The Schmidt-rank set in the spectral n-positivity criterion (Wolf Prop. 3.2)

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Verdict: scope-restriction (resolved).

1 The assertion

Let $\tau \in \mathcal{M}_{D'} \otimes \mathcal{M}_D$ be Hermitian, with spectral decomposition

$$\tau = \sum_i \nu_i |\phi_i\rangle\langle\phi_i|, \tag{1}$$

$$\rho_i = \mathrm{tr}_2(|\phi_i\rangle\langle\phi_i|)\,. \tag{2}$$

Here the eigenvectors ϕ_i are normalized, and ρ_i is their reduced density matrix on the first tensor factor. For a positive semidefinite matrix ρ, let $\|\rho\|_{(n)}$ denote its Ky–Fan n-norm, the sum of its n largest eigenvalues.

Let ν_0 be the smallest positive eigenvalue of τ, and let ν be its largest eigenvalue. Proposition 3.2 of Ref. [Wol12] states the lower bound

$$\inf_\psi \langle\psi|\tau|\psi\rangle \geq \nu_0 + \sum_{i:\nu_i \leq 0} (\nu_i - \nu_0)\|\rho_i\|_{(n)}. \tag{3}$$

This is Equation (3.7) of Ref. [Wol12]. If exactly one eigenvalue ν_- is non-positive and ρ_- is the reduced density matrix of a corresponding normalized eigenvector, the proposition also states

$$\inf_\psi \langle\psi|\tau|\psi\rangle \leq \nu + (\nu_- - \nu)\|\rho_-\|_{(n)}. \tag{4}$$

This is Equation (3.8) of Ref. [Wol12]. The local source is `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, line 153, including Equations (3.7) and (3.8). In both statements, the source says that the infimum is over normalized vectors "of Schmidt rank n."

2 The point at issue

For $1 \le n \le \min(D', D)$, define

$$S_{\le n} = \{\psi : \|\psi\| = 1,\ \mathrm{SR}(\psi) \le n\}, \tag{5}$$
$$S_{=n} = \{\psi : \|\psi\| = 1,\ \mathrm{SR}(\psi) = n\}. \tag{6}$$

Then

$$S_{=n} \subseteq S_{\le n}. \tag{7}$$

The phrase "Schmidt rank n" can denote either set.

The proof of Proposition 3.1 in Ref. [Wol12] fixes the intended reading. It expresses n-positivity of T by requiring

$$(T \otimes \mathrm{id}_n)(|\phi\rangle\langle\phi|) \ge 0 \tag{8}$$

for every $\phi \in \mathbb{C}^D \otimes \mathbb{C}^n$. Under the embedding $\mathbb{C}^n \subseteq \mathbb{C}^D$, these vectors have Schmidt rank at most n, not necessarily exactly n. Vectors of smaller Schmidt rank remain in the quantifier. The source makes the same identification later when it writes

$$|\psi\rangle = (\mathbb{1} \otimes X)^\dagger |\widetilde{\psi}\rangle, \tag{9}$$
$$\mathrm{rank}(X) = n, \tag{10}$$

and calls ψ a vector of Schmidt rank n, although this construction only implies $\mathrm{SR}(\psi) \le n$. The relevant local source is `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, Proposition 3.1 and its proof.

This bounded-rank interpretation also carries the operational content: n-positivity of a Hermitian map is equivalent to nonnegativity of the corresponding Choi quadratic form on vectors in $S_{\le n}$.

3 Comparison of the two readings

The lower bound is pointwise. Every $\psi \in S_{\le n}$ satisfies

$$\langle\psi|\tau|\psi\rangle \ge \nu_0 + \sum_{i:\nu_i \le 0} (\nu_i - \nu_0)\|\rho_i\|_{(n)}, \tag{11}$$

because

$$|\langle\phi_i|\psi\rangle|^2 \le \|\rho_i\|_{(n)}. \tag{12}$$

By (7), the same pointwise estimate holds on $S_{=n}$. The bounded-rank version of the lower bound is therefore stronger.

