Schmidt-number premise of Wolf's reduction criterion (Wolf eq. (3.18))

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Verdict: scope-restriction (resolved).

1 Source statement

For an integer $n \geq 1$, define the reduction map

$$T_n(X) = \mathrm{tr}(X)\mathbb{1} - \frac{1}{n}X. \tag{1}$$

The reduction-criterion discussion in `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 329–338, states that T_n is n-positive and self-dual [Wol12].

Let ρ be a bipartite state, and let ρ_1 and ρ_2 be its reduced density matrices on the first and second factors. Equation (3.18) of Ref. [Wol12] states that Schmidt number at most n implies

$$n\rho_1 \otimes \mathbb{1} \geq \rho, \tag{2}$$

$$n\mathbb{1} \otimes \rho_2 \geq \rho. \tag{3}$$

Writing $\mathrm{sn}(\rho)$ for the Schmidt number, the source assertion is

$$\mathrm{sn}(\rho) \leq n \Longrightarrow (n\rho_1 \otimes \mathbb{1} \geq \rho \text{ and } n\mathbb{1} \otimes \rho_2 \geq \rho)\,. \tag{4}$$

The source proof has two steps. A state of Schmidt number at most n is a convex combination of pure-state projectors whose vectors have Schmidt rank at most n. The n-positivity of T_n therefore gives

$$\mathrm{sn}(\rho) \leq n \Longrightarrow (T_n \otimes \mathrm{id})(\rho) \geq 0. \tag{5}$$

The algebraic identity

$$(T_n \otimes \mathrm{id})(\rho) = \mathbb{1} \otimes \rho_2 - \frac{1}{n}\rho \tag{6}$$

then implies

$$(T_n \otimes \mathrm{id})(\rho) \geq 0 \Longrightarrow n\mathbb{1} \otimes \rho_2 \geq \rho. \tag{7}$$

Applying the same argument after swapping the tensor factors gives (2).

2 The initially formalized operator step

The first formal theorems established only the implication in (7).[1] For a bipartite matrix ρ and $n \geq 1$, they prove

$$(T_n \otimes \mathrm{id})(\rho) \geq 0 \Longrightarrow n\mathbb{1} \otimes \rho_2 \geq \rho, \tag{8}$$

$$(T_n \otimes \mathrm{id})(\rho^F) \geq 0 \Longrightarrow n\rho_1 \otimes \mathbb{1} \geq \rho, \tag{9}$$

where ρ^F is obtained by swapping the tensor factors. The second hypothesis is the swapped form of $(\mathrm{id} \otimes T_n)(\rho) \geq 0$. The identity (6) is also formalized.[2]

3 Resolution of the Schmidt-number premise

The formal predicate `Matrix.HasSchmidtNumberLE` records that ρ is a finite sum of pure-state projectors whose vectors have Schmidt rank at most n.[3] This is the convex decomposition used in Ref. [Wol12], up to normalization. At $n = 1$, the predicate is equivalent to separability.[4]

For $1 \leq n < D$, the pure-state theorem states that

$$\mathrm{SR}(\psi) \leq n \Longrightarrow (T_n \otimes \mathrm{id})(|\psi\rangle\langle\psi|) \geq 0. \tag{10}$$

This is `Matrix.tensorMapId_tEta_posSemidef_of_hasSchmidtRankLE`. The proof writes ψ using a maximally entangled vector and a square right-factor matrix whose rank equals $\mathrm{SR}(\psi)$. It identifies the ampliation with a right-factor compression of the Choi matrix and applies the n-positivity of T_n.

Summing (10) over the pure terms in a Schmidt-number decomposition proves (5).[5] Composing that result with (8) and (9) yields Equation (3.18) of Ref. [Wol12] with its Schmidt-number premise.[6]

The required positivity threshold is

$$1 \leq n < D \Longrightarrow T_n \text{ is } n\text{-positive}. \tag{11}$$

This is formalized by `Matrix.isNPositiveMap_tEta_iff` in `QICLean/Channel/NPositivityChainStrict`; the reduction map is T_η at $\eta = n$ by `Matrix.reductionMap_eq_tEta`.

