Wolf Proposition 1.6: Complete Positivity from Positivity, Commutative Side

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Verdict: clarification (resolved).

1 Source statement

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $T : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Wolf's Proposition 1.6 states

$$\mathcal{A} \text{ commutative} \quad \text{or} \quad \mathcal{B} \text{ commutative} \Longrightarrow T \text{ completely positive.} \tag{1}$$

See Proposition 1.6 of Ref. [Wol12]. The local source is `Notes/WolfNoteTexSource/ch01_deconstructing_quantum.tex`, lines 600–612.

Wolf proves the finite-dimensional case. He reduces to the case in which $\mathcal{B}$ is commutative by using the equivalence between complete positivity of T and complete positivity of its dual T^*. For a positive block matrix (a_{ij}), the codomain argument simultaneously diagonalizes the pairwise-commuting family

$$\{T(a_{ij})\}_{i,j}. \tag{2}$$

The remark following the proof observes that the algebra structure of $\mathcal{A}$ is unused. The same proof therefore applies to a positive map from an operator system into a commutative C^*-algebra [Wol12, Proposition 1.6].

2 Matrix-algebra formulation

The ambient objects used for this proposition in Lean are linear endomorphisms of $M_D(\mathbb{C})$. The predicates `IsPositiveMap` and `IsCPMap` in `QICLean/Channel/Basic.lean` both fix

$$\mathcal{A} = \mathcal{B} = M_D(\mathbb{C}). \tag{3}$$

A literal requirement that the ambient codomain in Eq. 3 be commutative would force

$$D \leq 1. \tag{4}$$

That specialization is narrower than the argument in Proposition 1.6 and does not express the useful matrix-algebra case.

The formalization instead isolates the range property used by Wolf's proof. For a linear map $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$, the predicate `PositiveOnAbelian.HasCommutingRangeT` means

$$\forall X, Y, \qquad \text{Commute}(T(X))(T(Y)). \tag{5}$$

It is defined in `QICLean/Channel/Schwarz/PositiveOnAbelian/CompletePositivity.lean`. A pairwise-commuting range generates a commutative subalgebra of $M_D(\mathbb{C})$. Thus Eq. 5 says that T maps into a commutative matrix subalgebra, which is exactly the codomain property used by Wolf's proof and by his operator-system generalization [Wol12, Proposition 1.6].

3 Formalized codomain argument

The declaration `PositiveOnAbelian.isCPMap_of_isPositiveMap_of_hasCommutingRange` proves that positivity and a commuting range imply complete positivity:

$$T \text{ is positive} \quad \text{and} \quad \forall X, Y, \text{Commute}(T(X))(T(Y)) \Longrightarrow T \text{ is completely positive.} \tag{6}$$

The proof follows Wolf's route. It reduces complete positivity to nonnegativity of the block quadratic form associated with the Choi matrix and then applies the simultaneous-diagonalization lemma `PositiveOnAbelian.quadraticForm_nonneg_of_isPositiveMap_of_commuting_images`. The remaining wrapping gap after that lemma became sorry-free was closed by TNLean issue #5468.

The condition in Eq. 5 is implied by a commutative ambient codomain. For $D > 1$, it is strictly weaker than commutativity of the full codomain $M_D(\mathbb{C})$: a map can have commutative range even though the ambient matrix algebra is noncommutative. This weakening changes only the ambient presentation, not the source proof, because Wolf explicitly observes that the proof needs only a commutative target for the image under consideration [Wol12, Proposition 1.6].

4 Formalized domain argument

The declaration `PositiveOnAbelian.isCPMap_of_isPositiveMap_of_hasCommutingRange_adjoint` formalizes the commutative-domain case by the duality argument used in the source. Let `Matrix.traceAdjointMapT` denote the adjoint with respect to the trace pairing. The proof uses

$$T \text{ completely positive} \Longleftrightarrow \text{Adj}_{\text{tr}}(T) \text{ completely positive,} \tag{7}$$

where $\text{Adj}_{\text{tr}}(T)$ denotes `Matrix.traceAdjointMapT`. When this adjoint has commuting range, the codomain theorem in Eq. 6 applies to it, and Eq. 7 then gives complete positivity of T.

The declaration `PositiveOnAbelian.isCPMap_of_isPositiveMap_of_commutingRange_or` combines the two cases:

$$T \text{ is positive} \quad \text{and} \quad \big(T \text{ has commuting range or } \mathrm{Adj}_{\mathrm{tr}}(T) \text{ has commuting range}\big) \Longrightarrow T \text{ is completely positive} \tag{8}$$

5 Relationship to Proposition 1.6

The range-only formulation is the correct interpretation of Proposition 1.6 inside a development whose ambient domain and codomain are fixed as full matrix algebras. It neither adds a hypothesis absent from Wolf's argument nor removes a property used by that argument. Instead, it expresses commutativity at the level of the generated image algebra, where simultaneous diagonalization occurs.

Consequently, Eqs. 6 and 8 formalize Proposition 1.6 as Wolf proves and immediately generalizes it [Wol12, Proposition 1.6]. They are not merely corollaries obtained by forcing the entire ambient matrix algebra to be commutative.

6 Classification

The discrepancy was not mathematical. It concerned how to express a commutative domain or codomain inside a fixed full-matrix ambient type. The commuting-range formulation resolves that notational and structural clarification while preserving the complete-positivity conclusion and the source proof.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.