Wolf's PPT–Indecomposable Existence Equivalence:
Resolution of the Reverse Implication

2026-08-24

Verdict: scope-restriction (resolved).

1 The source equivalence

For fixed dimensions d and d', Proposition 3.5 of Ref. [Wol12] states

$$\begin{aligned}&\exists\, T : M_d(\mathbb{C}) \longrightarrow M_{d'}(\mathbb{C}) \text{ positive and indecomposable}\\ &\quad\Longleftrightarrow \exists\, \rho \in M_d(\mathbb{C}) \otimes M_{d'}(\mathbb{C}) : \rho \text{ is an entangled density operator and } \rho^{T_1} \geq 0.\end{aligned} \tag{1}$$

Immediately before that proposition, the source writes a decomposable witness as

$$W = P_1 + P_2^{T_1}, \tag{2}$$
$$P_1 \geq 0, \tag{3}$$
$$P_2 \geq 0. \tag{4}$$

This is Equation (3.15) of Ref. [Wol12]. Both the witness decomposition and the PPT condition use partial transpose on the first factor, whose dimension is d.[1]

The witness proof below uses the normalized form of Theorem 3 in Ref. [LKCH00]. In Wolf's first-factor convention, define

$$\mathcal{D}_1 := \left\{ aP_1 + (1-a)P_2^{T_1} : 0 \leq a \leq 1,\, P_1, P_2 \geq 0,\, \mathrm{tr}(P_1) = \mathrm{tr}(P_2) = 1 \right\}. \tag{5}$$

This is the compact convex set of normalized decomposable witnesses used in the separation argument.

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 276–303. Tracked by GitHub issue #22.

2 The rectangular formal statement

For $d, d' \geq 1$, the formalized theorem has the same input-to-output orientation as Proposition 3.5 of Ref. [Wol12]:

$$\begin{aligned}&\exists\, T : M_d(\mathbb{C}) \longrightarrow M_{d'}(\mathbb{C}) \text{ positive and indecomposable}\\ &\qquad \Longleftrightarrow \exists\, \rho \in M_d(\mathbb{C}) \otimes M_{d'}(\mathbb{C}) : \rho \geq 0,\ \mathrm{tr}(\rho) = 1,\ \rho^{T_1} \geq 0,\ \rho \text{ is not separable.}\end{aligned} \tag{6}$$

The fixed nonzero-dimensional theorem is `Matrix.exists_isIndecomposablePositiveMap_iff_exists_isPPT_not_isSeparable`. The declaration `Matrix.exists_isIndecomposablePositiveMap_iff_exists_isPPT_not_isSeparable_all_dimensions` states the same equivalence for arbitrary natural-number dimensions and handles the zero-dimensional boundary explicitly.[2]

The implication from a PPT entangled state to an indecomposable positive map is `Matrix.exists_isIndecomposablePositiveMap_of_isPPT_not_isSeparable`. It applies the rectangular positive-map criterion at level one. If the detecting map were decomposable, one could write

$$T = T_1 + T_2\theta, \tag{7}$$

$$(T \otimes \mathrm{id})(\rho) = (T_1 \otimes \mathrm{id})(\rho) + (T_2 \otimes \mathrm{id})(\rho^{T_1}), \tag{8}$$

$$(T \otimes \mathrm{id})(\rho) \geq 0, \tag{9}$$

contradicting detection.

3 The reverse implication

Let $T : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ be positive and indecomposable, where $d, d' \geq 1$. Let $T^* : M_{d'}(\mathbb{C}) \to M_d(\mathbb{C})$ be its trace-pairing adjoint, and define

$$W_0 := \tau(T^*) \in M_d(\mathbb{C}) \otimes M_{d'}(\mathbb{C}). \tag{10}$$

This is the output-first rectangular Choi matrix of T^*. Its rows and columns are indexed by $\mathrm{Fin}(d) \times \mathrm{Fin}(d')$, in that order. Therefore its first factor has dimension d, as in Equation (3.15) of Ref. [Wol12]; the construction does not exchange the tensor factors.

The trace adjoint of a positive map is positive. The rectangular Choi criterion at level one therefore gives

$$\mathrm{Re}\,\mathrm{tr}(W_0 |\psi\rangle\langle\psi|) \geq 0 \qquad \text{if } \mathrm{SR}(\psi) \leq 1. \tag{11}$$

Indecomposability implies $T \neq 0$. Positivity also implies that $T(\mathbb{1}_d) = 0$ would force $T = 0$. Hence $T(\mathbb{1}_d)$ is a nonzero positive operator. The rectangular

[2] Formal module: `QICLean/Channel/PPTIndecomposable`.