The upper bound uses a vector that maximizes the overlap with ϕ_-. The standard construction from the n largest eigenvalues of ρ_- has Schmidt rank

$$\mathrm{SR}(\psi_{\max}) = \min(n, \mathrm{rank}(\rho_-)), \tag{13}$$

$$|\langle\phi_-|\psi_{\max}\rangle|^2 = \|\rho_-\|_{(n)}. \tag{14}$$

If $\mathrm{rank}(\rho_-) < n$, then $\psi_{\max} \in S_{\leq n}$ but $\psi_{\max} \notin S_{=n}$. In this case,

$$\nu + (\nu_- - \nu)\|\rho_-\|_{(n)} = \nu_-. \tag{15}$$

No vector in $S_{=n}$ attains this value: an attaining vector would belong to the ν_- eigenspace, which is spanned by ϕ_- and contains no vector of Schmidt rank n.

Nevertheless, ϕ_- is a limit of vectors in $S_{=n}$ whenever $n \leq \min(D', D)$. Continuity of the quadratic form therefore gives

$$\inf_{\psi \in S_{\leq n}} \langle\psi|\tau|\psi\rangle = \inf_{\psi \in S_{=n}} \langle\psi|\tau|\psi\rangle. \tag{16}$$

Thus the two readings produce the same infimum bounds, although the exact-rank upper bound may be approached only by a limiting sequence.

4 The formalized statement

The formalization uses the normalized vectors of Schmidt rank at most n and states both inequalities for the infimum of their expectation values. [1] No hypothesis absent from Ref. [Wol12] is used.

The identity (16) is not formalized. A proof requires normalized vectors of exact Schmidt rank n and a rank-raising perturbation. For a normalized vector ψ and a product vector $u \otimes v$ orthogonal to its Schmidt vectors, set

$$\psi_\varepsilon = \sqrt{1-\varepsilon^2}\psi + \varepsilon\, u \otimes v. \tag{17}$$

The expectation changes according to

$$\begin{aligned}\langle\psi_\varepsilon|\tau|\psi_\varepsilon\rangle - \langle\psi|\tau|\psi\rangle &= 2\varepsilon\sqrt{1-\varepsilon^2}\Re(\langle\psi|\tau|u\otimes v\rangle) &(18)\\ &\quad + \varepsilon^2\left(\langle u\otimes v|\tau|u\otimes v\rangle - \langle\psi|\tau|\psi\rangle\right). &(19)\end{aligned}$$

[1] Formal declarations: `Matrix.schmidtRankLEExpectations`, `Matrix.IsHermitian.le_sInf_schmidtRankLEExpectations` for Equation (3.7) of Ref. [Wol12], `Matrix.IsHermitian.sInf_schmidtRankLEExpectations_le` for Equation (3.8) of Ref. [Wol12], and `Matrix.IsHermitian.sInf_schmidtRankLEExpectations_top` for the top index, all in `QICLean/Channel/NPositivitySpectralCriterion`.

The change is $O(\varepsilon)$ in general. The cross term in (18) vanishes only under an additional orthogonality condition involving τ, which the argument does not require. Iterating such perturbations and taking $\varepsilon \to 0$ proves the nontrivial direction of (16).

5 Verdict

The source phrase "vectors of Schmidt rank n" means vectors of Schmidt rank at most n, as the proof of Proposition 3.1 in Ref. [Wol12] shows. The formalized lower and upper bounds are therefore faithful to Equations (3.7) and (3.8) of Ref. [Wol12], and no additional hypothesis is introduced.

The residual scope restriction concerns only the exact-rank reading of the upper bound. The lower bound follows immediately from the bounded-rank result, whereas the upper bound requires the limiting argument leading to (16) rather than an exact-rank witness. That limiting argument is not formalized and is not needed for the source-facing bounded-rank statement. The Lean statement corresponding to Equation (3.8) of Ref. [Wol12] carries a "Scope restriction (Schmidt rank at most n)" marker pointing to this note.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.