equivalence for each match. Off-diagonal decay within the B -family makes f injective, hence bijective, and its inverse is the required permutation. $\square$

Lemma 9.4.2 (Permutation rigidity with asymptotically orthonormal overlaps). Lean ✓ Stmt Let $(A_j)_{j=1}^{g_A}$ and $(B_k)_{k=1}^{g_B}$ be two families of injective tensors of positive bond dimension satisfying the TP normalization $\sum_i (A_j^i)^\dagger A_j^i = \mathbb{1}$ and $\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$, whose self-overlaps converge to 1 and whose cross-overlaps within each family converge to 0. If, for all sufficiently large system sizes N ,

$$\text{span}_{\mathbb{C}}\{|V^{(N)}(A_j)\rangle : 1 \leq j \leq g_A\} = \text{span}_{\mathbb{C}}\{|V^{(N)}(B_k)\rangle : 1 \leq k \leq g_B\},$$

then $g_A = g_B$, and there is a permutation π of the common block index set such that A_j and $B_{\pi(j)}$ have the same bond dimension and are gauge-phase equivalent for each j .

Proof. ✓ Proof By Lemma E.1.10, both families are linearly independent for all sufficiently large N . Choose N large enough that both families are linearly independent and the assumed span equality holds. The common span then has the same dimension when computed from the A -family or the B -family, so $g_A = g_B$. After identifying the two index sets, Theorem 9.4.1 yields the permutation conclusion. $\square$

The two preceding results obtain the block matching from the asymptotic orthonormality of the overlaps. The same matching is obtained by the exact sector comparison of Section 9.8, which pairs the sectors of two canonical forms through a bijection fixed by their bond dimensions and gauge-phase equivalences. The proof of the Fundamental Theorem in Chapter 10 proceeds through the sector comparison.

The same overlap dichotomy also shows that the MPV states of distinct normal blocks are non-proportional at arbitrarily large lengths; this is an input to the exact sector matching later in the chapter.

Theorem 9.4.3 (Non-equivalent irreducible TP blocks are eventually non-proportional). [Lean](#)

[Lean](#) [✓ Stmt](#) *Let A and B be irreducible trace-preserving blocks whose self-overlaps converge to 1. If they are not gauge-phase equivalent, then for every lower bound N_0 there is $N \geq N_0$ such that $|V^{(N)}(A)\rangle$ is not proportional to $|V^{(N)}(B)\rangle$. Consequently, in an already separated same-dimensional BNT block family, any two distinct blocks have such non-proportional MPV states at arbitrarily large lengths.*

Proof. [✓ Proof](#) Suppose that for all sufficiently large N one had $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$. Then

$$O_{AA}(N) = |c_N|^2 O_{BB}(N), \quad (9.5)$$

$$O_{AB}(N) = c_N O_{BB}(N). \quad (9.6)$$

Since $O_{AA}(N) \rightarrow 1$ and $O_{BB}(N) \rightarrow 1$, the identity (9.5) gives $|c_N| \rightarrow 1$, and hence (9.6) gives $|O_{AB}(N)| \rightarrow 1$. The irreducible trace-preserving overlap decay theorem gives $O_{AB}(N) \rightarrow 0$ when $A \not\sim_{\text{gp}} B$, a contradiction. $\square$

9.5 Newton–Girard identities and power-sum recovery

The coefficients $c_N^{(j)} = \sum_q \mu_{j,q}^N$ that appear in the two-layer BNT expansion are power sums of the copy weights. Recovering the individual weights from these sums is a Newton–Girard problem: equal power sums determine equal characteristic polynomials, hence equal multisets of roots.

The general matrix statement is the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”]. It identifies a characteristic polynomial from the first n power-sum traces.

The matched-sector comparison below applies the power-sum identities to a single matched pair, after the BNT permutation and phase gauges have identified which basis tensors are compared: the coefficient $\sum_q \mu_{j,q}^N$ then equals $\sum_q (\nu_{j,q} e^{i\phi_j})^N$, and Newton–Girard recovers the weight multiset with multiplicity.

Theorem 9.5.1 (Finite-range equal power sums imply equal multisets). [Lean](#) [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#) *Let $\alpha, \beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$. If $\sum_i \alpha_i^k = \sum_i \beta_i^k$ for $1 \leq k \leq n$, then the multisets $\{\alpha_i\}$ and $\{\beta_i\}$ are equal.*

Two variants relax the equal-cardinality assumption. If $\alpha : \{0, \dots, m-1\} \rightarrow \mathbb{C}$ and $\beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$ have no zero entries and their power sums agree for $1 \leq k \leq \max(m, n)$, then $m = n$ and the multisets are equal. Without a nonzero hypothesis, the list form $\sum_{k=1}^{x_a} \lambda_{a,k}^N = \sum_{k=1}^{x_b} \lambda_{b,k}^N$ for $1 \leq N \leq \max(x_a, x_b)$ gives equality of the nonzero multisets; if it also holds at $N = 0$ then $x_a = x_b$ and the full lists agree. The nonzero variant is the one applied to the nonzero coefficients of a canonical form, which is how [CPGSV16, Lemma A.5] is used.

Proof. [✓ Proof](#) Apply the finite-range Newton–Girard theorem to $\text{diag}(\alpha)$ and $\text{diag}(\beta)$ and extract the roots of $\prod_i (X - \alpha_i)$ and $\prod_i (X - \beta_i)$. For unequal cardinalities, pad both lists with zeros to length $\max(m, n)$; the nonzero hypothesis makes the padded-zero counts $\max(m, n) - m$ and $\max(m, n) - n$, forcing $m = n$. Without that hypothesis, delete the zero entries first (unchanged for positive N); the equality at $N = 0$ gives $x_a = x_b$. $\square$

Corollary 9.5.2 (A nonzero finite family has a nonvanishing power sum). [Lean](#) [✓ Stmt](#) *Let $m \geq 1$ and let $\lambda_0, \dots, \lambda_{m-1} \in \mathbb{C}$ be nonzero. Then there is an exponent N with $1 \leq N \leq m$ such that $\sum_{q=0}^{m-1} \lambda_q^N \neq 0$. This is the consequence of the Appendix power-sum lemma used in the proof of Lemma L [CPGSV16, Appendix C.3, lines 1851–1858]; the finite-range comparison is [CPGSV16, Appendix A, lines 1155–1163].*

The copy retains the original coefficient μ_k , and the representatives $(C_j)_j$ form a basis of normal tensors for A .

Enumerate $\mathcal{G}$. The induced equivalence Φ between the dependent grouped block coordinates and the original flattened listed coordinates gives

$$\sum_{x \in \mathcal{G}} D_x = \sum_{k=1}^r D_k. \quad (9.13)$$

For $x = (j, q)$, let $\hat{\mu}_x$ be the original weight of that copy and let $\widehat{B}_x^i = \zeta_x C_j^i$, with the required dimension identification. Define the grouped tensor by

$$T_{\text{grp}}^i = \text{reindex}_{\Phi} \left(\bigoplus_{x \in \mathcal{G}} \hat{\mu}_x \widehat{B}_x^i \right). \quad (9.14)$$

If $G = \bigoplus_k X_k$ is the block-diagonal gauge, then, letter by letter,

$$\bigoplus_{k=1}^r \mu_k A_k^i = G T_{\text{grp}}^i G^{-1}, \quad (9.15)$$

and

$$A^i = U^\dagger G T_{\text{grp}}^i G^{-1} U. \quad (9.16)$$

Proof. ✓ Proof Enumerate the block indices by their phase classes and choose one normal representative in each class. The normal-tensor phase theorem gives the dimension equality, unit phase, and invertible gauge in (9.12). The BNT characterization proves that the representatives form a basis of normal tensors. Reindexing the dependent block coordinates gives Φ and (9.13). The exact heterogeneous block-diagonal reindexing identity identifies the grouped tensor with the representative copies in the original retained-coordinate space. Block-diagonal conjugation then gives (9.15); substitution into (9.9) gives (9.16) with the original coisometry U . In the notation of [CPGSV16, Proposition 4.13], these identities are

$$\bigoplus_{k=1}^r \mu_k A_k^i = G T_{\text{grp}}^i G^{-1}, \quad (9.17)$$

$$A^i = U^\dagger G T_{\text{grp}}^i G^{-1} U, \quad (9.18)$$

$$U U^\dagger = \mathbb{1}. \quad (9.19)$$

□

Definition 9.6.1.10 (BNT sector presentation). Lean Lean Lean Lean Lean ✓ Stmt A *BNT sector presentation* of a tensor A is a nonempty sector decomposition P such that every representative has a positive number of copies, every copy has nonzero weight, every representative C_j is normal, the representative matrix product vectors are eventually linearly independent, and, for every $N > 0$,

$$V^{(N)}(A) = V^{(N)}(P). \quad (9.20)$$

This records the nonzero canonical summands constructed at [CPGSV16, lines 217–246]: repeated copies and their weights remain explicit.

Theorem 9.6.1.11 (Uniqueness of BNT presentations). [Lean](#) [✓ Stmt](#) *Two BNT presentations of the same positive-length matrix product vector family have equally many representatives. After a permutation, corresponding representatives have equal bond dimension and are related by an invertible gauge and a unit-modulus phase.*

Proof. [✓ Proof](#) Apply Theorem 9.1.4 first to the direct-sum tensor of the first presentation, using the representatives of the second as a BNT, and then with the presentations interchanged. This gives a matching map in each direction. Eventual linear independence excludes gauge-phase-equivalent distinct representatives, so both maps are injective. Since the index sets are finite, they have equal cardinality, and either matching map is a bijection. Normal-tensor gauge rigidity gives the asserted gauge-phase relation for each matched pair. $\square$

Theorem 9.6.1.12 (Representative sector decomposition). [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *In the notation of Theorem 9.6.1.9, let $k(j, q)$ be the listed block corresponding to the class-copy pair (j, q) . The representatives C_j define a sector decomposition P_{rep} with copy weights*

$$\lambda_{j,q} = \nu_{k(j,q)} \zeta_{k(j,q)}. \quad (9.21)$$

Every copy weight in P_{rep} is nonzero. The representatives have simultaneous word-tuple separation at every sufficiently large length.

For every positive length N , the grouped tensor T_{grp} of Theorem 9.6.1.9 and the representative sector tensor generate the same matrix product vector. The original tensor has the same positive-length matrix product vectors as both:

$$V^{(N)}(T_{\text{grp}}) = V^{(N)}(P_{\text{rep}}). \quad (9.22)$$

$$V^{(N)}(A) = V^{(N)}(P_{\text{rep}}). \quad (9.23)$$

Hence P_{rep} is a BNT sector presentation of A whenever at least one listed block occurs.

Proof. [✓ Proof](#) The class-copy equivalence reindexes the listed blocks. (9.21) combines the original coefficient with the unit phase relating its block to C_j ; both factors are nonzero. The basis-of-normal-tensors condition and Theorem 9.3.4 give eventual simultaneous word-tuple separation for the representatives. Reindexing the finite sum by (j, q) gives (9.22). The original coisometric reconstruction and simultaneous block gauge preserve every positive-length closed-chain coefficient, yielding (9.23). $\square$

Definition 9.6.1.13 (BNT canonical form). [Lean](#) [✓ Stmt](#) *Given a sector decomposition P , these conditions define the normalized, left-canonical, aperiodic BNT surface used below. This is stronger than the abstract BNT of Definition 9.1.1 and the literal coefficient-form BNT of Definition 9.1.2: besides spanning and eventual linear independence, it fixes irreducible left-canonical representatives, their self-overlap normalization, gauge-phase separation, and the canonical-form weight normalization. It imposes no equal-modulus factorization or strict ordering on the raw weights. Explicitly, each basis block has positive bond dimension, is irreducible, and is left-canonical; the normalized self-overlap of every basis block converges to 1; the basis MPV family is eventually linearly independent; distinct same-dimension basis blocks are not gauge-phase equivalent after identifying equal bond dimensions; and the [CPGSV16, Section 2.3] canonical-form normalization holds,*

$$\begin{aligned} \forall (j, q), \quad & |\mu_{j,q}| \leq 1, \\ \exists (j_*, q_*), \quad & |\mu_{j_*,q_*}| = 1. \end{aligned}$$

The MPV expansion takes the raw two-layer form

$$V^{(N)}(P)_\sigma = \sum_{j=1}^g \left(\sum_{q=1}^{r_j} \mu_{j,q}^N \right) V^{(N)}(P_j)_\sigma, \quad (9.24)$$

matching [CPGSV16, Section 2.3] and [CPGSV21, Definition IV.2]. The canonical-form normalization admits every example of the source paper (in particular $C \oplus D$, $C \oplus (-C)$, $C \oplus (1/2)C$, and $C \oplus (-C) \oplus (1/2)C$) and is not derivable from the per-block normality hypotheses alone.

Definition 9.6.1.14 (Single-sector decomposition). [Lean](#) [✓ Stmt](#) For a tensor A , its single-sector decomposition has A as its only basis tensor, with one copy of weight 1.

Theorem 9.6.1.15 (Single-sector BNT canonical form). [Lean](#) [✓ Stmt](#) Let A have positive bond dimension, be irreducible and left-canonical, and satisfy $O_{AA}(N) \rightarrow 1$. Then its single-sector decomposition is a BNT canonical form.

Proof. [✓ Proof](#) The identity

$$O_{AA}(N) = \|V^{(N)}(A)\|^2 \rightarrow 1 \neq 0. \quad (9.25)$$

shows that $V^{(N)}(A)$ is nonzero for all sufficiently large N . A family consisting of one nonzero vector is linearly independent. All remaining conditions are immediate from the hypotheses and the unit weight. $\square$

Theorem 9.6.1.16 (Normal tensors have single-sector BNT presentations). [Lean](#) [✓ Stmt](#) Every normal tensor is gauge equivalent to a tensor whose single-sector decomposition is a BNT canonical form. The original tensor and the total tensor of this decomposition generate the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Choose the pure trace-preserving Perron gauge. Its tensor is irreducible and left-canonical, and primitivity gives self-overlap converging to one. The preceding theorem supplies the BNT canonical form. If P denotes the total tensor of its single-sector decomposition, then gauge invariance and the sector-decomposition formula give, for every $N > 0$ and word σ ,

$$V^{(N)}(A)_\sigma = V^{(N)}(B)_\sigma = 1^N V^{(N)}(B)_\sigma = V^{(N)}(P)_\sigma. \quad (9.26)$$

$\square$

Lemma 9.6.1.17 (Unit-modulus weight witness from the canonical-form normalization). [Lean](#) [✓ Stmt](#) Under the BNT canonical form, the [CPGSV16, Section 2.3] normalization provides indices (j_*, q_*) with $|\mu_{j_*, q_*}| = 1$.

Proof. [✓ Proof](#) This is the unit-modulus existential in the canonical-form normalization. $\square$

Corollary 9.6.1.18 (Basis blocks of a BNT canonical form are normal). [Lean](#) [✓ Stmt](#) If P is in BNT canonical form, then every basis block P_j is normal in the algebraic sense of Definition 2.4.4.

Proof. [✓ Proof](#) Each basis block has positive bond dimension, is irreducible and left-canonical, and has normalized self-overlap converging to 1. Theorem 6.9.3 applies to each block. $\square$

Theorem 9.6.1.19 (A BNT canonical form determines an algebraic BNT). [Lean](#) [✓ Stmt](#) If P is in BNT canonical form, then its basis blocks form an algebraic basis of normal tensors for the total tensor associated with P . Only this implication from canonical form to algebraic BNT is asserted.

Proof. ✓ **Proof** The preceding corollary gives normality of every basis block. The sector coefficient formula (9.8) expresses the total MPV as a linear combination of the basis MPVs at each positive length, and the BNT sector data give their eventual linear independence. $\square$

Theorem 9.6.1.20 (Positive blocking preserves BNT canonical form). Lean ✓ **Stmnt** *Let P be a BNT canonical form and let $p \geq 1$. Then the blocked sector decomposition $P^{[p]}$ is again a BNT canonical form.*

Proof. ✓ **Proof** For each representative A_j , normalized self-overlap gives peripheral period one, hence a primitive transfer map. Positive blocking therefore preserves irreducibility and left-canonical normalization. Moreover, $O_{A_j^{[p]} A_j^{[p]}}(N) = O_{A_j A_j}(Np) \rightarrow 1$. Let F_N be the linear isomorphism induced by concatenating blocked words, from the blocked configuration space at length N to the original one at length Np . For every representative, $F_N |V^{(N)}(A_j^{[p]})\rangle = |V^{(Np)}(A_j)\rangle$. Hence eventual linear independence is preserved. If two blocked representatives were gauge-phase equivalent, then for some nonzero ζ and every N , $|V^{(N)}(A_k^{[p]})\rangle = \zeta^N |V^{(N)}(A_j^{[p]})\rangle$. Writing these two vectors as f_k and f_j gives $1 \cdot f_k = \zeta^N \cdot f_j$ in an eventually linearly independent family, and therefore $k = j$. Finally, $|\mu_{j,q}^p| = |\mu_{j,q}|^p \leq 1$, and a unit-modulus weight remains of unit modulus after taking its p -th power. $\square$

Theorem 9.6.1.21 (Common exact quantum-Wielandt length for BNT representatives). Lean ✓ **Stmnt** *Let P be a BNT canonical form of total bond dimension D , with representatives A_j of bond dimensions D_j . For every $0 \leq j < g$, the representative A_j is D^4 -block injective: $S_{D^4}(A_j) = M_{D_j}(\mathbb{C})$.*

Proof. ✓ **Proof** The BNT hypotheses make A_j irreducible and left-canonical, with $O_{A_j A_j}(N) \rightarrow 1$. The preceding theorem therefore makes A_j normal, and quantum Wielandt gives $S_{D_j^4}(A_j) = M_{D_j}(\mathbb{C})$. By Theorem E.2.6, $D_j^4 \leq D^4$. Write $D^4 = D_j^4 + k$ and apply positive-length propagation k times. $\square$

Theorem 9.6.1.22 (CPSV simultaneous block-injectivity bound). Lean Lean ✓ **Stmnt** *Let P be a BNT canonical form with total bond dimension D . There is a positive integer $N \leq 3D^5$ such that its representatives have the simultaneous fixed-length span*

$$\text{span}\{(A_1^\omega, \dots, A_g^\omega) : |\omega| = N\} = \prod_{j=1}^g M_{D_j}(\mathbb{C}).$$

This is the block-injectivity estimate in [CPGSV16, lines 340–345].

Proof. ✓ **Proof** Every representative is block-injective at D^4 , and homogeneous-span propagation gives the same conclusion at $D^4 + 1$ and $3(D^4 + 1)$. If $g = 1$, take $N = D^4$. If $g \geq 2$, apply Lemma 9.3.3 with $L_0 = D^4$ and take

$$N = 3(g-1)(D^4 + 1).$$

The positive bond dimension of every representative gives $g \leq D$. Consequently $N \leq 3D^5$; the same estimate also covers the one-block case. The unit-modulus weight required by the BNT normalization ensures $D > 0$. $\square$

The matching arguments below read off three facts from a BNT canonical form: the raw coefficient identity, the cross-overlap decay between distinct basis blocks, and the combined-family linear independence.

Lemma 9.6.1.23 (Raw two-layer sector coefficient identity). Lean ✓ Stmt For every sector decomposition P , length N , and basis index j , $c_N^{(j)} = \sum_{q=1}^{r_j} \mu_{j,q}^N$. This is [CPGSV16, Section 2.3].

Proof. ✓ Proof The claim follows from the definition of the sector coefficient. $\square$

Lemma 9.6.1.24 (Cross-overlap decay between distinct basis blocks). Lean ✓ Stmt If P is a BNT canonical form and $j \neq k$, then $O_{P_j P_k}(N) \rightarrow 0$.

Proof. ✓ Proof Case on the bond dimensions. If they differ, the dimension-mismatch overlap-decay theorem applies. If they agree, the basis-distinctness clause rules out gauge-phase equivalence, so the equal-dimension overlap-decay theorem applies. $\square$

Lemma 9.6.1.25 (Combined-family eventual linear independence). Lean ✓ Stmt Let P and Q each be a BNT canonical form, and suppose every mixed overlap $O_{P_j Q_k}(N)$ tends to 0. Then the union family

$$\{|V^{(N)}(P_j)\}_{j=1}^{g_P} \cup \{|V^{(N)}(Q_k)\}_{k=1}^{g_Q}$$

is linearly independent for all sufficiently large N . This is [CPGSV16, Appendix A].

Proof. ✓ Proof Feed the per-family normalized self-overlaps, the within-family cross-overlap decay (Lemma 9.6.1.24), and the inter-family decay hypothesis into the two-family overlap-orthonormality criterion (Lemma E.1.9). $\square$

Example 9.6.1.26 ($C \oplus (1/2)C$, unequal-modulus admissibility). Lean ✓ Stmt For any tensor C , this construction forms a single-sector, two-copy decomposition with basis tensor C and raw weights $(1, 1/2)$. Thus the sector data themselves allow unequal-modulus copies. If C additionally has positive bond dimension, is irreducible and left-canonical, has normalized nonzero self-overlap, and satisfies the BNT independence clause, these data satisfy Definition 9.6.1.13; those hypotheses are not consequences of the construction alone.

9.6.2 Single-copy sector decompositions

A weighted block sum is the one-copy case of the sector decomposition used below. Equal or repeated sectors require the full coefficient formalism.

9.6.3 Prepared-block construction of BNT canonical forms

The construction below separates the after-blocking preparation of suitable blocks from the normalized-weight BNT construction: the prepared blocks are trace-preserving, primitive, irreducible, and injective; normalized weights then quotient gauge-phase-equivalent blocks into sectors to produce the canonical form of Definition 9.6.1.13.

Theorem 9.6.3.1 (Prepared BNT blocks after blocking). Lean ✓ Stmt Let A be an MPS tensor. Then there is a blocking length $p \geq 1$, nonzero weights μ_k , and blocks B_k over the p -blocked alphabet such that:

1. each block has positive bond dimension;
2. each block is trace-preserving, primitive, tensor-irreducible, and one-site injective;
3. $A^{[p]}$ and $\bigoplus_k \mu_k B_k$ generate the same MPV family at every positive length.

No weight-modulus normalization is asserted at this stage.

Proof. ✓ Proof Apply the common primitive irreducible block decomposition to (A, A) , giving a positive blocking length and a nonzero weighted family of trace-preserving primitive irreducible blocks with the positive-length MPV identity. Theorem E.3.2 supplies one common further blocking making every block one-site injective while preserving the other hypotheses; finally identify the iterated blocked alphabet with the direct one. $\square$

With the prepared blocks in hand, normalized weights produce a BNT canonical form by quotienting gauge-phase-equivalent blocks into sectors.

Theorem 9.6.3.2 (BNT canonical form of the phase-class quotient). Lean ✓ Stmt Let $(B_k)_{k=1}^r$ have positive bond dimensions and suppose that every B_k is trace-preserving, primitive, and irreducible. Assume $\mu_k \neq 0$ and $|\mu_k| \leq 1$ for every k , and $|\mu_{k_*}| = 1$ for some k_* . Partition the blocks into matrix-product-vector phase classes, choose one representative from each class, and retain every class member as a copy of its representative with the corresponding phase-adjusted weight. The resulting sector decomposition is in BNT canonical form.

Proof. ✓ Proof The chosen representatives inherit irreducibility and trace preservation, while primitivity gives normalized self-overlap. Distinct phase classes give gauge-phase-distinct representatives. The collapsed phase-class construction supplies the eventual linear independence and retains every original block as one copy. Its phase multipliers have unit modulus, so the adjusted weights preserve the assumed bounds and unit witness. These are precisely the fields of Definition 9.6.1.13. $\square$

Theorem 9.6.3.3 (BNT from normal blocks with normalized weights). Lean ✓ Stmt Let $r \in \mathbb{N}$, and for each $k = 1, \dots, r$ let $D_k \in \mathbb{N}$, $\mu_k \in \mathbb{C}$, and let B_k be an MPS tensor of physical dimension d and bond dimension D_k . Assume:

1. **positive bond dimensions:** $D_k > 0$ for all k ,
2. **TP normalization:** $\sum_i (B_k^i)^\dagger B_k^i = \mathbb{1}$,
3. **primitivity:** $\mathcal{E}_{B_k}$ is primitive for each k ,
4. **irreducibility:** each B_k is irreducible,
5. **nonzero weights:** $\mu_k \neq 0$ for all k ,
6. **bounded weights:** $|\mu_k| \leq 1$ for all k ,
7. **global unit witness:** $|\mu_{k_*}| = 1$ for some k .

Then there exists a sector decomposition P such that P and $\bigoplus_{k=1}^r \mu_k B_k$ generate the same MPV family at every positive length, and P satisfies the basis-of-normal-tensors conditions (Definition 9.1.2) together with the multiplicity, span, and weight-bound conditions of Definition 9.6.1.13.

Proof. ✓ Proof The phase-class quotient groups gauge-phase-equivalent blocks into sectors with unit-modulus per-copy scalars. Under the TP, primitive, and irreducible hypotheses the phase-class construction applies, and for distinct representatives $B_j, B_{j'}$ ($j \neq j'$) the cross-overlap decays,

$$\begin{aligned} \lim_{N \rightarrow \infty} O_{B_j B_{j'}}(N) &= 0, \\ \lim_{N \rightarrow \infty} O_{B_j B_j}(N) &= 1, \end{aligned}$$

which gives the eventual linear-independence condition with no injectivity required. The resulting sector decomposition inherits the span and multiplicity conditions, and the bounded-weight and global-unit hypotheses give the weight-norm conditions. $\square$

The BNT construction preserves total bond dimension because CF summands assigned to the same BNT representative have equal bond dimension. Indeed, suppose trace-preserving primitive irreducible NTs X and Y of positive bond dimension have MPV families related by $|V^{(N)}(Y)\rangle = \zeta^N |V^{(N)}(X)\rangle$ with $\zeta \neq 0$. The normalized self-overlaps give $|O_{XX}(N)| \rightarrow 1$. The phase relation gives the cross-overlap identity

$$\begin{aligned} O_{YX}(N) &= \zeta^N O_{XX}(N), \\ |O_{YX}(N)| &= |\zeta|^N |O_{XX}(N)|. \end{aligned}$$

Since the same normalization forces $|\zeta| = 1$, this gives $|O_{YX}(N)| \rightarrow 1$. If their bond dimensions differed, the rectangular overlap-decay theorem would give $O_{YX}(N) \rightarrow 0$, a contradiction.

Theorem 9.6.3.4 (Gauge realization of prepared phase-equivalent NTs). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimensions. If their matrix-product-vector families differ by a nonzero scalar power, then their bond dimensions are equal and, after identifying the bond spaces, X and Y are gauge-phase equivalent. Neither positivity of the scalar nor unitarity of the gauge is asserted.*

Proof. [✓ Proof](#) Lemma E.3.4 identifies the bond dimensions. If the resulting same-dimensional blocks were not gauge-phase equivalent, Theorem 9.4.3 would give a positive length at which their matrix product vectors are not proportional. This contradicts the assumed scalar-power relation. The normalized self-overlaps required there follow from Theorem C.5.2. $\square$

Lemma 9.6.3.5 (Gauge realization with the fixed phase scalar). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimensions. Fix $\zeta \in \mathbb{C} \setminus \{0\}$ and suppose that, for every positive length N and every physical word σ ,*

$$V^{(N)}(Y)_\sigma = \zeta^N V^{(N)}(X)_\sigma. \quad (9.27)$$

Then, after identifying the bond spaces, there exists $G \in \text{GL}_{D_Y}(\mathbb{C})$ such that, for every physical index i ,

$$Y^i = \zeta G X^i G^{-1}. \quad (9.28)$$

Proof. [✓ Proof](#) Theorem 9.6.3.4 gives an invertible G and a nonzero scalar η such that $Y^i = \eta G X^i G^{-1}$. By Theorem 2.3.18, $V^{(N)}(Y)_\sigma = \eta^N V^{(N)}(X)_\sigma$. Theorem C.5.2 gives consecutive positive lengths M and $M+1$ with nonzero coefficients of $V^{(M)}(X)$ and $V^{(M+1)}(X)$. Comparing the two scalar-power identities gives $\zeta^M = \eta^M$ and $\zeta^{M+1} = \eta^{M+1}$. Hence $\zeta = \eta$, which proves (9.28). $\square$

Theorem 9.6.3.6 (Dimension-preserving BNT form of prepared blocks). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 9.6.3.2, there is a BNT canonical-form sector decomposition P such that*

$$\begin{aligned} \mathcal{V}^+(P) &= \mathcal{V}^+ \left(\bigoplus_{k=1}^r \mu_k B_k \right), \\ D_{\text{tot}}(P) &= \sum_{k=1}^r D_k. \end{aligned} \quad (9.29)$$

Proof. ✓ Proof Use the phase-class sector decomposition from Theorem 9.6.3.2. Its flattened copies are in bijection with the original blocks, and phase-equivalent blocks have equal bond dimension by Theorem 9.6.3.4. Summing the copy dimensions over this bijection gives (9.29). $\square$

Theorem 9.6.3.7 (Exact reindexing of heterogeneous weighted block sums). Lean Lean ✓ Stmt Let $e : \{1, \dots, r\} \simeq \{1, \dots, r'\}$ satisfy $D_k = D'_{e(k)}$. Let Q_k be the coordinate permutation whose rows are indexed by the primed block and whose columns are indexed by the unprimed block. Suppose that, for every block and physical index,

$$\mu_k A_k^i = Q_k^\dagger \left(\mu'_{e(k)} B_{e(k)}^i \right) Q_k. \quad (9.30)$$

The induced equivalence E of flattened dependent block coordinates has a permutation matrix Q_E , with rows indexed by the primed coordinates and columns indexed by the unprimed coordinates. For every physical index i ,

$$\bigoplus_{k=1}^r \mu_k A_k^i = Q_E^\dagger \left(\bigoplus_{\ell=1}^{r'} \mu'_\ell B_\ell^i \right) Q_E. \quad (9.31)$$

Proof. ✓ Proof Evaluate both sides on two flattened block coordinates. When their block labels agree, (9.30) gives the matrix entry. When the labels differ, both block-diagonal entries vanish. This proves (9.31). $\square$

Theorem 9.6.3.8 (Exact BNT reconstruction of prepared normal blocks). Lean ✓ Stmt Under the hypotheses of Theorem 9.6.3.2, there are a BNT canonical-form sector decomposition P , an equivalence E of flattened bond coordinates, and $X \in \text{GL}_{D_{\text{tot}}(P)}(\mathbb{C})$. Write Q_E for the permutation matrix whose rows are indexed by the coordinates of P and whose columns are indexed by the original block coordinates. The first identity below holds, and the second holds for every physical index i :

$$D_{\text{tot}}(P) = \sum_{k=1}^r D_k, \quad (9.32)$$

$$\bigoplus_{k=1}^r \mu_k B_k^i = Q_E^\dagger (X P^i X^{-1}) Q_E. \quad (9.33)$$

Proof. ✓ Proof Partition the prepared blocks into matrix-product-vector phase classes. For the copy q in class j , retain the precise class scalar $\zeta_{j,q}$ chosen in the weight $\zeta_{j,q} \mu_{j,q}$. The normal-tensor phase theorem supplies an invertible $G_{j,q}$ with

$$B_{j,q}^i = \zeta_{j,q} G_{j,q} C_j^i G_{j,q}^{-1}. \quad (9.34)$$

Thus the original weighted block equals the corresponding sector copy after conjugation, since its sector weight is $\zeta_{j,q} \mu_{j,q}$. Assemble the $G_{j,q}$ into a block-diagonal matrix X and apply Theorem 9.6.3.7. This gives (9.33); summing the matched copy dimensions gives (9.32). $\square$

Lemma 9.6.3.9 (Gauge equivalence of direct sums from per-block gauge equivalence). Lean ✓ Stmt If each pair (A_k, B_k) is gauge equivalent, then the weighted block-diagonal tensors $\bigoplus_k \mu_k A_k$ and $\bigoplus_k \mu_k B_k$ are gauge equivalent.

