Wolf Theorem 6.14: Fixed-Point Classification Completion Record

2026-07-19

Verdict: open-gap (resolved).

1 The retraction classification

Proposition 1.5 of Ref. [Wol12] classifies a positive linear retraction onto a unital finite-dimensional $*$-subalgebra; its block formula appears in source Equation (1.40). The local source is `Notes/WolfNoteTexSource/ch01_deconstructing_quantum.tex`, lines 536–570.

The formal theorem `StarSubalgebra.exists_block_densities_of_positive_retraction` proves the complete proposition. It assumes that the map is positive, that its image lies in the subalgebra, and that it fixes every element of the subalgebra. The theorem constructs the unitary direct-sum representation, uses positivity to eliminate off-diagonal inputs, and produces a positive semidefinite trace-one matrix for each summand. It adds no complete-positivity, trace-preservation, or Schwarz hypothesis.

2 The fixed-point theorem

Let $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be positive and trace preserving, and let T^* denote its trace-pairing adjoint. Assume that T^* satisfies the Schwarz inequality. Theorem 6.14 of Ref. [Wol12] asserts the existence of a unitary U, a decomposition

$$\mathbb{C}^D = \mathcal{H}_0 \oplus \bigoplus_{k=1}^{K} \mathbb{C}^{d_k} \otimes \mathbb{C}^{m_k}, \tag{1}$$

and positive-definite density matrices $\rho_k \in M_{m_k}(\mathbb{C})$ such that source Equation (6.63) is

$$\mathrm{Fix}(T) = U \left(0 \oplus \bigoplus_{k=1}^{K} M_{d_k}(\mathbb{C}) \otimes \rho_k \right) U^*. \tag{2}$$

The summand $\mathcal{H}_0$ is a genuine zero summand: every fixed point vanishes on it. The local source for Theorem 6.14 is `Notes/WolfNotePDF/wolf_ch6_sections.txt`, lines 896–905.

3 Mean-ergodic retraction

Every positive trace-preserving matrix endomorphism has bounded forward orbits. Its mean-ergodic projection is positive and trace preserving, is idempotent, and has range exactly $\mathrm{Fix}(T)$. If T is unital, the projection is also unital.[1]

The trace-pairing adjoint of the mean-ergodic projection is a positive unital idempotent retraction onto $\mathrm{Fix}(T^*)$.[2] The proof uses the complementary decomposition for the original mean-ergodic projection and nondegeneracy of the trace pairing. It requires no bounded-orbit or spectral hypothesis for T^*.

4 Full-support block retraction

Assume first that T has a positive-definite fixed point. The Schwarz equality argument makes $\mathrm{Fix}(T^*)$ a unital $*$-algebra, and the adjoint mean-ergodic projection is a positive retraction onto this algebra. Applying Proposition 1.5 of Ref. [Wol12] gives the weighted partial-trace block form of the projection.[3] This formalizes the conditional-expectation step at local source lines 1483–1492.

The formalization orders each block as $\mathbb{C}^{m_k} \otimes \mathbb{C}^{d_k}$, so the adjoint fixed-point algebra is $\mathbb{1}_{m_k} \otimes M_{d_k}(\mathbb{C})$. The partial trace is therefore taken over the first, multiplicity factor:

$$P_T(B) = U \left(\bigoplus_k \sigma_k \otimes \mathrm{tr}_{m_k}((U^* B U)_{kk}) \right) U^*. \tag{3}$$

Using tr_{d_k} in (3) would reverse the factor convention and would be dimensionally inconsistent. The range is

$$\mathrm{ran}(P_T) = U \left(\bigoplus_k \sigma_k \otimes M_{d_k}(\mathbb{C}) \right) U^*, \tag{4}$$

and therefore equals $\mathrm{Fix}(T)$ under the full-support hypothesis.[4]

[1] Formal declarations: `IsPositiveMap.hasBoundedOrbits_of_tracePreserving`, `IsPositiveMap.meanErgodicProjection_isPositiveMap`, and `IsTracePreservingMap.meanErgodicProjection_isTracePreservingMap`.

[2] Formal declarations: `LinearMap.HasBoundedOrbits.range_traceAdjointMap_meanErgodicProjection` and `IsPositiveMap.traceAdjoint_meanErgodicProjection_isPositiveUnitalRetraction`.

[3] Formal declaration: `IsPositiveMap.exists_block_densities_of_adjoint_meanErgodicProjection` in `QICLean/Channel/FixedPoint/FullSupportBlockRetraction`.

[4] Formal declaration: `IsPositiveMap.exists_block_densities_of_meanErgodicProjection` in `QICLean/Channel/FixedPoint/TraceAdjointDensityBlocks`.

5 Maximal stationary support

For every positive trace-preserving map, define

$$\rho_0 = T_\infty(\mathbb{1}). \tag{5}$$

The matrix ρ_0 is positive semidefinite, and its support contains both the support and the range of every fixed point.[5] This supplies the maximal-support witness in the full generality of the source.

