Wolf Theorem "Asymptotic Convergence I" — Jordan-Scaling Qualifications

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Verdict: scope-restriction (open).

1 The source statement

For a linear map T, Wolf defines μ as the largest modulus of an eigenvalue of T in the open unit disc. He defines d_μ as the largest dimension of a Jordan block associated with an eigenvalue of modulus μ [Wol12, Chapter 8, lines 1221–1272]. The theorem "Asymptotic convergence I" compares the difference between the powers of the phase-weighted map $\widehat{T}$ and its peripheral part $\widehat{T}_\varphi$ with the scale

$$\mu^n n^{d_\mu - 1}. \tag{1}$$

More precisely, the source compares

$$\left\| \widehat{T}^n - \widehat{T}_\varphi^n \right\|_\infty \quad \text{with} \quad \mu^n n^{d_\mu - 1}. \tag{2}$$

The local source is `Notes/WolfNoteTexSource/ch08_distance_measures.tex`, lines 1221–1272, especially Equations (8.106)–(8.109) of Ref. [Wol12].

2 Empty and zero subperipheral spectra

If T has no eigenvalue in the open unit disc, neither the largest modulus μ nor the associated block size d_μ is defined in the printed statement of Ref. [Wol12]. The formal definition totalizes only the modulus by setting

$$\mu = 0. \tag{3}$$

On a nonempty subperipheral spectrum, finiteness shows that the defining supremum is an attained maximum strictly below one.[1]

[1] Formal declarations: `Module.End.subperipheralModulus`, `Module.End.exists_norm_eq_subperipheralModulus`, and `Module.End.subperipheralModulus_lt_one` in `QICLean/Analysis/SubperipheralSpectrum.lean`.

The equality in (3) does not imply that the subperipheral spectrum is empty. It means that every subperipheral eigenvalue is zero. A zero-eigenvalue Jordan block of dimension $D > 1$ is nilpotent, but its powers need not vanish when $1 \le n < D$. Consequently, an upper bound of the form

$$C_2 \mu^n n^{d_\mu - 1} = 0 \tag{4}$$

cannot control those initial powers.

A classical three-state transition gives a concrete example. Let state 1 map to itself, state 2 map to state 1, and state 3 map to state 2. The corresponding column-stochastic matrix is

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

For the standard basis vectors e_1, e_2, e_3, one has

$$P(e_3 - e_1) = e_2 - e_1 \neq 0, \tag{6}$$

$$P(e_2 - e_1) = 0. \tag{7}$$

Equations (6) and (7) exhibit a size-two Jordan chain at zero. The associated measure-and-prepare quantum channel is positive and trace preserving. Its phase-weighted peripheral map annihilates this chain. At $n = 1$, the left-hand side of Equation (8.107) of Ref. [Wol12] is therefore nonzero, whereas the printed right-hand side is zero.

The zero-modulus case is thus not merely a boundary case excluded by the source hypotheses. A correct zero-modulus statement must either begin at the nilpotence index or treat the finite initial range separately without the factor in (4). The source uses $n \in \mathbb{N}$ as a positive exponent in the proof at lines 1247–1266 of Ref. [Wol12]. In Lean, natural numbers include zero, so the zeroth power additionally requires an explicit convention for 0^0 and does not belong to that source argument.

For $0 < \mu \le 1$ and $d_\mu - 1 \le n$, the exact fixed-block consequence of Equation (8.104) of Ref. [Wol12] is formalized as a two-sided estimate for one supplied Jordan block.[2] This result proves the scale in (1) for a single block. It does not extend that formula across the zero-modulus boundary.

3 The block-diagonal comparison

Let J be a block-diagonal Jordan matrix, and let B be one of its blocks. At source line 1261, the notes assert

$$\|J^n\|_\infty \le \|B^n\|_\infty. \tag{8}$$

[2]Formal declaration: `Matrix.l2_opNorm_jordanBlock_pow_polynomial_bounds` in `QICLean/Analysis/JordanBlockAsymptotics.lean`.

This is the inequality printed for the block denoted by X in Ref. [Wol12]. In the Hilbert-space operator norm, an arbitrary chosen block instead satisfies

$$\|B^n\|_\infty \le \|J^n\|_\infty. \tag{9}$$

The intended eventual-dominance argument must select a block with largest subperipheral modulus and, among blocks tied in modulus, largest dimension. It must then prove that this block realizes the maximum of all block norms for sufficiently large n. Only that argument yields the eventual equality

$$\|J^n\|_\infty = \|B^n\|_\infty. \tag{10}$$

4 The Jordan condition infimum

Let L be a matrix similar to a Jordan matrix J, and choose an invertible matrix A such that

$$L = AJA^{-1}. \tag{11}$$

Submultiplicativity gives the basis-level comparison

$$\big(\|A\|_\infty \|A^{-1}\|_\infty\big)^{-1} \|J^n\|_\infty \le \|L^n\|_\infty, \tag{12}$$

$$\|L^n\|_\infty \le \|A\|_\infty \|A^{-1}\|_\infty \|J^n\|_\infty. \tag{13}$$

This comparison is formalized for the chosen basis.[3]

Equation (8.108) of Ref. [Wol12] instead uses the infimum κ_T over all Jordan changes of basis. Replacing the chosen factor in (12) and (13) by this infimum requires either attainment of the infimum or a limiting argument compatible with the selected normalized Jordan form. The imported Lean Jordan API does not currently provide such a result, so the formal declaration does not assume a minimizing basis.

Under the detailed-balance hypothesis in Equation (8.110) of Ref. [Wol12], the formal development constructs Wolf's chosen diagonalizing change of basis and its inverse:

$$A = S^{1/2}U, \tag{14}$$

$$A^{-1} = U^\dagger S^{-1/2}. \tag{15}$$

It also obtains the condition factor

$$\|A\|_\infty \|A^{-1}\|_\infty = \sqrt{\|S\|_\infty \|S^{-1}\|_\infty}. \tag{16}$$

These constructions and the norm calculation are formalized directly.[4] The formal statement deliberately does not rename the chosen factor in (16) as κ_T.

[3] Formal declaration: `Matrix.l2_opNorm_pow_similarity_bounds` in `QICLean/Analysis/JordanSimilarity.lean`.

[4] Formal declaration: `Matrix.PosDef.exists_sqrt_mul_unitary_diagonalization_of_detailedBalance` in `QICLean/Channel/DetailedBalance.lean`.

Completing the printed κ_T bound still requires the source-shaped infimum tracked in QICLean issue #297 and a proof that this infimum is bounded above by the condition factor of every admissible Jordan change of basis. The detailed-balance assembly is tracked in QICLean issue #298.

5 Verdict

Verdict: source qualifications and missing Jordan-form infrastructure. The shared modulus, the basis-level similarity inequalities in (12) and (13), and the positive-modulus single-block scaling are proved. The full statement for $\widehat{T}^n - \widehat{T}_\varphi^n$ still requires a constructive Jordan decomposition, largest-block data d_μ, the direct-sum norm maximum, eventual dominance, the zero-modulus and positive-exponent qualifications, and a justified passage to κ_T. Until these dependencies are supplied, Equations (8.106)–(8.109) of Ref. [Wol12] should not be marked formalized. The remaining work is tracked in QICLean issue #297.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.