Proof. ✓ Proof Choose per-block gauge matrices via the hypothesis; then apply Lemma F.4.3. $\square$

Theorem 9.6.3.10 (Positive-length MPVs under coisometric reconstruction). [Lean](#) [✓ Stmt](#) Let A and B be MPS tensors, possibly with different bond dimensions. Suppose that U satisfies $UU^\dagger = \mathbb{1}$ and $A^i = U^\dagger B^i U$ for every physical index i . Then A and B generate the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Multiply the reconstruction identities around a positive-length cycle. Each adjacent pair $UU^\dagger$ cancels, and cyclicity of the trace removes the remaining outer factors. $\square$

Theorem 9.6.3.11 (Sector BNT form of normalized CPSV canonical data). [Lean](#) [✓ Stmt](#) Let A have CPSV canonical-form data satisfying the separate normalization condition (8.25), and assume $A \neq 0$. Then there is a BNT canonical-form sector decomposition P such that

$$\begin{aligned} \mathcal{V}^+(A) &= \mathcal{V}^+(P), \\ D_{\text{tot}}(P) &= \sum_{k=1}^r D_k. \end{aligned} \tag{9.35}$$

Thus the dimension is the total dimension of the blocks, not the ambient bond dimension D . This statement records positive-length MPV equality; the exact coisometric reconstruction is stated next.

Proof. [✓ Proof](#) Apply Theorem 9.6.3.12. Its reconstruction writes $A^i = U^\dagger B^i U$, where $UU^\dagger = \mathbb{1}$ and $B^i = X P^i X^{-1}$. Theorem 9.6.3.10 shows that A and B generate the same matrix product vectors at every positive length. Theorem 2.3.17 gives the same conclusion for B and P . Transitivity yields $\mathcal{V}^+(A) = \mathcal{V}^+(P)$, while the dimension identity is already part of the exact reconstruction theorem. $\square$

Theorem 9.6.3.12 (Exact coisometric BNT reconstruction). [Lean](#) [✓ Stmt](#) Let A have CPSV canonical-form data satisfying the separate normalization condition (8.25), and assume $A \neq 0$. There are a BNT canonical-form sector decomposition P , a coisometry $U \in \mathbb{C}^{D_{\text{tot}}(P) \times D}$, and $X \in \text{GL}_{D_{\text{tot}}(P)}(\mathbb{C})$. The first two identities below hold, and the third holds for every physical index i :

$$D_{\text{tot}}(P) = \sum_{k=1}^r D_k, \tag{9.36}$$

$$UU^\dagger = \mathbb{1}, \tag{9.37}$$

$$A^i = U^\dagger (X P^i X^{-1}) U. \tag{9.38}$$

Proof. [✓ Proof](#) The canonical-form data give the exact reconstruction

$$A^i = V^\dagger \left(\bigoplus_{k=1}^r \mu_k A_k^i \right) V, \tag{9.39}$$

$$VV^\dagger = \mathbb{1}. \tag{9.40}$$

Put every normal block into a trace-preserving primitive irreducible gauge Y , and apply Theorem 9.6.3.8. Let Q be the permutation matrix of its flattened coordinate equivalence, so $QQ^\dagger = \mathbb{1}$. Reindexing Y by the same equivalence gives an invertible matrix W on the bond space of P and the identities

$$QY = WQ, \tag{9.41}$$

$$Y^{-1}Q^\dagger = Q^\dagger W^{-1}. \tag{9.42}$$

If X_0 is the prepared-block gauge, set $X = W^{-1}X_0$ and $U = QV$. Substitution in (9.39) gives (9.38), while

$$UU^\dagger = Q(VV^\dagger)Q^\dagger = \mathbb{1}.$$

This proves (9.37). The block enumeration gives (9.36). $\square$

Theorem 9.6.3.13 (Equal-ambient gauge equivalence for CPSV canonical forms). [Lean](#) [✓ Stmt](#) *Let A and B have the same ambient bond dimension D . Suppose each tensor has CPSV canonical-form data, both data satisfy the separate weight normalization condition (8.25), neither tensor is zero, and $\mathcal{V}^+(A) = \mathcal{V}^+(B)$. Then A and B are gauge equivalent: there is $G \in \text{GL}_D(\mathbb{C})$ such that, for every physical index i ,*

$$B^i = GA^iG^{-1}. \quad (9.43)$$

Unused ambient coordinates are allowed. The conclusion requires the common ambient dimension D and makes no claim for canonical forms with different ambient bond dimensions.

The cited sources supply the retained canonical-form comparison. The passage back from the retained tensors to the ambient tensors is the coisometric lifting of Theorem 8.8.5.

Proof. [✓ Proof](#) Apply Theorem 9.6.3.12 to obtain BNT canonical forms P, Q , coisometries U_A, U_B , and retained-space gauges X_A, X_B with

$$\begin{aligned} A^i &= U_A^\dagger X_A P^i X_A^{-1} U_A, \\ B^i &= U_B^\dagger X_B Q^i X_B^{-1} U_B. \end{aligned} \quad (9.44)$$

Theorem 9.6.3.10, gauge invariance, and the hypothesis give

$$\mathcal{V}^+(P) = \mathcal{V}^+(A) = \mathcal{V}^+(B) = \mathcal{V}^+(Q). \quad (9.45)$$

Theorem 10.4.2.1 therefore identifies the retained dimensions and gives Y such that $Q^i = YP^iY^{-1}$. After identifying the retained spaces, the two middle tensors in (9.44) are conjugate by $X_B Y X_A^{-1}$. Theorem 8.8.5 aligns the two coisometries inside the common ambient space and yields (9.43). $\square$

Combining the after-blocking preparation with the weight normalization of Theorem E.3.7 gives the arbitrary-input statement.

Theorem 9.6.3.14 (Normalized BNT form after blocking for a nonzero MPV family). [Lean](#) [✓ Stmt](#) *Let A be an MPS tensor whose matrix-product family has a nonzero coefficient at some positive length. Then there exist a blocking length $p \geq 1$, a positive real m , and a sector decomposition P over the blocked alphabet in the BNT canonical form of Definition 9.6.1.13, such that for every positive length N ,*

$$V^{(N)}(A^{[p]}) = m^N V^{(N)}(P). \quad (9.46)$$

The scalar m^N is the per-site normalization cost of the line-246 weight choice, which is realized rather than assumed. The nonzero-family hypothesis is the standing convention of [CPGSV16, Section 2.3]: the unit-modulus weight chosen there presupposes a nonempty weight family. The normalization and the proportional-MPV comparison are documented in [con26c].

Proof. ✓ Proof Choose a word w with $\text{tr}(A^w) \neq 0$. For every $p \geq 1$, there is a $q \geq 1$ such that $\text{tr}((A^w)^{pq}) \neq 0$. Otherwise Newton–Girard applied to $(A^w)^p$ and the zero matrix would show that $(A^w)^p$, and hence A^w , is nilpotent, contradicting $\text{tr}(A^w) \neq 0$. Repeating w exactly pq times gives a nonzero coefficient of the p -blocked tensor. Thus the prepared family at the blocking length supplied by Theorem 9.6.3.1 is nonempty.

Normalize its weights by their maximal modulus m (Theorem E.3.7). By the weight-scaling identity of Lemma E.3.8,

$$\begin{aligned} V^{(N)}(A^{[p]}) &= V^{(N)}\left(\bigoplus_k \mu_k B_k\right) = V^{(N)}\left(\bigoplus_k m \frac{\mu_k}{m} B_k\right), \\ &= m^N V^{(N)}\left(\bigoplus_k \frac{\mu_k}{m} B_k\right) = m^N V^{(N)}(P). \end{aligned}$$

where the last identity is the prepared supplier of Theorem 9.6.3.3 applied to the normalized weights, which gives the BNT canonical form P . $\square$

9.7 Coefficient comparison and copy-weight recovery

The exact matching of Section 9.8 produces, on each matched pair, a coefficient identity. This section turns that identity into equality of the per-copy weight multisets [CPGSV16, Appendix A]. No analytic non-decay estimate enters: the fixed-length linear independence isolates each sector coefficient exactly.

9.7.1 Matched-sector weight multiset equality

Given a matched pair $(j_0, \tilde{k}_0)$ and a nonzero scalar ζ with $c_N^{(j_0)}(P) = \zeta^N c_N^{(\tilde{k}_0)}(Q)$ past a threshold N_0 , this section records two conclusions: the multiset equality of the per-copy weights from power-sum recovery (Theorem E.2.5), and its upgrade to an explicit per-copy permutation. Together they give the $\mu_{j,q} = \nu_{j,q} e^{i\varphi_j}$ recovery of [CPGSV16, Appendix A]. In the equal-MPV case, Chapter 10 obtains this coefficient identity by expanding both total MPVs in the matched BNT basis and applying eventual linear independence (Lemma 10.2.2). For merely proportional total MPVs, a common length-dependent scalar remains in every matched coefficient identity and the scalar-free comparison need not hold; see Remark 10.3.6.

Lemma 9.7.1.1 (Matched-sector weight multiset equality). Lean ✓ Stmt *Let P, Q be sector decompositions with weights $\mu_{j,q}$ and $\nu_{k,q'}$, and fix indices $j_0, \tilde{k}_0$, and a nonzero scalar $\zeta \in \mathbb{C} \setminus \{0\}$. Suppose that for every $N > N_0$,*

$$c_N^{(j_0)}(P) = \zeta^N \cdot c_N^{(\tilde{k}_0)}(Q). \tag{9.47}$$

Then the per-sector multiplicities agree, $r_{j_0}(P) = r_{\tilde{k}_0}(Q)$, and the rescaled per-copy weight multisets coincide. This is [CPGSV16, Appendix A] read on a single matched sector.

Proof. ✓ Proof Expand both sides of the coefficient identity (9.47) into per-copy power sums $\sum_q \mu_{j_0,q}^N$ and $\sum_{q'} \nu_{\tilde{k}_0,q'}^N$, and distribute ζ^N through the right-hand sum to obtain, for every $N > N_0$,

$$\sum_q \mu_{j_0,q}^N = \sum_{q'} (\zeta \cdot \nu_{\tilde{k}_0,q'})^N.$$

Extend to all positive exponents and apply the unequal-cardinality multiset-recovery lemma (Theorem 9.5.1), using $\zeta \neq 0$ together with the weight non-vanishing assumption to ensure the input weights are nonzero. This yields the multiplicity match and the multiset equality. $\square$

Lemma 9.7.1.2 (Matched-sector weight-permutation extractor). Lean ✓ Stmt *Under the hypotheses of Lemma 9.7.1.1, the per-sector multiplicities agree and there is a permutation τ of the per-copy index set identifying the individual weights up to the gauge phase: for every q , $\nu_{k_0, \tau(q)} = \zeta^{-1} \cdot \mu_{j_0, q}$. This is the per-copy realisation of [CPGSV16, Appendix A]: the multiset equality from Lemma 9.7.1.1 is upgraded to a concrete indexing between the matched P - and Q -copies.*

Proof. ✓ Proof Invoke Lemma 9.7.1.1 to obtain the cardinality match and the multiset equality. From this multiset-map equality extract a permutation matching the elements pointwise (proved by induction on the cardinality). Compose with the cardinality identification and cancel ζ via $\zeta \neq 0$ to convert the gauge factor on the right of the pointwise identity to ζ^{-1} on the left. $\square$

9.8 Strong existential matching by exact linear independence

The exact matching reads [CPGSV16, Appendix A] as a statement at fixed length: equal MPV families force each basis sector of one form to pair with a basis sector of the other. The non-vanishing of the sector coefficients isolates the matched sector exactly, with no limit of a single-sector projection.

Theorem 9.8.1 (Exact single-sector matching). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family (at all positive lengths). For every sector j of P there exist a sector k of Q with matched bond dimension $\dim_P(j) = \dim_Q(k)$, a gauge-phase equivalence after identifying the matched bond dimensions, and a non-decaying cross-overlap. No per-sector unit-modulus hypothesis is required.*

Proof. ✓ Proof Equality at every positive length gives eventual nonzero proportionality with scalar 1. Apply Theorem E.2.1.1 to the chosen sector. $\square$

Theorem 9.8.2 (Strong existential matching, full-basis form). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family at every positive length. Then for every sector k of Q there exist a sector j of P with matched bond dimension $\dim_P(j) = \dim_Q(k)$, a gauge-phase equivalence after identifying the matched bond dimensions, and a non-decaying cross-overlap.*

Proof. ✓ Proof Convert equality to eventual nonzero proportionality with scalar 1 and apply Theorem E.2.1.2. $\square$

9.8.1 Bijective matching by symmetry

Applying the exact matching once more with P and Q swapped gives the reverse direction; the basis-distinctness clause forces injectivity both ways, so $g_a = g_b$ and the source relabelling $B_k = e^{i\phi_k} X_k A_{j_k} X_k^{-1}$ holds on the full basis [CPGSV16, Appendix A].

Theorem 9.8.1.1 (Bijective matching by symmetry). Lean ✓ Stmt *Let P, Q be two BNT canonical forms with the same MPV family at every positive length. Then there is a bijection $\beta : J(Q) \simeq J(P)$ such that for every $k \in J(Q)$ the bond dimensions match, the basis tensors $P_{\beta(k)}$ and Q_k are gauge-phase equivalent after identifying the matched bond dimensions, and the basis cross-overlap does not tend to zero.*

Proof.  **Proof** Apply the full-basis matcher in both directions, using symmetry of the same-MPV relation. Lemma [E.2.1.3](#) converts these two families of matches into the required bijection. $\square$

Chapter 10

Proof of the Fundamental Theorem

For BNT canonical forms P and Q , write the weighted direct sums over normal-tensor sectors as $P^i = \bigoplus_{j,q} \mu_{j,q} P_j^i$ and $Q^i = \bigoplus_{k,q} \nu_{k,q} Q_k^i$. The proportional theorem matches the normal sectors by a bijection β , unit phases ζ_k , and invertible matrices X_k satisfying

$$Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}.$$

It does not determine the copy weights. In the equal-MPV case, eventual linear independence gives

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q).$$

Finite power-sum comparison then gives $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Thus the sector phases do not become equal: each phase cancels against the inverse phase in its matched copy weight. The remaining block conjugacies assemble into $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$, and a sector-coordinate permutation yields the literal global gauge.

Throughout, “eventually” means “for all sufficiently large N ”. At a fixed length in that range, BNT linear independence detects exact coefficient equalities even when a weight of modulus less than 1 makes its powers decay.

10.1 Matched block maps for product algebras

The Fundamental Theorem compares the simple matrix summands selected by a product-algebra equivalence. The following tensor-network-facing statements record the component maps and their dimension consequence.

Theorem 10.1.1 (Paired component maps for products of matrix algebras). [Lean](#) [Lean](#) [Lean](#)

[✓ Stmt](#)

For a ring equivalence between products of full matrix algebras, the map induced between each pair of matched components is bijective.

Proof. [✓ Proof](#)

Apply Theorem [\[LTC26\]](#), “Paired component maps are bijections”] to the matched block ideals. □

Lemma F.1.2 applies to this MPV identity. Equivalently, the normalized self-overlaps satisfy

$$O_{Q_k Q_k}(N) = |\zeta_k|^{2N} O_{P_{\beta(k)} P_{\beta(k)}}(N),$$

and both outer sequences tend to 1, which forces $|\zeta_k| = 1$. $\square$

Lemma 10.2.2 (Full-basis coefficient identity from matched phases). Lean ✓ Stmt *Let P be a BNT canonical form, let Q be a sector decomposition, and assume P_{tot} and Q_{tot} have the same MPV family at every positive length. Suppose that there is a bijection $\beta : J(Q) \simeq J(P)$ and scalars $\zeta_k \in \mathbb{C}$ such that, for every $k \in J(Q)$, every positive length N , and every spin configuration σ of length N ,*

$$V^{(N)}(Q_k)_\sigma = \zeta_k^N V^{(N)}(P_{\beta(k)})_\sigma. \quad (10.2)$$

Then for every $k \in J(Q)$ there is N_0 such that, for all $N > N_0$, $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$. This is the coefficient-comparison step of [CPGSV16, Appendix MPV proof, lines 1187–1188] in full-basis form.

Proof. ✓ Proof For every N , expand the equality of total MPVs as

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = \sum_{k \in J(Q)} c_N^{(k)}(Q) |V^{(N)}(Q_k)\rangle.$$

The phase relation (10.2) gives $|V^{(N)}(Q_k)\rangle = \zeta_k^N |V^{(N)}(P_{\beta(k)})\rangle$, so reindexing the Q -sum by β onto the P -basis yields

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = \sum_{j \in J(P)} \zeta_{\beta^{-1}(j)}^N c_N^{(\beta^{-1}(j))}(Q) |V^{(N)}(P_j)\rangle.$$

For all sufficiently large N the P -basis MPV family $\{|V^{(N)}(P_j)\rangle\}_{j \in J(P)}$ is linearly independent, by the BNT eventual linear independence of P . Lemma E.1.11 then extracts the coefficient equalities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ from this eventual linear independence, for all sufficiently large N . $\square$

Definition 10.2.3 (Copy-weight matching for matched BNT sectors). Lean ✓ Stmt Let the sectors of two decompositions P and Q be matched by $\beta : J(Q) \simeq J(P)$, with sector phases ζ_k . A copy-weight matching consists of copy permutations τ_k such that, for all sectors k and copies q , $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. This is the copy-level conclusion of [CPGSV16, Appendix MPV proof, line 1188].

Definition 10.2.4 (Copy-weight matching from matched-sector coefficient identities). Lean ✓ Stmt The construction takes as input a sector matching $\beta : J(Q) \simeq J(P)$, nonzero sector phases ζ_k , and the eventual matched-sector coefficient identities

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q). \quad (10.3)$$

It returns the copy-weight matching obtained by choosing, in every sector, a copy permutation τ_k satisfying $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. This is the sectorwise construction in [CPGSV16, Appendix MPV proof, line 1188].

Proof. ✓ Proof Eventually the coefficient identity (10.3) in sector k reads

$$\sum_r \mu_{\beta(k),r}^N = \zeta_k^N \sum_q \nu_{k,q}^N = \sum_q (\zeta_k \nu_{k,q})^N.$$

Newton–Girard finite power-sum comparison (Lemma 9.7.1.2) applies sector by sector, producing a copy permutation τ_k with $\mu_{\beta(k),\tau_k(q)} = \zeta_k \nu_{k,q}$; since $\zeta_k \neq 0$ this is $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}$. $\square$

Theorem 10.2.5 (Equal-case sector and copy matching with differing bond dimensions). Lean

✓ Stmt *Let P and Q be BNT canonical forms, possibly with different sector counts and different bond dimensions before matching. If their total tensors generate the same MPV family at every positive length, then there is a bijection $\beta : J(Q) \simeq J(P)$ such that each matched basis pair has equal bond dimension and is gauge-phase equivalent. The matched copy multiplicities agree, and there are unit phases ζ_k and copy bijections τ_k satisfying*

$$\begin{aligned} Q_k^i &= \zeta_k X_k P_{\beta(k)}^i X_k^{-1}, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}. \end{aligned}$$

Only the global weight normalization in Definition 9.6.1.13 is assumed; no unit-modulus copy is required in each sector. This is the sector-data form of [CPGSV16, Corollary 2.11 and Appendix A, lines 1182–1188].

Proof. ✓ Proof First, Theorem 9.8.1.1 gives the sector bijection, the bond-dimension equalities, and the gauge-phase relations. Choose their phases ζ_k and use the normalized self-overlaps to obtain $|\zeta_k| = 1$. Second, Lemma 10.2.2 expands the equality of the two total MPVs in the matched P -basis. Eventual linear independence gives, for every k and all sufficiently large N ,

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q).$$

Third, expand both coefficients as finite power sums. Their eventual equality determines the two multisets of nonzero copy weights, so the multiplicities agree and a copy bijection τ_k realizes the identity $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}$. $\square$

10.3 The equal-MPV global gauge

For equal MPV families, the sector phases occur twice: in the matched tensor relation and with the inverse power in the matched copy weights. They therefore cancel before the block gauges are assembled.

Definition 10.3.1 (Flattened block-diagonal gauge). Lean Lean ✓ Stmt Given per-block gauges $X_k \in \text{GL}_{D_k}(\mathbb{C})$, form the block-diagonal element $\bigoplus_k X_k$ and then reindex the dependent block-coordinate space with the standard bond index $\{0, \dots, \sum_k D_k - 1\}$ of the assembled tensor. The result is an element of $\text{GL}_{\sum_k D_k}(\mathbb{C})$.

Theorem 10.3.2 (Global gauge from matched sectors and copy weights). Lean ✓ Stmt *Suppose that the sectors of two sector decompositions P and Q are matched by a bijection $\beta : J(Q) \simeq J(P)$, with bond-dimension equalities, copy permutations τ_k , nonzero phases ζ_k , and block gauges X_k . If, for every sector k , copy q , and physical index i ,*

$$\begin{aligned} Q_k^i &= \zeta_k X_k P_{\beta(k)}^i X_k^{-1}, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k),\tau_k(q)}, \end{aligned} \tag{10.4}$$

then, in the flattened Q -coordinates,

$$Q_{\text{flat}}^i = X P_{\text{matched}}^i X^{-1},$$

$$X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k).$$

This is the direct-sum gauge construction of [CPGSV16, Appendix MPV proof, lines 1189–1192], separated from the coefficient-comparison step that supplies the weight identities.

Proof. ✓ **Proof** For each matched copy,

$$\nu_{k,q} Q_k^i = (\zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}) (\zeta_k X_k P_{\beta(k)}^i X_k^{-1}) = \mu_{\beta(k), \tau_k(q)} X_k P_{\beta(k)}^i X_k^{-1}.$$

Thus every flattened weighted block is conjugated by its sector gauge X_k . Lemma F.4.3 combines these block conjugacies into the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$. $\square$

Lemma 10.3.3 (Equal-MPV matched copy-weight witnesses). Lean ✓ **Stmt** Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then there are a bijection $\beta : J(Q) \simeq J(P)$, bond-dimension equalities, unit-modulus phases ζ_k , and block gauges X_k such that $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. For the same phases there is a copy-weight matching: after a permutation of the copies inside each matched sector, $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$.

Proof. ✓ **Proof** The full-basis matching theorem gives the bijection and block gauges. The self-overlap comparison

$$\langle V_N(Q_k), V_N(Q_k) \rangle = |\zeta_k|^{2N} \langle V_N(P_{\beta(k)}), V_N(P_{\beta(k)}) \rangle$$

together with the BNT normalization limits on both sides forces $|\zeta_k| = 1$. The matched MPV phase identities also give the eventual coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$. Definition 10.2.4 applies the finite power-sum comparison in each sector and gives the copy-weight identities. $\square$

Theorem 10.3.4 (Full-basis global gauge in matched coordinates). Lean ✓ **Stmt** Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then there exist:

- a bijection $\beta : J(Q) \simeq J(P)$,
- bond-dimension equalities $\dim P_{\beta(k)} = \dim Q_k$,
- copy-number equalities $r_{\beta(k)}(P) = r_k(Q)$ and copy permutations τ_k ,
- unit-modulus phases ζ_k with $|\zeta_k| = 1$ and block gauges X_k ,

such that, for every sector k , copy q , and physical index i ,

$$Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1},$$

$$\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}.$$

Moreover, if $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$ is the flattened block-diagonal gauge, then, for every physical index i , the flattened Q -coordinates satisfy

$$Q_{\text{flat}}^i = X P_{\text{matched}}^i X^{-1}.$$

This is the explicit direct-sum gauge construction of [CPGSV16, Appendix MPV proof, lines 1189–1192].

Proof. ✓ Proof Lemma 10.3.3 gives β , the bond-dimension identifications, the phases ζ_k , the block gauges X_k , and copy permutations τ_k satisfying $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Theorem F.4.4 combines the block conjugacies into the flattened direct-sum conjugation by $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$. $\square$

Theorem 10.3.5 (Literal equal-MPV gauge after sector-coordinate permutation). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total tensors have the same MPV family at every positive length. Then the total bond dimensions agree; write their common value as D . There is*

$$Y \in \text{GL}_D(\mathbb{C}) \quad (10.5)$$

such that, for every physical index i ,

$$Q_{\text{tot}}^i = Y P_{\text{tot}}^i Y^{-1},$$

where P_{tot}^i and Q_{tot}^i are both regarded as matrices in $M_D(\mathbb{C})$ using the total bond-dimension equality. This is the BNT form of [CPGSV16, Corollary 2.11 and Appendix A, lines 1189–1192].

Proof. ✓ Proof Apply Theorem 10.3.4 to obtain the flattened gauge after the sectors of P have been reindexed by the bijection between basis blocks of P and Q . The sector-coordinate permutation ρ , induced by this bijection, the copy permutations, and the bond-dimension equalities, identifies the reindexed P -side direct sum with the total tensor P_{tot} , transported along the total bond-dimension equality. Composing the permutation gauge for ρ with the sector-permuted gauge gives the single matrix Y of (10.5). $\square$

Remark 10.3.6 (Proportional sector matching does not determine copy weights). The proportional theorem determines the normal sectors, but it does not determine the copy weights or imply conjugacy of the total tensors. Let C be a normalized normal tensor and consider the one-sector BNT forms

$$P = C \oplus \tfrac{1}{2}C, \quad Q = C \oplus \tfrac{1}{3}C.$$

Their positive-length MPV families are

$$V^{(N)}(P) = (1 + 2^{-N})V^{(N)}(C), \quad V^{(N)}(Q) = (1 + 3^{-N})V^{(N)}(C).$$

They are nonzero and proportional with

$$c_N = \frac{1 + 2^{-N}}{1 + 3^{-N}},$$

but $(c_N)_N$ is not a geometric sequence ξ^N . For a one-dimensional choice of C , the total matrices have eigenvalue multisets $\{1, \frac{1}{2}\}$ and $\{1, \frac{1}{3}\}$, so they are not conjugate, even after multiplication by a common phase.

Thus the scalar-free coefficient identity used in copy-weight recovery is an additional hypothesis in the proportional setting, not a consequence awaiting an analytic estimate. CPSV16 Theorem 2.10 asserts the sector matching proved in Theorem 10.4.1. The copy-weight recovery and global block gauge belong to the equal-MPV Corollary 2.11, where the proportionality scalar is identically 1.

10.4 The proportional and equal cases

Theorem 10.4.1 (Multi-block fundamental theorem for normal tensors). [Lean](#) [✓ Stmt](#) *Let P and Q be BNT canonical forms (Definition 9.6.1.13), with bases of normal tensors $(P_j)_{j=1}^{g_a}$ and $(Q_k)_{k=1}^{g_b}$. Every listed sector has a positive number of copies, and every copy weight is nonzero. If the total tensors $P_{\text{tot}}, Q_{\text{tot}}$ generate eventually nonzero proportional MPV families — for all sufficiently large N there is a nonzero scalar c_N with $V^{(N)}(P_{\text{tot}}) = c_N V^{(N)}(Q_{\text{tot}})$ — then $g_a = g_b$. After relabelling the basis of normal tensors of P by a bijection $\beta : J(Q) \simeq J(P)$, there are scalars $\zeta_k \in \mathbb{C}$ with $|\zeta_k| = 1$ and invertible matrices Y_k such that, for every k and every physical index i ,*

$$Q_k^i = \zeta_k Y_k P_{\beta(k)}^i Y_k^{-1}.$$

Thus the paired normal tensors have the same bond dimension ($\dim P_{\beta(k)} = \dim Q_k$) and the same gauge-phase class.

Proof. [✓ Proof](#) Apply Theorem 10.2.1. Its partition-and-substitution argument gives a bijection $\beta : J(Q) \simeq J(P)$, equal bond dimensions, invertible block gauges X_k , and phases ζ_k of unit modulus. Taking $Y_k = X_k$ gives exactly the asserted relations. $\square$

10.4.1 Unitary gauges on normalized sectors

Lemma 10.4.1.1 (Phase normalization for an irreducible gauge). [Lean](#) [✓ Stmt](#) *Let A and B be left-canonical tensors of common positive bond dimension, with B irreducible. If X is invertible, $\zeta \neq 0$, and $B^i = \zeta X A^i X^{-1}$ for every i , then $|\zeta| = 1$.*

Proof. [✓ Proof](#)

Let ρ be a nonzero positive fixed point of $\mathcal{E}_A$ and set $\sigma = X\rho X^\dagger$. Then σ is nonzero and positive, and $\mathcal{E}_B(\sigma) = |\zeta|^2 \sigma$. The map $\mathcal{E}_B$ also has a nonzero positive fixed point. Its irreducibility therefore forces $|\zeta|^2 = 1$. $\square$

Lemma 10.4.1.2 (Unitary replacement with the same phase). [Lean](#) [✓ Stmt](#) *Let A and B be left-canonical irreducible tensors of common positive bond dimension satisfying $B^i = \zeta X A^i X^{-1}$ for an invertible X and a nonzero scalar ζ . There is a unitary matrix U , with the same scalar ζ , such that $|\zeta| = 1$ and $B^i = \zeta U A^i U^\dagger$ for every i .*

Proof. [✓ Proof](#) Lemma 10.4.1.1 gives $|\zeta| = 1$. Substitution into the left-canonical identities gives

$$\sum_i A^{i\dagger} (X^\dagger X) A^i = X^\dagger X.$$

Thus $X^\dagger X$ is a positive-semidefinite fixed point of the adjoint transfer map of A . Irreducibility gives $X^\dagger X = c\mathbb{1}$ for some $c > 0$, so $X^{-1} = c^{-1} X^\dagger$ and $U = c^{-1/2} X$ is unitary. Moreover,

$$U A^i U^\dagger = c^{-1} X A^i X^\dagger = X A^i X^{-1}.$$

$\square$

Corollary 10.4.1.3 (Canonical Form II unitary refinement of proportional matching). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 10.4.1, there exist a bijection $\beta : J(Q) \simeq J(P)$, unitary matrices $U_k \in \text{U}(\dim Q_k)$, and phases $\zeta_k \in \mathbb{C}$ with $|\zeta_k| = 1$, such that $\dim P_{\beta(k)} = \dim Q_k$ and, for every k and every physical index i ,*

$$Q_k^i = \zeta_k U_k P_{\beta(k)}^i U_k^\dagger.$$

This is the proportional part of [CPGSV16, Corollary A.6, lines 1197–1199].

Proof. ✓ Proof The proportional sector-matching theorem gives a bijection β , invertible matrices X_k , and unit-modulus phases ζ_k with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. The matched tensors are left-canonical and irreducible. Hence Lemma F.8.1 gives unitary matrices U_k satisfying $X_k P_{\beta(k)}^i X_k^{-1} = U_k P_{\beta(k)}^i U_k^\dagger$, and therefore $Q_k^i = \zeta_k U_k P_{\beta(k)}^i U_k^\dagger$. $\square$

10.4.2 The equal case

Theorem 10.4.2.1 (Equal BNT canonical forms are globally conjugate). Lean ✓ Stmt *Let P and Q be BNT canonical forms (Definition 9.6.1.13) whose total tensors P_{tot} , Q_{tot} generate the same MPV family for every positive system size N . Every listed sector has a positive number of copies and every copy weight is nonzero. Then their total bond dimensions coincide; writing the common value as D , there is one invertible matrix $X \in \text{GL}_D(\mathbb{C})$ such that, for every physical index i ,*

$$Q_{\text{tot}}^i = X P_{\text{tot}}^i X^{-1}.$$

Proof. ✓ Proof This is Theorem 10.3.5. The BNT comparison of Chapter 9 provides the basis relabelling, the per-block gauge-phase equivalences, and the copy-weight multiset matching, which combine into the block-diagonal global gauge. The exact linear-independence matching needs no per-sector unit-modulus hypothesis, and the BNT canonical form needs no one-site injectivity. $\square$

Corollary 10.4.2.2 (Canonical Form II unitary refinement in the equal case). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total tensors generate the same MPV family at every positive length. Assume every listed sector has a positive number of copies and every copy weight is nonzero, as in Definition 9.6.1.13. Their total bond dimensions agree; writing the common value as D , there is a unitary matrix $U \in \text{U}(D)$ such that, for every physical index i ,*

$$Q_{\text{tot}}^i = U P_{\text{tot}}^i U^\dagger.$$

Proof. ✓ Proof For each matched sector, retain the phase in both identities

$$\begin{aligned} Q_k^i &= \zeta_k U_k P_{\beta(k)}^i U_k^\dagger, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}. \end{aligned}$$

For every matched copy (k, q) ,

$$\nu_{k,q} \zeta_k = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)} \zeta_k = \mu_{\beta(k), \tau_k(q)}.$$

Hence $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$. Repeating U_k on its matched copies gives

$$\begin{aligned} X &= \bigoplus_{k,q} U_k, \\ X^\dagger X &= \mathbb{1}. \end{aligned}$$

If Π is the permutation matrix restoring the original sector and copy order, then $\Pi^\dagger \Pi = \mathbb{1}$. Hence

$$\begin{aligned} U &= X \Pi, \\ U^\dagger U &= \mathbb{1}, \end{aligned}$$

and the permutation of matched coordinates gives

$$Q_{\text{tot}}^i = U P_{\text{tot}}^i U^\dagger.$$

$\square$

Chapter 11

Symmetries, Physical String Order, and Virtual-Boundary Nondecay

This chapter collects the symmetry-theoretic consequences of the MPS Fundamental Theorem [PGVWC07], following the symmetry analysis of [PGWS⁺08]. For injective tensors, equality of periodic MPV families under a physical index action determines virtual gauges, and those gauges define projective representations on the bond space. We keep this virtual covariance distinct from symmetry of an infinite physical state, and distinguish physical endpoint string order from virtual-boundary nondecay.

