The Square-Dimension Restriction in Wolf's Generic Normal Form

(restriction resolved 2026-08-04)

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Verdict: scope-restriction (resolved).

1 The assertion

Proposition 2.9 of Ref. [Wol12] gives a normal form for completely positive maps with full Kraus rank.[1] Let

$$T : \mathcal{M}_{d_1} \longrightarrow \mathcal{M}_{d_2} \tag{1}$$

be such a map. The proposition states that there are invertible completely positive maps $\Phi_i \in \mathcal{B}(\mathcal{M}_{d_i})$, each of Kraus rank one, such that

$$T' = \Phi_2 \circ T \circ \Phi_1, \tag{2}$$

$$T'(\mathbb{1}) \in \mathbb{C}\mathbb{1}, \tag{3}$$

$$T'^*(\mathbb{1}) \in \mathbb{C}\mathbb{1}. \tag{4}$$

Conditions (3) and (4) express double stochasticity in the source's sense. The input filtering Φ_1 acts on $\mathcal{M}_{d_1}$, and the output filtering Φ_2 acts on $\mathcal{M}_{d_2}$. Proposition 2.9 of Ref. [Wol12] imposes no relation between d_1 and d_2.

2 The point at issue

The original formal theorem covered only the equal-dimension specialization

$$d_1 = d_2 = D, \tag{5}$$

$$T : M_D(\mathbb{C}) \longrightarrow M_D(\mathbb{C}). \tag{6}$$

Both filterings in this theorem act on $M_D(\mathbb{C})$.[2]

[1] Local source: `Notes/WolfNoteTexSource/ch02_representations.tex`, lines 923–929.

[2] Formal declaration: `Wolf.exists_normal_form_generic` in `QICLean/Channel/LorentzNormalForm`.

For the square map in (6), the Choi matrix is

$$\tau = (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) \in M_D(\mathbb{C}) \otimes M_D(\mathbb{C}). \tag{7}$$

The original minimization therefore ranged over $\mathrm{SL}(D, \mathbb{C}) \times \mathrm{SL}(D, \mathbb{C})$. Its compactness and trace–determinant equality arguments were also stated only for the square space in (7).[3]

At $D = 2$, this square theorem supplies the full-Kraus-rank minimization result used in the diagonal case of the qubit Lorentz normal form. It does not prove the existence theorem combining the diagonal, non-diagonal, and singular cases. The scalar-normalization and Lorentz-orbit gap in that theorem remains recorded in `docs/paper-gaps/wolf_prop2_11_lorentz_scalar_filtering_gap.tex`.

3 The restricted statement

Before the rectangular generalization, the formal development proved

$$T : M_D(\mathbb{C}) \longrightarrow M_D(\mathbb{C}), \tag{8}$$

$$\tau = \mathrm{choi}(T) \succ 0, \tag{9}$$

$$\tau \succ 0 \Longrightarrow \exists\, \Phi_1, \Phi_2 : \ \Phi_2 \circ T \circ \Phi_1 \text{ is doubly stochastic.} \tag{10}$$

Here Φ_1 and Φ_2 are determinant-one filterings on $M_D(\mathbb{C})$. This is the specialization $d_1 = d_2$ of Proposition 2.9 of Ref. [Wol12]. It is correct but narrower than the source's rectangular statement.

4 Resolution

The rectangular generalization was completed on 2026-08-04 under TNLean issue #3616. It consists of two changes.

- The declaration `Wolf.infimum_is_attained` was generalized in place to positive-definite operators on $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_1}$ and filterings in $\mathrm{SL}(d_1, \mathbb{C}) \times \mathrm{SL}(d_2, \mathbb{C})$. The Frobenius-norm confinement now uses separate dimension constants for the two factors. The square caller remains unchanged because its dimensions unify.
- The rectangular proposition is formalized as `Wolf.exists_normal_form_generic_rect`. It applies to a completely positive map

  $$T : \mathcal{M}_{d_1} \longrightarrow \mathcal{M}_{d_2} \tag{11}$$

  in the rectangular Kraus sense `IsKrausCP`, assuming that its rectangular Choi matrix `ChoiRectangular.choiMatrix` is positive definite. It

[3] Formal declarations: `Wolf.infimum_is_attained` and `Matrix.PosSemidef.pow_card_mul_det_re_eq_trace_re_pow_iff`.

produces a Kraus-rank-one filtering Φ_1 on $\mathcal{M}_{d_1}$ and a Kraus-rank-one filtering Φ_2 on $\mathcal{M}_{d_2}$, each represented by a determinant-one matrix, such that

$$T' = \Phi_2 \circ T \circ \Phi_1, \tag{12}$$

$$T'(\mathbb{1}) \in \mathbb{C}\mathbb{1}, \tag{13}$$

$$T'^*(\mathbb{1}) \in \mathbb{C}\mathbb{1}. \tag{14}$$

These conditions are represented by `Wolf.DoublyStochasticRect`, using the trace-pairing adjoint.

5 Verdict

The scope restriction is resolved. Proposition 2.9 of Ref. [Wol12] is now formalized with independent input and output dimensions. The declaration `Wolf.exists_normal_form_generic` remains the unchanged equal-dimension specialization. The remaining gap concerns Proposition 2.11 of Ref. [Wol12], namely the qubit Lorentz classification, and is recorded in `docs/paper-gaps/wolf_prop2_11_lorentz_scalar_filtering_gap.tex`.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.