The nonempty-family condition in Wolf's Radon–Nikodym theorem

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Verdict: local-correction (resolved).

1 Source statement

Let $\{T_i\}_{i\in I}$ be a finite family of completely positive linear maps, and let T be another completely positive map, with

$$T_i, T : \mathcal{M}_{d'} \longrightarrow \mathcal{M}_d, \tag{1}$$
$$\sum_{i\in I} T_i = T. \tag{2}$$

Suppose that T has a supplied Stinespring representation

$$V : \mathbb{C}^d \longrightarrow \mathbb{C}^{d'} \otimes \mathbb{C}^r, \tag{3}$$
$$T(A) = V^\dagger (A \otimes \mathbb{1}_r) V. \tag{4}$$

Wolf's Radon–Nikodym theorem for quantum instruments states that there are positive operators $P_i \in \mathcal{M}_r$ satisfying

$$\sum_{i\in I} P_i = \mathbb{1}_r, \tag{5}$$
$$T_i(A) = V^\dagger (A \otimes P_i) V. \tag{6}$$

This statement appears in `Notes/WolfNoteTexSource/ch02_representations.tex`, lines 399–407, under the title "Radon–Nikodym for quantum instruments" in Ref. [Wol12]. The two preceding lines describe it as a corollary of the comparison theorem for ordered completely positive maps [Wol12].

The source does not state separately that the outcome set I is finite. The formal theorem interprets the unqualified sum in (2) as a finite sum, in accordance with the surrounding finite-dimensional development.

2 The nonempty-family boundary

The source-facing formal theorem assumes that the finite index type I is nonempty.[1] This condition does not strengthen the substantive instrument theorem. It records the boundary required for (5) to be a resolution of the identity by instrument components.

If I is empty, then (2) implies

$$T = 0. \tag{7}$$

The zero map admits the supplied Stinespring matrix $V = 0$. However, the conclusion would require

$$\sum_{i \in \varnothing} P_i = 0 \tag{8}$$

$$= \mathbb{1}_r. \tag{9}$$

This is false when the dilation dimension r is nonzero. If $r = 0$, the two sides agree, but nonemptiness is the uniform condition valid for every supplied dilation.

The formal theorem therefore states the source result for nonempty finite instruments and makes the rectangular Heisenberg dimensions explicit. The older declaration `IsCPMap.radon_nikodym_of_stinespring` remains the square-algebra specialization.

3 The formal proof

Choose rectangular Kraus families for the component maps and combine them into an outcome-labelled Stinespring representation $\widehat{V}$ of T. Apply the ordered-map comparison theorem to the reflexive relation $T \leq T$ and the two representations $\widehat{V}$ and V. This gives a contraction C such that

$$\widehat{V} = (\mathbb{1}_{d'} \otimes C)V, \tag{10}$$

$$C^\dagger C \leq \mathbb{1}_r. \tag{11}$$

Restrict the rows of C to the outcome i and call the resulting matrix C_i. Define

$$E_i = C_i^\dagger C_i. \tag{12}$$

Each E_i is positive and represents the corresponding component map. The row decomposition gives

$$\sum_{i \in I} E_i = C^\dagger C. \tag{13}$$

[1]Formal declaration: `IsKrausCP.radon_nikodym_of_stinespring`.

Define the residual operator

$$R = \mathbb{1}_r - C^\dagger C. \tag{14}$$

By (11), R is positive. Its Stinespring compression under V vanishes.

Nonemptiness is used exactly once. Choose $i_0 \in I$ and set

$$P_{i_0} = E_{i_0} + R, \tag{15}$$
$$P_i = E_i \qquad (i \neq i_0). \tag{16}$$

Then

$$\sum_{i \in I} P_i = \sum_{i \in I} E_i + R \tag{17}$$
$$= C^\dagger C + \mathbb{1}_r - C^\dagger C \tag{18}$$
$$= \mathbb{1}_r. \tag{19}$$

Because the compression of R vanishes, the adjustment does not change the component map assigned to i_0. This construction completes the identity resolution even when the supplied dilation is nonminimal.

If $d' = 0$, all source matrices vanish. The separate formal branch assigns $\mathbb{1}_r$ to one chosen outcome and zero to every other outcome. The square declaration is obtained by specializing the rectangular theorem.

4 Verdict

The nonempty case has no mathematical proof gap. The nonempty-family condition is a local correction to the well-posedness boundary of the printed finite-family statement in Ref. [Wol12]. An empty-index theorem would require a different conclusion, such as a restriction to zero dilation space or the omission of the identity resolution. Such a variant is not the instrument theorem used in the development.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.