11.1 Virtual Local Covariance and Projective Bond Actions

With a full group of on-site symmetries (rather than just a single unitary), the resulting gauges determine a projective representation on the bond space.

Definition 11.1.1 (Twisted tensor). [Lean](#) [✓ Stmt](#) Given a group representation $U : G \rightarrow \mathrm{GL}_d(\mathbb{C})$ on the physical index, the *g-twisted tensor* is defined, for each $i \in \{0, \dots, d-1\}$, by

$$\tilde{A}_g^i := \sum_{j=0}^{d-1} U(g)_{ij} A^j. \quad (11.1)$$

Definition 11.1.2 (MPV-family invariance under physical twists). [Lean](#) [✓ Stmt](#) The predicate called on-site symmetry requires $\mathcal{V}(A) = \mathcal{V}(\tilde{A}_g)$ for every $g \in G$, where $U : G \rightarrow \mathrm{GL}_d(\mathbb{C})$ acts on the physical index. This is equality of periodic MPV families and is the hypothesis used to construct virtual local gauges. It is not, by itself, the source notion that every reduced density operator of an infinite physical state is invariant under $U(g)^{\otimes N}$.

Lemma 11.1.3 (Functoriality of twisting). [Lean](#) [✓ Stmt](#) The twisted tensor satisfies the composition law

$$\tilde{A}_{gh} = \widetilde{(\tilde{A}_h)_g}, \quad (11.2)$$

i.e. twisting by gh is the same as first twisting by h and then by g .

Proof. ✓ Proof Expand both sides of (11.2) using (11.1) and $U(gh) = U(g)U(h)$, then swap the order of summation. □

Lemma 11.1.4 (Identity twist). Lean ✓ Stmt *Twisting by the identity group element is trivial: $\tilde{A}_1 = A$.*

Proof. ✓ Proof Since $U(1) = \mathbb{1}_d$, each component in (11.1) reduces to $\sum_j \delta_{ij} A^j = A^i$. □

Definition 11.1.5 (Blocked on-site action). Lean ✓ Stmt For a block of L physical sites, the induced action is the Kronecker-power representation

$$U^{[L]}(g) := U(g)^{\otimes L} \in \text{GL}_{d^L}(\mathbb{C}).$$

Lemma 11.1.6 (Twisting commutes with blocking). Lean ✓ Stmt *For every $g \in G$ and $L \geq 0$,*

$$\left(\widetilde{A^{[L]}}\right)_g^{U^{[L]}} = (\tilde{A}_g)^{[L]}. \quad (11.3)$$

Proof. ✓ Proof Decode a blocked physical index as a word of length L and use the word expansion from Lemma G.1.1.3. Reindexing the sum over words by the blocked index gives exactly the Kronecker-product matrix coefficients on the left of (11.3). □

Theorem 11.1.7 (MPV-family symmetry is preserved by blocking). Lean ✓ Stmt *If A has MPV-family invariance under U , then $A^{[L]}$ has MPV-family invariance under $U^{[L]}$ for every L .*

Proof. ✓ Proof Equation (11.3) identifies the twisted blocked tensor with the blocking of the twisted tensor. Equality of MPV families is preserved by blocking, so the defining equality for A yields the defining equality for $A^{[L]}$. □

Lemma 11.1.8 (Symmetric twists are gauge equivalent). Lean ✓ Stmt *If A is injective and on-site symmetric under U , then for every $g \in G$ the twisted tensor $\tilde{A}_g$ is gauge equivalent to A .*

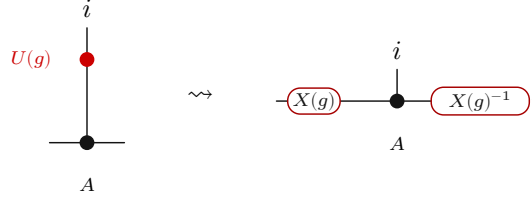
Proof. ✓ Proof On-site symmetry gives $\mathcal{V}(A) = \mathcal{V}(\tilde{A}_g)$, and the single-block Fundamental Theorem turns this equality of MPV families into gauge equivalence. □

Remark 11.1.9 (Fundamental-Theorem route for injective symmetry). The injective virtual-representation theorem is not proved from virtual-boundary nondecay. The on-site symmetry condition first identifies A and $\tilde{A}_g$ as the same MPV family; the single-block Fundamental Theorem gives the gauge in Lemma 11.1.8. Gauge uniqueness and the twist composition law then give Theorem 11.1.16. The virtual-boundary nondecay results later in this chapter use this virtual representation as an input.

Theorem 11.1.10 (Virtual symmetry equation). Lean ✓ Stmt *If A is injective and on-site symmetric under U , then for every $g \in G$ there exist an invertible matrix $X(g) \in \text{GL}_D(\mathbb{C})$ and a nonzero scalar $\varphi(g) \in \mathbb{C}^\times$ (in fact $\varphi = 1$ in the single-block case) such that, for every physical index i ,*

$$\tilde{A}_g^i = \varphi(g) X(g) A^i X(g)^{-1}. \quad (11.4)$$

Proof. ✓ Proof The physical action is transferred to the virtual level by the same local move that replaces the twisted tensor by a gauge-conjugated one:



Apply Lemma 11.1.8 to obtain $X(g)$, then set $\varphi(g) = 1$ in (11.4). $\square$

Lemma 11.1.11 (Gauge uniqueness up to scalar). [Lean](#) [✓ Stmt](#) *Let A be an injective tensor. If $X, Y \in \text{GL}_D(\mathbb{C})$ both satisfy $B^i = X A^i X^{-1} = Y A^i Y^{-1}$ for all i , then there exists a nonzero scalar $u \in \mathbb{C}^\times$ such that $Y = u \cdot X$.*

Proof. [✓ Proof](#) The matrix $Z = Y^{-1}X$ commutes with every A^i . Since A is injective, $\{A^i\}$ spans $M_D(\mathbb{C})$, so Z lies in the centre of $M_D(\mathbb{C})$. By Lemma 2.1.5, Z is a scalar matrix. Since Z is invertible, that scalar is nonzero, and $Y = u \cdot X$. $\square$

Theorem 11.1.12 (Gauge uniqueness from equality of MPV families). [Lean](#) [✓ Stmt](#) *Let A be an injective tensor and assume that A and B generate the same MPV family. If $X, Y \in \text{GL}_D(\mathbb{C})$ both satisfy $B^i = X A^i X^{-1} = Y A^i Y^{-1}$ for all i , then there is a nonzero scalar $u \in \mathbb{C}^\times$ such that $Y = u \cdot X$.*

Proof. [✓ Proof](#) This is exactly Lemma 11.1.11; the extra MPV hypothesis specifies the setting in which the two gauges are usually produced. $\square$

Definition 11.1.13 (Scalar 2-cochain). [Lean](#) [✓ Stmt](#) A scalar 2-cochain on a group G is a function $\omega : G \times G \rightarrow \mathbb{C}^\times$.

Definition 11.1.14 (Projective representation). [Lean](#) [✓ Stmt](#) Given a scalar 2-cochain ω , a projective representation of G on $\mathbb{C}^D$ is a map $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ satisfying, for all $g, h \in G$,

$$\rho(g)\rho(h) = \omega(g, h)\rho(gh). \quad (11.5)$$

Lemma 11.1.15 (Associativity forces the cocycle condition). [Lean](#) [✓ Stmt](#) *If ρ is a projective representation with factor system ω and $D > 0$, then ω satisfies the 2-cocycle identity*

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k). \quad (11.6)$$

Proof. [✓ Proof](#) Expand $\rho(g)(\rho(h)\rho(k))$ and $(\rho(g)\rho(h))\rho(k)$ using the projective multiplication law (11.5) and equate the results via matrix associativity. The hypothesis $D > 0$ allows cancellation of the invertible matrix factor, giving (11.6). $\square$

Theorem 11.1.16 (Virtual representation theorem for injective MPS). [Lean](#) [✓ Stmt](#) *Let A be an injective MPS tensor that is on-site symmetric under a group representation $U : G \rightarrow \text{GL}_d(\mathbb{C})$. Then there exist:*

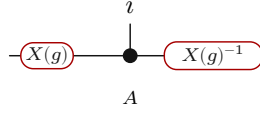
1. a scalar 2-cochain $\omega : G \times G \rightarrow \mathbb{C}^\times$, and
2. a map $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ satisfying, for all $g, h \in G$,

$$\rho(g)\rho(h) = \omega(g, h)\rho(gh).$$

such that, defining the gauge matrices $X(g) := \rho(g^{-1})$, one has, for every $g \in G$ and $i \in \{0, \dots, d-1\}$,

$$\tilde{A}_g^i = X(g) A^i X(g)^{-1}. \quad (11.7)$$

In tensor-network notation, for each fixed g this is the local gauge relation



with the black node denoting the tensor named in the surrounding formula and the two red side nodes acting on the virtual legs. In particular, ρ is a projective representation of G on the bond space $\mathbb{C}^D$.

Proof. ✓ Proof Since A is injective and on-site symmetric, each twisted tensor $\tilde{A}_g$ is gauge equivalent to A (Lemma 11.1.8). Choose $X(g) \in \text{GL}_D(\mathbb{C})$ satisfying (11.7) for all i .

By functoriality (Lemma 11.1.3), the product $X(h)X(g)$ also intertwines A with $\tilde{A}_{gh}$. Since $X(gh)$ does the same, gauge uniqueness (Lemma 11.1.11) gives a nonzero scalar $\omega'(h, g)$ such that

$$X(h)X(g) = \omega'(h, g)X(gh). \quad (11.8)$$

Defining $\rho(g) := X(g^{-1})$ converts (11.8) to the standard law (11.5), where $\omega(g, h) := \omega'(g^{-1}, h^{-1})$. □

Corollary 11.1.17 (2-cocycle identity for the factor system). Lean ✓ Stmt If $D > 0$, the factor system ω from Theorem 11.1.16 satisfies the 2-cocycle identity for all $g, h, k \in G$:

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k).$$

Proof. ✓ Proof Theorem 11.1.16 gives ρ and ω . By Lemma 11.1.15, the cocycle identity (11.6) follows from associativity of matrix multiplication and invertibility of the representation matrices. □

11.2 Cohomology Classes of Virtual Symmetry Cocycles

Two cocycles may differ by a coboundary yet represent the same projective-representation class. The following definitions and results make this precise and establish the quotient $H^2(G, \mathbb{C}^\times)$.

Definition 11.2.1 (Coboundary). Lean ✓ Stmt A scalar 2-cocycle ω is a *coboundary* if there exists $\varphi: G \rightarrow \mathbb{C}^\times$ such that, for all $g, h \in G$,

$$\omega(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}. \quad (11.9)$$

Definition 11.2.2 (Cohomologous cocycles). Lean ✓ Stmt Two cocycles ω_1, ω_2 are *cohomologous* (written $\omega_1 \sim \omega_2$) if there exists $\varphi: G \rightarrow \mathbb{C}^\times$ such that, for all $g, h \in G$,

$$\omega_1(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h). \quad (11.10)$$

Theorem 11.2.3 (Gauge independence of the cocycle class). [Lean](#) [✓ Stmt](#) If the bond dimension satisfies $D \geq 1$ and two projective representations ρ_1, ρ_2 (with cocycles ω_1, ω_2) both arise as virtual representations of the same injective MPS tensor A under an on-site symmetry U , then $\omega_2 \sim \omega_1$, and hence also $\omega_1 \sim \omega_2$ by symmetry.

Proof. [✓ Proof](#) Gauge uniqueness gives nonzero scalars $f(k)$ with $X_2(k) = f(k) X_1(k)$ for every physical twist k . Since the projective representation is $\rho_a(g) = X_a(g^{-1})$, setting $k = g^{-1}$ gives $\rho_2(g) = f(g^{-1})\rho_1(g)$. Equivalently, for each fixed g the two local gauges

$$\begin{array}{c} i \\ | \\ \boxed{X_1(g^{-1})} \text{---} \bullet \text{---} \boxed{X_1(g^{-1})^{-1}} \\ A \end{array} \quad \text{and} \quad \begin{array}{c} i \\ | \\ \boxed{X_2(g^{-1})} \text{---} \bullet \text{---} \boxed{X_2(g^{-1})^{-1}} \\ A \end{array}$$

describe the same twisted tensor, so they differ by a scalar. Set $\alpha(g) := f(g^{-1})$. Substituting $\rho_2(g) = \alpha(g)\rho_1(g)$ into the projective law (11.5) and cancelling the invertible matrix $X_1((gh)^{-1}) = \rho_1(gh)$ yields

$$\omega_2(g, h) = \alpha(g)\alpha(h)\alpha(gh)^{-1}\omega_1(g, h). \quad (11.11)$$

Thus ω_2 is the coboundary multiple of ω_1 with cochain α . This is precisely the orientation required by (11.10) for the stated relation $\omega_2 \sim \omega_1$. Equivalently,

$$\omega_1(g, h) = \alpha(g)^{-1}\alpha(h)^{-1}\alpha(gh)\omega_2(g, h),$$

so the inverse cochain also gives the symmetric conclusion $\omega_1 \sim \omega_2$. $\square$

Definition 11.2.4 (Multiplicative 2-cocycle condition). [Lean](#) [✓ Stmt](#) A scalar 2-cochain $\omega : G \times G \rightarrow \mathbb{C}^\times$ is a 2-cocycle if it satisfies, for all $g, h, k \in G$,

$$\omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k).$$

Theorem 11.2.5 (A projective factor system is a cocycle). [Lean](#) [✓ Stmt](#) If $D > 0$ and $\rho : G \rightarrow \text{GL}_D(\mathbb{C})$ obeys $\rho(g)\rho(h) = \omega(g, h)\rho(gh)$, then ω is a $\mathbb{C}^\times$ -valued 2-cocycle.

Proof. [✓ Proof](#) Associativity gives the scalar identity of Lemma 11.1.15. Because the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}$ is injective, the same identity holds in $\mathbb{C}^\times$, which is precisely Definition 11.2.4. $\square$

Definition 11.2.6 (Second cohomology quotient). [Lean](#) [✓ Stmt](#) The second cohomology set is the quotient

$$H^2(G, \mathbb{C}^\times) := Z^2(G, \mathbb{C}^\times)/\sim, \quad (11.12)$$

where $\sim$ is the coboundary-equivalence relation.

Definition 11.2.7 (Agreement of factor-system cohomology classes). [Lean](#) [✓ Stmt](#) The relation called projective equivalence holds precisely when the two factor systems are cohomologous. It compares their classes in $H^2(G, \mathbb{C}^\times)$; it does not assert an intertwining equivalence between the underlying representation spaces.

Theorem 11.2.8 (Factor-system class agreement). [Lean](#) [✓ Stmt](#) Let ρ_1, ρ_2 be projective representations with factor systems ω_1, ω_2 . Then

$$\rho_1 \sim_{\text{proj}} \rho_2 \iff \omega_1 \sim \omega_2. \quad (11.13)$$

Proof. [✓ Proof](#) This follows from the definition of projective equivalence as cohomology-class equality for factor systems. $\square$

11.2.1 Non-triviality via the commutator phase

For an abelian symmetry group, the antisymmetric commutator phase is invariant under a change of cocycle representative. A value not equal to 1 therefore witnesses non-triviality; no converse is asserted here.

Definition 11.2.1.1 (Commutator phase). [Lean](#) [✓ Stmt](#) The *commutator phase* of a 2-cochain ω at a pair g, h is $\beta_\omega(g, h) := \omega(g, h)\omega(h, g)^{-1}$. It is defined for any $\mathbb{C}^\times$ -valued 2-cochain; the cocycle condition is not needed.

Theorem 11.2.1.2 (The commutator phase is a class invariant). [Lean](#) [✓ Stmt](#) If $\omega_1 \sim \omega_2$ and $gh = hg$, then $\beta_{\omega_1}(g, h) = \beta_{\omega_2}(g, h)$.

Proof. [✓ Proof](#) Write $\omega_1(g, h) = \varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h)$ for a coboundary witness φ . Substituting into the ratio and using $gh = hg$ and the commutativity of $\mathbb{C}^\times$ gives

$$\begin{aligned}\beta_{\omega_1}(g, h) &= \frac{\varphi(g)\varphi(h)\varphi(gh)^{-1}\omega_2(g, h)}{\varphi(h)\varphi(g)\varphi(hg)^{-1}\omega_2(h, g)} \\ &= \frac{\omega_2(g, h)}{\omega_2(h, g)} = \beta_{\omega_2}(g, h),\end{aligned}\tag{11.14}$$

where the φ -factors cancel since $\varphi(gh) = \varphi(hg)$. $\square$

Definition 11.2.1.3 (Non-trivial cohomology class). [Lean](#) [✓ Stmt](#) A factor system ω has a *non-trivial class* when it is not cohomologous to the constant cocycle 1.

Theorem 11.2.1.4 (Commutator-phase non-triviality test). [Lean](#) [✓ Stmt](#) If a commuting pair g, h has $\beta_\omega(g, h) \neq 1$, then ω has a non-trivial class.

Proof. [✓ Proof](#) The trivial cocycle has commutator phase 1 at every pair, so by Theorem 11.2.1.2 and (11.14), any cocycle cohomologous to it has $\beta = 1$ at every commuting pair. Contrapositively, $\beta_\omega(g, h) \neq 1$ for one commuting pair forces ω to be non-cohomologous to the trivial cocycle. $\square$

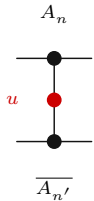
11.3 Physical Endpoint String Order and Virtual-Boundary Nondecay

The source notion of string order uses physical endpoint operators and a nonidentity unitary string [PGWS⁺08]. We also study a restricted virtual-boundary nondecay predicate and a virtual local-covariance relation. These notions are stated separately below; equivalence with physical state symmetry requires additional hypotheses.

Definition 11.3.1 (Twisted transfer map). [Lean](#) [✓ Stmt](#) Given an MPS tensor A and a physical-index matrix u , the *twisted transfer map* is

$$\mathcal{E}_u(X) := \sum_{n, n'} \langle n' | u | n \rangle A_n X A_{n'}^\dagger.\tag{11.15}$$

Diagrammatically one inserts the physical operator u on the doubled local transfer cell:

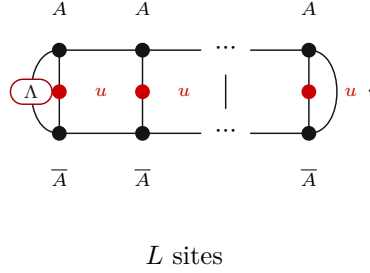


The upper and lower black nodes denote A_n and $A_n^\dagger$, and the red node labelled u couples the physical legs.

Definition 11.3.2 (Stationary-boundary block-twist functional). [Lean](#) [✓ Stmt](#) The *stationary-boundary block-twist functional* is

$$T_L(A, u, \Lambda) := \text{tr}(\Lambda \cdot \mathcal{E}_u^L(\mathbb{1})). \quad (11.16)$$

In tensor-network notation this is the length- L doubled chain with a copy of u inserted on each physical rung:



The boundary matrix Λ is contracted into the virtual boundary, the red nodes labelled u sit on the physical rungs, and the right boundary is traced.

Definition 11.3.3 (Periodic finite-chain symmetry overlap). [Not ready](#) For the periodic length- L matrix product vector $|\Psi_L(A)\rangle$, the source quantity is literally

$$R_L(u) := \langle \Psi_L(A) | u^{\otimes L} | \Psi_L(A) \rangle. \quad (11.17)$$

Identifying this periodic overlap with the normalized stationary-boundary functional $T_L(A, u, \Lambda)$ requires a thermodynamic-boundary and normalization hypothesis. Thus Definition 11.3.2 must not be read as the literal source definition of $R_L(u)$.

Definition 11.3.4 (Physical string correlator). [Lean](#) [✓ Stmt](#) For physical operators $x, y \in M_d(\mathbb{C})$, a twist $u \in M_d(\mathbb{C})$, and boundary state Λ , the transfer-form string correlator of [PGWS⁺08, display SOPMP, lines 176–181] is

$$S_N(x, y, u) := \text{tr}(\Lambda \mathcal{E}_x \mathcal{E}_u^N(\mathcal{E}_y(\mathbb{1}))). \quad (11.18)$$

This is the physical-endpoint expression corresponding to $\langle \Psi | x \otimes u^{\otimes N} \otimes y | \Psi \rangle$.

Definition 11.3.5 (Fixed-endpoint physical string order). [Lean](#) [✓ Stmt](#) For fixed physical operators x, y, u , the correlator has physical string order if there is $s > 0$ such that $|S_N(x, y, u)| \rightarrow s$.

Definition 11.3.6 (Physical endpoint string order). [Lean](#) [✓ Stmt](#) A tensor has physical string order in the sense of [PGWS⁺08, display SOP, lines 112–122] if there exist a unitary $u \neq \mathbb{1}$ and physical endpoint operators x, y for which Definition 11.3.5 holds. The comparison with arbitrary virtual boundary matrices is a separate endpoint- realizability statement.

Lemma 11.3.7 (Twisted spectral-radius bound and peripheral rigidity). [Not ready](#) This is Lemma 1 of [PGWS⁺08]. Let A represent a pure finitely correlated state in the canonical gauge $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, $\mathcal{E}_A^*(\Lambda) = \Lambda$, with $\Lambda > 0$ and $\text{tr}(\Lambda) = 1$. For every physical unitary u ,

$$\rho(\mathcal{E}_u) \leq 1.$$

Equality holds if and only if there are a unitary V and $\theta \in [0, 2\pi)$ satisfying the following identity for every j in an eigenbasis $u = \sum_j e^{i\theta_j} |\tilde{j}\rangle\langle\tilde{j}|$:

$$V^\dagger \tilde{A}_j = e^{i(\theta - \theta_j)} \tilde{A}_j V^\dagger. \quad (11.19)$$

Moreover, $\mathcal{E}_u$ has at most one eigenvalue of modulus one.

Proof. Let $\mathcal{E}_u(V) = \lambda V$. Multiplication by $\Lambda V^\dagger$, the trace, and Cauchy–Schwarz give

$$\begin{aligned} |\lambda| \operatorname{tr}(V \Lambda V^\dagger) &\leq \left(\sum_j \operatorname{tr}(V \tilde{A}_j^\dagger \Lambda \tilde{A}_j V^\dagger) \right)^{1/2} \left(\sum_j \operatorname{tr}(\tilde{A}_j^\dagger V \Lambda V^\dagger \tilde{A}_j) \right)^{1/2} \\ &= \operatorname{tr}(V \Lambda V^\dagger). \end{aligned} \quad (11.20)$$

Faithfulness of Λ yields $|\lambda| \leq 1$. Equality in Cauchy–Schwarz is equivalent to (11.19); conversely that relation makes $e^{i\theta}$ an eigenvalue. It also gives $\mathcal{E}_A(V^\dagger V) = V^\dagger V$. The fixed space of the pure canonical transfer map is one-dimensional, so $V^\dagger V = c\mathbb{1}$ for some $c > 0$. Replacing V by $c^{-1/2}V$ makes it unitary without changing the intertwining relation.

If V, V' are peripheral eigenmatrices with eigenvalues $e^{i\theta}, e^{i\theta'}$, respectively, the equality relations give the exact identity

$$\mathcal{E}_A(V^\dagger V') = e^{i(\theta' - \theta)} V^\dagger V'.$$

Purity leaves only the peripheral eigenvalue 1, hence $\theta' = \theta$ modulo 2π . The one-dimensional fixed space then gives $V^\dagger V' \in \mathbb{C}\mathbb{1}$, proving uniqueness of the peripheral eigenvalue and its eigendirection. $\square$

Theorem 11.3.8 (Physical endpoint criterion for string order). Not ready This is Theorem 1 of [PGWS⁺08]. A pure finitely correlated state has physical string order if and only if there exist a unitary $\tilde{u} \neq \mathbb{1}$, a unitary virtual matrix V , and physical indices n, m such that

$$\mathcal{E}_{\tilde{u}}(V) = V, \quad \operatorname{tr}(V \Lambda A_n A_m^\dagger) \neq 0. \quad (11.21)$$

The transfer criteria below assume one-site injectivity. The literal physical equivalence additionally requires that sufficiently long physical endpoint operators realize the needed virtual boundary matrices.

Proof. Lemma 11.3.7 shows that a nonzero limit requires a unique peripheral eigenvalue $e^{i\theta}$. If V and Y are the corresponding right and left eigenmatrices, then $Y = \Lambda V^\dagger$, and the leading term retains its length-dependent phase:

$$S_N(x, y, u) = e^{iN\theta} \operatorname{tr}(Y \mathcal{E}_y(\mathbb{1})) \operatorname{tr}(\Lambda \mathcal{E}_x(V)) + o(1).$$

In the eigenbasis $u = \sum_j e^{i\theta_j} |\tilde{j}\rangle\langle\tilde{j}|$, shift the string unitary by the peripheral phase,

$$\tilde{u} := e^{-i\theta} u = \sum_j e^{i(\theta_j - \theta)} |\tilde{j}\rangle\langle\tilde{j}|.$$

Then $\mathcal{E}_{\tilde{u}}(V) = V$. Choosing $y = x^\dagger \tilde{u}$ reduces nonvanishing to the second condition in (11.21); if it holds for n, m , take $x = |n\rangle\langle m|$. The reverse implication follows from the same phase-retaining factorization and the endpoint-realizability hypothesis. $\square$

Theorem 11.3.9 (Physical state symmetry and twisted spectral radius). Not ready This is Theorem 2 of [PGWS⁺08]. For a pure finitely correlated state and a unitary $u \neq \mathbb{1}$, the physical state is invariant under $u^{\otimes N}$ on every N -site reduced density operator if and only if

$$\rho(\mathcal{E}_u) = 1.$$

The theorem below gives the corresponding virtual local-covariance criterion for a one-site-injective tensor. Passing between that condition and invariance of every physical reduced state is an additional step.

Proof. If $\rho(\mathcal{E}_u) = 1$, Lemma 11.3.7 gives a unitary V satisfying the local intertwining relation. Substitution of that relation into the adjoint transfer map, together with $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$, gives $\mathcal{E}_A^\dagger(V\Lambda V^\dagger) = V\Lambda V^\dagger$. This positive fixed point has trace one, so uniqueness gives $V\Lambda V^\dagger = \Lambda$, equivalently $V^\dagger\Lambda V = \Lambda$, which yields the physical symmetry. Conversely, physical invariance gives, for length-independent boundary tensors L, R ,

$$D^{-2} \leq \text{tr}(\varrho_N^2) = \text{tr}[L(\mathcal{E}_u \otimes \mathcal{E}_{u^\dagger})^N(R)].$$

Exponential decay would contradict this bound, so $\rho(\mathcal{E}_u) = 1$. □

Definition 11.3.10 (Virtual local covariance with boundary invariance). Lean ✓ Stmt The virtual local-covariance predicate holds for A, u, Λ if there exist a unitary matrix V and a phase μ with $|\mu| = 1$ such that $V^\dagger\Lambda V = \Lambda$ and, for every physical index i ,

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger. \quad (11.22)$$

The unitary V intertwines the physical action of u with a virtual conjugation, and the boundary state Λ is invariant under V .

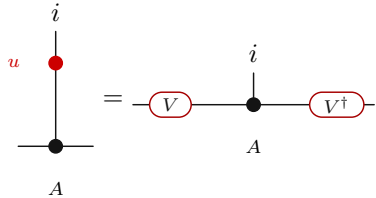
Definition 11.3.11 (Virtual-boundary nondecay). Lean ✓ Stmt For boundary matrices $X, Y \in M_D(\mathbb{C})$, set explicitly

$$R_L(u; X, Y) := \text{tr}(\Lambda X \mathcal{E}_u^L(Y)).$$

The virtual-boundary nondecay predicate holds for u and Λ if there exist X, Y and a positive constant $c > 0$ such that $c \leq \|R_L(u; X, Y)\|$ for every L . Thus some virtual-boundary sequence does not decay to zero.

Virtual-boundary nondecay is an existence statement over the virtual matrices X, Y , for a fixed twist u ; no claim is made for any particular boundary choice. Definition 11.3.6 records the physical-endpoint string-order notion from the source, which quantifies existentially over u, x, y and requires a positive limit rather than a lower bound uniform in the length. That paper also notes that the correlator for a particular endpoint choice can vanish even when the physical symmetry is present.

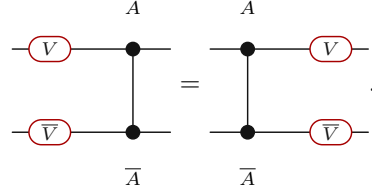
Definition 11.3.12 (Local physical intertwining). Lean ✓ Stmt The local physical intertwining relation is $\sum_j u_{ij} A^j = V A^i V^\dagger$. Locally, the physical action of u can therefore be pushed to the virtual legs:



Definition 11.3.13 (Transfer-map covariance). [Lean](#) [✓ Stmt](#) Transfer-map covariance is the identity that the transfer map commutes with virtual conjugation:

$$\mathcal{E}(VXV^\dagger) = V \mathcal{E}(X) V^\dagger. \quad (11.23)$$

In doubled-layer notation, the virtual operators slide through the local transfer cell:

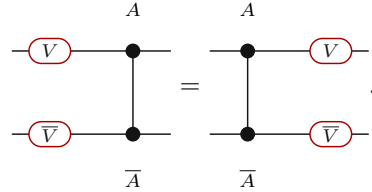


Definition 11.3.14 (Commutation in the doubled transfer picture). [Lean](#) [✓ Stmt](#) In the transfer-map language, the commutation of the doubled transfer matrix $E = \sum_i A_i \otimes \bar{A}_i$ with $V \otimes \bar{V}$ is

$$V \mathcal{E}_1(X) V^\dagger = \mathcal{E}_1(VXV^\dagger). \quad (11.24)$$

Theorem 11.3.15 (Transfer-map covariance and doubled commutation are equivalent). [Lean](#) [✓ Stmt](#) *Transfer-map covariance and commutation in the doubled transfer picture are equivalent for any matrix V .*

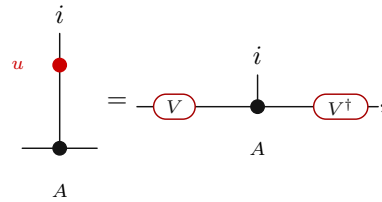
Proof. [✓ Proof](#) Both sides express the same local picture with opposite orientation:



The two formulas are the same identity read from opposite sides. □

Theorem 11.3.16 (Local physical intertwining implies transfer-map covariance). [Lean](#) [✓ Stmt](#) *If V and u are unitary and the local physical intertwining relation holds, then the transfer map is covariant.*

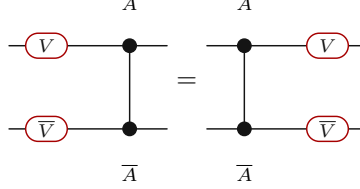
Proof. [✓ Proof](#) Starting from



one inserts $V^\dagger V = \mathbb{1}$ so that each local transfer cell is written in terms of conjugated Kraus operators, then replaces those operators using the intertwining relation. After that, the physical u -boxes disappear by the unitary Kraus-mixing identity. □

Theorem 11.3.17 (Transfer-map covariance implies local physical intertwining). [Lean](#) [✓ Stmt](#) Assume that A is one-site injective, $VV^\dagger = \mathbb{1}$, and the transfer map is covariant. Then there is u with $uu^\dagger = \mathbb{1}$ satisfying the local physical intertwining relation. In fact, covariance and $VV^\dagger = \mathbb{1}$ alone imply the conclusion; the injectivity hypothesis is unnecessary.

