Yamagami's Simultaneous-Singularity Boundary Count

2026-08-24

Verdict: clarification (resolved).

1 The cyclic function and its boundary

In the case denoted by $l = 1$, Yamagami studies a cyclic function on $\mathbb{Z}/N\mathbb{Z}$ [Yam93, pp. 524–525]. For a vector $x = (x_q)_{q\in\mathbb{Z}/N\mathbb{Z}}$, define

$$f_{N,m,s}(x) = \sum_{q\in\mathbb{Z}/N\mathbb{Z}} \frac{x_q}{(s-m)x_q + x_{q+1} + \cdots + x_{q+m}}. \tag{1}$$

All indices in (1) are cyclic. Throughout the boundary count, assume

$$N \geq 1, \tag{2}$$
$$m < N, \tag{3}$$
$$N \leq s. \tag{4}$$

These assumptions imply

$$m < s, \tag{5}$$
$$s - m > 0. \tag{6}$$

Let $\{x^{(r)}\}_{r\geq 1}$ be a sequence of strictly positive vectors that converges coordinatewise to a nonzero boundary vector $a = (a_q)_{q\in\mathbb{Z}/N\mathbb{Z}}$ with $a_q \geq 0$. If the limiting denominator at an index i vanishes, then

$$(s-m)a_i + a_{i+1} + \cdots + a_{i+m} = 0. \tag{7}$$

Every term on the left-hand side of (7) is nonnegative, and (6) makes the coefficient of a_i positive. Hence

$$a_i = a_{i+1} = \cdots = a_{i+m} = 0. \tag{8}$$

Because $a \neq 0$, cyclic traversal after this forward zero window eventually reaches a positive coordinate.

2 The calculation printed in the source

Yamagami first treats one indeterminate summand [Yam93, p. 525]. After a cyclic relabeling, the hypotheses are

$$a_N = a_1 = \cdots = a_m = 0, \tag{9}$$
$$a_{m+1} > 0, \tag{10}$$
$$a_{N-1} > 0. \tag{11}$$

The m summands with numerators indexed by $1, \ldots, m$ tend to zero. Every other positive summand is at most $1/(s-m)$. The printed estimates therefore give

$$\limsup_{r\to\infty} f_{N,m,s}(x^{(r)}) \leq \frac{N-m}{s-m}, \tag{12}$$
$$\frac{N-m}{s-m} \leq \frac{N}{s}. \tag{13}$$

For several indeterminate summands, Ref. [Yam93] records a cyclic zero-string condition. It selects an index j such that

$$a_j > 0, \tag{14}$$
$$a_{j-m-1} = a_{j-m} = \cdots = a_{j-1} = 0. \tag{15}$$

The paper says that the remaining argument is similar, but it does not print the finite first-positive construction or the simultaneous limsup calculation. Thus (9)–(13) constitute the printed single-singularity calculation, while the reduction of an arbitrary family of singular denominators to one global count is omitted.

3 The completed simultaneous count

Begin with the zero window in (8). Traverse the cyclic coordinates after $i+m$, and choose the first positive coordinate. Finiteness of $\mathbb{Z}/N\mathbb{Z}$ and the assumption $a \neq 0$ ensure that such a coordinate a_j exists. Minimality implies that every coordinate from the end of the zero window through $j-1$ vanishes. In particular, it gives (14) and (15).[1]

Define the cyclic index set

$$V = \{j-1, j-2, \dots, j-m\}. \tag{16}$$

The assumption (3) makes these m cyclic indices distinct. Therefore,

$$|V| = m, \tag{17}$$

$$|(\mathbb{Z}/N\mathbb{Z}) \setminus V| = N - m. \tag{18}$$

For $q = j - k \in V$, the limiting numerator a_q is zero, while the forward denominator contains the fixed positive coordinate a_j. The corresponding summand therefore tends to zero. Every summand indexed outside V remains nonnegative and is bounded above by $1/(s-m)$ along the positive approximants. Partitioning the finite sum once gives

$$\limsup_{r\to\infty} f_{N,m,s}(x^{(r)}) \leq |(\mathbb{Z}/N\mathbb{Z}) \setminus V| \, \frac{1}{s-m}, \tag{19}$$

$$\limsup_{r\to\infty} f_{N,m,s}(x^{(r)}) \leq \frac{N-m}{s-m}. \tag{20}$$

The assumption (4) gives the comparison in (13).[2]

This argument uses one global count. It neither assigns singular denominators to maximal zero blocks nor adds estimates obtained from separate blocks. Consequently, it applies without change when any number of limiting denominators vanish.

4 Scope of the resolution

The formal development also proves a conditional passage to the direct nonnegative value of (1), using the convention $0/0 = 0$ for real division. It assumes the corresponding inequality for every strictly positive vector. Given

[1] The finite first-positive step is formalized by `Yamagami.exists_hasSingularRun_of_zero_window`; its application to a zero denominator is `Yamagami.exists_hasSingularRun_of_forwardDenominator_eq_zero` in `QICLean/Analysis/YamagamiBoundary`.

[2] The reusable finite limsup estimate is `Yamagami.limsup_sum_le_card_mul_of_tendsto_zero_on`. The boundary-condition form of (20) and its comparison with N/s are `Yamagami.limsup_functional_le_sub_ratio_of_singularDenominator` and `Yamagami.limsup_functional_le_card_ratio_of_singularDenominator` in `QICLean/Analysis/YamagamiBoundary`.

a nonnegative vector x, it uses the uniform perturbation

$$y_q^{(r)} = x_q + \frac{1}{r+1}. \tag{21}$$

This conditional passage is formalized separately.[3] It does not supply the strictly positive interior inequality.

The regular-boundary recurrence on p. 525 of Ref. [Yam93] is formalized by `Yamagami.functional_le_deleteLastCoordinate`. The finite-coordinate Nowosad theorem [Now68] and the uniqueness assertion in Lemma 3 of Ref. [Yam93] are formalized by `Nowosad.finiteCoordinate_theorem_one_eight` and `Yamagami.lemma_three_localMax_isScalar`, respectively. The cyclic matrix package in `QICLean/Analysis/YamagamiCyclicMatrix` verifies the Fourier–Hessian hypotheses and identifies the generic Nowosad functional with the concrete cyclic functional `Yamagami.functional`.

The declaration `Yamagami.functional_card_le_one` assembles these ingredients. Its compact-superlevel argument produces a positive maximizer, and the Fourier–Hessian and Nowosad uniqueness theorem makes the maximizer scalar.

5 Verdict

The omitted simultaneous-singularity calculation is a resolved clarification. The formal lemma proves the global counting and limsup argument suggested by (14) and (15) without attributing the unprinted calculation to Ref. [Yam93]. This resolution supplies one boundary input to the subsequently completed cyclic inequality.

References

[Now68] Pedro Nowosad. Isoperimetric eigenvalue problems in algebras. *Communications on Pure and Applied Mathematics*, 21(5):401–465, 1968.

[Yam93] Shigeru Yamagami. Cyclic inequalities. *Proceedings of the American Mathematical Society*, 118(2):521–527, 1993.

[3] Formal declarations: `Yamagami.functional_le_of_strictlyPositive` and `Yamagami.functional_le_card_ratio_of_strictlyPositive` in `QICLean/Analysis/YamagamiBoundary`.