Complete-Positivity Scope of the Peripheral Cyclicity Theorem

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Verdict: scope-restriction (open).

1 The source theorem

Let $D \geq 1$, and let $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be an irreducible positive unital linear map satisfying the Schwarz inequality. Theorem 6.6 of Ref. [Wol12], stated at lines 244–252 of `Notes/WolfNotePDF/wolf_ch6_sections.txt`, describes the peripheral spectrum of T. Item (1) states that this spectrum is a finite cyclic group whose order m satisfies

$$m \leq D^2. \tag{1}$$

Items (2)–(4) establish nondegeneracy, the existence of a peripheral unitary, and a family of cyclic spectral projections, respectively [Wol12, Theorem 6.6]. Neither complete positivity nor a Kraus representation is a hypothesis of the source theorem.

2 The formal restriction

Let I be a finite index set, let $K_i \in M_D(\mathbb{C})$ for each $i \in I$, and define the Kraus map

$$E(X) = \sum_{i \in I} K_i X K_i^\dagger. \tag{2}$$

The available formal theorem is stated for this finite matrix family. It assumes that E is unital and irreducible and that the adjoint $E^\dagger$ has a positive-definite fixed point.[1]

[1] Formal declaration: `Kraus.peripheralEigenvalues_eq_range_primitiveRoot` in `QICLean/Channel/Peripheral/CyclicGroupKraus.lean`.

The finite Kraus representation in (2) makes E completely positive. Complete positivity is stronger than the positive Schwarz hypothesis in Theorem 6.6 of Ref. [Wol12]. The positive-definite fixed point of $E^\dagger$ is also additional signature-level data; the source theorem does not assume such a fixed point. In the finite-dimensional Kraus setting, this datum should instead be derived in two steps. Unitality of E makes $E^\dagger$ trace preserving, which yields a positive-semidefinite fixed point, and irreducibility then makes that fixed point positive definite.

The formal theorem proves the completely positive specialization of the cyclicity conclusion in Theorem 6.6(1) of Ref. [Wol12]. It does not prove the order bound in (1), and it does not state items (2)–(4). Separate finite-Kraus companion declarations establish nondegeneracy, peripheral unitaries, and cyclic projections under corresponding Kraus hypotheses. These declarations do not provide the abstract positive unital Schwarz-map formulation of the source theorem. The corresponding Blueprint entry therefore uses a specialization title.

3 Elimination plan

The first step is to remove the explicit fixed-point assumption from the finite-Kraus theorem. One must derive a positive-semidefinite fixed point of $E^\dagger$ from trace preservation and then use irreducibility to prove that the fixed point is positive definite.

The second step is to formulate peripheral eigenvectors and their multiplicative-domain relations for an abstract positive unital Schwarz map. The proof must establish the invertibility of peripheral eigenvectors and the closure of peripheral eigenvalues under multiplication without selecting Kraus operators. The finite-group argument used by the present cyclicity theorem then applies without change. After this abstract result is available, the finite-Kraus theorem should remain as a specialization rather than serve as the source-facing form of Theorem 6.6 of Ref. [Wol12].

4 Verdict

Verdict: scope restriction. The cyclicity declaration proves Theorem 6.6(1) of Ref. [Wol12] under the stronger complete-positivity hypothesis supplied by a finite Kraus representation. It also takes a positive-definite adjoint fixed point as explicit input and omits the bound in (1). Finite-Kraus companion specializations cover conclusions corresponding to items (2)–(4), but the abstract Schwarz-map formulation of all four items and the order bound in item (1) remain open at the source-facing interface.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.