Proof. [✓ Proof](#) Set $B^i := V A^i V^\dagger$. The transfer-map covariance relation



implies that the two d -element Kraus families B and A define the same transfer map. Rectangular Kraus freedom (the corresponding theorem in the companion quantum-channel volume [LTC26, “Rectangular Kraus freedom”]) converts $\mathcal{E}_B = \mathcal{E}_A$ into a unitary Kraus mixing matrix u with $B^i = \sum_j u_{ij} A^j$. $\square$

Theorem 11.3.18 (Twisted-transfer eigenvalues have modulus at most one). [Lean](#) [✓ Stmt](#) Let A be an injective MPS tensor with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and let u satisfy $uu^\dagger = \mathbb{1}$. If $\mathcal{E}_u(X) = \lambda X$ for some nonzero $X \in M_D(\mathbb{C})$, then $|\lambda| \leq 1$.

Proof. [✓ Proof](#) By Lemma G.4.2, the twisted transfer map is the mixed transfer operator F_{A,B_u} , as in (G.6). Because B_u is obtained from A by unitary mixing of the Kraus operators, the transfer maps of A and B_u coincide, so the two adjoint channels share a common positive definite fixed point. Gauging both families by that fixed point puts them in a common trace-preserving gauge. Theorem 6.3.1 then applies to the mixed transfer operator in that gauge, and similarity preserves the eigenvalue λ . $\square$

Theorem 11.3.19 (Modulus-one twisted eigenvalue implies gauge-phase equivalence). [Lean](#) [✓ Stmt](#) Let A be an injective MPS tensor with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, and let u satisfy $uu^\dagger = \mathbb{1}$. If

$$\mathcal{E}_u(X) = \lambda X \tag{11.25}$$

for some nonzero $X \in M_D(\mathbb{C})$ with $|\lambda| = 1$, then A is gauge-phase equivalent to the companion tensor B_u .

Proof. [✓ Proof](#) By Lemma G.4.2, the twisted transfer map is the mixed transfer operator F_{A,B_u} . Because B_u is obtained from A by unitary mixing of the Kraus operators, the transfer maps of A and B_u coincide, so the two adjoint channels share a common positive definite fixed point. Gauging both families by that fixed point puts them in a common trace-preserving gauge. In that gauge, (11.25) becomes a modulus-one peripheral eigenvalue for an irreducible mixed transfer operator, so Theorem 6.6.1 yields gauge-phase equivalence. Undoing the common trace-preserving gauge gives the claim for A and B_u . $\square$

Theorem 11.3.20 (Virtual-boundary nondecay gives companion gauge-phase equivalence). [Lean](#) [✓ Stmt](#) Let A be injective, let $\Lambda \in M_D(\mathbb{C})$, and assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If virtual-boundary nondecay holds for u with boundary parameter Λ , then A is gauge-phase equivalent to the companion tensor B_u .

Proof. ✓ Proof If $\rho(\mathcal{E}_u) < 1$, Lemma G.6.1 forces every virtual-boundary sequence to decay to zero, contradicting Definition 11.3.11. Hence $\rho(\mathcal{E}_u) \geq 1$. Theorem 11.3.18 shows that every eigenvalue has modulus at most 1, so some peripheral eigenvalue has modulus exactly 1. Applying Theorem 11.3.19 then gives the gauge-phase equivalence. $\square$

Theorem 11.3.21 (Gauge-phase equivalence with the companion tensor yields a virtual unitary). Lean ✓ Stmt *Let A be injective, assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If A is gauge-phase equivalent to the companion tensor B_u , then there exist a unitary V and a phase μ with $|\mu| = 1$ such that, for every physical index i ,*

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger. \quad (11.26)$$

Proof. ✓ Proof Start from an invertible intertwiner X and a nonzero gauge scalar ζ between A and B_u . The positive matrix $Q := XX^\dagger$ satisfies $\mathcal{E}_A(Q) = |\zeta|^2 Q$. Since $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, irreducibility forces the two positive eigenvalues to agree, so $|\zeta| = 1$ and Q is a positive fixed point of $\mathcal{E}_A$. Uniqueness of positive fixed points for an injective normalized tensor then makes Q a positive scalar multiple of the identity. Rescaling X by the square root of that scalar produces a unitary V , and the original intertwining relation becomes (11.26), with $|\mu| = 1$. $\square$

Theorem 11.3.22 (Virtual unitary gives a twisted-transfer eigenmatrix). Lean ✓ Stmt *Assume $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$ and that, for every physical index i , $\sum_j u_{ij} A^j = \mu V A^i V^\dagger$ with V unitary. Then V is an eigenmatrix of the twisted transfer map:*

$$\mathcal{E}_u(V) = \mu V. \quad (11.27)$$

Proof. ✓ Proof Expand $\mathcal{E}_u(V)$ using (11.15), insert the intertwining relation into each local term, and use $V^\dagger V = \mathbb{1}$ together with $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. $\square$

Theorem 11.3.23 (Boundary-state invariance of the virtual unitary). Lean ✓ Stmt *Let A be injective, assume $uu^\dagger = \mathbb{1}$, and let Λ be a positive definite boundary state with $\text{tr}(\Lambda) = 1$ and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. If V is a unitary and μ a phase satisfying, for every physical index i ,*

$$\sum_j u_{ij} A^j = \mu V A^i V^\dagger, \quad (11.28)$$

then

$$V^\dagger \Lambda V = \Lambda. \quad (11.29)$$

Proof. ✓ Proof Relation (11.28) first shows that $V \Lambda V^\dagger$ is another positive fixed point of the adjoint transfer map. It has the same trace as Λ , so uniqueness of the normalized positive fixed point gives $V \Lambda V^\dagger = \Lambda$. Multiplying by $V^\dagger$ on the left and by V on the right yields (11.29). $\square$

Lemma 11.3.24 (Virtual local covariance implies virtual-boundary nondecay). Lean ✓ Stmt *Assume $\text{tr}(\Lambda) = 1$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If A, u, Λ satisfy virtual local covariance, then virtual-boundary nondecay holds for u .*

Proof. ✓ Proof Choose the boundary matrices $X = V^\dagger$ and $Y = V$, where V is the unitary from the virtual local-covariance datum. By Theorem 11.3.22, one has $\mathcal{E}_u^L(V) = \mu^L V$. Therefore

$$R_L(u; V^\dagger, V) = \mu^L \text{tr}(\Lambda) = \mu^L,$$

whose norm is constantly 1. $\square$

Theorem 11.3.25 (Virtual local covariance iff a unitary peripheral eigenmatrix exists). [Lean](#) [✓ Stmt](#) Let A be injective, and assume $uu^\dagger = \mathbb{1}$, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then virtual local covariance holds if and only if there exist a unitary $V \in M_D(\mathbb{C})$ and a phase μ with $|\mu| = 1$ such that $\mathcal{E}_u(V) = \mu V$. By Theorem 11.3.18, this witness form is equivalent to the peripheral-spectrum condition $\rho(\mathcal{E}_u) = 1$.

Proof. [✓ Proof](#) The forward direction is Theorem 11.3.22. For the reverse direction, start from a unitary peripheral eigenmatrix. Theorem 11.3.19 gives gauge-phase equivalence with the companion tensor; then Theorem 11.3.21 yields a virtual unitary and phase satisfying the local covariance relation, and Theorem 11.3.23 gives the boundary-state invariance needed for virtual local covariance. $\square$

Theorem 11.3.26 (Virtual-boundary nondecay $\Leftrightarrow$ virtual local covariance). [Lean](#) [✓ Stmt](#) For an injective MPS, assume $uu^\dagger = \mathbb{1}$, $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then virtual-boundary nondecay for u holds if and only if A, u, Λ satisfy virtual local covariance.

Proof. [✓ Proof](#) If virtual-boundary nondecay holds, Theorem 11.3.20 gives a gauge-phase equivalence with the companion tensor. Theorem 11.3.21 extracts a virtual unitary and phase from that equivalence, and Theorem 11.3.23 shows that the same unitary preserves the boundary state, so one obtains virtual local covariance. Conversely, Lemma 11.3.24 turns virtual local covariance into a nondecaying virtual-boundary sequence. $\square$

Theorem 11.3.27 (Virtual unitary from virtual-boundary nondecay). [Lean](#) [✓ Stmt](#) For an injective MPS, let $\Lambda \in M_D(\mathbb{C})$, and assume $uu^\dagger = \mathbb{1}$ and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$. If virtual-boundary nondecay holds for u with boundary parameter Λ , then there exists a virtual unitary V and a unit-modulus scalar μ such that $\sum_j u_{ij} A^j = \mu V A^i V^\dagger$ for all i . Diagrammatically this is the same local intertwining relation, but only up to an overall unit scalar on the virtual side:

$$\begin{array}{c} i \\ | \\ \bullet \\ | \\ \bullet \\ A \end{array} \quad \text{with red line } u \text{ between dots} \quad = \quad \mu \quad \begin{array}{c} i \\ | \\ \text{---} V \text{---} \bullet \text{---} V^\dagger \text{---} \\ | \\ A \end{array}$$

Proof. [✓ Proof](#) Virtual-boundary nondecay first gives gauge-phase equivalence with the companion tensor by Theorem 11.3.20. Theorem 11.3.21 then normalizes the resulting intertwiner to a unitary and produces the phase μ . $\square$

11.4 SPT Phase Labels and Virtual-Boundary Nondecay

The predicate in Definition 11.4.1 compares virtual cocycle labels. The source classification below concerns symmetric paths that remain within the MPS and corresponding parent-Hamiltonian framework, with the endpoint representations embedded into a common physical space as in [SPGC11].

Definition 11.4.1 (Equality of virtual cocycle labels). [Lean](#) [✓ Stmt](#) For tensors A, B and a fixed physical representation $U : G \rightarrow \text{GL}_d(\mathbb{C})$, the predicate called *same SPT phase* asserts the existence of factor systems $\omega_A, \omega_B : G \times G \rightarrow \mathbb{C}^\times$ and projective bond representations ρ_A, ρ_B

such that

$$\begin{aligned}\widetilde{A}_g^i &= \rho_A(g^{-1})A^i\rho_A(g^{-1})^{-1}, \\ \widetilde{B}_g^i &= \rho_B(g^{-1})B^i\rho_B(g^{-1})^{-1},\end{aligned}\tag{11.30}$$

exactly, and $\omega_A \sim \omega_B$. Thus the predicate is equality of virtual factor-system classes. It neither quantifies over Hamiltonian paths nor states physical phase equivalence.

Remark 11.4.2 (Scalar and character conventions). These factor systems take values in $\mathbb{C}^\times$. For unitary bond representations one may choose unit-modulus representatives and obtain the physical classification label in $H^2(G, \text{U}(1))$; this normalization is not an equality $\mathbb{C}^\times = \text{U}(1)$. Moreover, Definition 11.4.1 requires exact tensor equations. A physical state symmetry may instead act on an MPS vector by a one-dimensional character $\chi(g)$, producing a scalar factor in the local tensor equation. Exact MPV-family equality and such a character are distinct hypotheses.

Theorem 11.4.3 (Classification along symmetric MPS parent-Hamiltonian paths). Not ready
For $p = 0, 1$, let A_p be an injective MPS tensor whose parent Hamiltonian H_p has a unique ground state and is symmetric under a unitary representation U^p of G on $\mathcal{H}_p$. Choose the endpoint phase gauges. After a common finite blocking, the embedded endpoints admit a continuous symmetric MPS path with corresponding translationally invariant, finite-range parent Hamiltonians that vary smoothly and remain uniformly gapped if and only if their virtual projective representations determine the same class in $H^2(G, \text{U}(1))$.

More precisely, for the forward construction there exist a path-sector Hilbert space $\mathcal{H}_{\text{path}}$ and a linear representation U^{path} such that the endpoints embed into

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_{\text{path}}, \quad U_g = U_g^0 \oplus U_g^1 \oplus U_g^{\text{path}}.$$

The converse quantifies over smooth paths whose unique ground states remain MPS and whose Hamiltonians are the corresponding parent Hamiltonians, as in [SPGC11, Section II.F, lines 929–953]. It does not assert invariance along an arbitrary gapped Hamiltonian path.

Proof. First deform each blocked injective MPS to its isometric form; the deformation preserves the symmetry relation

$$U_g \mathcal{P} = \mathcal{P}(V_g \otimes \overline{V}_g).$$

Along any continuous symmetric MPS path quantified in the theorem, the tensor and its induced virtual action V_g vary continuously. The cohomology class is discrete, so it cannot change along that path. This proves necessity within the MPS/parent-Hamiltonian framework.

Conversely, let the endpoint isometric forms have virtual actions V_g^0 and V_g^1 in the same cohomology class. Embed their bond spaces into the direct sum and interpolate between the maximally entangled bonds by

$$|\omega(\gamma)\rangle = (1 - \gamma) \sum_{i=1}^{D_0} |i, i\rangle + \gamma \sum_{i=D_0+1}^{D_0+D_1} |i, i\rangle.$$

The commuting parent Hamiltonians along this interpolation remain gapped. Their symmetry is

$$\begin{aligned}(V_g^0 \oplus V_g^1) \otimes (\overline{V}_g^0 \oplus \overline{V}_g^1) &= \widehat{U}_g^0 \oplus \widehat{U}_g^1 \oplus \widehat{U}_g^{\text{path}}, \\ \widehat{U}_g^{\text{path}} &= (V_g^0 \otimes \overline{V}_g^1) \oplus (V_g^1 \otimes \overline{V}_g^0).\end{aligned}\tag{11.31}$$

Equality of the cohomology classes is exactly what makes the cross-term action $\widehat{U}^{\text{path}}$ a linear representation. Conjugating back by the endpoint isometries gives the common direct-sum physical representation and the required symmetric gapped path. $\square$

Theorem 11.4.4 (Virtual-boundary nondecay for injective MPV-symmetric tensors). [Lean](#)

[✓ Stmt](#) *Let A be an injective tensor, symmetric under a unitary representation U . Assume $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, Λ is positive definite, $\text{tr}(\Lambda) = 1$, and $\mathcal{E}_A^\dagger(\Lambda) = \Lambda$. Then the virtual-boundary nondecay predicate holds for every group element g .*

Proof. [✓ Proof](#) The virtual representation theorem produces $\rho(g^{-1}) \in \text{GL}_D(\mathbb{C})$ satisfying $\sum_j U(g)_{ij} A^j = \rho(g^{-1}) A^i \rho(g^{-1})^{-1}$. A direct computation using $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$ shows that $\mathcal{E}_{U(g)}(\rho(g^{-1})) = \rho(g^{-1})$, so $\rho(g^{-1})$ is a nonzero eigenvector with eigenvalue 1. Theorem 11.3.19 gives gauge-phase equivalence with the companion tensor, and Theorem 11.3.21 extracts a virtual unitary and phase. By Theorem 11.3.23, this unitary preserves the boundary state, so Lemma 11.3.24 gives virtual-boundary nondecay. $\square$

Remark 11.4.5 (Virtual-boundary nondecay does not determine the SPT label). Theorem 11.4.4 applies to every injective symmetric tensor with canonical normalisations, irrespective of its cocycle class, so tensors in different SPT phases all satisfy virtual-boundary nondecay for every group element. This nondecay is equivalent to virtual local covariance (Theorem 11.3.26); the invariant that separates SPT phases is the cohomology class of the virtual representation cocycle (Theorem 11.4.3).

Part I

Appendices

Appendix A

Perron–Frobenius Theory for Channels and Transfer Maps: Supporting Results

This appendix supplies the canonical-gauge calculations used in Chapter 5.

A.1 Canonical-gauge algebra

For the right-canonical gauge in Theorem [LTC26, “Right-canonical (unital) gauge”], each summand becomes

$$\begin{aligned}(S^{-1}A^iS)(S^{-1}A^iS)^\dagger &= S^{-1}A^iSS^\dagger(A^i)^\dagger(S^\dagger)^{-1} \\ &= S^{-1}A^i\rho(A^i)^\dagger(S^\dagger)^{-1}.\end{aligned}$$

Hence

$$\sum_i A'^i(A'^i)^\dagger = S^{-1}\mathcal{E}_A(\rho)(S^\dagger)^{-1} = S^{-1}\rho(S^\dagger)^{-1} = \mathbb{1}.$$

For the left-canonical gauge in Theorem 5.4.2,

$$\begin{aligned}(SA^iS^{-1})^\dagger(SA^iS^{-1}) &= (S^\dagger)^{-1}(A^i)^\dagger S^\dagger SA^iS^{-1} \\ &= (S^\dagger)^{-1}(A^i)^\dagger \sigma A^iS^{-1},\end{aligned}$$

and therefore

$$\sum_i (A'^i)^\dagger A'^i = (S^\dagger)^{-1} \left(\sum_i (A^i)^\dagger \sigma A^i \right) S^{-1} = (S^\dagger)^{-1} \sigma S^{-1} = \mathbb{1}.$$

Finally, for the square-root gauge (5.1), self-adjointness of $\sigma^{1/2}$ gives

$$\begin{aligned}(B^i)^\dagger B^i &= \sigma^{-1/2}(A^i)^\dagger \sigma^{1/2} \sigma^{1/2} A^i \sigma^{-1/2} \\ &= \sigma^{-1/2}(A^i)^\dagger \sigma A^i \sigma^{-1/2}.\end{aligned}$$

Summing, substituting the adjoint fixed-point equation, and cancelling the outer square-root factors yields

$$\sum_i (B^i)^\dagger B^i = \sigma^{-1/2} \left(\sum_i (A^i)^\dagger \sigma A^i \right) \sigma^{-1/2} = \sigma^{-1/2} \sigma \sigma^{-1/2} = \mathbb{1}.$$

Lemma A.1.1 (Square-root trace-preserving gauge is a gauge equivalence). [Lean](#) [✓ Stmt](#) *If σ is positive definite and B is defined by (5.1), then A and B are gauge equivalent.*

Proof. [✓ Proof](#) The gauge matrix is $\sigma^{1/2}$, which is invertible because σ is positive definite. Thus (5.1) has the form of the gauge relation (2.5). $\square$

Appendix B

Peripheral Channel Structure and Transfer-Operator Gaps: Supporting Results

This appendix supports Chapter 6. It proves the algebraic mixed-transfer identities, an auxiliary trace-positivity result, the peripheral intertwiners behind the gauge-rigidity theorems, the Banach-algebra convergence facts behind the overlap-decay theorems, the rank-one Perron projection behind the primitive overlap limit, and the cyclic-decomposition machinery that removes peripheral periodicity.

B.1 Mixed-transfer powers and overlap traces

This section proves the self-transfer identity, the word expansion of the mixed transfer operator, and the trace identities used in Theorem 6.2.1 and Theorem 6.2.2. Recall the mixed transfer operator F_{AB} of Definition 6.1.1 and its rectangular form F_{AB}^{rect} of Definition 6.1.2.

Theorem B.1.1 (Self-transfer). Lean ✓ Stmt *Setting $B = A$ recovers the ordinary transfer map: $F_{AA} = \mathcal{E}_A$.*

Proof. ✓ Proof Substituting $B = A$ into (6.2) gives $F_{AA}(X) = \sum_i A^i X (A^i)^\dagger = \mathcal{E}_A(X)$ by (2.4). □

Lemma B.1.2 (Iterated mixed transfer). Lean ✓ Stmt *For every $X \in M_D(\mathbb{C})$ and $N \geq 0$,*

$$F_{AB}^N(X) = \sum_{\sigma \in \{0, \dots, d-1\}^N} A^\sigma X (B^\sigma)^\dagger, \quad (\text{B.1})$$

where $A^\sigma = A^{\sigma_1} \dots A^{\sigma_N}$ denotes the word evaluation of Definition 2.1.2.

Proof. ✓ Proof Induct on N . The base case is $F_{AB}^0(X) = X$. For the inductive step, apply F_{AB} to each summand in (B.1) and reindex using $\{0, \dots, d-1\}^{N+1} \cong \{0, \dots, d-1\} \times \{0, \dots, d-1\}^N$. □

Theorem 6.2.1 and Theorem 6.2.2 combine Lemma B.1.2 and Lemma [LTC26, “Trace expansion over matrix units”] the same way, over the square and rectangular matrix-unit bases respectively: expanding $\text{Tr}(T)$ by the trace-expansion formula of [LTC26, “Trace expansion over

matrix units”], applying (B.1) to each $T(E_{pq})$ for $T = F_{AB}^N$ or $T = (F_{AB}^{\text{rect}})^N$, and reassembling the sums over matrix-unit indices gives

$$\text{Tr}(T) = \sum_{\sigma} \text{tr}(A^{\sigma}) \overline{\text{tr}(B^{\sigma})} = \sum_{\sigma} V^{(N)}(A)_{\sigma} \overline{V^{(N)}(B)_{\sigma}} = O_{AB}(N).$$

B.2 Auxiliary trace positivity

B.3 Rectangular spectral-radius formulation

Lemma B.3.1 (Rectangular spectral radius bound). Lean ✓ Stmt For normalized tensors of bond dimensions D_1 and D_2 , $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) \leq 1$.

Proof. ✓ Proof This is the MPS formulation of the corresponding theorem in the companion quantum-channel volume [LTC26, “Spectral-radius bound for a trace-preserving mixed Kraus map”]: the rectangular mixed transfer is the mixed Kraus map, and the two tensor normalization identities are its trace-preserving hypotheses. $\square$

B.4 Peripheral intertwiners and gauge rigidity

This section proves the intertwining relations behind Theorem 6.6.1 and Theorem 6.6.4: transporting a modulus-one mixed-transfer eigenvector through separate unital gauges produces Kraus-level intertwiners, and these intertwiners force equal bond dimension once they are invertible.

Lemma B.4.1 (Gauged peripheral eigenvector yields Kraus intertwining). Lean ✓ Stmt Let A be a normalized tensor of bond dimension D_1 and B a normalized tensor of bond dimension D_2 . Suppose ρ_A, ρ_B are fixed points of the respective transfer maps and S_A, S_B are invertible matrices with $S_A S_A^{\dagger} = \rho_A$ and $S_B S_B^{\dagger} = \rho_B$ (for instance the positive definite Perron–Frobenius fixed points of Theorem 5.3.2 and their square roots, as in the right-canonical gauge of Theorem [LTC26, “Right-canonical (unital) gauge”]). Gauge to unital families $A'^i = S_A^{-1} A^i S_A$ and $B'^i = S_B^{-1} B^i S_B$. If $X \in M_{D_1 \times D_2}(\mathbb{C}) \setminus \{0\}$ satisfies $F_{AB}^{\text{rect}}(X) = \mu X$ with $|\mu| = 1$, then $X' = S_A^{-1} X (S_B^{\dagger})^{-1}$ is nonzero, the families A' and B' are unital, and, for every i ,

$$X'(B'^i)^{\dagger} = \mu(A'^i)^{\dagger} X', \quad A'^i X' = \mu X' B'^i. \quad (\text{B.2})$$

Proof. ✓ Proof Embed the gauged families and transported eigenvector into the block matrices

$$C^i := \begin{pmatrix} A'^i & 0 \\ 0 & B'^i \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & X' \\ 0 & 0 \end{pmatrix}, \quad (\text{B.3})$$

on $M_{D_1 + D_2}(\mathbb{C})$; the matrices $\{C^i\}$ form a unital Kraus family, and $E_C(Y) = \mu Y$. The positive definite fixed points of the adjoint maps give a faithful weighted trace for the block map. Applying the Kadison–Schwarz inequality to Y and pairing its positive semidefinite gap with this fixed point shows that the gap vanishes. The Kraus relation of the corresponding theorem in the companion quantum-channel volume [LTC26, “KS equality implies Kraus commutation”] then gives $Y(C^i)^{\dagger} = \mu(C^i)^{\dagger} Y$; applying the same argument to the complementary product gives $C^i Y = \mu Y C^i$. Expanding these block relations gives (B.2). $\square$

Lemma B.4.2 (Intertwining yields gauge-phase equivalence). Lean ✓ Stmt Let A and B be tensors of the same bond dimension. Suppose invertible gauges take them to A' and B' , and suppose there are an invertible matrix X' and a scalar μ with $|\mu| = 1$ such that $A'^i X' = \mu X' B'^i$ for every physical index i . Then A and B are gauge-phase equivalent.

Proof. ✓ Proof Compose the two gauge matrices with the intertwiner to obtain one invertible matrix conjugating A to a scalar multiple of B . $\square$

Combining these two lemmas proves Theorem 6.6.1: Lemma B.4.1 supplies the intertwining relations (B.2) for the separately gauged, unital families. The Kadison–Schwarz equality used inside its own proof makes $X'X'^\dagger$ and $X'^\dagger X'$ nonzero positive semidefinite fixed points of the gauged transfer maps; under irreducibility, Theorem 5.1.1 upgrades these to positive definite matrices, so the transported intertwiner X' is invertible. Lemma B.4.2 then converts the invertible intertwiner, together with the two gauge matrices, into a single invertible matrix realizing the gauge-phase relation $B^i = \bar{\mu} X A^i X^{-1}$ via the Skolem–Noether theorem (Theorem [LTC26, “Skolem–Noether”]).

Lemma 6.4.1 gives the companion kernel-invariance argument, independent of irreducibility: if a modulus-one rectangular intertwiner (6.6) exists between injective tensors of bond dimensions D_1 and D_2 , then the first relation shows that $\ker(X)$ is invariant under every $(B^i)^\dagger$; since the $\{B^i\}$ span the full matrix algebra, so do the $\{(B^i)^\dagger\}$, so $\ker(X)$ is trivial and $D_2 \leq D_1$. Taking adjoints of the second relation and applying the same argument to $X^\dagger$ shows that $\ker(X^\dagger)$ is invariant under every $(A^i)^\dagger$ and hence trivial, giving $D_1 \leq D_2$. Combined with Lemma B.4.1, this proves $\rho_{\text{spec}}(F_{AB}^{\text{rect}}) < 1$ whenever A and B are injective tensors of unequal bond dimension, which is exactly Theorem 6.4.2.

B.5 Spectral-radius decay and overlap limits

This section collects the Banach-algebra convergence facts underlying Theorem 6.5.4, Theorem 6.5.5, and the block-separation consequences of the transfer-operator gap.

Theorem 6.4.3 applies Lemma [LTC26, “Powers vanish below unit spectral radius”] to the rectangular mixed transfer operator and then evaluates its powers at a fixed matrix. Theorem 6.5.4 combines the gap $\rho_{\text{spec}}(F_{AB}) < 1$ of Theorem 6.3.6 with the equal-bond-dimension specialization of this rectangular decay theorem. Since Tr is continuous on the finite-dimensional matrix-endomorphism algebra, this gives $\text{Tr}(F_{AB}^N) \rightarrow 0$ term by term over the finitely many matrix-unit entries in the trace-expansion formula of [LTC26, “Trace expansion over matrix units”]; combined with the overlap–trace identity (6.4), this proves Theorem 6.5.5.

Theorem B.5.1 (Cross-correlation decay). Lean ✓ Stmt *If A and B are injective normalized tensors that are not gauge-phase equivalent, then $\text{tr}(F_{AB}^N(X)) \rightarrow 0$ as $N \rightarrow \infty$ for every $X \in M_D(\mathbb{C})$.*

Proof. ✓ Proof Theorem 6.5.4 gives $F_{AB}^N(X) \rightarrow 0$ in operator norm, and continuity of the trace gives the conclusion. $\square$

Theorem B.5.2 (Self-correlation persists). Lean ✓ Stmt *If ρ is a fixed point of $\mathcal{E}_A$, then $\text{tr}(\mathcal{E}_A^N(\rho)) = \text{tr}(\rho)$ for every $N \geq 0$. In particular, this applies to the distinguished Perron–Frobenius fixed point from Theorem 5.3.2.*

Proof. ✓ Proof Since $\mathcal{E}_A(\rho) = \rho$, one has $\mathcal{E}_A^N(\rho) = \rho$ for every N . Taking the trace proves the claim. $\square$

B.6 Rank-one Perron projection

This section proves the rank-one Perron limit (Theorem 6.10.1) underlying the primitive overlap convergence (Theorem 6.10.2). Let $E : M_D(\mathbb{C}) \rightarrow M_D(\mathbb{C})$ be trace-preserving with a fixed point

$\rho \neq 0$, $\text{tr}(\rho) \neq 0$. Write $P := P_\rho$ for the fixed-point projection of the corresponding definition in the companion quantum-channel volume [LTC26, “Fixed-point projection”],

$$P(X) = \frac{\text{tr}(X)}{\text{tr}(\rho)} \rho,$$

and $N := E - P$ for the complementary part. The power decomposition (the corresponding theorem in the companion quantum-channel volume [LTC26, “Power decomposition”]) gives $E^n = P + N^n$ for $n \geq 1$.

Under the hypothesis $\rho_{\text{spec}}(N) < 1$ of Theorem 6.10.1, Lemma [LTC26, “Powers vanish below unit spectral radius”] gives $N^n \rightarrow 0$, so $\text{Tr}(E^n) = \text{Tr}(P) + \text{Tr}(N^n) \rightarrow \text{Tr}(P) = 1$ by Lemma [LTC26, “Trace of the fixed-point projection”]. This is exactly the mechanism recorded in Theorem 6.10.1.

When $E = \mathcal{E}_A$ is the transfer map of a normalized MPS tensor and ρ is its Perron–Frobenius fixed point (Theorem 5.3.2, Chapter 5), the overlap–trace identity (6.4) with $A = B$ gives $O_{AA}(N) = \text{Tr}(\mathcal{E}_A^N)$, so Theorem 6.10.1 yields the primitive self-overlap limit of Theorem 6.10.2. The hypothesis $\rho_{\text{spec}}(N) < 1$ is exactly the complementary transfer-map gap that the complementary spectral-radius theorem for primitive maps (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]) supplies: for a primitive E all eigenvalues bounded by 1 in modulus and no trace-zero fixed point, every eigenvalue of $N = E - P$ has modulus strictly less than 1, and since $M_D(\mathbb{C})$ is finite-dimensional this bounds the spectral radius $\rho_{\text{spec}}(N)$ itself.

Theorem B.6.1 (Complementary spectral-radius gap for irreducible primitive tensors). [Lean]
✓ Stmt *Let A be an irreducible normalized MPS tensor whose transfer map $\mathcal{E}_A$ is primitive. Then there is a nonzero positive semidefinite fixed point ρ of $\mathcal{E}_A$ with $\text{tr}(\rho) \neq 0$ such that, for $P := P_\rho$ and $N := \mathcal{E}_A - P$, $\rho_{\text{spec}}(N) < 1$.*

Proof. ✓ Proof The channel fixed-point existence theorem (Theorem 5.3.1) supplies a nonzero positive semidefinite fixed point ρ . Apply Theorem [LTC26, “Complementary gap at a prescribed fixed point”] to $\mathcal{E}_A$ and ρ . □

This concretely instantiates the hypothesis of Theorem 6.10.1 and, via Theorem 6.10.2, gives the primitive self-overlap limit for every irreducible peripherally primitive normalized MPS tensor.

B.7 Periodicity removal

This section proves the cyclic-decomposition machinery of Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”] and the divisibility statements of Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”] and Theorem [LTC26, “Transfer-map period divides bond dimension”]: normalizing a peripheral eigenvector to a unitary, forming its discrete Fourier (cyclic) spectral projections, and tracking how the channel shifts the resulting corners.

Combining these lemmas proves Theorem [LTC26, “Cyclic decomposition for irreducible finite Kraus maps”]: Lemma [LTC26, “Normalized peripheral unitary”] supplies a unitary generator of exact order m for a primitive m -th peripheral root, and Lemma [LTC26, “Cyclic projections from a peripheral unitary”] builds the m Fourier projections out of its powers, using Lemma [LTC26, “Powers on a peripheral-unitary orbit”] to identify how the channel shifts them.

B.7.1 Cyclic corners and restricted primitivity

Corner preservation, the associated corner subspace, irreducibility of the restricted map, and corner rank are defined in the companion quantum-channel volume [LTC26, “Corner preservation” and the following corner definitions]. We use those definitions here in the MPS periodicity-removal argument.

B.7.2 Group structure and divisibility

Theorem [LTC26, “Peripheral eigenspaces are one-dimensional”] follows the same pattern: given a peripheral unitary eigenvector U for γ (Lemma [LTC26, “Peripheral unitary eigenvector”]), the multiplicative-domain identity gives $E(XU^\dagger) = XU^\dagger$ for any γ -eigenvector X , so the scalar fixed-point lemma for irreducible unital maps (Lemma [LTC26, “Scalar fixed points for irreducible unital maps”]) gives $XU^\dagger = c\mathbb{1}$, hence $X = cU$ and the eigenspace is one-dimensional.

