

Wolf Proposition 1.5: Factor and Coordinate Direct-Sum Scope Restrictions

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Verdict: scope-restriction (resolved).

1 Source statement

Wolf's Proposition 1.5 classifies positive linear retractions onto arbitrary unital finite-dimensional $\ast$ -subalgebras [Wol12, Proposition 1.5]. After unitary conjugation, such a subalgebra has the block form

$$\mathcal{A} = \bigoplus_{k=1}^K M_{d_k}(\mathbb{C}) \otimes \mathbf{1}_{m_k}. \quad (1)$$

The proposition states that a positive retraction onto $\mathcal{A}$ is a direct sum of weighted partial traces, one on each summand; see source Equation (1.40) [Wol12, Equation (1.40)]. Its proof first shows that the retraction removes inputs between distinct central summands and produces no output in an incorrect summand. It then proves the weighted-partial-trace formula on each factor separately. The local source is `Notes/WolfNoteTexSource/ch01_deconstructing_quantum.tex`, lines 536–570.

2 The one-factor theorem

The declaration `Matrix.exists_density_of_positive_retraction_onto_right_factor` proves the complete factor argument for a positive retraction whose range is

$$\mathbf{1}_m \otimes M_d(\mathbb{C}). \quad (2)$$

It introduces no hypothesis absent from the one-factor specialization of Proposition 1.5. In particular, the representing matrix ρ is required only to satisfy

$$\rho \geq 0, \quad (3)$$

$$\mathrm{tr}(\rho) = 1. \quad (4)$$

The theorem therefore gives a faithful one-factor case of the source result, but by itself it does not establish the direct-sum classification in Eq. 1.

3 The coordinate direct-sum construction

The declaration `Matrix.coordinateDirectSumConditionalExpectation_isKrausCTP_and_isConditionalExpectation` constructs the normalized trace-preserving choice simultaneously on all summands in displayed direct-sum coordinates. In QICLean’s tensor-factor order, its range is

$$\bigoplus_{k=1}^K \mathbb{1}_{m_k} \otimes M_{d_k}(\mathbb{C}). \quad (5)$$

This order is obtained from Eq. 1 by interchanging the two tensor factors in each summand.

The coordinate theorem performs the coordinatewise partial traces and removes the off-diagonal blocks. It does not, by itself, conjugate an arbitrarily represented subalgebra into the coordinates of Eq. 5, nor does it corestrict the ambient matrix-valued map to that subalgebra. It is therefore a coordinate construction rather than the full unitarily transported classification of Proposition 1.5 [Wol12].

4 Resolution of the direct-sum argument

The declaration `StarSubalgebra.exists_block_densities_of_positive_retraction` combines the one-factor theorem with the finite-dimensional block decomposition. Its proof proceeds as follows:

1. It conjugates and reindexes the ambient matrices using the unitary block data.
2. It proves that the positive retraction annihilates inputs between distinct central summands.
3. It proves that an input supported in one summand has no output in a different summand.
4. It applies `Matrix.exists_density_of_positive_retraction_onto_right_factor` to each diagonal summand.
5. It conjugates the resulting direct sum of weighted partial traces back to the original subalgebra.

The second and third steps formalize the off-diagonal positivity arguments that precede source Equation (1.42) [Wol12, Equation (1.42)]. For the central projection p_k of the k th summand, these arguments use the complementary projection

$$1 - p_k. \quad (6)$$

They eliminate cross-summand inputs and wrong-summand outputs before the factor theorem is applied.

The resulting theorem supplies positive trace-one block densities ρ_k and reconstructs the source classification as a unitarily transported direct sum of weighted partial traces. The distinction between classifying a given retraction and constructing a fixed-point projection later in the development remains recorded in `docs/paper-gaps/wolf_theorem6_14_fixed_point_projection_gap.tex`.

5 Classification

The one-factor theorem and the coordinate direct-sum construction were individually scope-restricted relative to Wolf's Proposition 1.5. The theorem `StarSubalgebra.exists_block_densities_of_positive_retraction` now combines the missing block, support, and unitary-transport arguments. The scope restriction is therefore resolved, and the formal development covers the full finite-dimensional classification stated in Proposition 1.5 [Wol12, Proposition 1.5].

References

- [Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.