SIC–POVM Equality Families: the One-Dimensional Degeneracy

2026-08-21

Verdict: local-correction (resolved).

1 Source passage

Let $P_1, \ldots, P_n \in M_d(\mathbb{C})$ be positive semidefinite matrices with $d \leq n$ and

$$\operatorname{tr}(P_i^2) = 1 \qquad (1 \leq i \leq n). \tag{1}$$

The proposition entitled "SIC POVMs" in Chapter 2 of Ref. [Wol12] proves

$$\sum_{i \neq j} |\operatorname{tr}(P_i P_j)|^2 \geq \frac{n(n-d)^2}{(n-1)d^2}. \tag{2}$$

At equality, the proof gives

$$P_i^2 = P_i, \tag{3}$$

$$\operatorname{rank}(P_i) = 1, \tag{4}$$

$$\sum_i P_i = \frac{n}{d}\mathbb{1}, \tag{5}$$

$$\operatorname{tr}(P_i P_j) = \frac{n-d}{(n-1)d} \qquad (i \neq j). \tag{6}$$

The source then asserts that every equality family is linearly independent [Wol12]. This assertion appears at lines 816–823 of the local Chapter 2 source file.

2 The one-dimensional counterexample

The linear-independence conclusion fails when $d = 1$ and $n > 1$. In one dimension, positivity and (1) force

$$P_i = [1] \qquad (1 \leq i \leq n). \tag{7}$$

For this family, each off-diagonal overlap equals one, so

$$\sum_{i \neq j} |\mathrm{tr}(P_i P_j)|^2 = n(n-1), \tag{8}$$

$$\frac{n(n-d)^2}{(n-1)d^2} = \frac{n(n-1)^2}{n-1} \tag{9}$$

$$= n(n-1). \tag{10}$$

Thus the family attains equality in (2). It is linearly dependent whenever $n > 1$ because all its members are equal.

3 The corrected coefficient argument

Assume that an equality family satisfies a linear relation

$$\sum_i c_i P_i = 0. \tag{11}$$

Taking the trace and using (3) and (4) gives

$$\sum_i c_i = 0. \tag{12}$$

Pairing (11) with P_j in the trace inner product and using (6) yields

$$0 = c_j + \frac{n-d}{(n-1)d} \sum_{i \neq j} c_i \tag{13}$$

$$= \left(1 - \frac{n-d}{(n-1)d}\right) c_j. \tag{14}$$

For $n > 1$, the prefactor satisfies

$$1 - \frac{n-d}{(n-1)d} = \frac{n(d-1)}{(n-1)d}. \tag{15}$$

It vanishes exactly when $d = 1$. Therefore every equality family is linearly independent under the corrected condition $d > 1$.

When $n = 1$, the bound (2) is not stated because its right-hand side has denominator $n - 1$. The singleton family is nevertheless linearly independent whenever its sole matrix has unit purity, because that matrix is nonzero.

4 Formalization status

The overlap bound and its equality characterization are formalized in `QICLean/Algebra/SICPOVMBound`. The declaration `Matrix.sicPOVM_offDiag_overl` ap_sq_bound proves (2), and Matrix.sicPOVM_offDiag_overlap_sq_eq_iff proves the full equality characterization (3)–(6).

The corrected linear-independence result is divided into two declarations. The theorem Matrix.sicPOVM_linearIndependent_of_overlap_bound_eq covers $d \geq 2$, and Matrix.singleton_linearIndependent_of_trace_sq_eq_one covers the singleton case. The specialized SICPOVM API then provides the $n = d^2$ operator-basis and channel consequences.

5 Verdict

The overlap bound and equality characterization from Ref. [Wol12] are formalized without an artificial restriction on d. The final linear-independence claim is false for $d = 1$ and $n > 1$. The corrected conclusion assumes $d > 1$, with the singleton case handled separately. This is a local source correction, and it is fully formalized.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.