The Abstract Schwarz-Map Hypotheses in Wolf Theorem 6.12

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Verdict: scope-restriction (open).

1 Source statement

A Schwarz map in Chapter 5 of Ref. [Wol12] is a positive unital matrix map E satisfying source Equation (5.2):

$$E(A^\dagger A) - E(A^\dagger)E(A) \succeq 0 \qquad \text{for every } A. \tag{1}$$

This definition appears immediately before Example 5.3, at local source lines 347–352.

Theorem 6.12 of Ref. [Wol12] assumes a unital Schwarz map and an arbitrary full-rank fixed point of its trace-pairing adjoint. Its algebraic conclusion is

$$\mathrm{Fix}(E) = \{X : E(X) = X\}, \tag{2}$$
$$\mathrm{Fix}(E) \text{ is a unital } *\text{-subalgebra.} \tag{3}$$

The proof is at local source lines 1395–1417. It invokes Proposition 6.9 on positive fixed points, at local source lines 1161–1192, to replace the full-rank witness by a positive-definite fixed point before applying the weighted-trace argument. The complete source theorem also invokes the explicit block realization in source Equation (1.39) of Ref. [Wol12]. This note concerns only the $*$-algebra conclusion in (3).

The source theorem does not repeat the word "positive" because positivity is part of the Chapter 5 definition of a Schwarz map. Making positivity explicit in a formal signature does not strengthen the theorem.

2 Weighted Schwarz equality

Assume that $\rho > 0$ satisfies

$$E^*(\rho) = \rho, \tag{4}$$

and let $X \in \mathrm{Fix}(E)$. Positivity of E implies

$$E(X^\dagger) = X^\dagger. \tag{5}$$

Define the Schwarz gap by

$$\begin{align}
\Delta_X &= E(X^\dagger X) - E(X^\dagger)E(X), \tag{6}\\
\Delta_X &\succeq 0. \tag{7}
\end{align}$$

The trace-pairing identity gives the calculation in source Equation (6.60) of Ref. [Wol12]:

$$\begin{align}
\mathrm{tr}(\rho\Delta_X) &= \mathrm{tr}\big(\rho E(X^\dagger X)\big) - \mathrm{tr}\big(\rho E(X^\dagger)E(X)\big)\,, \tag{8}\\
&= \mathrm{tr}\big(E^*(\rho)X^\dagger X\big) - \mathrm{tr}\big(\rho X^\dagger X\big)\,, \tag{9}\\
&= 0. \tag{10}
\end{align}$$

Because ρ is positive definite and Δ_X is positive semidefinite, (10) implies

$$\Delta_X = 0. \tag{11}$$

Repeating the argument for $X^\dagger$ gives equality in both one-sided Schwarz inequalities. The abstract multiplicative-domain theorem then yields, for fixed points X and Y,

$$\begin{align}
E(XY) &= E(X)E(Y), \tag{12}\\
&= XY. \tag{13}
\end{align}$$

This proves closure under multiplication; the preceding identities give closure under adjoints, and the unital hypothesis supplies the identity.

The argument is formalized for arbitrary positive unital Schwarz maps once a positive-definite invariant weight is supplied explicitly.[1]

3 Comparison with the Kraus specialization

The earlier declaration `Kraus.fixedPointsStarSubalgebra` assumed a Kraus representation and is located in `QICLean/Channel/FixedPoint/Algebra`. It is mathematically correct, but complete positivity is stronger than the Schwarz-map hypothesis of Theorem 6.12 in Ref. [Wol12]. The declaration remains a useful specialization of the post-reduction lemma, but it is not the literal formalization of the source theorem.

[1] Formal declarations: `SchwarzMap.schwarz_equality_of_mem_fixedPoints`, `SchwarzMap.mem_multiplicativeDomain_of_mem_fixedPoints`, and `SchwarzMap.fixedPointsStarSubalgebra` in `QICLean/Channel/FixedPoint/AbstractAlgebra`.

4 Verdict

The earlier Kraus-only attribution was a *scope restriction*, and the abstract lemma removes that restriction. A second scope restriction remains. The formal signature assumes the positive-definite invariant weight in (4), whereas Theorem 6.12 assumes only a full-rank fixed point and derives a faithful positive witness through Proposition 6.9 [Wol12, Theorem 6.12 and Proposition 6.9]. The current theorem is therefore the source-faithful post-reduction algebra lemma, not yet a literal formalization of Theorem 6.12.

Eliminating the remaining restriction requires formalizing the proposition on positive fixed parts for the abstract positive trace-preserving trace adjoint, deriving a positive-definite fixed point from the full-rank witness, and then applying the present lemma. The explicit trace-adjoint realization from source Equation (1.39) of Ref. [Wol12] also remains open at this level of generality. The downstream Schrödinger-picture density-block classification, support reduction, and zero-block assembly required for Theorem 6.14 are complete; see `docs/paper-gaps/wolf_theorem6_14_fixed_point_projection_gap.tex`.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.