Let $\rho \succeq 0$ be any fixed point and let Q be its support projection. Every positive matrix in the corner $QM_D(\mathbb{C})Q$ is mapped back into that corner. Decomposing an arbitrary matrix into four positive matrices extends the result to the entire corner. Choose an isometry V satisfying

$$V^*V = \mathbb{1}, \tag{6}$$

$$VV^* = Q. \tag{7}$$

The compressed map is

$$\widetilde{T}(X) = V^*T(VXV^*)V. \tag{8}$$

It is positive and trace preserving and has a positive-definite fixed point. It also satisfies source Equation (6.52) of Ref. [Wol12]:

$$T(VXV^*) = V\widetilde{T}(X)V^*. \tag{9}$$

When $\rho = \rho_0$, maximality of the support gives source Equation (6.51) of Ref. [Wol12]:

$$\mathrm{Fix}(T) = V\,\mathrm{Fix}(\widetilde{T})V^*. \tag{10}$$

Thus every ambient fixed point vanishes on the complementary block.

For a general stationary ρ, the auxiliary fixed-space equivalence assumes that $QXQ = X$ for every $X \in \mathrm{Fix}(T)$. This is the hypothesis named `hmax` in `IsPositiveMap.fixedPoint_iff_exists_fixedPoint_stationarySupportCompression`. The source-faithful theorem `IsPositiveMap.exists_maximalSupportCompression` exposes no such hypothesis: it sets $\rho_0 = T_\infty(\mathbb{1})$ and derives support maximality from `IsPositiveMap.exists_maximalSupport_fixedPoint`.[6]

6 Schwarz compression and positive-definite blocks

The trace adjoint commutes with isometric compression:

$$\widetilde{T}^*(A) = V^*T^*(VAV^*)V. \tag{11}$$

[5] Formal declaration: `IsPositiveMap.exists_maximalSupport_fixedPoint` in `QICLean/Channel/FixedPoint/MaximalSupportBasic`.

[6] Formal declarations: `IsPositiveMap.map_supported_on_fixedPoint_support` and `IsPositiveMap.exists_maximalSupportCompression` in `QICLean/Channel/FixedPoint/StationarySupportRestriction`.

The Schwarz defect of $\widetilde{T}^*$ is the compression of the Schwarz defect of T^* plus a positive term supported on $\mathbb{1} - VV^*$. Therefore, the Schwarz hypothesis passes from T^* to $\widetilde{T}^*$ using only the positivity of T already assumed in Theorem 6.14 of Ref. [Wol12].

The full-support theorem applied to $\widetilde{T}$ gives

$$\mathrm{Fix}(\widetilde{T}) = U \left(\bigoplus_k \sigma_k \otimes M_{d_k}(\mathbb{C}) \right) U^*, \tag{12}$$

where each σ_k is positive semidefinite and has trace one. Let ρ_{full} be the positive-definite fixed point retained by the compression theorem. Then

$$U^* \rho_{\mathrm{full}} U = \bigoplus_k \sigma_k \otimes X_k. \tag{13}$$

Every principal block $\sigma_k \otimes X_k$ is positive definite. Taking the two partial traces gives

$$\mathrm{tr}_{m_k}(\sigma_k \otimes X_k) = X_k, \tag{14}$$

$$\mathrm{tr}_{d_k}(\sigma_k \otimes X_k) = \mathrm{tr}(X_k)\sigma_k. \tag{15}$$

The first identity implies that X_k is positive definite, so $\mathrm{tr}(X_k) > 0$. The second then implies that σ_k is positive definite. This argument changes no block dimension and introduces no additional kernel summand.[7]

7 Ambient transport

Let U_c be the compressed block unitary and define

$$W = VU_c. \tag{16}$$

The map W is an isometry from the compressed space into the original space. Compare it with the canonical inclusion of $\mathbb{C}^n$ into $\mathbb{C}^{D-n} \oplus \mathbb{C}^n$. The two inclusions have the same Gram matrix. A finite-dimensional unitary extension therefore gives an ambient unitary U satisfying

$$WAW^* = U(0_{D-n} \oplus A)U^* \qquad \text{for every } A \in M_n(\mathbb{C}). \tag{17}$$

Combining (10), (12), and (17) gives

$$\mathrm{Fix}(T) = U \left(0_{D-n} \oplus \bigoplus_k \sigma_k \otimes M_{d_k}(\mathbb{C}) \right) U^*. \tag{18}$$

[7] Formal declarations: `Matrix.traceAdjointMap_stationarySupportCompression`, `IsSchwarzMap.traceAdjointMap_stationarySupportCompression`, and `IsPositiveMap.exists_block_densities_of_maximalSupportCompression` in `QICLean/Channel/FixedPoint/SupportCompressedDensityBlocks`. The positive-definite partial-trace results are `Matrix.PosDef.partialTraceRight`, `Matrix.PosDef.partialTraceLeft`, and `Matrix.PosDef.left_of_kronecker_of_trace_eq_one` in `QICLean/Channel/PartialTrace`.

The first block is exactly the orthogonal complement of maximal stationary support. It is the only zero summand because every σ_k is positive definite.[8]

8 Conclusion

Wolf's Theorem 6.14 is now formalized with its full source hypotheses and conclusion [Wol12, Theorem 6.14]. The theorem assumes exactly that T is positive and trace preserving and that T^* satisfies the Schwarz inequality. Its fixed-point space has the single complementary zero summand asserted in source Equation (6.63). This document therefore records a resolved paper gap.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[8] Formal declarations: `Matrix.exists_unitary_zero_extension_eq` in `QICLean/Algebra/MatrixGramUnitary`, and `IsPositiveMap.exists_fixedPoints_densityBlocks_with_zero` in `QICLean/Channel/FixedPoint/WolfTheorem614`.