Yamagami Lemma 6: The Strict-Disk Endpoint at Stride One

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Verdict: false-source (resolved).

1 The source statement

Let C be the permutation matrix for the cyclic shift on $\mathbb{Z}/n\mathbb{Z}$. For integers l, m, n, s in the range

$$1 \leq m \leq n - l, \tag{1}$$

$$n \leq s, \tag{2}$$

Lemma 6 of Ref. [Yam93] considers the cyclic matrix

$$S_{n,m,s} = (s - m)I + C^l + C^{l+1} + \cdots + C^{l+m-1}. \tag{3}$$

Its row sum is s. The lemma states that every eigenvalue $\lambda \neq s$ lies in the open disc

$$\left|\lambda - \frac{s}{2}\right| < \frac{s}{2}. \tag{4}$$

This is the spectral condition used by Corollary 5 of Ref. [Yam93, pp. 523–524].

2 The false stride-one endpoint

Choose the stride-one endpoint

$$l = 1, \tag{5}$$

$$m = n - 1, \tag{6}$$

$$s = n. \tag{7}$$

These values satisfy (1) and (2). The matrix in (3) becomes

$$S_{n,n-1,n} = I + C + C^2 + \cdots + C^{n-1}. \tag{8}$$

Every nontrivial standard character of $\mathbb{Z}/n\mathbb{Z}$ is an eigenvector of (8) with eigenvalue zero. For such an eigenvalue,

$$\left|0 - \frac{n}{2}\right| = \frac{n}{2}. \tag{9}$$

Equation (9) contradicts the strict inequality in (4). At this endpoint, the strict triangle-inequality step in the proof of Lemma 6 of Ref. [Yam93] also loses all strictness.

3 The correction used by QICLean

The Choi-type application requires only stride one with $m \leq n-2$. Under the assumptions

$$n \geq 3, \tag{10}$$
$$s \geq n, \tag{11}$$
$$m \leq n-2, \tag{12}$$

the formal development proves (4) for every nonzero standard character.[1] The proof follows the two cases used at $s = n$ in Lemma 6 of Ref. [Yam93] and then applies the source's translation argument from n to s.

The omitted endpoint $m = n-1$ also satisfies the open-disc condition when $s > n$. Its eigenvalue on every nontrivial character is

$$\lambda = s - n. \tag{13}$$

For this eigenvalue, the open-disc condition is equivalent to

$$\left|s - n - \frac{s}{2}\right| < \frac{s}{2}, \tag{14}$$
$$0 < n < s. \tag{15}$$

Thus the corrected stride-one alternatives are

$$m \leq n-2 \qquad \text{with } s \geq n, \tag{16}$$
$$m = n-1 \qquad \text{with } s > n. \tag{17}$$

[1] Formal declaration: `Yamagami.norm_symbol_sub_half_lt_half` in `QICLean/Analysis/CyclicReciprocal`.

4 Scope

Verdict: false source endpoint, resolved for the required stride-one range. This correction concerns only the open-disc condition in Lemma 6 of Ref. [Yam93]; it does not by itself prove Yamagami's cyclic reciprocal inequality.

The finite Nowosad theorem [Now68], the uniqueness assertion in Lemma 3 of Ref. [Yam93], and the simultaneous-singularity boundary count are now formalized. The cyclic matrix package in `QICLean/Analysis/YamagamiCyclicMatrix` proves invertibility, positive semidefiniteness of the Hessian with kernel $\mathbb{R}\mathbf{1}$, and the bridge from the generic Nowosad functional to the concrete cyclic functional `Yamagami.functional`. The regular-boundary recurrence on p. 525 of Ref. [Yam93] is formalized by `Yamagami.functional_le_deleteLastCoordinate`.

The declaration `Yamagami.functional_card_le_one` assembles these ingredients. Its compact-superlevel argument produces a positive maximizer, and the Fourier–Hessian and Nowosad uniqueness theorem makes that maximizer scalar. The endpoint correction is therefore a resolved prerequisite of the completed positivity theorem.

References

[Now68] Pedro Nowosad. Isoperimetric eigenvalue problems in algebras. *Communications on Pure and Applied Mathematics*, 21(5):401–465, 1968.

[Yam93] Shigeru Yamagami. Cyclic inequalities. *Proceedings of the American Mathematical Society*, 118(2):521–527, 1993.