Schur Complement TFAE — Missing Left-Support Condition

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Verdict: false-source (resolved).

1 Source statement

Let P and R be positive semidefinite complex matrices, and let Q be a complex matrix of compatible dimensions. Theorem 5.2 of Ref. [Wol12] states that the following conditions are equivalent:

1. The block matrix
$$B = \begin{pmatrix} P & Q \\ Q^\dagger & R \end{pmatrix} \tag{1}$$
is positive semidefinite.
2. The right-support condition
$$\ker(R) \subseteq \ker(Q) \tag{2}$$
holds, and
$$P \geq Q R^{+} Q^\dagger, \tag{3}$$
where R^{+} is the pseudoinverse, equal to the inverse on the support of R.
3. The condition in (2) holds, and
$$\left\| P^{-1/2} Q R^{-1/2} \right\|_\infty \leq 1, \tag{4}$$
where the inverse square roots vanish on the corresponding kernels.

The third condition in Theorem 5.2 of Ref. [Wol12] is false when P is singular.

2 Counterexample and correction

Take one-dimensional matrices

$$P = 0, \tag{5}$$
$$Q = 1, \tag{6}$$
$$R = 1. \tag{7}$$

Then $\ker(R) \subseteq \ker(Q)$, and the support inverse gives

$$P^{-1/2} Q R^{-1/2} = 0, \tag{8}$$

so the printed contraction condition (4) holds. However, for $x = (1, -1)^\mathsf{T}$,

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1. \tag{9}$$

Thus the block matrix is not positive semidefinite.

The missing hypothesis is the left-support condition

$$\ker(P) \subseteq \ker(Q^\dagger). \tag{10}$$

3 The Schur-complement equivalence

The formal development defines the block matrix (1), proves its self-adjointness for Hermitian diagonal blocks, and defines the pseudoinverse Schur complement

$$S = P - QR^+Q^\dagger. \tag{11}$$

The corresponding declarations are `SchurComplement.blockMatrix`, `SchurComplement.blockMatrix_isHermitian`, and `SchurComplement.schurComplement` in `QICLean/Channel/Schwarz/SchurComplement`.

Conditions one and two of Theorem 5.2 in Ref. [Wol12] are equivalent under the source hypotheses:

$$B \geq 0 \iff \left(\ker(R) \subseteq \ker(Q) \text{ and } P - QR^+Q^\dagger \geq 0\right). \tag{12}$$

This is `SchurComplement.blockMatrix_posSemidef_iff`. The forward implication combines `SchurComplement.ker_subset_of_block_psd` with quadratic-form minimization at

$$y = -R^+Q^\dagger x. \tag{13}$$

The reverse implication uses the completed-square identity

$$\begin{pmatrix} x \\ y \end{pmatrix}^\dagger \begin{pmatrix} P & Q \\ Q^\dagger & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (R^+Q^\dagger x + y)^\dagger R (R^+Q^\dagger x + y) \tag{14}$$
$$+ x^\dagger (P - QR^+Q^\dagger) x. \tag{15}$$

This identity is formalized as `SchurComplement.schur_complement_quadratic_form`. Its proof uses

$$Q\,\mathrm{supp}(R) = Q, \tag{16}$$

$$RR^+ = \mathrm{supp}(R), \tag{17}$$

$$R^+R = \mathrm{supp}(R), \tag{18}$$

$$(R^+)^\dagger = R^+. \tag{19}$$

4 The corrected contraction condition

The scalar counterexample is formalized as `SchurComplement.wolf_condition_three_not_sufficient`. To state the corrected condition, define

$$K = P^{-1/2} Q R^{-1/2}. \tag{20}$$

Assume both support conditions (2) and (10). Then

$$KK^\dagger = P^{-1/2}(QR^+Q^\dagger)P^{-1/2}. \tag{21}$$

Consequently,

$$P - QR^+Q^\dagger \geq 0 \iff \|K\|_\infty \leq 1. \tag{22}$$

This equivalence is formalized as `SchurComplement.schurComplement_posSemidef_iff_contraction`; its block-matrix form is `SchurComplement.blockMatrix_posSemidef_iff_contraction`.

5 Verdict

Conditions one and two of Theorem 5.2 in Ref. [Wol12] are correct as printed and are formalized without additional hypotheses. Condition three is false as printed, as (5)–(9) show. Adding the necessary left-support condition (10) restores the three-way equivalence. Thus TNLean issue #5501 is resolved by correcting the source statement rather than formalizing the false original assertion.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.