4 The rectangular forward theorem

The general forward step allows independent dimensions. Let $T : M_d(\mathbb{C}) \to M_r(\mathbb{C})$ be n-positive, and let $\psi \in \mathbb{C}^d \otimes \mathbb{C}^k$ have Schmidt rank at most n. Write

[1] Formal declarations: `Matrix.reductionCriterion_left` and `Matrix.reductionCriterion_right` in `QICLean/Channel/ReductionCriterion`.

[2] Formal declaration: `Matrix.tensorMapId_tEta_eq`.

[3] The declaration is in `QICLean/Channel/SchmidtNumber`.

[4] Formal declaration: `Matrix.hasSchmidtNumberLE_one_iff_isSeparable`.

[5] Formal declaration: `Matrix.HasSchmidtNumberLE.tensorMapId_tEta_posSemidef`.

[6] Formal declarations: `Matrix.reductionCriterion_left_of_hasSchmidtNumberLE` and `Matrix.reductionCriterion_right_of_hasSchmidtNumberLE`.

ψ using a normalized maximally entangled vector and a rectangular right-factor matrix $X \in M_{d\times k}(\mathbb{C})$ satisfying

$$\operatorname{rank}(X) = \mathrm{SR}(\psi). \tag{12}$$

The ampliation of $|\psi\rangle\langle\psi|$ is a right-factor compression of the rectangular Choi matrix of T. Its quadratic form pulls back to a Choi quadratic form on a vector of Schmidt rank at most $\operatorname{rank}(X)$. Hence

$$\mathrm{SR}(\psi) \le n \Longrightarrow (T \otimes \mathrm{id}_k)(|\psi\rangle\langle\psi|) \ge 0. \tag{13}$$

Summing over a Schmidt-number decomposition gives

$$\mathrm{sn}(\rho) \le n \Longrightarrow (T \otimes \mathrm{id}_k)(\rho) \ge 0. \tag{14}$$

These results are formalized by `Matrix.tensorMapId_posSemidef_of_hasSchmidtRankLE` and `Matrix.HasSchmidtNumberLE.tensorMapId_posSemidef` in `QICLean/Channel/SchmidtNumber`.

The older padding constructions `Matrix.tensorMapId_posSemidef_of_hasSchmidtRankLE_general` and `Matrix.HasSchmidtNumberLE.tensorMapId_posSemidef_general` remain as compatibility declarations in `QICLean/Channel/SchmidtNumberFactors`. The latter now delegates to the direct rectangular theorem. Neither declaration is needed for the source-facing positive-map characterization.

5 Classification

The bipartite-dimension scope restriction is resolved. The general implication in (14) holds for every n-positive map with independent input, output, and bystander dimensions through `Matrix.HasSchmidtNumberLE.tensorMapId_posSemidef`.

The specialized reduction criterion is also dimension-general. The inequalities

$$n\mathbb{1} \otimes \rho_2 \ge \rho, \tag{15}$$
$$n\rho_1 \otimes \mathbb{1} \ge \rho \tag{16}$$

are proved on a general space $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ by `Matrix.reductionCriterion_left_of_hasSchmidtNumberLE_general` and `Matrix.reductionCriterion_right_of_hasSchmidtNumberLE_general`. These theorems compose the rectangular forward result with `Matrix.reductionCriterion_left` and `Matrix.reductionCriterion_right`. The square declarations `Matrix.reductionCriterion_left_of_hasSchmidtNumberLE` and `Matrix.reductionCriterion_right_of_hasSchmidtNumberLE` remain the specializations with $d = d'$.

The only residual scope concerns the positivity threshold of T_n: the left inequality requires $1 \le n < d$, and the right inequality requires $1 \le n < d'$, as in Equation (3.11) of Ref. [Wol12]. The complete-positivity endpoints $n = d$ and $n = d'$, respectively, are documented in `docs/paper-gaps/wolf_t_eta_top_index_scope.tex`. The dimension restriction recorded in this note is fully eliminated.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.