Exact-Rank Wording in Wolf's Choi and Witness Criteria

2026-08-24

Verdict: false-source (resolved).

1 The printed criterion

Let $T : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ be linear. Define its Choi matrix by

$$\tau_T = (T \otimes \mathrm{id}_d)(|\Omega_d\rangle\langle\Omega_d|), \tag{1}$$
$$\tau_T \in M_{d'}(\mathbb{C}) \otimes M_d(\mathbb{C}). \tag{2}$$

Proposition 3.1 of Ref. [Wol12] assumes $k \le d$ and includes the condition

$$\langle\psi|\tau_T|\psi\rangle \ge 0 \qquad \text{whenever } \mathrm{SR}(\psi) = k, \tag{3}$$

where $\mathrm{SR}(\psi)$ denotes the Schmidt rank. This is condition (3) in the printed proposition. The local source uses "Schmidt-rank n" in both the statement and the final sentence of the proof.[1]

2 The dimension defect

Every $\psi \in \mathbb{C}^{d'} \otimes \mathbb{C}^d$ satisfies

$$\mathrm{SR}(\psi) \le \min\{d', d\}. \tag{4}$$

If $d' < k \le d$, no vector has Schmidt rank exactly k. The condition (3) is then vacuous and cannot characterize k-positivity.

For example, set

$$d = 2, \tag{5}$$
$$d' = 1, \tag{6}$$
$$k = 2, \tag{7}$$
$$T(X) = -\operatorname{tr}(X). \tag{8}$$

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 89–115. Tracked by QICLean issue #24.

There is no Schmidt-rank-two vector in $\mathbb{C}^1 \otimes \mathbb{C}^2$, so the printed quadratic-form condition holds. However, applying $T \otimes \mathrm{id}_2$ to a nonzero product projector produces a negative rank-one matrix. Thus T is not 2-positive.

The final step of the printed proof has the same defect. A vector of the form

$$\psi = (\mathbb{1} \otimes X)^\dagger \widetilde{\psi} \tag{9}$$

satisfies

$$\mathrm{SR}(\psi) \leq \mathrm{rank}(X), \tag{10}$$

but equality need not hold. The compression argument directly proves a bounded-rank criterion, not the printed exact-rank criterion.

3 The corrected statement

The dimension-generic Choi criterion is

$$T \text{ is } k\text{-positive} \iff \langle\psi|\tau_T|\psi\rangle \geq 0 \quad \text{for every } \psi \text{ with } \mathrm{SR}(\psi) \leq k. \tag{11}$$

This is the condition proved by the compression argument. Factoring the coefficient matrix of ψ through $\mathbb{C}^k$ gives

$$\psi = R_X^\dagger \eta, \tag{12}$$

and positivity of the compressed Choi matrix

$$R_X \tau_T R_X^\dagger \geq 0 \tag{13}$$

implies the quadratic-form inequality in (11).

For $d > 0$, the formal theorem permits independent input and output dimensions, including $k = 0$ and $d' = 0$.[2] The projection formulation additionally assumes $k \leq d$, which is precisely the condition required to choose a rank-k projection on the input factor.

If $k \leq d'$ as well, the exact-rank condition follows from the bounded-rank condition by a separate density argument: coefficient matrices of rank exactly k are dense among coefficient matrices of rank at most k, and the quadratic form is continuous. This is not the proof printed in Proposition 3.1 of Ref. [Wol12], and the formal criterion does not use it.

[2]Formal declarations: `ChoiJamiolkowski.isNPositiveMap_iff_forall_hasSchmidtRankLE_choiMatrix_quadraticForm_nonneg_rectangular` and `ChoiJamiolkowski.isNPositiveMap_iff_forall_rankProjection_choiMatrix_sandwich_posSemidef_rectangular` in `QICLean/Channel/Schwarz/ChoiCompression`.

4 The same defect in Proposition 3.3

Let $\rho \in M_d(\mathbb{C}) \otimes M_{d'}(\mathbb{C})$ be a density operator. Proposition 3.3 of Ref. [Wol12] states that ρ has Schmidt number at least $n+1$ if and only if there is a Hermitian operator W such that

$$\operatorname{tr}(W\rho) < 0, \tag{14}$$

$$\langle\psi|W|\psi\rangle \geq 0 \qquad \text{whenever } \operatorname{SR}(\psi) = n. \tag{15}$$

The source defines S_n as the density operators of Schmidt number at most n and separates ρ from S_n. Its converse identifies S_n with the closed convex hull of the same printed exact-rank set.[3]

This exact-rank wording fails when $n > \min\{d, d'\}$. No vector then has Schmidt rank exactly n, so (15) is vacuous. Taking

$$W = -\mathbb{1} \tag{16}$$

gives

$$\operatorname{tr}(W\rho) = -1 < 0 \tag{17}$$

for every density operator, although no such state can have Schmidt number at least $n+1$. The printed equivalence is therefore false whenever its test set is empty.

The dimension-generic correction is

$$\operatorname{sn}(\rho) > n \iff \exists\, W = W^\dagger : \operatorname{tr}(W\rho) < 0 \text{ and } \langle\psi|W|\psi\rangle \geq 0 \text{ whenever } \operatorname{SR}(\psi) \leq n. \tag{18}$$

This is the separating-hyperplane criterion for S_n, because each pure projector in a defining decomposition of S_n has Schmidt rank at most n. It is formalized by `Matrix.not_hasSchmidtNumberLE_iff_exists_witness` in `QICLean/Channel/EntanglementWitness`. It also supplies the witness used in the rectangular positive-map characterization `Matrix.hasSchmidtNumberLE_iff_forall_isNPositiveMap_tensorMapId_posSemidef` in `QICLean/Channel/PositiveMapDetection`.

When $n \leq \min\{d, d'\}$, density and continuity again recover the exact-rank test from the bounded-rank test. This additional argument is not the separating-hyperplane proof printed in Proposition 3.3 of Ref. [Wol12], and the formal theorem does not use it.

5 Conclusion

The exact-rank wording in Proposition 3.1 of Ref. [Wol12] is false under the printed hypothesis $k \leq d$. The same wording in Proposition 3.3 is false when

[3] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 229–243. Tracked by QICLean issue #23.

$n > \min\{d, d'\}$. Replacing exact Schmidt rank by Schmidt rank at most the specified bound gives dimension-generic Choi and witness criteria and matches the respective compression and convex-separation proofs. The formal statements and the Blueprint use this corrected wording; they do not attribute an additional argument to the source.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.