

Wolf Theorem 6.4: Positive-Map Scope and the Zero-Map Boundary

2026-08-29

Verdict: false-source (resolved).

1 Source statement

Theorem 6.4 of Ref. [Wol12] concerns a positive linear map

$$T : M_D(\mathbb{C}) \longrightarrow M_D(\mathbb{C}) \tag{1}$$

with spectral radius r . It states that irreducibility is equivalent to the following three properties:

1. r is an ordinary nondegenerate eigenvalue of T ;
2. there is an $X > 0$ satisfying $T(X) = rX$;
3. there is a $Y > 0$ satisfying $T^*(Y) = rY$.

Here T^* is the adjoint for the bilinear trace pairing

$$\mathrm{tr}(T^*(Y)Z) = \mathrm{tr}(YT(Z)) . \tag{2}$$

The exact local source is `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 698–721. No complete-positivity or Kraus hypothesis occurs in the statement or proof.

The word “nondegenerate” is geometric: the ordinary complex eigenspace at r has dimension one. The source does not assert algebraic simplicity or a one-dimensional generalized eigenspace.

2 Formal statement

The witness structure `WolfSpectralProperties` and predicate `HasWolfSpectralProperties` in [QIClean/Channel/Irreducible/PositiveMapSpectralCharacterization](#) record a real number $r > 0$ and matrices $X, Y > 0$ such that

$$\varrho(T) = r, \quad (3)$$

$$T(X) = rX, \quad (4)$$

$$\text{traceAdjointMap}(T)(Y) = rY, \quad (5)$$

$$\dim_{\mathbb{C}} \ker(T - r\mathbf{1}) = 1. \quad (6)$$

Thus the left eigenvector is source-facing and does not depend on a choice of Kraus family.

For a nonzero positive map on a nonzero full matrix algebra, the declaration `wolf_theorem_6_4` proves

$$T \text{ is irreducible} \iff T \text{ has the properties (3)–(6)}. \quad (7)$$

The Lean hypotheses are `IsPositiveMapT`, $T \neq 0$, and `NeZeroD`. There is no complete-positivity assumption.

3 Necessary boundary correction

The printed statement does not exclude the zero map, although its displayed condition requires

$$T(X) = rX > 0. \quad (8)$$

On $M_1(\mathbb{C})$, the zero map is positive and irreducible: the only orthogonal projections are 0 and $\mathbf{1}$. Its spectral radius is zero, so (8) is impossible. The printed equivalence is therefore false at this boundary.

The hypothesis $T \neq 0$ is the uniform correction. If $D > 1$, irreducibility already excludes the zero map. The assumption `NeZeroD` separately excludes the empty matrix algebra, in which positive-definite density witnesses are unavailable.

4 Source-faithful proof

For the forward implication, `hasWolfSpectralProperties_of_irreducible_positive` applies the positive-map form of Theorem 6.3 to T . It obtains

$X > 0$, the ordinary one-dimensional r -eigenspace, and the spectral-radius identity. The trace-adjoint consequence of Theorem 6.3 supplies $Y > 0$ at the same eigenvalue.

For the reverse implication, set $S = \sqrt{Y}$ and define exactly the transform used at local source lines 712–714,

$$T' = r^{-1} \text{similarityMap}(S^{-1}, T), \quad T'(Z) = r^{-1} S T(S^{-1} Z S^{-1}) S. \quad (9)$$

Positivity is preserved by congruence and positive rescaling. Trace preservation follows directly from (2): for $W = S^{-1} Z S^{-1}$,

$$\text{tr}(T'(Z)) = r^{-1} \text{tr}(Y T(W)) = r^{-1} \text{tr}(T^*(Y) W) = \text{tr}(Z). \quad (10)$$

Moreover, $X' = S X S > 0$ is fixed by T' .

If T' preserved a proper corner $P M_D(\mathbb{C}) P$, the positive trace-preserving Cesàro theorem `IsPositiveMap.exists_fixed_density_of_preserves_compression` would produce a stationary density matrix τ in that corner. Pulling τ back through S^{-1} gives an ordinary r -eigenvector of T . The one-dimensional eigenspace therefore makes τ a scalar multiple of X' , which is impossible because $X' > 0$ has full support whereas τ lies in a proper corner. This is precisely the contradiction at local source lines 715–720.

The earlier finite-Kraus package remains available through `HasSpectralProperties`. The declaration `hasSpectralProperties_iff_hasWolfspectralProperties_of_cp` proves that, for a nonzero completely positive map, it is equivalent to the source-general package. It is therefore a specialization rather than the statement of Theorem 6.4 itself.

5 Verdict

Verdict: resolved, with one necessary source correction. The former Kraus/complete-positive restriction and the former uniqueness-only-among-PSD formulation have been removed from the source-facing theorem. Ordinary eigenspace nondegeneracy and the trace-pairing left eigenvector are explicit. The additional hypothesis $T \neq 0$ is necessary only because the printed equivalence fails for the irreducible zero map on $M_1(\mathbb{C})$.

References

- [Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.