Wolf Theorem 6.3: Positive-Map Scope and Boundary Corrections

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Verdict: false-source (resolved).

1 Source statement

Theorem 6.3 of Ref. [Wol12] concerns an irreducible positive map

$$T : M_D(\mathbb{C}) \longrightarrow M_D(\mathbb{C}). \tag{1}$$

It does not assume complete positivity. The exact local source is `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 604–680.

For each nonzero $X \succeq 0$, source Equations (6.29)–(6.30) define

$$r(X) = \sup\{a \in \mathbb{R} : T(X) - aX \succeq 0\}, \tag{2}$$
$$\widetilde{r}(X) = \inf\{a \in \mathbb{R} : T(X) - aX \preceq 0\}. \tag{3}$$

The theorem identifies the global lower and upper values, constructs a strictly positive eigenvector, proves that its ordinary eigenspace is one-dimensional, compares every positive eigenvalue with the distinguished value, and identifies that value with the spectral radius [Wol12, Theorem 6.3].

2 Positive-map scope

The declaration `exists_wolfTheorem63_of_irreducible_positive` in `QICLean/Channel/Irreducible/SpectralRadius` proves all four conclusions for an arbitrary irreducible positive map on a nonzero full matrix algebra. It produces a density matrix $X > 0$ and a scalar $r \geq 0$ satisfying

$$T(X) = rX, \tag{4}$$
$$\varrho(T) = r. \tag{5}$$

The ordinary complex eigenspace at r has dimension one. Every positive eigenvalue admitting a nonzero positive semidefinite eigenvector equals r. The global lower value is a maximum and the corrected global upper value is a minimum, and both are attained at X.

The source-form declaration `exists_wolfTheorem63_of_irreducible_positive_of_ne_zero` adds the hypothesis $T \neq 0$ and obtains

$$r > 0. \tag{6}$$

The structural growth theorem `growth_posDef_of_irreducible`, the trace-adjoint theorem `IsIrreducibleMap.traceAdjointMap`, and the spectral-radius theorem `spectralRadius_eq_of_posDef_eigenvector_of_positive` assume only positivity. Earlier completely positive declarations remain available as specializations.

Here “non-degenerate” means that the ordinary eigenspace is one-dimensional, exactly as established by the boundary argument at local source lines 653–661. No assertion is made about algebraic multiplicity or the generalized eigenspace.

3 One-dimensional zero-map boundary

The printed theorem does not exclude the zero map, but its proof begins at line 640 with a positive lower value, and item 2 asserts

$$T(X) = rX > 0. \tag{7}$$

This assertion is false in dimension one. On $M_1(\mathbb{C})$, the zero map is positive and irreducible under both the source and repository definitions because the only orthogonal projections are 0 and $\mathbb{1}$. Its spectral radius is zero, and it has no strictly positive eigenvalue.

The assumption-free corrected theorem therefore separates the conclusions

$$X > 0, \tag{8}$$

$$T(X) = rX, \tag{9}$$

$$r \geq 0. \tag{10}$$

The printed conclusion $r > 0$ becomes valid after adding $T \neq 0$. If $D > 1$, irreducibility already excludes the zero map, but the uniform hypothesis records the exact missing boundary.

4 Correction to the upper extremum

At local source lines 617–619, the printed global definitions and comparison are

$$r := \sup_{X \succeq 0} r(X), \tag{11}$$

$$\widetilde{r} := \sup_{X \succeq 0} \widetilde{r}(X), \tag{12}$$

$$r \geq \widetilde{r}. \tag{13}$$

This is not the Collatz–Wielandt pair used by the theorem. The corrected upper quantity is

$$\widetilde{r} = \inf_{\substack{X \succeq 0 \\ X \neq 0}} \widetilde{r}(X). \tag{14}$$

Equivalently, the infimum may be taken over density matrices, as in the compact normalization used by the proof. For an irreducible positive map, the lower supremum and upper infimum are attained at the same positive-definite eigenvector and are equal. The reversed inequality at line 619 is part of the same local source error.

Irreducibility does not imply pointwise equality $r(X) = \widetilde{r}(X)$ for every nonzero $X \succeq 0$. Consider $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$T(Z) = \mathrm{tr}(Z)\mathbb{1}. \tag{15}$$

This map is positive and irreducible: for every nonzero proper projection P, the matrix $T(P) = \mathbb{1}$ is not supported on the range of P. At

$$X = \mathrm{diag}(1, 2), \tag{16}$$

the lower and upper feasibility conditions give

$$r(X) = \min\{3, 3/2\} = 3/2, \tag{17}$$

$$\widetilde{r}(X) = \max\{3, 3/2\} = 3. \tag{18}$$

Equality at a positive eigenvector is sufficient for the theorem.

The declarations `lowerCollatzWielandtValue` and `upperCollatzWielandtValue` define the two pointwise functionals in the extended reals. Their values at the excluded zero matrix are

$$r(0) = +\infty, \tag{19}$$

$$\widetilde{r}(0) = -\infty. \tag{20}$$

If the defining set for the upper functional is empty, its infimum is $+\infty$. For every nonzero $X \succeq 0$, the declaration `lowerCollatzWielandtValue_le_upperCollatzWielandtValue` proves

$$r(X) \leq \widetilde{r}(X). \tag{21}$$

The declaration `lower_and_upperCollatzWielandtValue_eq_of_eigenvector` proves equality at every positive eigenvector. Thus the formal result matches the use at local source lines 649–651 without asserting false pointwise equality for arbitrary positive semidefinite matrices.

5 Source-faithful proof route

The formal proof follows the order of the source argument [Wol12, Theorem 6.3]. Lower and upper feasibility are first defined for every real scalar as in (2) and (3). After normalization to density matrices, zero is lower feasible and negative lower candidates cannot increase the supremum. The maximizing search can therefore be restricted to a compact interval. Positivity and the trace inequality make all upper candidates nonnegative. These observations produce a lower maximizer without changing either source extremum.

Source Equation (6.31) of Ref. [Wol12], together with the positive-map growth theorem, makes the maximizer positive definite and forces its lower residual to vanish:

$$T(X) - rX = 0. \tag{22}$$

Source Equation (6.32) of Ref. [Wol12] is then applied after decomposing an arbitrary matrix into Hermitian and skew-Hermitian parts. This proves that the ordinary eigenspace is one-dimensional.

If $r = 0$, positivity and $X > 0$ imply that a map satisfying $T(X) = 0$ is the zero map, after which the remaining conclusions are immediate. In the branch $r > 0$, the complementary-projection argument at local source lines 604–606 shows that T^* is irreducible. A positive-definite Perron eigenvector for T^* supplies source Equation (6.33) of Ref. [Wol12]. Pairing with this vector proves uniqueness of positive eigenvalues and, after pairing with an upper residual, the corrected upper minimum.

Finally, conjugation by $X^{1/2}$ and rescaling by r^{-1} produce the positive unital map used by Wolf. Proposition 6.1 of Ref. [Wol12] identifies its spectral radius as one, which gives (5).

6 Verdict

Verdict: resolved, with two source corrections. The former complete-positivity restriction has been removed: Theorem 6.3 is formalized for arbitrary positive maps. At the one-dimensional zero-map boundary, the assumption-free theorem requires $r \geq 0$; the printed conclusion $r > 0$ requires the additional hypothesis $T \neq 0$. The second global extremum at local source lines 617–619 is an infimum rather than a supremum, and the printed comparison is correspondingly invalid. The pointwise functionals use extended values, their zero and empty-set boundaries are explicit, and their equality is asserted only at positive eigenvectors.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.