Top Schmidt-rank index in the n-positivity threshold of T_η (Wolf Chapter 3)

2026-06-24

Verdict: scope-restriction (resolved).

1 Source statement

Let $D \geq 1$. Chapter 3 of Ref. [Wol12] studies the maps $T_\eta : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ defined for $\eta \in \mathbb{R}_+$ by source Equation (3.11):

$$T_\eta(\rho) = \mathrm{tr}(\rho)\mathbb{1} - \frac{\rho}{\eta}. \tag{1}$$

The formal map is written $\mathrm{tr}(\rho)\mathbb{1} - \eta^{-1}\rho$. It agrees with (1) for $\eta > 0$ and is also defined at $\eta = 0$.

The Choi–Jamiolkowski operator used in the source is

$$\tau_\eta = D^{-1}\mathbb{1} - \eta^{-1}|\Omega\rangle\langle\Omega|. \tag{2}$$

All positive eigenvalues of τ_η equal $1/D$. Its unique nonpositive eigenvalue and the corresponding reduced density matrix are

$$\nu_- = \frac{1}{D} - \frac{1}{\eta}, \tag{3}$$

$$\rho_- = \frac{\mathbb{1}}{D}. \tag{4}$$

Consequently, the Ky–Fan n-norm satisfies

$$\|\rho_-\|_{(n)} = \frac{n}{D}. \tag{5}$$

The source concludes that

$$T_\eta \text{ is } n\text{-positive} \iff \eta \geq n, \qquad 1 \leq n \leq D. \tag{6}$$

At $n = D$, n-positivity is complete positivity. The threshold therefore makes every inclusion in source Equation (3.3) strict:

$$\mathcal{T}_{\mathrm{cp}} = \mathcal{T}_D \subseteq \mathcal{T}_{D-1} \subseteq \cdots \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1. \tag{7}$$

These claims appear in Chapter 3 of Ref. [Wol12].[1]

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex:181--193`, equation label `eq:ch3:3.11`.

2 Point at issue

The initial formal statement proved only

$$T_\eta \text{ is } k\text{-positive} \iff \eta \geq k, \qquad 1 \leq k < D. \tag{8}$$

It omitted the top index $k = D$. This restriction was inherited from the formal version of Wolf's maximal-overlap principle, Lemma 3.1 [Wol12, Lemma 3.1]. For normalized vectors ψ of Schmidt rank at most k, that principle identifies

$$\sup_\psi |\langle\Omega|\psi\rangle|^2 = \|\rho\|_{(k)}. \tag{9}$$

The formal theorem was available only for $1 \leq k < D$ because it depended on a Ky–Fan maximum principle stated for orthogonal projections of rank exactly $k < D$. At $k = D$, the Ky–Fan norm is the full trace and the maximizing projection is $\mathbb{1}$, but that boundary case was absent from the formal maximum principle.

The resulting strictness witness, a k-positive map that is not $(k+1)$-positive, was therefore available only when

$$1 \leq k, \tag{10}$$
$$k + 1 < D. \tag{11}$$

This range proves every strict inclusion in (7) except

$$\mathcal{T}_D \subsetneq \mathcal{T}_{D-1}, \tag{12}$$

which compares complete positivity with $(D-1)$-positivity and requires the threshold at $k = D$.

3 Resolution

The endpoint $k = D$ is proved without the restricted maximal-overlap or Ky–Fan principles. On $M_D(\mathbb{C})$, D-positivity is equivalent to complete positivity. The corresponding formal equivalence is `isCPMap_iff_isNPositiveMap_card`. Complete positivity of T_η is equivalent to positive semidefiniteness of (2).

For every vector ψ, the quadratic form of the Choi operator is

$$\langle\psi|\tau_\eta|\psi\rangle = D^{-1}\|\psi\|^2 - \eta^{-1}|\langle\Omega|\psi\rangle|^2. \tag{13}$$

Because $|\Omega\rangle$ is normalized, Cauchy–Schwarz gives

$$|\langle\Omega|\psi\rangle|^2 \leq \|\psi\|^2, \tag{14}$$

with equality at $\psi = \Omega$. Hence

$$\tau_\eta \succeq 0 \iff \eta \geq D, \tag{15}$$
$$T_\eta \text{ is } D\text{-positive} \iff \eta \geq D. \tag{16}$$

The declaration `Matrix.isNPositiveMap_tEta_iff` proves (8), and `Matrix.isNPositiveMap_tEta_card_iff` proves (16). Both are in `QICLean/Channel/NPositivityChainStrict`. Together they establish the full source range in (6).

The witness `Matrix.exists_isNPositiveMap_not_succ` remains stated for $k+1 < D$. The endpoint thresholds at $k = D-1$ and $k = D$ nevertheless imply

$$\mathcal{T}_D \subsetneq \mathcal{T}_{D-1}, \tag{17}$$

although this strict inclusion is not recorded as a separate witness declaration.

4 Classification

Resolved. The declarations `Matrix.isNPositiveMap_tEta_iff` and `Matrix.isNPositiveMap_tEta_card_iff` now cover the complete source range $1 \leq k \leq D$ from source Equation (3.11) [Wol12, Chapter 3]. The top index uses the Choi-operator argument rather than the restricted Ky–Fan route. No hypothesis foreign to the source is introduced.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.