Resolution of the Choi-Type Positivity Scope (Wolf Example 3.1)

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Verdict: scope-restriction (resolved).

1 Source statement

Let $\mathcal{M}_d = M_d(\mathbb{C})$, let $D\colon \mathcal{M}_d \to \mathcal{M}_d$ be the projection onto the diagonal matrices, and let U_{k0} be the cyclic shift by k coordinates. Example 3.1 in Chapter 3 of Ref. [Wol12] introduces the Choi-type map

$$T_C(X) = (d-n)D(X) - X + \sum_{k=1}^{n} D(U_{k0} X U_{k0}^{\dagger}). \tag{1}$$

This is Equation (3.20) of Ref. [Wol12].[1] The source states that T_C is positive and indecomposable whenever

$$1 \le n \le d-2. \tag{2}$$

It imposes no hypothesis that the coordinates of an input vector be nonzero.

2 Endpoint statements

The endpoint arguments give independent proofs of two slices of the positivity claim.

For the lower endpoint $n = 1$, the development proves that T_C maps every positive semidefinite input to a positive semidefinite output in every dimension $d \ge 3$. The scalar input is the boundary-inclusive cyclic reciprocal inequality

$$\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{x_i}{(d-1)x_i + x_{i-1}} \le 1, \qquad x_i \ge 0, \tag{3}$$

[1] Local source: `Notes/WolfNoteTexSource/ch03_positive_not_completely.tex`, lines 357–365.

where a summand with zero denominator is defined to be zero. In the interior of the nonnegative cone, define

$$y_i = \frac{x_{i-1}}{x_i}, \tag{4}$$

$$\prod_{i\in\mathbb{Z}/d\mathbb{Z}} y_i = 1. \tag{5}$$

Clearing denominators in (3) produces an expression in the elementary symmetric sums $e_k(y)$. The arithmetic–geometric mean inequality gives

$$e_k(y) \geq \binom{d}{k}, \tag{6}$$

and the resulting binomial expression vanishes for the constant family. This is the proof route of Theorem 1 and Lemmas 2–3 of Ref. [TT88, pp. 309–311]: Lemma 3 gives the reciprocal estimate, and Theorem 1 applies the rank-one diagonal-minus-projector criterion from Lemma 2.

The argument in Ref. [TT88, pp. 309–311] assumes strictly positive coordinates. If some x_i vanishes, at most $d-1$ summands in (3) are nonzero, and every nonzero summand is at most $1/(d-1)$. This proves the boundary case as well.[2] A spectral decomposition of a positive semidefinite matrix into rank-one positive semidefinite summands then extends the rank-one result to positivity of the map.

For the upper endpoint $n = d-2$, the development proves positivity on every positive semidefinite input for all $d \geq 3$. This endpoint includes the case $(d, n) = (3, 1)$.[3] The backward window $x_{i-1}, \ldots, x_{i-n}$ then omits exactly x_i and x_{i+1}. Define

$$T = \sum_j x_j, \tag{7}$$

$$a_i = T + x_i - x_{i+1}. \tag{8}$$

The cyclic estimate becomes a special case of the permutation inequality

$$\sum_i \frac{x_i}{T + x_i - x_{\sigma(i)}} \leq 1, \qquad x_i \geq 0, \tag{9}$$

where σ is an arbitrary permutation.

Set $\delta_i = x_i - x_{\sigma(i)}$. The pair bound $x_i + x_{\sigma(i)} \leq T$ gives

$$\frac{x_i}{T + \delta_i} \leq \frac{x_i T - x_i\delta_i + \delta_i^2/2}{T^2}. \tag{10}$$

[2] Formal declarations: `Matrix.choiType_cyclic_reciprocal_one_of_nonneg`, `Matrix.choiTypeMap_vecMulVec_posSemidef_one`, and `Matrix.choiTypeMap_isPositiveMap_one` in `QICLean/Channel/ChoiTypeMap/Positivity`.