The divisibility statements of Theorem [LTC26, “Transfer-map peripheral eigenvalues form a cyclic group”] and Theorem [LTC26, “Transfer-map period divides bond dimension”] follow from Lemma [LTC26, “Cyclic projections from a peripheral unitary”] and Theorem [LTC26, “Rank equals trace for an orthogonal projection”]: the m cyclic Fourier projections $P_0, \dots, P_{m-1}$ sum to $\mathbb{1}$ and are cyclically permuted by E , so trace preservation gives $\text{tr}(P_{k+1}) = \text{tr}(P_k)$ for every k , hence $D = \text{tr}(\mathbb{1}) = \sum_{k=0}^{m-1} \text{tr}(P_k) = m \text{tr}(P_0)$. Since the trace of an orthogonal projection is a natural number (Theorem [LTC26, “Rank equals trace for an orthogonal projection”]), $m \mid D$.

This periodicity-removal chain — unitary normalization, Fourier projections, cyclic Kraus shift (Lemma 6.8.1.1), the blocked cyclic corners of this section, and restricted primitivity — is exactly the machinery behind the period-removing blocking theorems of Chapter 8: [Wol12, Theorem 6.6] supplies the cyclic peripheral spectrum of the adjoint transfer map, and Theorem 8.3.1.1 (Section 8.3.1) and Theorem 8.11.1 apply it to realize the cyclic-sector decomposition of a blocked irreducible tensor with rectangular support isometries.

Appendix C

Wielandt Bound: Supporting Results

This appendix supports Chapter 7. It collects exact-word, cumulative-span, augmentation, blocking, complementary-gap, and one-step padding results used by the main quantitative argument.

C.1 Exact-word and block-injectivity support

This section proves the exact-word identities and low-dimensional facts used in Lemma 7.1.1.20, Theorem 7.1.1.21, Theorem 7.5.2, and Theorem 7.9.1.

Lemma C.1.1 (Word span products). [Lean](#) [✓ Stmt](#) *One has $S_m(A)S_n(A) \subseteq S_{m+n}(A)$: the product of word spans at lengths m and n is contained in the word span at length $m+n$.*

Proof. [✓ Proof](#) Every product $A^{w_1}A^{w_2}$ with $|w_1| = m$ and $|w_2| = n$ equals $A^{w_1w_2}$, where $|w_1w_2| = m+n$. $\square$

Lemma C.1.2 (Zero-length word span). [Lean](#) [✓ Stmt](#) *If $D \geq 2$, then $S_0(A) \neq M_D(\mathbb{C})$.*

Proof. [✓ Proof](#) The zero-length products consist only of the identity, so $S_0(A) = \text{span}_{\mathbb{C}}\{1\}$ has dimension 1. Since $\dim M_D(\mathbb{C}) = D^2 \geq 4$, this span cannot equal $M_D(\mathbb{C})$. $\square$

Corollary C.1.3 (0-block injectivity). [Lean](#) [✓ Stmt](#) *If $D \geq 1$ and A is 0-block injective, then $D = 1$.*

Proof. [✓ Proof](#) By Lemma 7.1.1.20, 0-block injectivity gives $S_0(A) = M_D(\mathbb{C})$. Lemma C.1.2 excludes $D \geq 2$, so the nonzero bond dimension must be 1. $\square$

Lemma C.1.4 (Exact words around a nonzero middle matrix). [Lean](#) [✓ Stmt](#) *If A is L -block injective and $X \neq 0$, then*

$$\text{span}_{\mathbb{C}}\{A^u X A^v : |u| = |v| = L\} = M_D(\mathbb{C}).$$

Proof. [✓ Proof](#) By L -block injectivity, the length- L word products span $M_D(\mathbb{C})$. Bilinearity of $(R, S) \mapsto RXS$ reduces the claim to Lemma [LTC26, “Two-sided span from one nonzero matrix”], with R and S chosen among those word products. $\square$

C.2 Cumulative-span and spectral linear algebra

This section supplies the finite-dimensional stabilization and spectral facts used in Theorem 7.2.2, Theorem 7.3.2, and Theorem 7.4.8.

The next two lemmas connect irreducible matrix actions with the cumulative word span used throughout Chapter 7.

Remark C.2.1 (Burnside input for cumulative spanning). The passage from irreducible tensors to full algebra spanning first passes through the irreducible action on $\mathbb{C}^D$. Burnside’s theorem then says that the resulting irreducible subalgebra of $M_D(\mathbb{C})$ is all of $M_D(\mathbb{C})$. Once the algebra span is full, finite-dimensionality gives a single cumulative span that already equals $M_D(\mathbb{C})$.

Lemma C.2.2 (Irreducible action gives an irreducible tensor). Lean ✓ Stmt *If the letters of A act irreducibly on $\mathbb{C}^D$, then A has no nontrivial invariant orthogonal projection.*

Proof. ✓ Proof If P were a nontrivial invariant orthogonal projection, then its range would be invariant under every letter of A . Irreducibility of the action forces this range to be either 0 or all of $\mathbb{C}^D$, so P must be 0 or $\mathbb{1}$, a contradiction. □

Lemma C.2.3 (The algebra span reaches a finite cumulative span). Lean ✓ Stmt *If the generated unital algebra satisfies $\text{alg}(A) = M_D(\mathbb{C})$, then there exists N such that $T_N(A) = M_D(\mathbb{C})$.*

Proof. ✓ Proof Every element of $\text{alg}(A)$ lies in some cumulative span $T_n(A)$: the generators lie in $T_1(A)$, scalar multiples of the identity lie in $T_0(A)$, and the family $\{T_n(A)\}_{n \geq 0}$ is closed under addition and under multiplication with lengths adding. Since $M_D(\mathbb{C})$ is finite-dimensional, the ascending chain $T_0(A) \subseteq T_1(A) \subseteq \dots$ stabilizes, so one can choose a single N with $T_N(A) = M_D(\mathbb{C})$. □

Theorem C.2.4 (Cumulative span reaches $M_D(\mathbb{C})$ — explicit bound). Lean ✓ Stmt *Let $D > 0$. If A is normal, then $T_{D^2}(A) = M_D(\mathbb{C})$.*

Proof. ✓ Proof By Theorem 7.1.1.19, if $T_n \neq T_{n+1}$, then $\dim T_{n+1} > \dim T_n$. Since $\dim T_n \leq D^2$ by Lemma 7.1.1.17, after at most D^2 steps either $T_n = M_D(\mathbb{C})$ or T_n has stabilised. Stabilisation before reaching $M_D(\mathbb{C})$ contradicts normality, which requires $S_{L_0} = M_D(\mathbb{C})$ for some L_0 and hence $T_{L_0} = M_D(\mathbb{C})$. □

Theorem C.2.5 (Nonzero trace product). Lean ✓ Stmt *Let $D > 0$. If A is normal, then there exists a word w of length at most D^2 such that $\text{tr}(A^w) \neq 0$. This is the weaker D^2 corollary of [SPGWC10, Lemma 1]. The sharp source statement is Theorem 7.2.4.*

Proof. ✓ Proof By Theorem 7.2.1, $\mathbb{1} \in T_{D^2}(A)$, so $\mathbb{1}$ is a linear combination of word products of length at most D^2 . At least one has nonzero trace because $\text{tr}(\mathbb{1}) = D \neq 0$. □

Theorem C.2.6 (Eigenvalue extraction). Lean ✓ Stmt *Let $D > 0$. If A is normal, then there exist a word w_0 with $|w_0| \leq D^2$, a nonzero scalar $\mu \neq 0$, and a nonzero vector $\varphi \neq 0$ such that $A^{w_0}\varphi = \mu\varphi$.*

Proof. ✓ Proof By Theorem C.2.5, there is a word w_0 with $\text{tr}(A^{w_0}) \neq 0$ and $|w_0| \leq D^2$. By Theorem [LTC26, “Nonzero trace implies eigenvector”], A^{w_0} has a nonzero eigenvector. □

Theorem C.2.7 (Vector span stabilisation). Lean ✓ Stmt *If $K_n(A, \varphi) = K_{n+1}(A, \varphi)$, then $K_m(A, \varphi) = K_n(A, \varphi)$ for every $m \geq n$.*

Proof. ✓ Proof The argument is the same as for Theorem 7.1.1.18: left multiplication by A^i cannot escape the stabilised span. □

Lemma C.2.8 (Vector span from matrix span). [Lean](#) [✓ Stmt](#) If $T_N(A) = M_D(\mathbb{C})$ and $\varphi \neq 0$, then $K_N(A, \varphi) = \mathbb{C}^D$.

Proof. [✓ Proof](#) Every matrix $M \in M_D(\mathbb{C})$ lies in $T_N(A)$, so in particular every rank-one matrix sending φ to any target vector v lies in $T_N(A)$. Applying such matrices to φ recovers all of $\mathbb{C}^D$. $\square$

C.3 One-step augmentation

This section proves the one-step augmentation results used in Theorem 7.4.2 and Theorem 7.4.6.

Definition C.3.1 (One-step augmentation). [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with Kraus family $\{A^i\}_{i=0}^{d-1}$, and let $X \in M_D(\mathbb{C})$. The *one-step augmentation* of A by X is the tensor $\tilde{A}$ with physical dimension $d+1$ whose first Kraus operator is X and whose remaining Kraus operators are those of A :

$$\begin{aligned}\tilde{A}^0 &= X, \\ \tilde{A}^{i+1} &= A^i.\end{aligned}$$

Lemma C.3.2 (Kraus operators lie in the one-step span). [Lean](#) [✓ Stmt](#) Every Kraus operator A^i belongs to the one-step span $S_1(A)$.

Proof. [✓ Proof](#) This is the length-one word with letter i . $\square$

Lemma C.3.3 (Redundant one-step augmentation). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) Let $X \in S_1(A)$, and let $\tilde{A}$ be the one-step augmentation of A by X . Then $S_n(\tilde{A}) = S_n(A)$ for every $n \geq 0$.

Proof. [✓ Proof](#) The new generator X is already in $S_1(A)$, and each original generator remains in the augmented family. Hence the one-step spans agree. Multiplying word spans and arguing by induction on the word length gives equality at every length. $\square$

Lemma C.3.4 (Normality and Kraus rank under one-step augmentation). [Lean](#) [Lean](#) [✓ Stmt](#) Let $X \in S_1(A)$, and let $\tilde{A}$ be the one-step augmentation of A by X . If A is normal, then $\tilde{A}$ is normal, and $\text{kr}(\tilde{A}) = \text{kr}(A)$.

Proof. [✓ Proof](#) Since all exact word spans are unchanged by Lemma C.3.3, eventual full word span for A gives eventual full word span for $\tilde{A}$. The equality of Kraus ranks is the equality of the dimensions of the common one-step span. $\square$

Theorem C.3.5 (Wielandt case (2), invertible generator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be a normal MPS tensor with bond dimension $D > 0$, and suppose that some Kraus operator A^{i_0} is invertible. Then

$$\begin{aligned}S_{D^2 - \text{kr}(A) + 1}(A) &= M_D(\mathbb{C}), \\ \iota(A) &\leq D^2 - \text{kr}(A) + 1.\end{aligned}$$

Proof. [✓ Proof](#) The dimensions of the exact word spans cannot decrease, because left multiplication by the invertible Kraus operator is injective. If this dimension growth stopped before the full matrix algebra was reached, the same argument at every later length would contradict normality. Since $\dim S_1(A) = \text{kr}(A)$ and $\dim M_D(\mathbb{C}) = D^2$, the full algebra is reached by length $D^2 - \text{kr}(A) + 1$. The index bound follows from the definition of $\iota(A)$. $\square$

Corollary C.3.6 (Wielandt case (2), invertible Kraus operator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If some Kraus operator A^{i_0} is invertible, then

$$S_{D^2 - \text{kr}(A) + 1}(A) = M_D(\mathbb{C}),$$

$$\iota(A) \leq D^2 - \text{kr}(A) + 1.$$

Proof. [✓ Proof](#) Apply the one-step case to $X = A^{i_0}$, which belongs to $S_1(A)$ by Lemma C.3.2. $\square$

Corollary C.3.7 (Wielandt case (3), non-invertible Kraus operator). [Lean](#) [Lean](#) [✓ Stmt](#) Let A be an MPS tensor with bond dimension $D \geq 1$, satisfying $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, and suppose that A is primitive in the sense of Definition 7.1.1.3. If some Kraus operator A^{i_0} is non-invertible and has an eigenvector $\varphi \neq 0$ with nonzero eigenvalue μ , then

$$S_{D^2}(A) = M_D(\mathbb{C}),$$

$$\iota(A) \leq D^2.$$

Proof. [✓ Proof](#) Apply the one-step case to $X = A^{i_0}$, which belongs to $S_1(A)$ by Lemma C.3.2. $\square$

C.4 Blocking and fixed-length spanning

This section proves the blocking and fixed-length constructions used in Theorem 7.4.8, Theorem 7.5.5, and Theorem 7.9.1.

Theorem C.4.1 (Blocking preserves normality). [Lean](#) [✓ Stmt](#) If A is normal and $L > 0$, then the blocked tensor $A^{[L]}$ is also normal.

Proof. [✓ Proof](#) Choose N with $S_N(A) = M_D(\mathbb{C})$. Then $\mathbb{1} \in S_N(A)$. For any $M \in S_N(A)$, write $\mathbb{1} = \sum_r c_r A^{w_r}$ with each $|w_r| = N$. Then

$$M = M\mathbb{1} = \sum_r c_r M A^{w_r} \in S_N(A) S_N(A) \subseteq S_{2N}(A). \quad (\text{C.1})$$

By Lemma C.1.1, iterating (C.1) gives $S_N(A) \subseteq S_{kN}(A)$ for every $k \geq 1$. Taking $k = L$ yields $M_D(\mathbb{C}) = S_N(A) \subseteq S_{NL}(A)$. By Lemma C.4.2, $S_{NL}(A) \subseteq S_N(A^{[L]})$, so $S_N(A^{[L]}) = M_D(\mathbb{C})$. $\square$

Lemma C.4.2 (Chunking: word span embeds into blocked word span). [Lean](#) [✓ Stmt](#) One has $S_{nL}(A) \subseteq S_n(A^{[L]})$ for all n, L .

Proof. [✓ Proof](#) Every length- nL word in the original alphabet can be split into n consecutive chunks of length L , each of which is a single blocked Kraus operator by (2.13). The claim follows by induction on n , using Lemma C.1.1. $\square$

Theorem C.4.3 (Word eigenvector gives blocked eigenvector). [Lean](#) [✓ Stmt](#) If $A^{w_0} \varphi = \mu \varphi$ for a word w_0 of length L , then $(A^{[L]})^{j_0} \varphi = \mu \varphi$, where j_0 is the encoded blocked index corresponding to w_0 .

Proof. [✓ Proof](#) By (2.13), $(A^{[L]})^{j_0} = A^{w_0}$. $\square$

Theorem C.4.4 (Blocked fixed-length matrix spanning). [Lean](#) [✓ Stmt](#) Suppose A has bond dimension $D > 0$, is normal, and $L > 0$, and we have:

1. a word w_0 of length L with eigenvector $A^{w_0}\varphi = \mu\varphi$ ($\mu \neq 0$, $\varphi \neq 0$),
2. a word τ_0 of length L with transpose eigenvector $(A^{\tau_0})^T\psi = \nu\psi$ ($\nu \neq 0$, $\psi \neq 0$),
3. a rank-one element $\varphi\psi^T \in S_m(A^{[L]})$ for some m .

Then there exists N such that $S_N(A) = M_D(\mathbb{C})$.

Remark. The proof gives the explicit bound $N = ((D-1) + m + (D-1))L$; the theorem states only the existential conclusion.

Proof. ✓ Proof Set $B = A^{[L]}$, which is normal by Theorem C.4.1. The word eigenvectors transfer to single-index eigenvectors of B and B^T by Theorem C.4.3. Eigenvector spreading and padding then give $H_{D-1}(B, \varphi) = H_{D-1}(B^T, \psi) = \mathbb{C}^D$.

For each standard basis vector e_j , choose $R_j \in S_{D-1}(B^T)$ with $R_j\psi = e_j$. Reversing each word after transposition shows that $R_j^T \in S_{D-1}(B)$, and therefore $(\varphi\psi^T)R_j^T = \varphi e_j^T$ belongs to $S_{m+D-1}(B)$. Theorem 7.5.2, applied with $n = D-1$ and $m' = m + D-1$, now gives $S_{(D-1)+m+(D-1)}(B) = M_D(\mathbb{C})$. Finally, Lemma 7.5.4 transfers this identity to $S_{((D-1)+m+(D-1))L}(A) = M_D(\mathbb{C})$. $\square$

Theorem C.4.5 (Rank-one extraction for blocked tensor). Lean ✓ Stmt Suppose A has bond dimension $D > 0$, is normal, and $N_0 \geq 1$ with $S_{N_0}(A) = M_D(\mathbb{C})$. Then there exist words σ_0, τ_0 of length N_0 , nonzero vectors φ, ψ , nonzero scalars μ, ν , and $m \in \mathbb{N}$ such that:

1. $A^{\sigma_0}\varphi = \mu\varphi$,
2. $(A^{\tau_0})^T\psi = \nu\psi$,
3. $\varphi\psi^T \in S_m(A^{[N_0]})$.

Proof. ✓ Proof Set $B = A^{[N_0]}$. By (2.13) and $S_{N_0}(A) = M_D(\mathbb{C})$, one has $S_1(B) = M_D(\mathbb{C})$. Hence $\mathbb{1} \in S_1(B)$, so not all blocked generators can have trace zero. Choose j_0 with $\text{tr}(B^{j_0}) \neq 0$. By Theorem [LTC26, “Nonzero trace implies eigenvector”], B^{j_0} has a nonzero eigenvector φ with eigenvalue $\mu \neq 0$. Writing j_0 as the blocked index corresponding to a word σ_0 of length N_0 gives $A^{\sigma_0}\varphi = \mu\varphi$. Applying the same argument to the transpose blocked tensor gives a word τ_0 of length N_0 , a nonzero vector ψ , and $\nu \neq 0$ with $(A^{\tau_0})^T\psi = \nu\psi$. Since A is normal and $N_0 \geq 1$, the blocked tensor B is normal by Theorem C.4.1. Therefore there exists M with $S_M(B) = M_D(\mathbb{C})$. In particular, $\varphi\psi^T \in S_M(B)$. Taking $m = M$ proves (3). $\square$

Theorem C.4.6 (Fixed-length word-span saturation via blocking). Lean ✓ Stmt If A is a normal MPS tensor with bond dimension $D > 0$, then there exists N such that $S_N(A) = M_D(\mathbb{C})$. This is a blocked fixed-length construction under the stated normality hypothesis. The quantitative result of [SPGWC10, Lemma 2(b)] is Theorem 7.4.5.

Proof. ✓ Proof By normality, some $N_0 > 0$ has $S_{N_0}(A) = M_D(\mathbb{C})$. Theorem C.4.5 produces eigenvectors and a rank-one element $\varphi\psi^T \in S_m(A^{[N_0]})$. Theorem C.4.4 then gives $S_{((D-1)+m+(D-1))N_0}(A) = M_D(\mathbb{C})$. $\square$

C.5 Complementary-gap consequences

This section proves the complementary-gap consequences used in Lemma 7.7.2.1, Lemma 7.7.4.1, and Theorem 7.7.4.2.

Theorem C.5.1 (Primitive overlap convergence). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor, then $O_{AA}(N) \rightarrow 1$.*

Proof. [✓ Proof](#) The claim follows from the self-overlap convergence under a complementary transfer-map gap (Theorem 6.10.2). $\square$

Theorem C.5.2 (Peripheral primitivity and irreducibility give normalized overlap). [Lean](#) [✓ Stmt](#) *Let A have positive bond dimension and satisfy $\sum_i (A^i)^\dagger A^i = \mathbb{1}$. If A is tensor-irreducible and the transfer map is primitive, then $O_{AA}(N) \rightarrow 1$.*

Proof. [✓ Proof](#) Irreducible Perron–Frobenius theory supplies a nonzero positive fixed point. Peripheral primitivity gives a strict spectral-radius bound on the complementary transfer map. Thus A is a primitive MPS tensor, and Theorem C.5.1 applies. $\square$

Lemma C.5.3 (Primitive MPS tensors have nonzero word products). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor, then for every $n \geq 0$ there exists a word w of length exactly n with $A^w \neq 0$.*

Proof. [✓ Proof](#) The PSD fixed point ρ of $\mathcal{E}_A$ satisfies $\mathcal{E}_A^n(\rho) = \rho \neq 0$. Since $\mathcal{E}_A^n(\rho) = \sum_{|w|=n} A^w \rho (A^w)^\dagger$, at least one summand is nonzero, hence some $A^w \neq 0$. $\square$

Lemma C.5.4 (Iterated transfer does not vanish). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor, then $\mathcal{E}_A^n \neq 0$ for all $n \geq 0$.*

Proof. [✓ Proof](#) By Lemma C.5.3, there exists a word w of length n with $A^w \neq 0$. The expansion $\mathcal{E}_A^n(\mathbb{1}) = \sum_{|u|=n} A^u (A^u)^\dagger$ contains the nonzero positive semidefinite summand $A^w (A^w)^\dagger$, so $\mathcal{E}_A^n(\mathbb{1}) \neq 0$. $\square$

Lemma C.5.5 (Fixed-point uniqueness from a complementary transfer-map gap). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor with PSD fixed point ρ , then every fixed point σ of $\mathcal{E}_A$ is proportional to ρ : $\sigma = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$.*

Proof. [✓ Proof](#) Set $\sigma' = \sigma - \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$. Then $\text{tr}(\sigma') = 0$ and $(\mathcal{E}_A - P_\rho)(\sigma') = \sigma'$. If $\sigma' \neq 0$, then 1 is an eigenvalue of $\mathcal{E}_A - P_\rho$, contradicting the complementary transfer-map gap $\rho_{\text{spec}}(\mathcal{E}_A - P_\rho) < 1$. $\square$

Lemma C.5.6 (Complement powers tend to zero). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor with PSD fixed point ρ , then $(\mathcal{E}_A - P_\rho)^n \rightarrow 0$ in operator norm.*

Proof. [✓ Proof](#) The spectral radius of $\mathcal{E}_A - P_\rho$ is strictly less than 1 by the complementary transfer-map gap hypothesis. Apply the general result that powers of an operator with spectral radius < 1 tend to zero. $\square$

Lemma C.5.7 (Primitive tensors with positive-definite fixed point have irreducible transfer map). [Lean](#) [✓ Stmt](#) *If A is a primitive MPS tensor with positive-definite fixed point ρ , then the transfer map $\mathcal{E}_A$ is irreducible.*

The proof uses fixed-point uniqueness (Lemma C.5.5): every PSD fixed point is proportional to ρ , so Wolf’s criterion for irreducibility (a positive-definite fixed point together with uniqueness of positive fixed points implies irreducibility) applies directly.

Proof. [✓ Proof](#) By Lemma C.5.5, if $\sigma \geq 0$ and $\mathcal{E}_A(\sigma) = \sigma$, then $\sigma = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)} \rho$. Since $\rho > 0$, Wolf’s uniqueness criterion for irreducibility applies. $\square$

C.6 Identity in the one-step span

This section proves the one-step padding conversions used in Theorem 7.8.2. It also records why cumulative spanning alone does not imply normality.

Remark C.6.1 (Identity-in-the-one-step-span condition). ✓ Stmt The *one-step padding condition* is $\mathbb{1} \in S_1(A)$: the identity matrix lies in the span of the Kraus operators $\{A^i\}_{i=0}^{d-1}$. This is an additional hypothesis. The normalization $\sum_i (A^i)^\dagger A^i = \mathbb{1}$ gives a quadratic identity and does not imply $\mathbb{1} \in S_1(A)$. Theorem 7.7.4.2 obtains normality from a positive-definite fixed point and a complementary gap without this padding condition.

Remark C.6.2 (Counterexample: cumulative spanning does not imply normality). ✓ Stmt In $M_2(\mathbb{C})$, let $A^0 = E_{12}$ and $A^1 = E_{21}$ be the off-diagonal matrix units. Then $\text{alg}(A) = M_2(\mathbb{C})$ and $T_2(A) = M_2(\mathbb{C})$, but the word spans alternate: $S_n(A)$ contains only off-diagonal matrices for odd n and only diagonal matrices for even n . Hence $S_n(A) \neq M_2(\mathbb{C})$ for every n , and A is not normal. The one-step padding condition fails because $\mathbb{1} \notin S_1(A) = \text{span}_{\mathbb{C}}\{E_{12}, E_{21}\}$.

Theorem C.6.3 (Word-span monotonicity from one-step padding). Lean ✓ Stmt If $\mathbb{1} \in S_1(A)$, then $S_n(A) \leq S_{n+1}(A)$ for every n .

Proof. ✓ Proof For any $M \in S_n(A)$, one has $M = M\mathbb{1} \in S_n(A)S_1(A) \subseteq S_{n+1}(A)$ by Lemma C.1.1. □

Theorem C.6.4 (Cumulative span equals word span under one-step padding). Lean ✓ Stmt If $\mathbb{1} \in S_1(A)$, then $T_n(A) = S_n(A)$ for every n .

Proof. ✓ Proof By Theorem C.6.3, the word spans form the chain $S_0(A) \leq S_1(A) \leq \dots \leq S_n(A)$. Therefore their sum $T_n(A) = S_0(A) + \dots + S_n(A)$ equals the largest term $S_n(A)$. □

Theorem C.6.5 (Cumulative spanning plus one-step padding implies normality). Lean ✓ Stmt If $T_N(A) = M_D(\mathbb{C})$ and $\mathbb{1} \in S_1(A)$, then A is normal.

Proof. ✓ Proof Theorem C.6.4 gives $S_N(A) = T_N(A) = M_D(\mathbb{C})$. □

Appendix D

Canonical Form Reduction: Supporting Results

This appendix supports Chapter 8. It collects the projection, gauge, blocking, compression, and reindexing arguments used by the main reduction while keeping the source-named canonical-form statements and the principal reductions in the chapter.

Throughout, $\mathcal{E}_A(X) = \sum_i A^i X (A^i)^\dagger$, $A^{[p]}$ denotes blocking by p , and $V^{(N)}(A)_\sigma = \text{tr}(A^{\sigma_1} \dots A^{\sigma_N})$. Weighted direct sums are written $\bigoplus_k \mu_k A_k$.

D.1 Similarities, strict splitting, and zero blocks

Similarity invariance and strict dimension decrease justify the induction used in Theorem 8.1.1.2; the final result removes blocks that vanish at every positive length.

Theorem D.1.1 (Unitary conjugation preserves the MPV family). [Lean](#) [✓ Stmt](#) *If U is a unitary matrix, then A and $U^\dagger A U$ generate the same MPV family.*

Proof. [✓ Proof](#) By cyclicity of the trace, $\text{tr}((U^\dagger A^{i_1} U) \dots (U^\dagger A^{i_N} U)) = \text{tr}(A^{i_1} \dots A^{i_N})$. □

Theorem D.1.2 (Nontrivial invariant projection gives strict two-block reduction). [Lean](#) [✓ Stmt](#) *Under the hypotheses of Theorem 8.1.2, assume in addition that $P \neq 0$ and $P \neq \mathbb{1}$. Then there exist n, m with $n + m = D$, $n < D$, and $m < D$, together with block tensors A_1, A_2 , such that A and $A_1 \oplus A_2$ generate the same MPV family in the sense of Definition 2.3.3.*

Proof. [✓ Proof](#) Apply Theorem 8.1.2. In the diagonal form of P , the 1-eigenspace is nonempty because $P \neq 0$, and the 0-eigenspace is nonempty because $P \neq \mathbb{1}$. Hence both block sizes are positive. Since $n + m = D$, this implies $n < D$ and $m < D$. □

Theorem D.1.3 (Nonzero irreducible block decomposition). [Lean](#) [✓ Stmt](#) *Every MPS tensor A admits a finite family of nonzero irreducible blocks $(B_k)_{k=1}^r$, each with at least one nonzero Kraus operator and positive bond dimension. For every $N \geq 1$ and every σ ,*

$$V^{(N)}(A)_\sigma = V^{(N)}\left(\bigoplus_{k=1}^r B_k\right)_\sigma. \quad (\text{D.1})$$

Moreover, $\sum_{k=1}^r D_k \leq D$.

Proof. ✓ Proof Apply the irreducible block decomposition (Theorem 8.1.1.2) and retain the blocks having a nonzero Kraus operator. For every all-zero irreducible block C_j and every $N \geq 1$, $V^{(N)}(C_j)_\sigma = 0$, so (D.1) holds. The dimension inequality follows by restricting the structural identity $\sum_j D_j = D$ to the retained subset. $\square$

D.2 Projector closure and isometric corner decompositions

The results below turn reducing projections into support isometries, compress tensors to their corners, and reconstruct closed-chain traces from an orthogonal family of corners. They support Theorem 8.7.2 and Theorem 8.7.4.

Theorem D.2.1 (Invariant projections are reducing under projector closure). Lean ✓ Stmt Suppose that A has invariant-projector closure and that P is an orthogonal projection satisfying $(\mathbb{1} - P)A^iP = 0$ for all i . Then P reduces every tensor matrix: $PA^i = A^iP$ for all i .

Proof. ✓ Proof The lower-corner relation gives $(\mathbb{1} - P)A^i = (\mathbb{1} - P)A^i(\mathbb{1} - P)$. Applying projector closure to $\mathbb{1} - P$ gives $A^i(\mathbb{1} - P) = (\mathbb{1} - P)A^i(\mathbb{1} - P)$, hence the upper corner $PA^i(\mathbb{1} - P)$ also vanishes. Therefore $PA^i = PA^iP = A^iP$. $\square$

Theorem D.2.2 (Invariant-projector closure gives a reducing projection). Lean ✓ Stmt If A has invariant-projector closure and admits a nontrivial invariant projection, then it admits a nontrivial orthogonal projection commuting with every A^i .

Proof. ✓ Proof Apply Theorem D.2.1 to the projection supplied by Definition 8.1.3. $\square$

Lemma D.2.3 (Support isometry of an orthogonal projection). Lean ✓ Stmt

Every orthogonal projection P on $\mathbb{C}^D$ factors through an isometry: there are $n \in \mathbb{N}$ and $V \in \mathbb{C}^{D \times n}$ such that

$$\begin{aligned} V^\dagger V &= \mathbb{1}, \\ VV^\dagger &= P, \\ n &= \text{tr}(P). \end{aligned} \tag{D.2}$$

Proof. ✓ Proof Since P is Hermitian and idempotent, its eigenvalues are 0 and 1. Writing $n = \text{tr}(P)$, unitary diagonalization gives

$$P = U \text{diag}(\underbrace{1, \dots, 1}_n, 0, \dots, 0)U^\dagger.$$

The matrix $V \in \mathbb{C}^{D \times n}$ formed by the first n columns of U satisfies (D.2). $\square$

The rectangular Gram-equality criterion that turns $B^\dagger B = A^\dagger A$ into a unitary U with $B = UA$ is a generic algebra fact, proved in the companion quantum-channel volume [LTC26, “Rectangular Gram equality gives a unitary”].

Lemma D.2.4 (Corner tensors inherit projector closure). Lean ✓ Stmt Let A have invariant-projector closure, and let V be an isometry, $V^\dagger V = \mathbb{1}$, whose range projection $VV^\dagger$ commutes with every A^i . Then the corner tensor $B^i = V^\dagger A^i V$ again has invariant-projector closure.

Proof. ✓ Proof Let Q be an orthogonal projection with $QB^i = QB^iQ$ for all i . Then $VQV^\dagger$ is an orthogonal projection. Commutation of $VV^\dagger$ with each A^i gives $V^\dagger A^i = B^i V^\dagger$, so left-invariance of Q yields

$$(VQV^\dagger)A^i = VQB^iV^\dagger = VQB^iQV^\dagger = (VQV^\dagger)A^i(VQV^\dagger).$$

Projector closure for A yields $A^i(VQV^\dagger) = (VQV^\dagger)A^i(VQV^\dagger)$, and compressing both sides between $V^\dagger$ and V returns $B^iQ = QB^iQ$. $\square$

Lemma D.2.5 (Transfer map of an intertwined corner). Lean Lean ✓ Stmt Let V satisfy $A^iV = VB^i$ for all i . Then, for every X ,

$$\mathcal{E}_A(VXV^\dagger) = V\mathcal{E}_B(X)V^\dagger. \quad (\text{D.3})$$

If moreover $V^\dagger V = \mathbb{1}$, then every eigenvalue of $\mathcal{E}_B$ is an eigenvalue of $\mathcal{E}_A$.

