Top-index scope of the maximal-overlap lemma (Wolf Lemma 3.1)

2026-06-24

Verdict: scope-restriction (resolved).

1 Source statement

Let D', D be finite dimensions, and let $\phi \in \mathbb{C}^{D'} \otimes \mathbb{C}^{D}$ be a normalized vector. Define its reduced density matrix on the first tensor factor by

$$\rho = \mathrm{tr}_2(|\phi\rangle\langle\phi|). \tag{1}$$

For $1 \le n \le D'$, let $\|\rho\|_{(n)}$ denote the Ky Fan n-norm, the sum of the n largest eigenvalues of ρ. Wolf's Lemma 3.1 states

$$\sup_{\substack{\psi \text{ normalized} \\ \mathrm{SchmidtRank}(\psi) \le n}} |\langle\phi|\psi\rangle|^2 = \|\rho\|_{(n)}. \tag{2}$$

The local source is `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 119–128; see Lemma 3.1 of Ref. [Wol12].

Because ρ acts on $\mathbb{C}^{D'}$, its Ky Fan n-norm is defined throughout the range $1 \le n \le D'$. In the application to n-positivity in Wolf's Proposition 3.2, the Schmidt-rank index has the same range [Wol12, Proposition 3.2]. In particular, the endpoint $n = D'$ belongs to the source statement.

2 Originally formalized range

The theorem `Matrix.maximalSchmidtOverlap_eq_kyFanNorm` establishes Eq. 2 and strengthens the supremum assertion to attainment through Lean's `IsGreatest` predicate.[1] Its hypotheses originally covered

$$1 \le n < D'. \tag{3}$$

Thus the formal theorem omitted only the endpoint $n = D'$.

[1] Formal declaration: `Matrix.maximalSchmidtOverlap_eq_kyFanNorm` in `QICLean/Channel/MaximalOverlap`.

3 The endpoint

At $n = D'$, every vector in $\mathbb{C}^{D'} \otimes \mathbb{C}^D$ has Schmidt rank at most D', so the Schmidt-rank constraint is vacuous. The normalized choice $\psi = \phi$ gives

$$|\langle\phi|\phi\rangle|^2 = 1. \tag{4}$$

Cauchy–Schwarz gives the matching upper bound

$$|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle\langle\psi|\psi\rangle = 1 \tag{5}$$

for every normalized ψ. Since ρ is a density matrix,

$$\|\rho\|_{(D')} = \mathrm{tr}(\rho) = 1. \tag{6}$$

Equations 4–6 prove the source identity at the endpoint.

The original restriction came from the underlying maximum principle for Ky Fan norms.[2] That theorem characterizes the Ky Fan k-norm as a maximum over orthogonal projections of rank exactly k only when $k < D'$. At $k = D'$, the only rank-D' orthogonal projection is the identity, and the variational value reduces to

$$\mathrm{Re}\,\mathrm{tr}(A). \tag{7}$$

The variational statement degenerates at the endpoint; it does not fail.

4 Resolution

The companion theorem `Matrix.maximalSchmidtOverlap_eq_kyFanNorm_top` now proves the endpoint $n = D'$. It uses the vacuous Schmidt-rank constraint, the maximizing vector $\psi = \phi$, the upper bound in Eq. 5, and the trace identity in Eq. 6.

Together, the declarations `Matrix.maximalSchmidtOverlap_eq_kyFanNorm` and `Matrix.maximalSchmidtOverlap_eq_kyFanNorm_top` cover

$$1 \leq n \leq D', \tag{8}$$

which is the full range of Wolf's Lemma 3.1 [Wol12, Lemma 3.1].

5 Classification

The former omission was a scope restriction in the formalization, not an error in the source theorem. It is resolved by the endpoint companion theorem, exactly as anticipated by the plan to treat the identity projection separately.

[2] Formal declaration: `Matrix.IsHermitian.kyFanNorm_eq_sup_trace`.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.