Top-index scope of the spectral n-positivity criterion (Wolf Prop. 3.2)

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Verdict: scope-restriction (resolved).

1 Source statement

Let $\tau \in \mathcal{M}_{D'} \otimes \mathcal{M}_D$ be Hermitian, with normalized eigenvectors ϕ_i and spectral data

$$\tau = \sum_i \nu_i |\phi_i\rangle\langle\phi_i|, \tag{1}$$

$$\rho_i = \mathrm{tr}_2(|\phi_i\rangle\langle\phi_i|)\,. \tag{2}$$

Here ρ_i acts on the first factor $\mathbb{C}^{D'}$. Let ν_0 be the smallest positive eigenvalue of τ, let ν be its largest eigenvalue, and let $\|\rho_i\|_{(n)}$ denote its Ky–Fan n-norm.

Proposition 3.2 of Ref. [Wol12] gives

$$\inf_\psi \langle\psi|\tau|\psi\rangle \geq \nu_0 + \sum_{i:\nu_i \leq 0} (\nu_i - \nu_0)\|\rho_i\|_{(n)}. \tag{3}$$

This is Equation (3.7) of Ref. [Wol12]. If a single eigenvalue ν_- is non-positive and ρ_- is the reduced density matrix of its eigenvector, the proposition also gives

$$\inf_\psi \langle\psi|\tau|\psi\rangle \leq \nu + (\nu_- - \nu)\|\rho_-\|_{(n)}. \tag{4}$$

This is Equation (3.8) of Ref. [Wol12]. The local source is `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 153–179.

Because ρ_i acts on $\mathbb{C}^{D'}$, its Ky–Fan norm is naturally indexed by $1 \leq n \leq D'$. Proposition 3.1 of Ref. [Wol12] restricts the positivity index to $1 \leq n \leq D$, as recorded in the same local source at lines 90–92. Consequently, the endpoint $n = D'$ belongs to the source range exactly when $D' \leq D$. If $D' > D$, that endpoint extends beyond the source range. The formal statements nevertheless hold for every $n \geq 1$, and hence cover every index used by the source.

2 Formalized statement below the top index

For $1 \leq n < D'$, the formal per-vector theorems prove that every normalized vector ψ of Schmidt rank at most n satisfies the lower bound, and that some such vector satisfies the upper bound.[1]

The infimum formulations use the set of normalized vectors of Schmidt rank at most n.[2] The interpretation of the source phrase "vectors of Schmidt rank n" is discussed in `docs/paper-gaps/wolf_prop_3_2_schmidt_rank_reading.tex`.

3 The top index

At $n = D'$, every normalized vector satisfies the Schmidt-rank constraint, and each reduced density has full Ky–Fan sum

$$\|\rho_i\|_{(D')} = \mathrm{tr}(\rho_i) \tag{5}$$
$$= 1. \tag{6}$$

Let $\nu_{\min}$ denote the smallest eigenvalue of τ. The spectral expansion gives, for every normalized ψ,

$$\langle\psi|\tau|\psi\rangle = \sum_i \nu_i \left|\langle\phi_i|\psi\rangle\right|^2, \tag{7}$$
$$\sum_i \left|\langle\phi_i|\psi\rangle\right|^2 = 1, \tag{8}$$
$$\langle\psi|\tau|\psi\rangle \geq \nu_{\min}. \tag{9}$$

Evaluating at a normalized least-eigenvalue eigenvector gives

$$\inf_{\|\psi\|=1} \langle\psi|\tau|\psi\rangle = \nu_{\min}. \tag{10}$$

At the top index, the right-hand side of the lower bound in Equation (3.7) of Ref. [Wol12] becomes

$$\nu_0 + \sum_{i:\nu_i \leq 0} (\nu_i - \nu_0)\|\rho_i\|_{(D')} = \nu_0 + \sum_{i:\nu_i \leq 0} (\nu_i - \nu_0). \tag{11}$$

This quantity is at most $\nu_{\min}$ and can be strictly smaller. For the spectrum

[1] Formal declarations: `Matrix.IsHermitian.spectral_lower_bound` and `Matrix.IsHermitian.exists_le_spectral_upper_bound` in `QICLean/Channel/NPositivitySpectralCriterion`.

[2] Formal declarations: `Matrix.IsHermitian.le_sInf_schmidtRankLEExpectations` for Equation (3.7) of Ref. [Wol12], `Matrix.IsHermitian.sInf_schmidtRankLEExpectations_le` for Equation (3.8) of Ref. [Wol12], and `Matrix.IsHermitian.sInf_schmidtRankLEExpectations_top` for the top index, all in `QICLean/Channel/NPositivitySpectralCriterion`.

$(-2, -1, 3)$ with $\nu_0 = 3$,

$$\nu_0 + \sum_{i:\nu_i \le 0} (\nu_i - \nu_0) = 3 + (-2 - 3) + (-1 - 3) \tag{12}$$

$$= -6, \tag{13}$$

$$\nu_{\min} = -2. \tag{14}$$

Thus the top-index lower bound follows from (9), but it is not identical to the Rayleigh inequality.

Under the hypothesis of Equation (3.8) of Ref. [Wol12], ν_- is the unique non-positive eigenvalue and therefore equals $\nu_{\min}$. Using (6), its right-hand side is

$$\nu + (\nu_- - \nu)\|\rho_-\|_{(D')} = \nu + (\nu_- - \nu) \tag{15}$$

$$= \nu_- \tag{16}$$

$$= \nu_{\min}. \tag{17}$$

A normalized least-eigenvalue eigenvector realizes this value. The endpoint proof therefore uses the Rayleigh expansion and Parseval's identity, not the Ky–Fan machinery required for $n < D'$.

The top-index minimum is formalized by `Matrix.IsHermitian.spectral_lower_bound_top` and `Matrix.IsHermitian.exists_spectral_lower_bound_top`. The top-index infimum form of the lower bound is `Matrix.IsHermitian.le_sInf_schmidtRankLEExpectations_top`; the upper bound is the corresponding half of `Matrix.IsHermitian.sInf_schmidtRankLEExpectations_top`.

4 Classification

The endpoint is resolved. Equations (3.7) and (3.8) of Ref. [Wol12] are formalized at every index $n \ge 1$, and hence throughout the source range $1 \le n \le D$, regardless of the relation between D and D'. The declarations `Matrix.IsHermitian.spectral_lower_bound` and `Matrix.IsHermitian.exists_le_spectral_upper_bound` cover $1 \le n < D'$. The declarations `Matrix.IsHermitian.spectral_lower_bound_top` and `Matrix.IsHermitian.exists_spectral_lower_bound_top` cover $n \ge D'$ by the endpoint argument in (7)–(10). The infimum form has the same coverage, with `Matrix.IsHermitian.le_sInf_schmidtRankLEExpectations_top` supplying the top-index lower bound. No hypothesis absent from Ref. [Wol12] is used.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.