Proof. ✓ Proof Expanding the left-hand side of (D.3) termwise, the intertwining $A^iV = VB^i$ and its adjoint $V^\dagger(A^i)^\dagger = (B^i)^\dagger V^\dagger$ move both factors past V and $V^\dagger$. For the eigenvalue statement, if $\mathcal{E}_B(X) = \mu X$ with $X \neq 0$, then $\mathcal{E}_A(VXV^\dagger) = \mu VXV^\dagger$, and $VXV^\dagger \neq 0$ because compression with the isometry gives $V^\dagger(VXV^\dagger)V = X$. $\square$

Lemma D.2.6 (Matrix product vectors split along an isometric partition). Lean ✓ Stmt Let $V_1, \dots, V_r$ be isometries, $V_k^\dagger V_k = \mathbb{1}$, with $\sum_k V_k V_k^\dagger = \mathbb{1}$, and suppose $A^i V_k = V_k B_k^i$ for some tensors $B_1, \dots, B_r$. Then $\mathcal{V}(A) = \mathcal{V}(\bigoplus_{k=1}^r B_k)$.

Proof. ✓ Proof The intertwining relation propagates from letters to words, so $A^w V_k = V_k B_k^w$ for every word w . Inserting the partition of unity $\sum_k V_k V_k^\dagger = \mathbb{1}$ into the trace and cycling gives

$$\text{tr}(A^w) = \sum_{k=1}^r \text{tr}(V_k^\dagger A^w V_k) = \sum_{k=1}^r \text{tr}(B_k^w),$$

which is the matrix product vector of the direct sum. $\square$

Lemma D.2.7 (Canonical inclusion of a dependent direct summand). Lean Lean Lean Lean ✓ Stmt Let $E_k : H_k \rightarrow \bigoplus_l H_l$ be the canonical inclusion and let $B = \bigoplus_l B_l$. For distinct k and ℓ , the corresponding inclusions have orthogonal ranges. The identities are

$$\begin{aligned} E_k^\dagger E_k &= \mathbb{1}, \\ E_k^\dagger E_\ell &= 0, \\ BE_k &= E_k B_k, \\ E_k^\dagger B &= B_k E_k^\dagger, \\ E_k^\dagger BE_k &= B_k, \\ B &= \sum_k E_k B_k E_k^\dagger. \end{aligned} \quad (\text{D.4})$$

In particular, B commutes with the range projection $E_k E_k^\dagger$.

Proof. ✓ Proof The identities in (D.4) follow directly from the defining block-diagonal entries. The relations $BE_k = E_k B_k$ and $E_k^\dagger B = B_k E_k^\dagger$ imply that B commutes with $E_k E_k^\dagger$. $\square$

Lemma D.2.8 (Canonical hypotheses descend to dependent direct summands). [Lean](#) [✓ Stmt](#) Suppose that, after a change of coordinates, A is the dependent direct sum of tensors B_k . If A has invariant-projector closure and no nontrivial periodic vectors, then every B_k has both properties. The assertion also applies to summands of dimension zero.

Proof. [✓ Proof](#) Conjugate the canonical inclusion of the k -th summand back to the original bond coordinates of A . This is an isometry which intertwines B_k with A on both sides, and its range projection commutes with every letter of A . Projector closure therefore descends by exact compression. A periodic vector of B_k would embed through the same isometry as a periodic vector of A . $\square$

Lemma D.2.9 (Closed-chain equality from an exact isometric decomposition). [Lean](#) [✓ Stmt](#) Suppose that isometries V_k satisfy

$$\begin{aligned} V_k^\dagger A^i &= (\mu_k B_k^i) V_k^\dagger, \\ A^i &= \sum_k V_k (\mu_k B_k^i) V_k^\dagger. \end{aligned} \tag{D.5}$$

Then A and $\bigoplus_k \mu_k B_k$ have the same matrix product vectors at every positive length.

Proof. [✓ Proof](#) Expand the first letter by the reconstruction formula in (D.5). In each summand, move the remaining word through $V_k^\dagger$ using the intertwining relation. Cyclicity of the trace places $V_k^\dagger V_k = \mathbb{1}$ next to the weighted block word, leaving precisely its closed-chain trace. $\square$

D.3 Perron gauges and preservation under conjugation

This section records the positive Perron eigenpair, its spectral normalization, and the behavior of irreducibility and eigenvalues under the corresponding gauge.

Lemma D.3.1 (Perron eigenpair of a nonzero irreducible tensor). [Lean](#) [✓ Stmt](#) Let B be an irreducible MPS tensor of positive bond dimension, not all of whose matrices vanish. Then there are $\rho > 0$ and $r > 0$ with $\mathcal{E}_B(\rho) = r\rho$.

Proof. [✓ Proof](#)

Irreducibility of the tensor makes the transfer map an irreducible completely positive map, and a nonzero matrix makes it nonzero, so it does not annihilate any nonzero PSD matrix. The Perron–Frobenius theorem for positive maps (Theorem [LTC26, “Positive-eigenvalue specialization”]) gives a nonzero PSD eigenvector with eigenvalue $r > 0$, and irreducibility upgrades it to a positive-definite one ([Wol12, Theorem 6.3(2)–(3)]). $\square$

Lemma D.3.2 (Spectral normalization of an irreducible block). [Lean](#) [✓ Stmt](#) Let B be an irreducible MPS tensor of positive bond dimension with $\mathcal{E}_B(\rho) = r\rho$ for some $\rho > 0$ and $r > 0$, and suppose every eigenvalue of $\mathcal{E}_B$ of modulus r equals r . Then $r^{-1/2}B$ is a normal tensor.

Proof. [✓ Proof](#) The rescaling identity $\mathcal{E}_{r^{-1/2}B}(X) = r^{-1}\mathcal{E}_B(X)$ gives $\mathcal{E}_{r^{-1/2}B}(\rho) = r^{-1}r\rho = \rho$, so ρ is a positive-definite fixed point of $\mathcal{E}_{r^{-1/2}B}$. The rescaled map remains irreducible and completely positive, so Theorem [LTC26, “Spectral radius is the Perron eigenvalue”] identifies its spectral radius with the eigenvalue one. The eigenvalues of modulus one of $\mathcal{E}_{r^{-1/2}B}$ are the eigenvalues of $\mathcal{E}_B$ of modulus r , divided by r ; by hypothesis the only such eigenvalue is one, so the peripheral eigenvalue set is $\{1\}$. Irreducibility is unchanged by a nonzero rescaling. $\square$

Lemma D.3.3 (Range factorization of the support projection). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#)

The support projection π of a positive semidefinite matrix ρ factors as $\pi = \rho W$ for a spectral pseudo-inverse W , witnessing that the range of π is the range of ρ . For a rectangular matrix Y , the support projection π of $YY^\dagger$ fixes Y , $\pi Y = Y$; and whenever $AY = YG$ for some matrix G —that is, A maps the range of Y into itself—the one-sided invariance $(\mathbb{1} - \pi)A\pi = 0$ holds.

Proof. [✓ Proof](#)

In the spectral decomposition $\rho = U \text{diag}(\lambda) U^\dagger$, take $W = U \text{diag}(\lambda^+) U^\dagger$, where λ^+ inverts the positive eigenvalues and vanishes on the kernel. Then $\rho W = U \text{diag}(\mathbf{1}_{\lambda>0}) U^\dagger = \pi$. From $\pi(YY^\dagger) = YY^\dagger$, the matrix $((\mathbb{1} - \pi)Y)((\mathbb{1} - \pi)Y)^\dagger$ vanishes, hence $\pi Y = Y$. Finally,

$$(\mathbb{1} - \pi)A\pi = (\mathbb{1} - \pi)AY(Y^\dagger W) = (\mathbb{1} - \pi)Y G(Y^\dagger W) = 0.$$

□

Theorem D.3.4 (Irreducibility under rescaled conjugation). [Lean](#) [✓ Stmt](#) Let B be an irreducible tensor, let X be invertible, and let $c \neq 0$. Then the rescaled conjugated tensor with letters $cX^{-1}B^vX$ is irreducible. In particular, the rescaled unital gauge of an irreducible tensor is irreducible; this is the normalization of [CPGSV16, Section 2.3] applied to an irreducible corner.

Proof. [✓ Proof](#) An invariant orthogonal projection $P \notin \{0, \mathbb{1}\}$ of the conjugated tensor makes the columns of $Y = XP$ span a subspace invariant under every letter of B : $B^vY = YG_v$, where $G_v = X^{-1}B^vXP$. The support projection of $YY^\dagger$ is then a nontrivial invariant orthogonal projection of B by Lemma D.3.3: it is nonzero because $Y \neq 0$, and it is not the identity because any nonzero vector in the kernel of P transports through $(X^\dagger)^{-1}$ to the kernel of $YY^\dagger$. This contradicts irreducibility of B . □

Theorem D.3.5 (Unital Schwarz data for an irreducible corner). [Lean](#) [✓ Stmt](#) Let B be an irreducible tensor with a positive definite transfer eigenvector, $\mathcal{E}_B(\rho) = r\rho$, where $\rho > 0$ and $r > 0$. Then the rescaled unital gauge $K^v = r^{-1/2}\rho^{-1/2}B^v\rho^{1/2}$ is a unital Kraus family, is an irreducible tensor with an irreducible transfer map, and its adjoint map has a positive definite fixed point. These are the hypotheses of the cyclic decomposition of the peripheral spectrum, [Wol12, Theorem 6.6].

Proof. [✓ Proof](#) Unitality is Theorem 8.8.2; irreducibility is Theorem D.3.4. The adjoint of a unital family is trace-preserving, so its transfer map is a channel and has a positive semidefinite fixed point by the Cesàro argument; irreducibility of the adjoint map upgrades the fixed point to a positive definite one. □

Lemma D.3.6 (Eigenvalue transport to the unital gauge). [Lean](#) [✓ Stmt](#) In the setting of Theorem D.3.5, every transfer eigenvalue μ of B becomes the transfer eigenvalue μ/r of the rescaled unital gauge: the gauge conjugates the transfer map by the congruence with $\rho^{1/2}$ and divides it by r . In particular, the peripheral eigenvalues $e^{2\pi i q/p}$ described in [CPGSV16, Section 2.3] arise from the transfer eigenvalues of modulus r .

Proof. [✓ Proof](#) If $\mathcal{E}_B(Z) = \mu Z$, then $Z' = \rho^{-1/2}Z\rho^{-1/2}$ is nonzero and satisfies $\mathcal{E}_K(Z') = (\mu/r)Z'$ by direct computation. □

D.4 Blocking identities and primitive stability

These identities control transfer-map powers, weighted direct sums, and positive-length MPV equalities under repeated blocking. They are used in the common-length reduction of Theorem 8.3.1.4.

Blocking by a length p commutes with weighted direct sums up to the p th power of each weight: for nonzero weights (μ_k) and blocks (B_k) ,

$$\left(\bigoplus_k \mu_k B_k \right)^{[p]} \sim \bigoplus_k \mu_k^p B_k^{[p]}, \quad (\text{D.6})$$

where $\sim$ is equality of MPV families. When two block families are reblocked to a common length $p = m_k e_k$, the p -blocked alphabet is identified with the e_k -fold tensor power of the m_k -blocked alphabet, so each sector reblocked through its own period agrees with the directly p -blocked block.

Theorem D.4.1 (Common blocking period for a block family). Lean Stmt *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension MPS tensors, each admitting a blocking period p_k making its transfer map primitive. Then $p = \text{lcm}(p_1, \dots, p_r)$ is a common period: $\mathcal{E}_{B_k^{[p]}}$ is primitive for every k .*

Proof. Proof Each p_k divides p , so primitivity of $\mathcal{E}_{B_k^{[p_k]}}$ carries over to $\mathcal{E}_{B_k^{[p]}}$ via the divisibility identity for transfer-map powers. □

Theorem D.4.2 (Common primitive blocking for irreducible left-canonical blocks). Lean Stmt *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension, left-canonical, tensor-irreducible MPS tensors. Then there is a single blocking length $p > 0$ such that $\mathcal{E}_{B_k^{[p]}}$ is primitive for every k .*

Proof. Proof Apply Theorem 8.9.2.1 to each block, then apply Theorem D.4.1 to the resulting finite family of positive periods. Their least common multiple is positive and makes every blocked transfer map primitive. □

Theorem D.4.3 (Quasi-idempotence under physical blocking). Lean Stmt *Let A be an MPS tensor and let $c \in \mathbb{C}$. If*

$$\mathcal{E}_A^2 = c \mathcal{E}_A,$$

then, for every integer $L \geq 0$,

$$\mathcal{E}_{A^{[L]}}^2 = c^L \mathcal{E}_{A^{[L]}}.$$

Proof. Proof The transfer map of $A^{[L]}$ is $\mathcal{E}_A^L$. Therefore

$$(\mathcal{E}_A^L)^2 = (\mathcal{E}_A^2)^L = (c \mathcal{E}_A)^L = c^L \mathcal{E}_A^L,$$

which is the required identity. □

Lemma D.4.4 (Adjoint primitivity equivalence). Lean Lean Stmt *Let E be a linear endomorphism of a finite-dimensional complex inner product space. Then E is primitive if and only if its adjoint $E^\dagger$ is primitive. Equivalently, the eigenvalues λ of E with $|\lambda| = 1$ form the singleton $\{1\}$ if and only if the same holds for $E^\dagger$, because adjunction sends every eigenvalue to its complex conjugate.*

Lemma D.4.5 (Primitivity under blocking by a multiple). [Lean](#) [✓ Stmt](#) *Let A have positive bond dimension and let $p_0, p \geq 1$ with $p_0 \mid p$. If $\mathcal{E}_{A^{[p_0]}}$ is primitive, then $\mathcal{E}_{A^{[p]}}$ is primitive.*

Theorem D.4.6 (Left-canonical normalization under blocking). [Lean](#) [✓ Stmt](#) *If a finite matrix family satisfies*

$$\sum_i (A^i)^\dagger A^i = \mathbb{1},$$

then every blocked family satisfies the same identity.

Proof. [✓ Proof](#) Expand the blocked matrices as products indexed by words and sum one letter at a time. $\square$

Theorem D.4.7 (Irreducibility from a faithful fixed point). [Lean](#) [✓ Stmt](#)

Let E be a channel with a positive-definite fixed point. If all nonzero positive semidefinite fixed points of E are proportional, then E is irreducible.

Proof. [✓ Proof](#) A nontrivial invariant projection would produce a positive semidefinite fixed point not proportional to the faithful one. $\square$

Lemma D.4.8 (Fixed-point criteria for irreducibility after primitive blocking). [Lean](#) [Lean](#)

[Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) *Let A be left-canonical, tensor-irreducible, and have primitive transfer map. Then A has primitive fixed-point data (ρ, P_ρ, N) with $\rho > 0$ and $N^n \rightarrow 0$. For every blocking length P , the tensor $A^{[P]}$ is left-canonical, its transfer map is $\mathcal{E}_A^P$, and ρ is fixed by this blocked map. For every $n \geq 1$,*

$$\mathcal{E}_A^n = P_\rho + N^n. \tag{D.7}$$

Finally, a channel with a positive-definite fixed point whose positive semidefinite fixed points are all proportional is irreducible as a map; irreducibility of a transfer map implies tensor irreducibility.

Proof. [✓ Proof](#) Peripheral primitivity, tensor irreducibility, and left-canonical normalization first give primitive fixed-point data, and tensor irreducibility makes its positive semidefinite fixed point positive definite. Blocking preserves the left-canonical identity, replaces the transfer map by its positive power, and therefore preserves the fixed point. The defining complementary-power identity gives (D.7). The channel fixed-point criterion then gives map irreducibility from uniqueness of positive semidefinite fixed points, and the final implication converts map irreducibility to tensor irreducibility. $\square$

Theorem D.4.9 (Blocking preserves irreducibility for primitive irreducible blocks). [Lean](#) [✓ Stmt](#)

Let A be left-canonical, irreducible, primitive, of positive bond dimension. For every $P > 0$, the blocked tensor $A^{[P]}$ is irreducible: there is no orthogonal projection Q with $Q \neq 0$, $Q \neq \mathbb{1}$ such that $(\mathbb{1} - Q)(A^{[P]})^\alpha Q = 0$ for every P -blocked index α .

Proof. [✓ Proof](#) Lemma D.4.8 gives primitive fixed-point data with $\rho > 0$. In particular, if $P_\rho(X) = \text{tr}(X)\rho / \text{tr}(\rho)$ and $N := \mathcal{E}_A - P_\rho$, then

$$\begin{aligned} \mathcal{E}_A &= P_\rho + N, \\ N^n &\rightarrow 0. \end{aligned}$$

The primitive fixed-point theorem supplies this complementary-power convergence.

The blocked transfer map is the corresponding positive power:

$$\mathcal{E}_{A^{[P]}}(X) = \mathcal{E}_A^P(X). \quad (\text{D.8})$$

Hence ρ remains a positive-definite fixed point after blocking. Let $\sigma \geq 0$ be any fixed point of $\mathcal{E}_A^P$, set

$$c = \frac{\text{tr}(\sigma)}{\text{tr}(\rho)},$$

$$\sigma' = \sigma - c\rho.$$

Then $\text{tr}(\sigma') = 0$, so $P_\rho(\sigma') = 0$, while $\mathcal{E}_A^P(\sigma') = \sigma'$. Since the fixed-point projection and the complement annihilate one another, iteration gives, for every $k \geq 1$,

$$\sigma' = \mathcal{E}_A^{Pk}(\sigma') = N^{Pk}(\sigma').$$

But $N^{Pk}(\sigma') \rightarrow 0$ as $k \rightarrow \infty$, hence $\sigma' = 0$. Thus every positive semidefinite fixed point of the blocked map is proportional to ρ .

Blocking preserves the left-canonical identity, so Theorem 4.2.6 makes the blocked transfer map a channel. Its positive-definite fixed point and the uniqueness just proved imply map irreducibility by Lemma D.4.8. The final implication in that lemma then shows that $A^{[P]}$ is tensor-irreducible. $\square$

Theorem D.4.10 (Extra blocking of primitive irreducible sector blocks). [Lean](#) [✓ Stmt](#) *Let A have positive bond dimension and be trace-preserving, primitive, and tensor-irreducible. For every $k > 0$,*

$$\sum_{\alpha} ((A^{[k]})^{\alpha})^{\dagger} (A^{[k]})^{\alpha} = \mathbb{1},$$

$$\sigma_{\partial}(\mathcal{E}_{A^{[k]}}) = \{1\}, \quad (\text{D.9})$$

and $A^{[k]}$ is tensor-irreducible. This k is a later common-refinement or injectivity length, distinct from the period-removal length that produced the sector block.

Proof. [✓ Proof](#) Trace preservation follows from blocking a trace-preserving tensor, primitivity from the transfer-power identity (D.8), and irreducibility from Theorem D.4.9. $\square$

Lemma D.4.11 (Blocking the weighted block sum). [Lean](#) [✓ Stmt](#) *For nonzero weights $(\mu_k)_{k=1}^r$, blocks $(B_k)_{k=1}^r$, and any $p \geq 1$, the blocking identity (D.6) holds, where $\sim$ denotes generation of the same MPV family at every length.*

Lemma D.4.12 (Positive-length equality survives blocking). [Lean](#) [✓ Stmt](#) *If two tensors have the same MPV coefficients at every positive length, then so do their blocked tensors after any positive blocking length.*

Theorem D.4.13 (Eventual nonzero proportionality survives blocking). [Lean](#) [✓ Stmt](#) *If two tensors have eventually nonzero proportional MPV coefficients, then so do their blocked tensors after any positive blocking length.*

Proof. [✓ Proof](#) If proportionality holds from length N_0 onward, a blocked chain of length $N \geq N_0$ expands to an original chain of length $Np \geq N_0$. Use the original proportionality scalar at length Np . $\square$

Lemma D.4.14 (Positive-length blocking of a weighted block sum). [Lean](#) [✓ Stmt](#) If A and $\bigoplus_k \mu_k B_k$ have the same MPV coefficients at every positive length, then after any positive blocking length p , $A^{[p]}$ and $\bigoplus_k \mu_k^p B_k^{[p]}$ have the same MPV coefficients at every positive length.

Lemma D.4.15 (Positive-length equality of powered block sums). [Lean](#) [✓ Stmt](#) If $\bigoplus_j \mu_j^A A_j$ and $\bigoplus_k \mu_k^B B_k$ have the same MPV coefficients at every positive length, then after any positive common blocking length p , $\bigoplus_j (\mu_j^A)^p A_j^{[p]}$ and $\bigoplus_k (\mu_k^B)^p B_k^{[p]}$ have the same MPV coefficients at every positive length.

Lemma D.4.16 (Nonzero weights survive blocking). [Lean](#) [✓ Stmt](#) If $\mu \in \mathbb{C}$ is nonzero and $p \geq 1$, then $\mu^p \neq 0$; blocking preserves the nonzero-weight hypothesis of a weighted block decomposition.

Lemma D.4.17 (Iterated blocking at the transfer-map level). [Lean](#) [✓ Stmt](#) For any MPS tensor A and $m, n \geq 0$,

$$\mathcal{E}_{(A^{[m]})^{[n]}} = \mathcal{E}_{A^{[mn]}}. \quad (\text{D.10})$$

D.5 Support compression and cyclic-sector families

Support isometries first realize the cyclic corners. The remaining results place finitely many cyclic-sector families over one blocked alphabet and identify their weighted nonzero parts.

Theorem D.5.1 (Supported corner compression with a support isometry). [Lean](#) [✓ Stmt](#)

Let A be a tensor normalized on the support of an orthogonal projection P , so that $PA^iP = A^i$ and $\sum_i (A^i)^\dagger A^i = P$. Then there are a corner tensor C , a linear isomorphism $\varphi : M_{\dim C}(\mathbb{C}) \xrightarrow{\sim} PM_D(\mathbb{C})P$, and a rectangular isometry $V : \mathbb{C}^{\dim C} \rightarrow \mathbb{C}^D$ such that

$$\begin{aligned} V^\dagger V &= \mathbb{1}, \\ VV^\dagger &= P, \\ \varphi(X) &= V XV^\dagger. \end{aligned} \quad (\text{D.11})$$

Moreover, $\varphi(C^i) = A^i$ for every physical letter, and for every word $\sigma = (\sigma_1, \dots, \sigma_N)$,

$$V^{(N)}(C)_\sigma = \text{tr}(PA^{\sigma_1} \dots A^{\sigma_N}). \quad (\text{D.12})$$

Proof. [✓ Proof](#)

Apply Theorem [LTC26](#), “Compression of a supported finite Kraus family” to the family $(A^i)_i$. It gives C , φ , and V satisfying [\(D.11\)](#) and $\varphi(C^i) = A^i$. It remains to identify the coefficients of the associated words. For each word $\sigma = (\sigma_1, \dots, \sigma_N)$, cyclicity of the trace and $VV^\dagger = P$ give

$$V^{(N)}(C)_\sigma = \text{tr}(C^{\sigma_1} \dots C^{\sigma_N}) = \text{tr}(V^\dagger A^{\sigma_1} \dots A^{\sigma_N} V) = \text{tr}(PA^{\sigma_1} \dots A^{\sigma_N}),$$

which proves [\(D.12\)](#). □

Theorem D.5.2 (Commuting projections give sector isometries). [Lean](#) [✓ Stmt](#) Let a left-canonical tensor A commute with a family of orthogonal projections (P_k) summing to $\mathbb{1}$. Then

A decomposes into compressed sector tensors (C_k) , with a unit-weight direct-sum MPV decomposition, corner isomorphisms $\varphi_k : M_{\dim C_k}(\mathbb{C}) \xrightarrow{\sim} P_k M_D(\mathbb{C}) P_k$, and rectangular isometries V_k satisfying

$$\begin{aligned} V_k^\dagger V_k &= \mathbb{1}, \\ V_k V_k^\dagger &= P_k, \\ \varphi_k(X) &= V_k X V_k^\dagger. \end{aligned} \tag{D.13}$$

The sector letters obey $\varphi_k(C_k^i) = P_k A^i P_k$.

Proof. ✓ Proof Apply Theorem D.5.1 to the compression of A on each sector P_k . Orthogonality and $\sum_k P_k = \mathbb{1}$ give the direct-sum decomposition. For every word $\sigma = (\sigma_1, \dots, \sigma_N)$, the partition of unity and (D.12) give

$$V^{(N)}(A)_\sigma = \text{tr} \left(\left(\sum_k P_k \right) A^{\sigma_1} \dots A^{\sigma_N} \right) = \sum_k \text{tr}(P_k A^{\sigma_1} \dots A^{\sigma_N}) = \sum_k V^{(N)}(C_k)_\sigma.$$

This is the unit-weight matrix-product-vector identity. □

Theorem D.5.3 (Adjoint-fixed projections give sector isometries). Lean ✓ Stmt Let A be left-canonical and let (P_k) be orthogonal projections summing to $\mathbb{1}$ and fixed by the adjoint transfer map. Then the same sector decomposition and support isometries of Theorem D.5.2 exist. In particular, the adjoint fixed-point condition $\mathcal{E}_A^\dagger(P_k) = P_k$ gives $P_k A^i = A^i P_k$, which gives the corner-letter identity $\varphi_k(C_k^i) = P_k A^i P_k$.

Proof. ✓ Proof The fixed-point identity $\mathcal{E}_A^\dagger(P_k) = P_k$, together with left-canonicity $\sum_i (A^i)^\dagger A^i = \mathbb{1}$, places P_k in the multiplicative domain of $\mathcal{E}_A^\dagger$, forcing $P_k A^i = A^i P_k$ for every letter A^i . The result then follows from Theorem D.5.2. □

Theorem D.5.4 (Cyclic sectors after removing zero blocks). Lean ✓ Stmt If A and B generate the same MPV family at every positive length, then after removing the all-zero blocks and applying the trace-preserving gauge there are weighted nonzero-block tensors $\bigoplus_j \mu_j^A A_j$ and $\bigoplus_k \mu_k^B B_k$ such that, for every $N \geq 1$ and σ ,

$$V^{(N)}(A)_\sigma = V^{(N)} \left(\bigoplus_j \mu_j^A A_j \right)_\sigma, \tag{D.14}$$

$$V^{(N)}(B)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k^B B_k \right)_\sigma. \tag{D.15}$$

The two nonzero parts agree:

$$V^{(N)} \left(\bigoplus_j \mu_j^A A_j \right)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k^B B_k \right)_\sigma. \tag{D.16}$$

Each nonzero block has primitive irreducible cyclic sectors: for each j , there is $m_j \geq 1$ with a unit-weight sector decomposition $A_j^{[m_j]} \sim \bigoplus_{u=1}^{m_j} C_{j,u}^A$, and similarly for B_k .

Proof. [✓ Proof](#) Apply Theorem 8.2.1.1 separately to A and B . Its positive-length conclusion gives (D.14) and (D.15). Chaining these identities through $\mathcal{V}^+(A) = \mathcal{V}^+(B)$ gives (D.16). Finally apply Theorem 8.3.1.3 to every trace-preserving irreducible nonzero-weight block. $\square$

Lemma D.5.5 (Common reblocking of cyclic-sector families). [Lean](#) [✓ Stmt](#) *Given a finite family $(B_k)_{k=1}^r$, each with primitive irreducible cyclic sectors, there is a common blocking length p such that each $B_k^{[p]}$ admits a unit-weight direct-sum decomposition over a single common blocked alphabet $d^{[p]}$ into left-canonical blocks with primitive transfer maps, tensor irreducibility, and positive bond dimensions.*

Lemma D.5.6 (Common reblocking at a prescribed multiple). [Lean](#) [✓ Stmt](#) *With hypotheses as in Lemma D.5.5, and given any common multiple p of the per-block period-removal lengths (m_k) , the common reblocking can be performed at the prescribed length p .*

Lemma D.5.7 (Structural properties of the common reblocked sectors). [Lean](#) [✓ Stmt](#) *Each sector tensor in the common reblocked cyclic-sector family of Lemma D.5.5 is trace-preserving, $\sum_i (B^i)^\dagger B^i = \mathbb{1}$, has primitive transfer map, $\sigma_\partial(\mathcal{E}_B) = \{1\}$, is tensor-irreducible, and has positive bond dimension.*

Lemma D.5.8 (Common-alphabet flattening). [Lean](#) [✓ Stmt](#) *With $p = m_k e_k$ for each k , the iterated blocked tensor $(B_k^{[m_k]})^{[e_k]}$ agrees with $B_k^{[p]}$ in MPV family, after the canonical identification of the p -blocked alphabet $d^{[p]}$ with the e_k -fold tensor power of the m_k -blocked alphabet $d^{[m_k]}$.*

Lemma D.5.9 (Direct blocking in the common alphabet). [Lean](#) [✓ Stmt](#) *For every original block B_k , direct blocking at the common length $p = m_k e_k$ agrees, as an MPV family, with the same block transported to the common blocked alphabet. Write $\widetilde{B}_k$ for this common-alphabet image. Then*

$$V^{(N)}(B_k^{[p]})_\sigma = V^{(N)}(\widetilde{B}_k)_\sigma.$$

Lemma D.5.10 (Reindexed nonzero-part identity). [Lean](#) [✓ Stmt](#) *The powered nonzero-part decomposition $\bigoplus_k \mu_k^p B_k^{[p]}$ carries over under the common-alphabet identification of Lemma D.5.8: the common-alphabet sectors have the same nonzero-part decomposition as the direct p -blocked tensors.*

D.6 Prepared normal-form consequences

The first theorem derives the overlap condition from a prepared primitive family. The second does not construct primitive blocks from an arbitrary tensor: it only orders a family that already satisfies all primitive, irreducible, trace-preserving, nonzero-weight, and positive-dimension hypotheses.

Theorem D.6.1 (Self-overlap is derived in normal canonical form). [Lean](#) [✓ Stmt](#) *If (μ_k, A_k) satisfies the normal canonical form conditions, then $O_{A_k A_k}(N) \rightarrow 1$ for every block k .*

Proof. [✓ Proof](#) Fix a block k . Irreducibility and the TP normalization give a nonzero positive semidefinite fixed point ρ of $\mathcal{E}_{A_k}$ with fixed-point projector P ; primitivity gives the complementary gap $\rho_{\text{spec}}(\mathcal{E}_{A_k} - P) < 1$ (the corresponding theorem in the companion quantum-channel volume [LTC26, “Complementary transfer-map gap for primitive maps”]). Theorem 6.10.2 then yields $O_{A_k A_k}(N) \rightarrow 1$. $\square$

Theorem D.6.2 (Reordering prepared primitive blocks into normal canonical form). [Lean](#)
 $\checkmark$ *Stmt* Suppose A generates the same MPV family as $\bigoplus_k \mu_k A_k$, and that each A_k is irreducible, satisfies $\sum_i (A_k^i)^\dagger A_k^i = \mathbb{1}$, and has primitive transfer map, all $\mu_k \neq 0$, and all bond dimensions are positive. Then there exist a blocking length $p > 0$, weights (ν_k) , and blocked tensors (B_k) such that $A^{[p]}$ generates the same MPV family as $\bigoplus_k \nu_k B_k$, and (ν_k, B_k) satisfies the normal canonical form conditions. The weight moduli need not be distinct; equal-modulus blocks are allowed.

Proof. $\checkmark$ *Proof* The given blocks are already primitive, so take $p = 1$. Reorder the blocks by non-increasing weight modulus, with ties allowed, and set $\nu_k := \mu_k$ and $B_k := A_k$ in that order. Irreducibility, TP normalization, primitivity, nonzero weights, and positive bond dimensions are preserved, so the reordered family satisfies Definition 8.9.2 and generates the same MPV family as A . $\square$

Remark D.6.3 (Scope of Theorem D.6.2). Theorem D.6.2 assumes all six block hypotheses in advance and produces the normal canonical form by reordering. It is not the arbitrary-input existence theorem of [CPGSV16, Section 2.3]; that primitive-block reduction, including its structural bond-dimension bound, is Theorem 8.11.1.2 in Section 8.11.1.

Appendix E

Basis of Normal Tensors: Supporting Results

This appendix supports Chapter 9. It proves the elementary normal-tensor consequences, Gram-matrix criteria, finite-dimensional change-of-basis facts, sector-blocking identities, direct-sum separation lemmas, and prepared-family bookkeeping used by the main BNT characterization, matching, and block-injectivity results.

E.1 Elementary normal-tensor and Gram-matrix support

Throughout this section, $V^{(N)}(A)$ denotes the length- N matrix product vector of a tensor A , and $O_{AB}(N)$ denotes the corresponding mixed overlap. We first record elementary consequences of the two notions of normality used in Chapter 9, and then prove the finite-family linear-independence criteria used in the BNT characterization and matching arguments.

