Wolf Proposition 6.2 — Trivial Jordan Blocks for Peripheral Eigenvalues

2026-08-05

Verdict: scope-restriction (resolved).

1 The assertion

Let $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a positive linear map that is either trace-preserving or unital. Proposition 6.2 of Ref. [Wol12] states that every eigenvalue λ of T with $|\lambda| = 1$ has equal geometric and algebraic multiplicities. Equivalently, every Jordan block associated with λ is one-dimensional.

The proof in Ref. [Wol12] uses two facts. First, positivity together with trace preservation or unitality makes the iterates of T uniformly bounded in operator norm. Second, powers of a nontrivial Jordan block whose eigenvalue has modulus one grow without bound. These facts are incompatible.

2 Bounded orbits

The bounded-orbit result is formalized under both alternatives in Proposition 6.2 of Ref. [Wol12].[1] Both proofs reduce an arbitrary matrix to the positive-semidefinite case using the same lemma.[2]

For a positive semidefinite input, trace preservation fixes the trace of every iterate. Unitality instead bounds each iterate in the Loewner order by a scalar multiple of the identity and then bounds its trace. The common scalar estimate used in Ref. [Wol12] is

$$\begin{aligned} \operatorname{tr}(A\,T(B)) &\le \|A\|_\infty \|B\|_\infty \operatorname{tr}(\mathbb{1}\,T(\mathbb{1})) && (1) \\ &= d\|A\|_\infty \|B\|_\infty. && (2) \end{aligned}$$

[1] Formal declarations: `IsPositiveMap.hasBoundedOrbits_of_tracePreserving` in `QICLean/Channel/FixedPoint/MeanErgodicProjection`, and `IsPositiveMap.hasBoundedOrbits_of_unital` in `QICLean/Channel/Peripheral/JordanBlocks`.

[2] Formal declaration: `IsPositiveMap.hasBoundedOrbits_of_posSemidef_orbit_bounded` in `QICLean/Channel/FixedPoint/MeanErgodicProjection`.

Under trace preservation,

$$\operatorname{tr}(T(\mathbb{1})) = \operatorname{tr}(\mathbb{1}) \tag{3}$$
$$= d. \tag{4}$$

Under unitality,

$$T(\mathbb{1}) = \mathbb{1}, \tag{5}$$
$$\operatorname{tr}(\mathbb{1}\, T(\mathbb{1})) = \operatorname{tr}(\mathbb{1}) \tag{6}$$
$$= d. \tag{7}$$

Thus (2) holds under either hypothesis. The formal proof follows this direct argument and does not pass through the adjoint map.

3 The Jordan-block argument

Suppose that $X, Y \in M_d(\mathbb{C})$ and $\lambda \in \mathbb{C}$ satisfy

$$(T - \lambda\mathbb{1})X = Y, \tag{8}$$
$$(T - \lambda\mathbb{1})Y = 0, \tag{9}$$
$$Y \neq 0. \tag{10}$$

These relations describe a rank-two generalized eigenvector. Expanding $T = \lambda\mathbb{1} + (T - \lambda\mathbb{1})$ and using (8) and (9) gives

$$T^n X = \lambda^n X + n\lambda^{n-1} Y. \tag{11}$$

This identity is formalized by `IsPositiveMap.pow_apply_rank_two_genEig`.

When $|\lambda| = 1$, the reverse triangle inequality applied to (11) yields

$$\|T^n X\| \geq n\|Y\| - \|X\|. \tag{12}$$

Because $Y \neq 0$, the right-hand side of (12) is unbounded as $n \to \infty$. This contradicts the bounded-orbit theorem under either trace preservation or unitality. Higher-rank generalized eigenvectors reduce to the rank-two case by induction on the length of the generalized-eigenvector chain.

4 Verdict

Both alternatives in Proposition 6.2 of Ref. [Wol12] are formalized. The shared binomial expansion is `IsPositiveMap.pow_apply_rank_two_genEig`; the rank-two prohibitions and the full generalized-eigenvector reductions are `IsPositiveMap.no_rank_two_genEigenvector_of_tracePreserving`, `IsPositiveMap.no_rank_two_genEigenvector_of_unital`, `IsPositiveMap.peripheral_Jordan_trivial_of_tracePreserving`, and `IsPositiveMap.peripheral_Jordan_trivial_of_unital`, all in `QICLean/Channel/Peripheral/JordanBlocks`.

The unital case was formerly recorded as a scope restriction while a proof through the trace adjoint was under consideration. That route is unnecessary: the estimate in (1)–(2) applies directly under either hypothesis. The unital and trace-preserving cases are therefore proved in parallel. The work is tracked in TNLean issue #3984 and closed in TNLean issue #5447.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.