Choi trace identity gives

$$\operatorname{tr}(W_0) = \frac{1}{d'} \operatorname{tr}(T(\mathbb{1}_d)) \,, \tag{12}$$

$$\operatorname{tr}(W_0) > 0. \tag{13}$$

Define the normalized witness

$$W := \frac{W_0}{\operatorname{tr}(W_0)}. \tag{14}$$

It is Hermitian, block-positive, and has trace one.

The trace-adjoint rectangular Choi correspondence identifies decomposability of T with membership of W_0 in the cone $P_1 + P_2^{T_1}$ from Equation (3.15) of Ref. [Wol12]. Positive rescaling preserves this membership. Since T is indecomposable,

$$W \notin \mathcal{D}_1. \tag{15}$$

Strictly separate W from the compact convex set $\mathcal{D}_1$ in the real vector space of complex matrices. Represent the separating real-linear functional by a Hermitian matrix and subtract a multiple of the trace functional. Because W and every $D \in \mathcal{D}_1$ have trace one, there is a Hermitian matrix R such that

$$\operatorname{Re}\operatorname{tr}(WR) < 0, \tag{16}$$

$$\operatorname{Re}\operatorname{tr}(DR) \geq 0 \qquad \text{for every } D \in \mathcal{D}_1. \tag{17}$$

At the endpoint $a = 1$ in (5), the second inequality holds for every positive trace-one matrix P. Scaling extends it to every $P \geq 0$, and self-duality of the positive cone gives

$$R \geq 0. \tag{18}$$

At the endpoint $a = 0$, the inequality holds for Q^{T_1} whenever Q is positive and has trace one. First-factor transpose is self-adjoint for the real trace pairing:

$$\operatorname{Re}\operatorname{tr}\bigl(Q^{T_1} R\bigr) = \operatorname{Re}\operatorname{tr}\bigl(Q R^{T_1}\bigr) \,. \tag{19}$$

Scaling and self-duality therefore give

$$R^{T_1} \geq 0. \tag{20}$$

The transpose remains on the first factor throughout the argument.

The strict inequality (16) implies $R \neq 0$. Since $R \geq 0$,

$$\operatorname{tr}(R) > 0. \tag{21}$$

Define

$$\rho := \frac{R}{\operatorname{tr}(R)}. \tag{22}$$

Then

$$\rho \geq 0, \tag{23}$$
$$\operatorname{tr}(\rho) = 1, \tag{24}$$
$$\rho^{T_1} \geq 0. \tag{25}$$

Moreover,

$$\operatorname{Re}\operatorname{tr}(W\rho) < 0. \tag{26}$$

Block positivity of W makes its pairing nonnegative on every separable operator. Thus (26) proves that ρ is entangled.

The separation and normalization steps are formalized by `Matrix.exists_posSemidef_isPPT_negative_separator_of_not_isDecomposableWitness` and `Matrix.exists_isPPT_not_isSeparable_of_not_isDecomposableWitness`. The complete map-to-state implication is `Matrix.IsIndecomposablePositiveMap.exists_isPPT_not_isSeparable`. The compactness and convexity of $\mathcal{D}_1$, the first-factor trace identity, and the rectangular Choi correspondence are respectively `Matrix.isCompact_setOf_isNormalizedDecomposableWitness`, `Matrix.convex_setOf_isNormalizedDecomposableWitness`, `Matrix.trace_partialTransposeLeft_mul_re`, and `ChoiRectangular.isDecomposablePositiveMap_iff_choiMatrix_traceAdjointMap_isDecomposableWitness`.

This proof does not use closedness of the full unbounded cone or define decomposability by duality. It separates from the compact trace-one set and derives the two positivity conditions from its endpoints, following the normalized witness argument of Ref. [LKCH00].

4 The zero-dimensional boundary

If $d = 0$, every map with domain $M_0(\mathbb{C})$ is zero. If $d' = 0$, every map with codomain $M_0(\mathbb{C})$ is also zero. In either case the map is decomposable, so no indecomposable positive map exists.

The corresponding tensor product is empty, and every matrix on it has trace zero. Hence no density operator exists. Both sides of (6) are false. This proves the arbitrary-dimensional statement without a hidden nonzero-dimensional assumption.

5 Verdict

The scope restriction is resolved. Both implications in Proposition 3.5 of Ref. [Wol12] are formalized with independent dimensions d and d', the source orientation $M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$, the trace-adjoint rectangular Choi witness on $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, and partial transpose on the first factor. The fixed nonzero-dimensional equivalence and the explicit zero-dimensional boundary are proved. No source-level gap remains in this proposition.

References

[LKCH00] Maciej Lewenstein, Barbara Kraus, J. Ignacio Cirac, and Pawel Horodecki. Optimization of entanglement witnesses. *Physical Review A*, 62:052310, 2000.

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.