The False Converse in Wolf's Pauli-Block Truncation Proposition

2026-08-23

Verdict: false-source (resolved).

1 The printed equivalence

Let $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a Hermiticity-preserving linear map. In the Pauli basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_0 = \mathbb{1}$, define its transfer matrix by

$$\widehat{T}_{ij} = \frac{1}{2}\operatorname{tr}(\sigma_i T(\sigma_j))\,, \qquad 0 \le i, j \le 3. \tag{1}$$

Write this matrix in block form as

$$\widehat{T} = \begin{pmatrix} \widehat{T}_{00} & r^{\mathsf{T}} \\ v & \Delta \end{pmatrix}, \tag{2}$$

$$\Delta = (\widehat{T}_{ij})_{i,j=1}^{3}. \tag{3}$$

The vectors r and v form the off-diagonal blocks. This notation agrees with the Bloch-ball affine representation $x \mapsto v + \Delta x$ in Equation (2.39) of Ref. [Wol12].

Proposition 2.10 of Ref. [Wol12] defines the Pauli-block truncation by

$$\widehat{T}' := \widehat{T}_{00} \oplus \Delta, \tag{4}$$

$$\widehat{T}' = \begin{pmatrix} \widehat{T}_{00} & 0 \\ 0 & \Delta \end{pmatrix}. \tag{5}$$

The proposition claims that T' is positive, respectively completely positive, if and only if T has the corresponding property.[1]

2 What the proof establishes

Define the Pauli time-reversal map $\Theta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ and its transfer matrix D by

$$\Theta(X) = \operatorname{tr}(X)\mathbb{1} - X, \tag{6}$$

$$D = \operatorname{diag}(1, -1, -1, -1). \tag{7}$$

[1] Local source: `Notes/WolfNoteTexSource/ch02_representations.tex`, lines 984–990.

The transfer matrix of Θ is D. Consequently,

$$T' = \frac{T + \Theta \circ T \circ \Theta}{2}, \tag{8}$$

$$\widehat{T}' = \frac{\widehat{T} + D\widehat{T}D}{2}, \tag{9}$$

$$\frac{\widehat{T} + D\widehat{T}D}{2} = \widehat{T}_{00} \oplus \Delta. \tag{10}$$

The proof of Proposition 2.10 in Ref. [Wol12] uses this average to establish

$$T \text{ positive} \Longrightarrow T' \text{ positive}, \tag{11}$$

$$T \text{ completely positive} \Longrightarrow T' \text{ completely positive}. \tag{12}$$

Double time reversal preserves positivity and complete positivity in this setting, and both cones are convex. This argument proves only the forward implications (11) and (12); it proves neither converse.[2]

The formalization follows this averaging argument. It defines Pauli time reversal, its diagonal transfer coefficients, and the truncation through `Wolf.pauliTimeReversal`, `Wolf.pauliTimeReversalDiagonal`, and `Wolf.pauliBlockDiagonalTruncation`. The transfer-matrix identities are `Wolf.pauliTransferEntry_timeReversal_comp` and `Wolf.pauliTransferEntry_pauliBlockDiagonalTruncation`. The forward preservation results are `Wolf.pauliBlockDiagonalTruncation_isPositiveMap` and `Wolf.pauliBlockDiagonalTruncation_isCPMap`.[3]

3 Counterexample to both converses

Define

$$T(X) = \operatorname{tr}(X) \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{13}$$

This map is Hermiticity preserving. It is not positive because

$$T(\mathbb{1}/2) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \tag{14}$$

$$T(\mathbb{1}/2) \not\geq 0. \tag{15}$$

Its transfer matrix has $\widehat{T}_{00} = 1$ and $\Delta = 0$; its only nonzero off-diagonal data lie in the column v. Truncation therefore gives

$$\widehat{T}' = \operatorname{diag}(1, 0, 0, 0), \tag{16}$$

$$T'(X) = \frac{\operatorname{tr}(X)}{2} \mathbb{1}. \tag{17}$$

[2]Local source: `Notes/WolfNoteTexSource/ch02_representations.tex`, lines 992–998.
[3]All declarations are in `QICLean/Channel/LorentzNormalForm/PauliBlockTruncation`.

The map in (17) is the completely depolarizing channel and is completely positive. Thus T' can be completely positive while T is not even positive. The converse fails for both positivity and complete positivity.

4 Conclusion

The "if and only if" assertion in Proposition 2.10 of Ref. [Wol12] is false as printed. Its proof establishes only the forward implications (11) and (12), which are the statements formalized in Lean. The map (13) shows that no converse follows from Pauli-block truncation. Both printed equivalences should be replaced by forward implications.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.