Lemma E.1.1 (Positive bond dimensions in a coefficient-form BNT). [Lean](#) [✓ Stmt](#) *Every block in a coefficient-form basis of normal tensors has positive bond dimension.*

Proof. [✓ Proof](#) Eventual linear independence makes the matrix product vector of each block nonzero at some positive length. A tensor with zero bond dimension has zero matrix product vector at every length, which is a contradiction. $\square$

Lemma E.1.2 (Bond dimension one implies irreducibility). [Lean](#) [✓ Stmt](#) *Every tensor with a one-dimensional bond space is irreducible.*

Proof. [✓ Proof](#) The only orthogonal projections on a one-dimensional complex vector space are zero and the identity. $\square$

Lemma E.1.3 (Normality in bond dimension one). [Lean](#) [✓ Stmt](#) *A tensor with one-dimensional bond space and identity transfer map is a normal tensor.*

Proof. [✓ Proof](#) Irreducibility follows from the preceding lemma. The identity transfer map has spectral radius one, and its only eigenvalue is one. $\square$

Lemma E.1.4 (Perron gauges preserve primitivity). [Lean](#) [✓ Stmt](#) *Let σ be positive definite. The transfer map of the Perron gauge determined by σ is primitive if and only if the original transfer map is primitive.*

Proof. ✓ **Proof** The two transfer maps are similar through the invertible congruence $X \mapsto \sigma^{-1/2} X \sigma^{-1/2}$. Similar linear maps have the same spectrum, so the condition that 1 is the only eigenvalue on the unit circle is unchanged. $\square$

Lemma E.1.5 (Preservation of an algebraic BNT under positive-length MPV equality). Lean ✓ **Stmt** *Let A and A' have the same matrix product vectors at every positive length. If $(A_j)_{j=1}^g$ is a basis of normal tensors for A , then the same family is a basis of normal tensors for A' .*

Proof. ✓ **Proof** Substitute the positive-length MPV equality into the spanning identity for A . Normality and eventual linear independence depend only on the family $(A_j)_{j=1}^g$. $\square$

Corollary E.1.6 (Positive-length geometric proportionality phase). Lean ✓ **Stmt** *Let A and B be normal tensors of positive, possibly different, bond dimensions. Suppose that, for every positive N , their matrix product vectors are proportional by a nonzero scalar. Then there is $a \in \mathbb{C}$, with $|a| = 1$, such that $|V^{(N)}(A)\rangle = a^N |V^{(N)}(B)\rangle$ for every positive N . This is the positive-length one-normal-block consequence of [CPGSV16, Corollary A.3 and Theorem 2.10].*

Proof. ✓ **Proof** Positive-length nonzero proportionality implies eventual nonzero proportionality. Apply Theorem 9.2.12. $\square$

Lemma E.1.7 (Eventual linear independence from finite-index overlap orthonormality). Lean ✓ **Stmt** *Let I be a finite index set and let $(A_i)_{i \in I}$ be a finite family of tensors such that $O_{A_i A_i}(N) \rightarrow 1$ for each i , and $O_{A_i A_j}(N) \rightarrow 0$ whenever $i \neq j$. Then the MPV states $\{|V^{(N)}(A_i)\rangle\}_{i \in I}$ are linearly independent for all sufficiently large N .*

Proof. ✓ **Proof** The Gram matrix of the family $\{|V^{(N)}(A_i)\rangle\}_{i \in I}$ converges entrywise to the identity matrix. Hence for all sufficiently large N it is invertible, so the MPV states are linearly independent. $\square$

The same conclusion is often more convenient with an explicit threshold rather than the eventual quantifier.

Theorem E.1.8 (Existential BNT independence from overlap limits). Lean ✓ **Stmt** *If the self-overlaps of a finite block family tend to 1 and the cross-overlaps of distinct blocks tend to 0, then the MPV states are linearly independent for all sufficiently large system sizes.*

Proof. ✓ **Proof** This is Lemma E.1.10, restated with an explicit threshold N_0 . $\square$

Lemma E.1.9 (Eventual linear independence for two families). Lean ✓ **Stmt** *Let $(A_j)_{j=1}^{g_A}$ and $(B_k)_{k=1}^{g_B}$ be finite families of tensors. Suppose the overlaps inside each family are asymptotically orthonormal, and suppose every mixed overlap $O_{A_j B_k}(N)$ tends to 0. Then the combined MPV family*

$$\{|V^{(N)}(A_j)\rangle\}_{j=1}^{g_A} \cup \{|V^{(N)}(B_k)\rangle\}_{k=1}^{g_B}$$

is linearly independent for all sufficiently large N .

Proof. ✓ **Proof** Apply the finite-index Gram-matrix criterion to the disjoint union of the two index sets. The within-family hypotheses give the diagonal and off-diagonal limits inside each block, and the mixed overlap hypothesis gives the off-diagonal limits between the two blocks. $\square$

The following two linear-algebra consequences package the eventual Gram-matrix criterion and its coefficient-extraction corollary.

Lemma E.1.10 (Eventual linear independence from overlap orthonormality). [Lean](#) [Lean](#) [✓ Stmt](#) Let $(A_j)_{j=1}^g$ be a finite family of tensors such that $O_{A_j A_j}(N) \rightarrow 1$ for each j and $O_{A_i A_j}(N) \rightarrow 0$ for $i \neq j$. Then the MPV states $\{|V^{(N)}(A_j)\rangle\}_{j=1}^g$ are linearly independent for all sufficiently large N .

Proof. [✓ Proof](#) Apply Lemma E.1.7 with $I = \{1, \dots, g\}$. □

Eventual linear independence is also what converts an equality of two finite linear combinations into equality of their coefficients; this is the mechanism behind the coefficient identities of Chapter 10.

Lemma E.1.11 (Eventual coefficient extraction from linear independence). [Lean](#) [✓ Stmt](#) If a finite family of vectors is linearly independent for all sufficiently large N , and two finite linear combinations of that family are equal for all sufficiently large N , then the corresponding coefficients are equal for all sufficiently large N .

Proof. [✓ Proof](#) Move the two linear combinations to one side. Linear independence forces each coefficient difference to vanish at every sufficiently large length. □

When two BNT families must be compared against each other, the same Gram-matrix criterion applies to their disjoint union, provided the mixed overlaps also decay. This is the form used in the exact sector matching of Section 9.8.

E.2 Change of basis, power sums, and sector bookkeeping

This section collects the finite-dimensional linear algebra behind alternative BNT comparisons, the extrapolation identities for finite sums of geometric sequences, and the bookkeeping identities for sector decompositions. A sector has copy weights $\mu_{j,q} \neq 0$, coefficient $c_{N,j} = \sum_q \mu_{j,q}^N$, representative bond dimension D_j , and total bond dimension $D_{\text{tot}} = \sum_j r_j D_j$.

Lemma E.2.1 (Invertible change of basis). [Lean](#) [✓ Stmt](#) Let $v_1, \dots, v_g$ and $w_1, \dots, w_g$ be two linearly independent families in a complex vector space with the same span. Then there exists an invertible matrix $U \in \text{GL}_g(\mathbb{C})$ such that $w_j = \sum_{i=1}^g U_{ij} v_i$ for every j .

Proof. [✓ Proof](#) Because the two families span the same subspace, each w_j has a unique expansion in the basis $\{v_i\}_{i=1}^g$; these coefficients form the columns of U . If $Ua = 0$, then the corresponding linear relation among the w_j is trivial because the w_j are linearly independent. Thus U is injective, hence invertible. □

Lemma E.2.2 (Eventual change of basis from equal spans). [Lean](#) [✓ Stmt](#) If two finite families of blocks have asymptotically orthonormal overlaps within each family, and if their MPV spans agree for all sufficiently large system sizes N , then for all sufficiently large N there exists an invertible matrix U_N expressing the MPV states of one family as linear combinations of the MPV states of the other.

Proof. [✓ Proof](#) By Lemma E.1.10, both families are linearly independent for all sufficiently large N . Hence, for all sufficiently large N , both families are linearly independent and the assumed span equality holds. Lemma E.2.1 applies at each such N . □

Lemma E.2.3 (Eventual vanishing of a finite geometric sum). [Lean](#) [✓ Stmt](#) Let $w_1, \dots, w_n$ be nonzero complex numbers. If $\sum_i c_i w_i^N = 0$ for every sufficiently large N , then the same equality holds for every non-negative exponent.

Proof. [✓ Proof](#) Induct on the number of terms. Subtracting w_1 times the equality at exponent N from the equality at exponent $N+1$ gives $\sum_{i=2}^n c_i(w_i - w_1)w_i^N = 0$. The induction hypothesis yields this equality at every exponent, so the original sum $S_N = \sum_i c_i w_i^N$ satisfies $S_{N+1} = w_1 S_N$ for every N . Since S_N eventually vanishes and $w_1 \neq 0$, it follows that $S_0 = 0$ and hence $S_N = 0$ for every N . $\square$

Lemma E.2.4 (Eventual power-sum equality holds at every exponent). [Lean](#) [✓ Stmt](#) Let $a_1, \dots, a_m$ and $b_1, \dots, b_n$ be nonzero complex numbers. If their power sums agree for every sufficiently large exponent, then $\sum_i a_i^k = \sum_j b_j^k$ for every non-negative exponent k .

Proof. [✓ Proof](#) Set $w_l = a_l$ for $l \leq m$ and $w_{m+l} = b_l$ for $l \leq n$, with coefficients $c_l = 1$ for $l \leq m$ and $c_l = -1$ for $l > m$. Then $\sum_{l=1}^{m+n} c_l w_l^k = \sum_i a_i^k - \sum_j b_j^k$. The hypothesis makes this expression zero for every sufficiently large k . Lemma E.2.3 makes it zero for every k , which is the required power-sum equality. $\square$

Theorem E.2.5 (Equal power sums imply equal multisets). [Lean](#) [✓ Stmt](#) Let $\alpha, \beta : \{0, \dots, n-1\} \rightarrow \mathbb{C}$. If $\sum_i \alpha_i^k = \sum_i \beta_i^k$ for all $k \geq 1$, then the multisets $\{\alpha_i\}$ and $\{\beta_i\}$ are equal.

Proof. [✓ Proof](#) Construct diagonal matrices $\text{diag}(\alpha)$ and $\text{diag}(\beta)$. The power-sum hypothesis gives $\text{tr}(\text{diag}(\alpha)^k) = \text{tr}(\text{diag}(\beta)^k)$. By the corresponding theorem in the companion quantum-channel volume [LTC26, “Newton–Girard trace recursion”], the characteristic polynomials agree: $\prod_i (X - \alpha_i) = \prod_i (X - \beta_i)$. Extracting roots gives multiset equality. $\square$

Theorem E.2.6 (Representative dimensions are bounded by the total bond dimension). [Lean](#) [Lean](#) [✓ Stmt](#) Let the representative A_j have bond dimension D_j , and let D be the bond dimension of the flattened sector tensor. Then $D_j \leq D$. If every D_j is positive and there are g representatives, then also $g \leq D$.

Proof. [✓ Proof](#) Since $r_j > 0$, choose one copy of A_j . Its contribution D_j is one non-negative summand in the total dimension of the flattened copy family, so $D_j \leq D$.

For the second assertion, every representative has multiplicity $r_j \geq 1$, and each copy has positive dimension $D_j > 0$. Therefore

$$g = \sum_j 1 \leq \sum_j r_j \leq \sum_j r_j D_j = D.$$

$\square$

Theorem E.2.7 (Blocked sector coefficients and assembled tensor). [Lean](#) [Lean](#) [✓ Stmt](#) The coefficient of the blocked decomposition satisfies

$$c_{P^{[p]},j}^{(N)} = \sum_q (\mu_{j,q}^p)^N = c_{P,j}^{(Np)}. \quad (\text{E.1})$$

Moreover, blocking the tensor assembled from P gives the same MPV family as the tensor assembled from the blocked sector decomposition.

Proof. [✓ Proof](#) The coefficient identity in (E.1) is $(\mu^p)^N = \mu^{pN}$. The tensor identity is Theorem D.4.11 applied to the flattened copy family. $\square$

Lemma E.2.8 (Existence of an MPV phase class). [Lean](#) [✓ Stmt](#) A nonempty finite family of blocks has at least one MPV phase class.

Proof. ✓ Proof Choose one original block index. Its occurrence in the phase-class enumeration supplies a phase class containing it. $\square$

Lemma E.2.9 (Regrouping a matrix sum by MPV phase classes). Lean ✓ Stmt *Let F_k be a finite family of square matrices indexed by the original blocks. If $k = (j, q)$ enumerates the copies in the MPV phase classes, then $\sum_j \sum_q F_{j,q} = \sum_k F_k$.*

Proof. ✓ Proof Evaluate both sides at a pair of matrix indices and apply the bijective enumeration of the finite phase-class quotient. $\square$

Definition E.2.10 (Single-copy sector decomposition). Lean ✓ Stmt Given nonzero weights $(\mu_k)_{k=1}^r$ and blocks $(B_k)_{k=1}^r$, the *single-copy sector decomposition* has r sectors, one basis block B_k per sector, multiplicity 1, and sector weight μ_k . As a finite sum,

$$\sum_{k=1}^r \mu_k^N V^{(N)}(B_k)_\sigma = V^{(N)} \left(\bigoplus_k \mu_k B_k \right)_\sigma. \quad (\text{E.2})$$

Lemma E.2.11 (Single-copy sector decomposition preserves the represented family). Lean ✓ Stmt *The single-copy sector decomposition of $(\mu_k, B_k)_k$ represents the same MPS family as $\bigoplus_k \mu_k B_k$.*

Proof. ✓ Proof Expanding the sector decomposition gives the left-hand side of (E.2), which is the finite block-sum expression for $\bigoplus_k \mu_k B_k$. $\square$

Lemma E.2.12 (Sector-decomposition coefficient is not eventually zero). Lean Lean ✓ Stmt *For any sector decomposition P and any sector j , the coefficient $c_{P,j}^{(N)} = \sum_q \mu_{j,q}^N$ is not eventually zero in N . More generally, for every positive integer p , the subsequence $c_{P,j}^{(pN)}$ is not eventually zero.*

Proof. ✓ Proof If the coefficient vanished for every $N \geq M$, the finite geometric-sum extrapolation would make it vanish at every non-negative exponent. However, at exponent zero, $c_{P,j}^{(0)} = \sum_q \mu_{j,q}^0 = r_j > 0$, which is impossible. For the subsequence, apply the same argument to the nonzero bases $\mu_{j,q}^p$. $\square$

E.2.1 Proportional matching and finite bijection extraction

The equal-MPV matching theorems in Chapter 9 are specializations of an eventual proportionality argument. We record that general mechanism and the finite-set construction used to package matches in both directions.

Theorem E.2.1.1 (Exact sector matching under eventual proportionality). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total MPV families are eventually nonzero and proportional. For every sector j of P , there is a sector k of Q with equal bond dimension, gauge-phase-equivalent basis tensor after the dimension identification, and non-decaying cross-overlap.*

Proof. ✓ Proof Suppose the chosen sector decays against every sector of Q , and separate all such P -sectors from the remaining ones. For each remaining sector, choose a Q -sector with non-decaying overlap. Both blocks are normal by the canonical-form hypotheses. Theorem 9.2.9 rules out its decay alternative and supplies equal bond dimensions and a gauge-phase relation with a unit phase. Gauge-phase covariance therefore expresses that P -sector MPV as a geometric scalar multiple of the chosen Q -sector MPV.

Substitute these relations into the eventual proportionality identity. The combined-family Gram criterion makes the decaying residue together with the Q family eventually linearly independent. Exact coefficient extraction then makes the chosen sector coefficient eventually zero, contradicting its power-sum nonvanishing. Thus a non-decaying partner exists. A second application of Theorem 9.2.9 rules out decay and gives the required dimension equality and gauge-phase equivalence; no modulus-one overlap limit is assumed. $\square$

Theorem E.2.1.2 (Full-basis matching under eventual proportionality). Lean Stmt *Under the same hypotheses, every sector of Q has an equal-dimensional, gauge-phase-equivalent partner in P with non-decaying cross-overlap.*

Proof. Proof Reverse the eventual proportionality and apply Theorem E.2.1.1 to each sector of Q ; symmetry of the overlap converts the resulting witness to the stated orientation. $\square$

Lemma E.2.1.3 (Bijection from matches in both directions). Lean Stmt *Suppose every sector of Q has an equal-dimensional, gauge-phase-equivalent partner in P with non-decaying overlap, and the converse holds from P to Q . Then the forward partners may be chosen as a bijection between the two finite sector sets, preserving all three properties.*

Proof. Proof Basis separation makes each partner map injective: two sectors with the same partner would be gauge-phase equivalent to one another. The injections in both directions force equal finite cardinalities, so the forward map is bijective. $\square$

E.3 Direct-sum separation and prepared-family bookkeeping

We now prove the supporting separation and regrouping results used by the source-facing block-injectivity and supplier theorems in Chapter 9. The direct-sum argument follows the ground-space uniqueness mechanism of [PGVWC07]: disjoint local MPS spaces give pair-trace selectors, products of those selectors isolate one BNT block, and an injective prefix supplies an arbitrary matrix on that block. The remaining results identify dimensions while phase-equivalent prepared blocks are grouped into sectors and normalize the resulting finite weight family.

Lemma E.3.1 (Three-block separation of distinct BNT representatives). Lean Lean Lean Stmt *Let $A^1, \dots, A^r$ be irreducible left-canonical representatives with normalized self-overlaps, and suppose that distinct same-dimensional representatives are not gauge-phase equivalent. Fix $L_0 > 0$ such that every block is injective at lengths $L_0, L_0 + 1$, and $3(L_0 + 1)$. Then each ordered pair $k \neq j$ is trace-separated at the common length $S = 3(L_0 + 1)$: the map*

$$(c_\omega)_{|\omega|=S} \mapsto \left(\sum_{|\omega|=S} c_\omega A_\omega^k, \sum_{|\omega|=S} c_\omega A_\omega^j \right)$$

is onto $M_{D_k}(\mathbb{C}) \oplus M_{D_j}(\mathbb{C})$. In particular, one may prescribe an arbitrary matrix on either block while making the same word combination vanish on the other block.

Proof. Proof Put $L = L_0 + 1$ and $S = 3L$. For a block A , write

$$\Gamma_n^A(X) = \sum_{|w|=n} \text{tr}(XA_w)|w\rangle, \quad \mathcal{G}_n(A) = \text{im } \Gamma_n^A.$$

Block injectivity at L and S makes both boundary maps injective; in particular,

$$\dim \mathcal{G}_L(A^q) = D_q^2 \quad (1 \leq q \leq r). \quad (\text{E.3})$$

Fix $k \neq j$ and first order the pair so that $D_j \leq D_k$. Suppose that a nonzero vector belongs to $\mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j)$. There are boundary matrices X and Y such that

$$0 \neq \psi = \Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(Y). \quad (\text{E.4})$$

Injectivity at S gives $X \neq 0$ and $Y \neq 0$. Split every length- S word as uvw with $|u| = |v| = |w| = L$. Equality E.4 gives

$$\text{tr}(XA_u^k A_v^k A_w^k) = \text{tr}(YA_u^j A_v^j A_w^j). \quad (\text{E.5})$$

Because the length- L words of A^k span $M_{D_k}(\mathbb{C})$ and $X \neq 0$, the two-sided ideal generated by X is all of $M_{D_k}(\mathbb{C})$:

$$\text{span}\{A_u^k X A_v^k : |u| = |v| = L\} = M_{D_k}(\mathbb{C}). \quad (\text{E.6})$$

Indeed, expand arbitrary left and right matrix units in length- L words; a nonzero entry of X then produces every matrix unit. Thus, for each $Z \in M_{D_k}(\mathbb{C})$, choose coefficients a_{uv} with $Z = \sum_{u,v} a_{uv} A_u^k X A_v^k$ and set $R(Z) = \sum_{u,v} a_{uv} A_u^j Y A_v^j$. Cyclicity of the trace and E.5, applied to the word vwu , yield

$$\begin{aligned} \text{tr}(Z A_w^k) &= \sum_{u,v} a_{uv} \text{tr}(X A_v^k A_w^k A_u^k) \\ &= \sum_{u,v} a_{uv} \text{tr}(Y A_v^j A_w^j A_u^j) = \text{tr}(R(Z) A_w^j). \end{aligned}$$

Hence $\Gamma_L^{A^k}(Z) = \Gamma_L^{A^j}(R(Z))$ for every Z , and therefore

$$\mathcal{G}_L(A^k) \subseteq \mathcal{G}_L(A^j). \quad (\text{E.7})$$

Equations E.3 and E.7 give

$$D_k^2 = \dim \mathcal{G}_L(A^k) \leq \dim \mathcal{G}_L(A^j) = D_j^2.$$

Together with $D_j \leq D_k$, this forces $D_j = D_k$ and equality in E.7. This proves the unequal-dimension branch immediately: if $D_j < D_k$, a nonzero intersection is impossible. Reversing the initial ordering treats $D_k < D_j$ in the same way.

It remains to exclude the intersection when $D_j = D_k$. Gauge-phase separation and Theorem 9.4.3 give, beyond any prescribed threshold, a chain length $N > L$ for which the two periodic MPVs are not proportional. On the other hand, equality $\mathcal{G}_L(A^k) = \mathcal{G}_L(A^j)$ gives identical translated length- L constraints on the N -site ring. The parent-Hamiltonian uniqueness mechanism identifies the corresponding chain ground spaces with the one-dimensional MPV spans:

$$\mathcal{G}_{N,L}(A^k) = \mathbb{C} V^{(N)}(A^k), \quad \mathcal{G}_{N,L}(A^j) = \mathbb{C} V^{(N)}(A^j).$$

Equality of the local spaces therefore forces these two MPV lines to be equal, contradicting the choice of N . Consequently, for every distinct pair, including the unequal-dimensional pairs,

$$\mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j) = \{0\}. \quad (\text{E.8})$$

Finally, let Φ_{kj} be the displayed length- S word-evaluation map from coefficient families to $M_{D_k}(\mathbb{C}) \oplus M_{D_j}(\mathbb{C})$. Pair the target by

$$\langle (X, Y), (M, N) \rangle = \text{tr}(XM) + \text{tr}(YN).$$

A pair (X, Y) annihilates $\text{im}(\Phi_{kj})$ precisely when

$$\text{tr}(XA_w^k) + \text{tr}(YA_w^j) = 0 \quad (|w| = S),$$

or equivalently $\Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(-Y)$. There is therefore a linear map

$$\Theta : \text{im}(\Phi_{kj})^\perp \longrightarrow \mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j), \quad \Theta(X, Y) = \Gamma_S^{A^k}(X) = \Gamma_S^{A^j}(-Y).$$

It is injective because both boundary maps are injective. Conversely, if ψ lies in the intersection, there are unique X and Z with $\Gamma_S^{A^k}(X) = \psi = \Gamma_S^{A^j}(Z)$; the pair $(X, -Z)$ annihilates $\text{im}(\Phi_{kj})$ and maps to ψ . This construction is the inverse of Θ , so the two spaces are isomorphic:

$$\text{im}(\Phi_{kj})^\perp \cong \mathcal{G}_S(A^k) \cap \mathcal{G}_S(A^j). \quad (\text{E.9})$$

The right-hand side vanishes by E.8, and hence so does the annihilator. The trace pairing is nondegenerate and all spaces are finite-dimensional, so $\text{im}(\Phi_{kj})^\perp = \{0\}$ implies that Φ_{kj} is surjective. Surjectivity first gives an arbitrary value on one block and zero on an anchor block; the selector construction then chooses $(1, 0)$ for the remaining pairs, and products of these homogeneous selectors give the simultaneous word-tuple span in Lemma 9.3.3. $\square$

Theorem E.3.2 (Common injective blocking for primitive irreducible families). [Lean](#) [✓ Stmt](#) *Let $(B_k)_{k=1}^r$ be a finite family of positive-bond-dimension MPS tensors, each trace-preserving, primitive, and tensor-irreducible. Then there is a single $L \geq 1$ such that each $B_k^{[L]}$ is one-site injective, and this blocking preserves trace preservation, primitivity, and irreducibility for every block.*

Proof. [✓ Proof](#) For each block, Theorem 7.9.1 gives a positive length at which it becomes one-site injective; the product of these lengths is a common length. Injectivity persists under positive further blocking, and Theorem D.4.10 preserves the other hypotheses. $\square$

Lemma E.3.3 (Total dimension after grouping by BNT representatives). [Lean](#) [✓ Stmt](#) *Suppose CF summands are grouped by BNT representative, every original copy weight is nonzero, and every summand assigned to a representative has that representative's bond dimension. Then the total bond dimension of the resulting BNT sector decomposition is the sum of the original block dimensions.*

Proof. [✓ Proof](#) The collapsed decomposition counts one representative dimension for each assigned copy. Let $e(j, q)$ denote the original CF summand assigned to the q th copy of representative P_j . Since $(j, q) \mapsto e(j, q)$ reindexes the original summands and each assigned summand has the representative's bond dimension,

$$\sum_{j=1}^g \sum_{q=1}^{r_j} \dim(P_j) = \sum_{j=1}^g \sum_{q=1}^{r_j} D_{e(j, q)} = \sum_{k=1}^r D_k.$$

This is the equality between the collapsed total dimension and the original direct-sum dimension. $\square$

Lemma E.3.4 (Prepared phase-equivalent NTs have the same bond dimension). [Lean](#) [✓ Stmt](#) *Let X and Y be trace-preserving, primitive, irreducible NTs of positive bond dimension. If their MPV families differ by a nonzero scalar power, then X and Y have the same bond dimension.*

Proof. [✓ Proof](#) Choose $\zeta \neq 0$ with $|V^{(N)}(Y)\rangle = \zeta^N |V^{(N)}(X)\rangle$ for every positive length N . The normalized self-overlaps give $|O_{XX}(N)| \rightarrow 1$, and the phase relation gives

$$\begin{aligned} O_{YX}(N) &= \zeta^N O_{XX}(N), \\ |O_{YX}(N)| &= |\zeta|^N |O_{XX}(N)|. \end{aligned}$$

Since the same normalization forces $|\zeta| = 1$, this gives $|O_{YX}(N)| \rightarrow 1$. If the two bond dimensions differed, the rectangular overlap-decay theorem would force $O_{YX}(N) \rightarrow 0$, contradicting the preceding limit. $\square$

Lemma E.3.5 (Prepared BNT representative classes preserve bond dimension). [Lean](#) [✓ Stmt](#) *In a prepared family of positive-bond-dimension trace-preserving primitive irreducible NTs, every CF summand assigned to a BNT representative has that representative's bond dimension.*

Proof. [✓ Proof](#) Apply Lemma E.3.4 to each representative and each assigned summand. $\square$

Lemma E.3.6 (Prepared collapsed BNT total dimension). [Lean](#) [✓ Stmt](#) *For prepared trace-preserving primitive irreducible NTs of positive bond dimension with all copy weights nonzero, grouping CF summands by BNT representative preserves the total bond dimension: the collapsed sector decomposition has total bond dimension equal to the sum of the original block dimensions.*

Proof. [✓ Proof](#) Every assigned summand shares the bond dimension of its representative, by the equal-dimension fact above. The displayed regrouping identity in Lemma E.3.3 then identifies the collapsed sum over pairs (j, q) with $\sum_{k=1}^r D_k$, the original direct-sum dimension. $\square$

Theorem E.3.7 (Projective weight normalization). [Lean](#) [✓ Stmt](#) *For nonzero weights $\mu_1, \dots, \mu_r$ with $r \geq 1$ there is a positive real m such that every normalized weight satisfies*

$$\begin{aligned} \left| \frac{\mu_k}{m} \right| &\leq 1, \\ \frac{\mu_k}{m} &\neq 0, \end{aligned}$$

and at least one normalized weight has unit modulus, $|\mu_{k_0}/m| = 1$. This is the normalization that [CPGSV16, line 246] chooses for the canonical-form weights.

Proof. [✓ Proof](#) Take $m = \max_k |\mu_k|$; the maximum over a nonempty finite family is attained, positive because every weight is nonzero, an upper bound for every modulus, and equal to the modulus of the attaining weight. $\square$

Lemma E.3.8 (Common scalar rescaling of block weights). [Lean](#) [✓ Stmt](#) *Multiplying every block weight in a block-diagonal MPS tensor by a scalar c multiplies every length- N MPV coefficient by c^N .*

Proof. [✓ Proof](#) Expand the block-diagonal MPV coefficient as a finite sum over blocks and factor c^N out of the sum. $\square$

Appendix F

Proof of the Fundamental Theorem: Supporting Results

This appendix supports Chapter 10. It proves phase normalization, coefficient extraction from matched gauges, direct-sum conjugation, the exact consequences of length-dependent proportionality, coordinate reindexing, and unitary refinements. The first part concerns coefficients and block gauges; the second identifies matched coordinates with the literal total bond space.

F.1 Matched phases and coefficient extraction

This section isolates two elementary steps used by the main proof. First, the sector partners obtained in both directions determine a finite bijection. Second, normalization of the matched normal tensors forces every gauge scalar to be a phase.

Theorem F.1.1 (Bijective matching under eventual proportionality). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total MPV families are eventually nonzero and proportional. There is a bijection $\beta : J(Q) \simeq J(P)$ such that every Q_k has an equal-dimensional, gauge-phase-equivalent partner $P_{\beta(k)}$ satisfying*

$$O_{P_{\beta(k)} Q_k}(N) \not\rightarrow 0.$$

Proof. ✓ Proof Apply the full-basis matching theorem first from Q to P and then to the reverse proportionality from P to Q . If P_j and $P_{j'}$ had the same partner Q_k , then symmetry and transitivity would give

$$P_j \sim_{\text{gp}} Q_k \sim_{\text{gp}} P_{j'} \implies P_j \sim_{\text{gp}} P_{j'},$$

contradicting BNT basis separation. Both partner maps are therefore injective. The two finite sector sets have equal cardinality, so the forward map is bijective and retains the displayed overlap limit together with the dimension and gauge data. $\square$

Lemma F.1.2 (Unit phase from matched normalized BNT blocks). Lean ✓ Stmt *Let P and Q be BNT canonical forms. If, for every positive length N and word σ , a sector P_j and a sector Q_k have MPV families related by*

$$V^{(N)}(Q_k)_\sigma = \zeta^N V^{(N)}(P_j)_\sigma, \tag{F.1}$$

then $|\zeta| = 1$.

Proof. ✓ Proof The BNT normalization gives self-overlap limits of modulus 1 for both sectors. The MPV identity (F.1) gives, for every positive N ,

$$O_{Q_k Q_k}(N) = |\zeta|^{2N} O_{P_j P_j}(N).$$

Since $O_{Q_k Q_k}(N) \rightarrow 1$ and $O_{P_j P_j}(N) \rightarrow 1$, taking limits forces $|\zeta|^{2N} \rightarrow 1$ and hence $|\zeta| = 1$. $\square$

F.2 Coefficient identities from gauge-phase equalities

The main coefficient theorem takes the matched phases as input. The next result shows how those phases are selected from gauge-phase equivalences before the linear-independence argument is applied.

Lemma F.2.1 (Full-basis coefficient identity from matched sectors). Lean ✓ Stmt *Let P and Q be BNT canonical forms with the same MPV family at every positive length, matched by a bijection $\beta : J(Q) \simeq J(P)$ with bond-dimension equalities and a gauge-phase equivalence between Q_k and $P_{\beta(k)}$ for every k . Then each sector carries a unit-modulus phase ζ_k , and eventually*

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q). \quad (\text{F.2})$$

This is the equal-MPV coefficient comparison of [CPGSV16, Appendix MPV proof, lines 1187–1188].

Proof. ✓ Proof For each matched sector, the gauge-phase equivalence gives $V^{(N)}(Q_k)_\sigma = \zeta_k^N V^{(N)}(P_{\beta(k)})_\sigma$. Lemma F.1.2 gives $|\zeta_k| = 1$. Applying Lemma 10.2.2 to the same phases gives $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ for all sufficiently large N . $\square$

F.3 The scalar-threaded proportional identity

For proportional total MPVs, coefficient extraction retains the one scalar that relates the two total vectors at each length. This identity is exact, but it does not imply the scalar-free identity needed to recover copy weights; the counterexample in Remark 10.3.6 shows that no such implication is possible.

