Wolf Proposition 6.1 — The Russo–Dye Factor

2026-08-23

Verdict: scope-restriction (resolved).

1 The assertion

Let $d \geq 1$, and let $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be positive. Proposition 6.1 of Ref. [Wol12] states

$$\varrho(T) \leq \|T(\mathbb{1})\|_\infty. \tag{1}$$

This is Equation (6.2) of the source. If T is unital or trace preserving, then 1 is an eigenvalue, $\varrho(T) = 1$, and every eigenvalue lies in the closed unit disk.[1]

The proof invokes the Russo–Dye estimate

$$\|T(X)\|_\infty \leq \|T(\mathbb{1})\|_\infty \, \|X\|_\infty. \tag{2}$$

This occurs at line 84 of the local source. If $X \neq 0$ and

$$T(X) = \lambda X, \tag{3}$$

then

$$|\lambda| \|X\|_\infty = \|T(X)\|_\infty, \tag{4}$$

$$\|T(X)\|_\infty \leq \|T(\mathbb{1})\|_\infty \, \|X\|_\infty, \tag{5}$$

$$|\lambda| \leq \|T(\mathbb{1})\|_\infty. \tag{6}$$

The final inequality is Equation (6.3) of Ref. [Wol12]. Maximizing over the eigenvalue moduli gives (1).

2 The formalization

The sharp coefficient-one theorem is formalized in `QICLean/Channel/Peripheral/SpectralRadius`. Its principal declarations are as follows.

- `IsPositiveMap.norm_apply_le_norm_map_one_mul_norm` proves (2) for every matrix X.

[1] Local source: `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 73–91.

- `IsPositiveMap.eigenvalue_norm_le_norm_map_one` proves (6).
- `IsPositiveMap.spectralRadius_le_nnnorm_map_one` proves (1).
- `eigenvalue_one_of_map_one_eq_one`, `IsPositiveMap.eigenvalue_norm_le_one_of_map_one_eq_one`, and `IsPositiveMap.spectralRadius_eq_one_of_map_one_eq_one` prove the three unital conclusions without assuming complete positivity.

The trace-preserving results remain independent sibling theorems. The declaration `IsPositiveMap.eigenvalue_norm_le_one_of_tracePreserving` in `QICLean/Channel/Determinant/Bound` gives the closed-unit-disk bound, and `IsPositiveMap.eigenvalue_one_exists_of_tracePreserving` produces a nonzero positive-semidefinite fixed point. The declaration `IsPositiveMap.spectralRadius_eq_one_of_tracePreserving` combines these results to prove the trace-preserving spectral-radius equality.

3 The proof route

The pinned Mathlib version contains neither a standalone Russo–Dye theorem nor a theorem identifying the operator-norm unit ball with the closed convex hull of the unitaries. The formal proof instead derives (2) from Theorem 5.6 of Ref. [Wol12], formalized as `KadisonSchwarz.schwarz_inequality_commuting_dominant_operator`. This is an alternative derivation of the estimate used in the source, not a formalization of the convex-hull theorem.

Set

$$c := \|T(\mathbb{1})\|_\infty. \tag{7}$$

For $c > 0$, define

$$S := c^{-1}T. \tag{8}$$

The map S is positive and subunital. If $\|A\|_\infty \leq 1$, then

$$A^\dagger A \leq \mathbb{1}. \tag{9}$$

Applying Theorem 5.6 of Ref. [Wol12] with dominant operator $\mathbb{1}$ gives

$$S(A)^\dagger S(A) \leq S(\mathbb{1}), \tag{10}$$
$$S(\mathbb{1}) \leq \mathbb{1}, \tag{11}$$
$$S(A)^\dagger S(A) \leq \mathbb{1}. \tag{12}$$

The C*-identity then yields

$$\|S(A)\|_\infty \leq 1. \tag{13}$$

The case $c = 0$ is separate. The same inequality gives

$$T(A)^\dagger T(A) \le T(\mathbb{1}), \tag{14}$$
$$T(\mathbb{1}) = 0, \tag{15}$$
$$T(A)^\dagger T(A) \le 0, \tag{16}$$
$$T(A) = 0. \tag{17}$$

Rescaling an arbitrary nonzero X then proves (2) in all cases. The argument does not decompose X into four positive parts and introduces no factor four.

The formal hypotheses match the source: T is positive and $d \ge 1$. The norm is Mathlib's matrix C*-operator norm, corresponding to $\|\cdot\|_\infty$ in Ref. [Wol12]; it is neither the Frobenius norm nor the row-sum matrix norm.

4 Verdict

The former scope restriction is resolved. Equations (6.2) and (6.3), and the unital conclusions of Proposition 6.1 in Ref. [Wol12], are formalized for arbitrary positive matrix maps with sharp coefficient one and without a complete-positivity assumption. The formal proof derives the source's norm estimate from Theorem 5.6 of the same reference.[2]

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.

[2]Tracking: GitHub issue #36.