Lemma F.3.1 (Proportional matched-phase coefficient identity, scalar-threaded). Lean ✓ Stmt *Let P be a BNT canonical form and Q a sector decomposition whose total tensors generate eventually nonzero proportional MPV families. Suppose a bijection $\beta : J(Q) \simeq J(P)$ and scalars ζ_k satisfy the matched MPV phase identities (10.2) for every sector k . Then there is a single eventually-nonzero scalar sequence c_N and a threshold N_0 such that, for all $N > N_0$ and every $k \in J(Q)$,*

$$c_N^{(\beta(k))}(P) = c_N \zeta_k^N c_N^{(k)}(Q). \quad (\text{F.3})$$

The scalar c_N is the same for every sector k : it is the single global proportionality scalar of [CPGSV16, Theorem thm1, lines 1167–1170] at length N .

Proof. ✓ Proof For each N on the proportionality tail, expand both total MPVs in their sector bases and use the global proportionality scalar c_N :

$$\sum_{j \in J(P)} c_N^{(j)}(P) |V^{(N)}(P_j)\rangle = c_N \sum_{k \in J(Q)} c_N^{(k)}(Q) |V^{(N)}(Q_k)\rangle.$$

The matched phase identities (10.2) give $|V^{(N)}(Q_k)\rangle = \zeta_k^N |V^{(N)}(P_{\beta(k)})\rangle$; because every sector is matched, reindexing the Q -sum by β rewrites the right-hand side entirely in the P -basis. The BNT eventual linear independence of the P -basis and Lemma E.1.11 then extract the coefficient identities (F.3) for all sufficiently large N . $\square$

F.4 Direct-sum conjugation

We now prove the general block-matrix identities used by the global gauge in Section 10.3. These statements do not choose or match sectors; they only combine conjugacies that have already been obtained.

Definition F.4.1 (Invertible matrix associated to a unitary). Lean Stmt A unitary matrix U determines an invertible matrix whose inverse is $U^\dagger$.

Lemma F.4.2 (Unitary block-diagonal gauges). Lean Stmt If every block gauge U_k is unitary, then the flattened block-diagonal gauge $U = \bigoplus_k U_k$ is unitary.

Proof. Proof The products are computed blockwise:

$$UU^\dagger = \bigoplus_k U_k U_k^\dagger = \bigoplus_k \mathbb{1} = \mathbb{1}.$$

$\square$

Lemma F.4.3 (Direct-sum conjugation by the flattened gauge). Lean Lean Stmt If, for every block k and physical index i , $B_k^i = X_k A_k^i X_k^{-1}$, then the weighted direct sums satisfy

$$\bigoplus_k \mu_k B_k^i = X \left(\bigoplus_k \mu_k A_k^i \right) X^{-1},$$

where X is the flattened block-diagonal gauge from Definition 10.3.1. Hence the two assembled tensors are gauge equivalent.

Proof. Proof Multiplication of block-diagonal matrices is computed blockwise. The scalar weight μ_k commutes with the conjugation on the k th block, and the reindexing from dependent block coordinates to the flattened bond index preserves products and inverses. $\square$

Theorem F.4.4 (Global gauge from a copy-weight matching). Lean Stmt Suppose that the sectors of P and Q are matched, with bond-dimension equalities, nonzero phases, and block gauges satisfying $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. If the same phases also give a copy-weight matching, then the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$. This is only a reformulation of Theorem 10.3.2.

Proof. Proof The copy-weight matching supplies $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$. Together with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$, these are exactly the two hypotheses (10.4) of Theorem 10.3.2. $\square$

F.5 Additional proportional consequences

The following implications are useful when copy weights satisfy extra relations. They are not conclusions of CPSV16 Theorem 2.10. In particular, their additional hypotheses fail for the proportional BNT forms in Remark 10.3.6.

Theorem F.5.1 (Additional proportional global gauge under copy-weight matching). [Lean](#) [✓ Stmt](#) Let P and Q be BNT canonical forms whose total tensors generate eventually nonzero proportional MPV families. Then the proportional sector-matching theorem gives β , the bond-dimension equalities, the unit phases ζ_k , and the block gauges X_k . If one additionally assumes a copy-weight matching for these same phases, then the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$.

Proof. [✓ Proof](#) The proportional sector-matching theorem supplies β , X_k , and unit phases ζ_k satisfying $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. Since $|\zeta_k| = 1$, each ζ_k is nonzero. A copy-weight matching gives $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$, so Theorem F.4.4 gives the direct-sum gauge. $\square$

Theorem F.5.2 (Additional proportional global gauge under coefficient identities). [Lean](#) [✓ Stmt](#) Let P and Q be BNT canonical forms whose total tensors generate eventually nonzero proportional MPV families. Then there are a bijection $\beta : J(Q) \simeq J(P)$, bond-dimension equalities, unit-modulus phases ζ_k , and block gauges X_k such that $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$ for every sector k and physical index i . If one additionally assumes that these same phases satisfy the matched-sector coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ for all sufficiently large N , then they determine a copy-weight matching and hence the flattened Q -tensor is obtained from the matched-coordinate P -tensor by the direct-sum gauge $X = \bigoplus_k (\mathbf{1}_{r_k} \otimes X_k)$.

Proof. [✓ Proof](#) The sector-matching theorem gives β , X_k , and $|\zeta_k| = 1$ with $Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}$. Since $\zeta_k \neq 0$, the coefficient identities $c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q)$ produce $\nu_{k,q} = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}$ by Definition 10.2.4. Theorem F.5.1 then applies to this copy-weight matching. $\square$

F.6 Matched flattened coordinates

Let $\beta : J(Q) \simeq J(P)$ match the basis sectors and let τ_k match the copies in sector k . Flatten the pair (k, q) to the copy index of Q . The matched P -data on that index are the weight and tensor

$$\tilde{\mu}_{k,q} = \mu_{\beta(k), \tau_k(q)}, \quad \tilde{P}_{k,q} = P_{\beta(k)},$$

where the second expression is read after the matched bond-dimension identification. The block gauge repeats X_k on every copy q .

Definition F.6.1 (Matched weights, tensors, and repeated gauges). [Lean](#) [Lean](#) [Lean](#) [✓ Stmt](#) The matched flattened data are the functions $\tilde{\mu}$, $\tilde{P}$, and $(k, q) \mapsto X_k$ defined above on the flattened copy coordinates of Q .

Lemma F.6.2 (Equality of the matched total bond dimensions). [Lean](#) [✓ Stmt](#) If matched sectors have equal bond dimensions and matched copy sets are bijective, then P and Q have the same total bond dimension.

Proof. [✓ Proof](#) Reindex first by the sector bijection and then by each copy bijection. Every summand $\dim P_{\beta(k)}$ becomes $\dim Q_k$, so

$$\sum_{j \in J(P)} r_j(P) \dim P_j = \sum_{k \in J(Q)} r_k(Q) \dim Q_k.$$

$\square$

Definition F.6.3 (Sector and bond-coordinate equivalences). [Lean](#) [Lean](#) [✓ Stmt](#) The sector bijection, copy bijections, and bond-dimension equalities induce an equivalence of the dependent triples (j, q, a) and (k, q', a') . After flattening the dependent bond coordinate, this gives an equivalence of the standard total bond-index sets.

Proof. [✓ Proof](#) Send (k, q, a) to $(\beta(k), \tau_k(q), a)$, using the bond-dimension equality to identify the last coordinate. The inverse uses β^{-1} and τ_k^{-1} . Both composites are the identity componentwise. Conjugating by the standard equivalence between dependent triples and a finite interval gives the flattened bond-coordinate equivalence. $\square$

Theorem F.6.4 (Equal-MPV gauge with all matched witnesses). [Lean](#) [✓ Stmt](#) *Under the hypotheses of the equal-MPV theorem, one may choose simultaneously the sector bijection, bond-dimension equalities, copy bijections, phases, block gauges, total-dimension equality, and the flattened global gauge. They satisfy the matched block and weight identities of Theorem 10.3.4, together with the conjugation of Q by the matched-coordinate P -tensor.*

Proof. [✓ Proof](#) Theorem 10.3.4 supplies all witnesses except the displayed equality of total dimensions. Apply Lemma F.6.2 to its sector and copy bijections. The resulting equality identifies the total bond spaces, while the global conjugation equation is unchanged. $\square$

F.7 Permutation gauges and literal coordinates

The matched P -tensor has the correct blocks but lists them in Q -order. A permutation matrix restores the literal order of P . We record its invertible and unitary forms before proving the coordinate conversion used in Theorem 10.3.5.

Definition F.7.1 (Invertible permutation gauge). [Lean](#) [✓ Stmt](#) For a permutation ρ of a finite bond-index set, let Π_ρ be its permutation matrix. The corresponding element of the general linear group is denoted G_ρ .

Lemma F.7.2 (Matrix and inverse identities for a permutation gauge). [Lean](#) [Lean](#) [✓ Stmt](#) *The matrix of G_ρ is Π_ρ , and the matrix of its inverse is $\Pi_{\rho^{-1}}$.*

Proof. [✓ Proof](#) The first identity is the definition of the permutation gauge. Since the inverse permutation reverses the coordinate relabelling, $G_\rho^{-1} = G_{\rho^{-1}}$, which gives the second identity. $\square$

Every row and every column of Π_ρ contains exactly one nonzero entry, equal to 1. Hence

$$\Pi_\rho \Pi_{\rho^{-1}} = \mathbb{1}, \quad \Pi_\rho^\dagger = \Pi_{\rho^{-1}}.$$

Together with Lemma F.7.2, these identities show that $G_\rho^\dagger = G_\rho^{-1}$, so the permutation gauge is unitary.

Lemma F.7.3 (Permutation conjugation reindexes both coordinates). [Lean](#) [✓ Stmt](#) *For a permutation ρ and a square matrix M , conjugation by the permutation matrix reindexes both coordinates by ρ :*

$$\Pi_\rho M \Pi_{\rho^{-1}} = (M_{\rho(i), \rho(j)})_{i,j}. \quad (\text{F.4})$$

Equivalently, the right-hand side is the submatrix of M along $\rho \times \rho$.

Proof. [✓ Proof](#) Left multiplication by Π_ρ selects row $\rho(i)$ in row i , and right multiplication by $\Pi_{\rho^{-1}}$ selects column $\rho(j)$ in column j . The two selections give the displayed submatrix. $\square$

The coordinate equivalence of Definition F.6.3 induces the required permutation ρ after the total-dimension identification. A convention with inverse indices merely replaces the chosen permutation by ρ^{-1} ; we use the convention in (F.4) to match the formal coordinate identity. Applying that identity to P_{tot}^i gives the matched-coordinate tensor. Composing G_ρ with the block-diagonal gauge therefore gives the literal gauge of Theorem 10.3.5. This also explains why the total-dimension cast and the sector permutation must occur in that order.

F.8 Unitary witness refinements

The main text proves that a specified gauge between left-canonical irreducible sectors can be replaced by a unitary without changing its phase. We record the existential form and then repeat the construction over every matched copy.

Lemma F.8.1 (Unitary gauge for left-canonical irreducible tensors). Lean ✓ Stmt *Let A and B be left-canonical irreducible tensors of common positive bond dimension that are gauge-phase equivalent. Then the gauge can be taken unitary: there are a unitary matrix U and a scalar ζ with $|\zeta| = 1$ such that, for every physical index i ,*

$$B^i = \zeta U A^i U^\dagger.$$

Proof. ✓ Proof Unpack the gauge-phase equivalence as $B^i = \zeta X A^i X^{-1}$ with X invertible and $\zeta \neq 0$. Lemma 10.4.1.2 replaces X by a unitary U while retaining this same phase and proves $|\zeta| = 1$. $\square$

Lemma F.8.2 (Unitary gauge in matched sector coordinates). Lean ✓ Stmt *Let P and Q be BNT canonical forms whose total tensors generate the same MPV family at every positive length. There are a sector bijection β , copy bijections τ_k , phases ζ_k , sector unitaries U_k , and a unitary block-diagonal matrix X such that*

$$\begin{aligned} Q_k^i &= \zeta_k U_k P_{\beta(k)}^i U_k^\dagger, \\ \nu_{k,q} &= \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)}, \\ |\zeta_k| &= 1. \end{aligned}$$

The matched sectors and copies have equal bond dimensions and multiplicities. If P_{match} denotes the corresponding reordered block assembly, then $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$.

Proof. ✓ Proof Apply Lemma 10.4.1.2 to each matched sector, retaining the phase in the copy-weight identity. Repeating U_k on every copy gives $X = \bigoplus_{k,q} U_k$, which is unitary by Lemma F.4.2. For every matched copy (k, q) ,

$$\nu_{k,q} Q_k^i = \zeta_k^{-1} \mu_{\beta(k), \tau_k(q)} \zeta_k U_k P_{\beta(k)}^i U_k^\dagger = \mu_{\beta(k), \tau_k(q)} U_k P_{\beta(k)}^i U_k^\dagger.$$

These are precisely the blocks of $Q_{\text{flat}}^i = X P_{\text{match}}^i X^{-1}$. $\square$

Appendix G

Symmetries and Virtual-Boundary Nondecay: Supporting Results

This appendix supports Chapter 11. It collects permutation and gauge bookkeeping, factor-system identities, stationary-boundary twist calculations, common trace-preserving gauges, and the finite-dimensional limit arguments used in the virtual-boundary nondecay results. The notation is introduced locally in each part, and every reference to the main development is given by its stable label.

The physical index set is $I = \{0, \dots, d-1\}$, the bond algebra is $M_D(\mathbb{C})$, and G denotes a group. All sums below are finite. The first section treats permutation actions on I ; the second isolates scalar uniqueness for two gauges of the same injective tensor; the third records the elementary factor-system bookkeeping used in the cohomology discussion of Chapter 11.

G.1 Permutation reindexing and identity bookkeeping

The linear-combination twist $\tilde{A}_g^i = \sum_j U(g)_{ij} A^j$ is the standard form used in the virtual representation theorem. The following alternative formulation replaces the matrix action by a permutation of the physical index; it is useful for symmetries that act by reordering basis states rather than by a general linear map.

Definition G.1.1 (Permutation symmetry datum). [Lean](#) [✓ Stmt](#) An *on-site symmetry datum* for a group G on a physical space of dimension d consists of a matrix-valued map $u : G \rightarrow M_d(\mathbb{C})$ and a physical-index action $\sigma : G \rightarrow \text{End}(\{0, \dots, d-1\})$.

Definition G.1.2 (Permutation-twisted tensor). [Lean](#) [✓ Stmt](#) Given an on-site symmetry datum (u, σ) , the *permutation-twisted tensor* at group element g is

$$(A^{\sigma_g})^i := A^{\sigma(g)(i)}. \tag{G.1}$$

In tensor-network notation the local tensor is unchanged and only the physical leg is relabelled:

$$\begin{array}{ccc}
\begin{array}{c} i \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_g} & \begin{array}{c} \sigma(g)(i) \\ | \\ \bullet \\ | \\ A \end{array} .
\end{array}$$

Lemma G.1.3 (Identity permutation twist). [Lean](#) [✓ Stmt](#) If $\sigma(1) = \text{id}$, then $(A^{\sigma_1})^i = A^i$ for all i .

Proof. [✓ Proof](#) The claim follows from $\sigma(1)(i) = i$ for all i . $\square$

Composition order. The linear-combination twist (Lemma 11.1.3) composes as $\widetilde{(\widetilde{A_h})_g} = \widetilde{A_{gh}}$, applying h first then g . The permutation twist below uses the same left-action convention $\sigma(gh) = \sigma(g) \circ \sigma(h)$, so $(A^{\sigma_g})^{\sigma_h} = A^{\sigma_{gh}}$; the inner permutation $\sigma(h)$ acts first on the index.

Lemma G.1.4 (Composition of permutation twists). [Lean](#) [✓ Stmt](#) If $\sigma(gh) = \sigma(g) \circ \sigma(h)$ for all $g, h \in G$, then

$$A^{\sigma_{gh}} = (A^{\sigma_g})^{\sigma_h}. \quad (\text{G.2})$$

Proof. [✓ Proof](#) Expanding both sides gives $(A^{\sigma_{gh}})^i = A^{\sigma(g)(\sigma(h)(i))} = ((A^{\sigma_g})^{\sigma_h})^i$. Diagrammatically this is just two consecutive relabellings of the same local tensor:

$$\begin{array}{ccccc}
\begin{array}{c} i \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_h} & \begin{array}{c} \sigma(h)(i) \\ | \\ \bullet \\ | \\ A \end{array} & \xrightarrow{\sigma_g} & \begin{array}{c} \sigma(g)(\sigma(h)(i)) \\ | \\ \bullet \\ | \\ A \end{array} .
\end{array}$$

$\square$

G.1.1 Linear physical-index rotations

The same bookkeeping applies to a general matrix acting on the physical index.

Definition G.1.1.1 (Physical-index rotation). [Lean](#) [✓ Stmt](#) Given a matrix $M \in M_d(\mathbb{C})$ and an MPS tensor A , the *physical-index rotation* is the tensor $(A_M)^i := \sum_j M_{ij} A^j$. When the matrix is a unitary u , the rotation is written A_u .

Lemma G.1.1.2 (Matrix product vector under physical rotation). [Lean](#) [✓ Stmt](#) For a length- N physical configuration $s = (s_0, \dots, s_{N-1})$, physical rotation satisfies

$$\text{tr}((A_M)^{s_0} \dots (A_M)^{s_{N-1}}) = \sum_{t \in \{0, \dots, d-1\}^N} \left(\prod_{n=0}^{N-1} M_{s_n, t_n} \right) \text{tr}(A^{t_0} \dots A^{t_{N-1}}). \quad (\text{G.3})$$

Proof. [✓ Proof](#) Substitute $(A_M)^{s_n} = \sum_{t_n} M_{s_n, t_n} A^{t_n}$ at each site, distribute the finite sums through the ordered matrix product, and use linearity of the trace to obtain (G.3). $\square$

Lemma G.1.1.3 (Word expansion for a twisted tensor). [Lean](#) [✓ Stmt](#) For a word $b : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}$,

$$A_g^{b_0} \dots A_g^{b_{L-1}} = \sum_{v : \{0, \dots, L-1\} \rightarrow \{0, \dots, d-1\}} \left(\prod_{k=0}^{L-1} U(g)_{b_k v_k} \right) A^{v_0} \dots A^{v_{L-1}}. \quad (\text{G.4})$$

Proof. [✓ Proof](#) Substitute $A_g^{b_k} = \sum_{v_k} U(g)_{b_k v_k} A^{v_k}$ into the ordered product. Repeated distributivity gives one summand for each function v , scalar coefficients commute past the matrices and multiply, and the remaining ordered product is $A^{v_0} \dots A^{v_{L-1}}$. $\square$

G.2 Gauge ratios and scalar uniqueness

The virtual representation theorem (Theorem 11.1.16) relies on the fact that composing two gauge transformations yields another gauge up to a scalar factor.

Lemma G.2.1 (Gauge ratio commutes with tensor matrices). [Lean](#) [Stmnt](#) Diagrammatically, the hypothesis is that the two local gauges

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} i \\ | \\ \bullet \\ B \end{array} & = & \begin{array}{c} i \\ | \\ \text{---} \text{---} X \text{---} \bullet \end{array} \\
 \end{array} \\
 \text{and} \\
 \begin{array}{ccc}
 \begin{array}{c} i \\ | \\ \bullet \\ B \end{array} & = & \begin{array}{c} i \\ | \\ \text{---} \text{---} Y \text{---} \bullet \end{array} \\
 \end{array}
 \end{array}$$

produce the same target tensor B^i from the same source tensor A^i . If $X, Y \in \text{GL}_D(\mathbb{C})$ both conjugate an MPS tensor A to the same tensor B (i.e. $B^i = XA^iX^{-1} = YA^iY^{-1}$ for all i), then $Y^{-1}X$ commutes with every A^i .

Proof. [Proof](#) Direct matrix calculation: conjugating both gauge relations and composing gives $(Y^{-1}X)A^i = A^i(Y^{-1}X)$. $\square$

Lemma G.2.2 (Gauge ratio is scalar for injective tensors). [Lean](#) [Stmnt](#) Under the same hypotheses, if A is additionally injective, then $Y^{-1}X = \lambda \mathbb{1}_D$ for some scalar $\lambda \in \mathbb{C}$.

Proof. [Proof](#) The same two-gauge picture shows that the ratio $Y^{-1}X$ acts invisibly on the local tensor. By Lemma G.2.1, $Y^{-1}X$ commutes with all A^i . Since $\{A^i\}$ spans $M_D(\mathbb{C})$, the matrix $Y^{-1}X$ commutes with all of $M_D(\mathbb{C})$. By Lemma 2.1.5, it is a scalar matrix. $\square$

Theorem G.2.3 (Projective multiplication law for gauge matrices). [Lean](#) [Stmnt](#) Let A be an injective MPS tensor with on-site symmetry under a group representation $U : G \rightarrow \text{GL}_d(\mathbb{C})$. For each $g \in G$, let $X(g) \in \text{GL}_D(\mathbb{C})$ satisfy $\tilde{A}_g^i = X(g)A^iX(g)^{-1}$ for all i . Then for all $g, h \in G$, there exists $c(g, h) \in \mathbb{C}$ such that $X(h)X(g) = c(g, h)X(gh)$. In fact $c(g, h) \neq 0$ since both sides are invertible, so $c(g, h) \in \mathbb{C}^\times$. Diagrammatically, the two candidate gauges are competing local replacements for the same twisted tensor:

$$\begin{array}{ccc}
 \begin{array}{c} i \\ | \\ \text{---} \text{---} X(h)X(g) \text{---} \bullet \end{array} & = & \begin{array}{c} i \\ | \\ \text{---} \text{---} X(gh) \text{---} \bullet \end{array} \\
 A & & A
 \end{array}$$

and both sides conjugate A^i to the same twisted tensor $\tilde{A}_{gh}^i$.

Proof. [Proof](#) By the composition law $\tilde{A}_{gh} = \widetilde{(\tilde{A}_h)_g}$ (Lemma 11.1.3), both $X(h)X(g)$ and $X(gh)$ conjugate A to $\tilde{A}_{gh}$. Lemma G.2.2 then gives the scalar factor. $\square$

G.3 Cocycle equivalence and coboundaries

Let $\omega_1, \omega_2 : G \times G \rightarrow \mathbb{C}^\times$ be arbitrary scalar 2-cochains. The definitions of coboundary and cohomology are Definitions 11.2.1 and 11.2.2.

Theorem G.3.1 (Cohomology relation is an equivalence relation). [Lean](#) [✓ Stmt](#) *The relation “cohomologous to” is reflexive, symmetric, and transitive on the set of $\mathbb{C}^\times$ -valued scalar 2-cochains, and therefore also on the subset of cocycles.*

Proof. [✓ Proof](#) Reflexivity: take $\varphi = 1$. Symmetry: replace φ by φ^{-1} . Transitivity: compose the witnesses φ and ψ pointwise in (11.10). $\square$

Lemma G.3.2 (Coboundary iff cohomologous to the trivial cocycle). [Lean](#) [✓ Stmt](#) *A scalar 2-cochain ω is a coboundary if and only if $\omega \sim 1$, where $1(g, h) = 1$ for all g, h .*

Proof. [✓ Proof](#) Both directions follow by absorbing or introducing the factor $1(g, h) = 1$ via $x \cdot 1 = x$ in (11.10). $\square$

G.4 Stationary-boundary twist support

Fix an MPS tensor $A = (A^i)_{i=0}^{d-1}$, a physical matrix u , and a stationary boundary matrix $\Lambda \in M_D(\mathbb{C})$. The twisted transfer map $\mathcal{E}_u$ is Definition 11.3.1. The next constructions are virtual-boundary functionals; they should not be confused with the physical endpoint correlator of Definition 11.3.4.

Definition G.4.1 (Twisted companion tensor). [Lean](#) [✓ Stmt](#) For an MPS tensor A and a physical-index matrix u , define the companion tensor B_u by

$$B_u^n := \sum_{n'=0}^{d-1} \overline{u_{n'n}} A^{n'}. \quad (\text{G.5})$$

Lemma G.4.2 (Twisted transfer as a mixed transfer operator). [Lean](#) [✓ Stmt](#) *The twisted transfer map of A is the mixed transfer operator of the pair (A, B_u) :*

$$\mathcal{E}_u = F_{A, B_u}. \quad (\text{G.6})$$

Proof. [✓ Proof](#) Expand both definitions and rearrange the double sum. $\square$

Lemma G.4.3 (The companion tensor has the same transfer map). [Lean](#) [✓ Stmt](#) *If $uu^\dagger = \mathbb{1}_d$, then for every $X \in M_D(\mathbb{C})$ the companion family B_u satisfies*

$$\mathcal{E}_{B_u}(X) = \mathcal{E}_A(X).$$

Proof. [✓ Proof](#) the corresponding theorem in the companion quantum-channel volume [LTC26, “Unitary freedom in Kraus operators”] applies to the unitary combination defining B_u . Equivalently, expand $B_u^n = \sum_{n'} \overline{u_{n'n}} A^{n'}$. The coefficient of $A^j X (A^k)^\dagger$ in $\mathcal{E}_{B_u}(X)$ is

$$\sum_n \overline{u_{jn}} u_{kn} = (uu^\dagger)_{kj} = \delta_{kj}.$$

All off-diagonal terms vanish and the diagonal terms sum to $\mathcal{E}_A(X)$. $\square$

Definition G.4.4 (Virtual-boundary twist functional). [Lean](#) [✓ Stmt](#) For a boundary state Λ and boundary matrices $X, Y \in M_D(\mathbb{C})$, the *virtual-boundary twist functional* is

$$R_L(u; X, Y) := \text{tr}(\Lambda X \mathcal{E}_u^L(Y)). \quad (\text{G.7})$$

The special case $X = Y = \mathbb{1}$ is the stationary-boundary block-twist functional of Definition 11.3.2. The matrices X, Y act on the virtual level. In the physical string correlator of [PGWS⁺08] the string of twists is instead flanked by local operators x, y acting on physical sites; expanding such operators in the transfer picture produces particular virtual matrices, and [PGWS⁺08] argues that for injective tensors, physical operators on sufficiently many sites produce every virtual matrix. The results below are stated directly in this virtual-boundary form.

Lemma G.4.5 (Virtual-boundary nondecay prevents convergence to zero). [Lean](#) [✓ Stmt](#) If virtual-boundary nondecay holds for A, u, Λ , then for some boundary matrices X, Y the sequence $R_L(u; X, Y)$ does not tend to 0 as $L \rightarrow \infty$.

Proof. [✓ Proof](#) A uniform lower bound $c \leq \|R_L(u; X, Y)\|$ is incompatible with convergence to 0. $\square$

G.5 Kraus mixing, transfer expansions, and a common TP gauge

The spectral estimates in the main chapter compare A with B_u . They are placed in one trace-preserving gauge obtained from their common transfer map.

Lemma G.5.1 (Unitary Kraus mixing). [Lean](#) [✓ Stmt](#) If u is unitary, then replacing Kraus operators $\{A_i\}$ by their unitary mixtures $\{\sum_j u_{ij} A_j\}$ preserves the channel:

$$\sum_i \left(\sum_j u_{ij} A_j \right) Y \left(\sum_j u_{ij} A_j \right)^\dagger = \sum_i A_i Y A_i^\dagger. \quad (\text{G.8})$$

Proof. [✓ Proof](#) This is exactly the usual invariance of a Kraus decomposition under a unitary change of Kraus operators; see the corresponding theorem in the companion quantum-channel volume [LTC26, “Unitary freedom in Kraus operators”]. $\square$

Definition G.5.2 (Common trace-preserving gauge for a twisted pair). [Lean](#) [✓ Stmt](#) A *common trace-preserving gauge setup* for (A, u) consists of a tensor B equal to the companion B_u and satisfying $\mathcal{E}_B(X) = \mathcal{E}_A(X)$ for every $X \in M_D(\mathbb{C})$. It also includes a positive definite matrix σ fixed by the adjoint transfer map of B , its positive square root S , and gauged tensors A', B' satisfying

$$\begin{aligned} B &= B_u, & \mathcal{E}_B(X) &= \mathcal{E}_A(X), & \mathcal{E}_{B^\dagger}(\sigma) &= \sigma, & \sigma &> 0, \\ S &= \sigma^{1/2}, & S^\dagger &= S, & A'^i &= S A^i S^{-1}, & B'^i &= S B^i S^{-1}. \end{aligned} \quad (\text{G.9})$$

In addition,

$$\begin{aligned} S S^{-1} &= S^{-1} S = \mathbb{1}, & S^\dagger (S^\dagger)^{-1} &= \mathbb{1}, & (S^{-1})^\dagger &= S^{-1}, \\ \sum_i (A'^i)^\dagger A'^i &= \mathbb{1}, & \sum_i (B'^i)^\dagger B'^i &= \mathbb{1}, \end{aligned} \quad (\text{G.10})$$

together with irreducibility of both A' and B' . These identities complete the common trace-preserving gauge data.

Definition G.5.3 (Construction of the common trace-preserving gauge). [Lean](#) [✓ Stmt](#) Let $D > 0$. If A is injective, $uu^\dagger = \mathbb{1}_d$, and $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, define the common trace-preserving gauge setup for (A, u) by choosing the unique positive definite stationary state of the adjoint channel and using the data in Definition G.5.2.

Proof. [✓ Proof](#) Injectivity makes $\mathcal{E}_A$ irreducible. The adjoint channel therefore has a positive definite stationary state σ . By Lemma G.4.3, A and B_u have the same transfer map, hence the same adjoint fixed point. Put $S = \sigma^{1/2}$. Positivity makes S invertible, and direct substitution gives

$$\begin{aligned} \sum_i (SA^i S^{-1})^\dagger (SA^i S^{-1}) &= \mathbb{1}, \\ \sum_i (SB_u^i S^{-1})^\dagger (SB_u^i S^{-1}) &= \mathbb{1}. \end{aligned} \tag{G.11}$$

Similarity preserves irreducibility, so the two gauged families have all the asserted properties. $\square$

Theorem G.5.4 (Twisted eigenvalues survive the common gauge). [Lean](#) [✓ Stmt](#) If $V \neq 0$ and $\mathcal{E}_u(V) = \lambda V$, then λ is an eigenvalue of the mixed transfer map $F_{A', B'}$ in any common trace-preserving gauge setup.

Proof. [✓ Proof](#) By Lemma G.4.2, the original equation is $F_{A, B_u}(V) = \lambda V$. Multiplying on the left and right by S and using $S^\dagger = S$ gives

$$F_{A', B'}(SVS) = \lambda SVS.$$

Since S is invertible, $SVS \neq 0$. $\square$

G.6 Boundary and limit support

Lemma G.6.1 (Spectral radius < 1 forces virtual-boundary decay). [Lean](#) [✓ Stmt](#) If $\rho(\mathcal{E}_u) < 1$, then, as $L \rightarrow \infty$, for all boundary matrices $X, Y, \Lambda \in M_D(\mathbb{C})$ one has $R_L(u; X, Y) \rightarrow 0$.

Proof. [✓ Proof](#) In finite dimension, $\rho(\mathcal{E}_u) < 1$ implies $\mathcal{E}_u^L \rightarrow 0$ in operator norm. Composing with the fixed trace pairing $Z \mapsto \text{tr}(\Lambda X Z)$ gives the stated scalar convergence. $\square$

G.7 Routine universality corollary

The following consequence uses only the virtual MPV-family symmetry predicate and the canonical fixed-point hypotheses from the main chapter.

Theorem G.7.1 (Virtual-boundary nondecay agrees for two injective MPV-symmetric tensors). [Lean](#) [✓ Stmt](#) If A and B are injective, both on-site symmetric under a unitary representation U , and if $\mathcal{E}_A(\mathbb{1}) = \mathbb{1}$, $\mathcal{E}_B(\mathbb{1}) = \mathbb{1}$, Λ_A and Λ_B are positive definite, $\text{tr}(\Lambda_A) = 1$, $\text{tr}(\Lambda_B) = 1$, $\mathcal{E}_A^\dagger(\Lambda_A) = \Lambda_A$, and $\mathcal{E}_B^\dagger(\Lambda_B) = \Lambda_B$, then for every $g \in G$, virtual-boundary nondecay holds for A with twist $U(g)$ if and only if it holds for B . In particular, since Definition 11.4.1 implies on-site symmetry for both tensors, this theorem applies to tensors in the same SPT phase; it applies equally to symmetric tensors in different SPT phases.

Proof. [✓ Proof](#) Both directions follow from Theorem 11.4.4, which shows that virtual-boundary nondecay holds universally for injective symmetric tensors with canonical normalisations. $\square$

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