[3] Formal declarations: `Matrix.perm_reciprocal_sum_le_one`, `Matrix.choiTypeMap_vecMulVec_posSemidef_sub_two`, and `Matrix.choiTypeMap_isPositiveMap_sub_two` in `QICLean/Channel/ChoiTypeMap`.

For a permutation,

$$\sum_i x_i\delta_i = \frac{1}{2}\sum_i \delta_i^2. \tag{11}$$

Summing (10) and using (11) proves (9). This cyclic estimate is due to Ando, as recorded in the introduction of Ref. [Yam93].

3 Resolution by Yamagami's variational argument

The former scope restriction is resolved by the nonnegative cyclic reciprocal estimate

$$\sum_{i\in\mathbb{Z}/d\mathbb{Z}} \frac{x_i}{(d-n)x_i+\sum_{k=1}^n x_{i-k}} \le 1, \qquad x_i \ge 0, \quad 2\le n\le d-3, \tag{12}$$

again with zero-denominator summands defined to be zero. The corresponding forward-window inequality is proved at the cardinal parameter. Index negation converts it to (12); applying the result to $x_i = |v_i|^2$ proves positivity on rank-one positive semidefinite inputs. Spectral decomposition then proves positivity of T_C for every $d \ge 3$ and $1 \le n \le d-2$.[4]

The middle range requires more than the endpoint arguments. The case $n = 1$ is Theorem 1 of Ref. [TT88, pp. 309–311], with Lemmas 2–3 supplying the proof route used here. Yamagami proves the full range $1 \le n \le d-1$ by a variational argument [Yam93]. The argument combines Nowosad's uniqueness theorem for critical points of the functional

$$x \longmapsto \sum_i \frac{x_i}{(Sx)_i} \tag{13}$$

on the positive cone, a spectral condition for the Hessian at the constant vector, and a projective-boundary analysis that counts vanishing summands [Now68, Yam93].

3.1 The spectral and Hessian ingredients

Let C denote the cyclic shift matrix. For integers $d \ge 3$ and $1 \le m \le d-2$, and for $s \ge d$, define

$$S = (s-m)I + C + \cdots + C^m. \tag{14}$$

[4]Formal declarations: `Yamagami.functional_card_le_one`, `Matrix.choiTypeRankOneWeight_reciprocal_sum_middle`, `Matrix.choiTypeMap_vecMulVec_posSemidef_middle`, and `Matrix.choiTypeMap_isPositiveMap` in `QICLean/Channel/ChoiTypeMap/YamagamiPositivity`.

For every nontrivial standard character, the corresponding eigenvalue of S lies in the open disk stated in Lemma 6 of Ref. [Yam93, pp. 523–524].[5] This is the corrected stride-one portion needed for Wolf's parameter range.

Lemma 6 of Ref. [Yam93, pp. 523–524] is printed for $1 \le m \le d - l$ and $s \ge d$. At $l = 1$, $m = d - 1$, and $s = d$, however,

$$S = I + C + \cdots + C^{d-1} \tag{15}$$

has eigenvalue zero on every nontrivial character. The resulting disk bound is an equality rather than the printed strict inequality. The endpoint becomes valid when $s > d$, while $m \le d - 2$ is the exact range required here. The false printed endpoint and its correction are recorded separately in Ref. [con26a].

Define the Hessian matrix

$$H = s(S + S^{\mathsf{T}}) - 2S^{\mathsf{T}}S. \tag{16}$$

In the range $d \ge 3$, $1 \le m \le d - 2$, and $s \ge d$, the matrix S has nonnegative entries, every row and column sum is s, and S is invertible. Moreover,

$$H \succeq 0, \tag{17}$$

$$\ker H = \mathbb{R}\mathbf{1}. \tag{18}$$

The proof diagonalizes S and H by the finite Fourier transform and proves strict positivity away from the constant character.[6]

Let $f_{d,m,s}$ denote Yamagami's cyclic functional, and let $\lambda_{S^{-1}}$ denote the corresponding inverse-operator functional. The previously missing identity is

$$f_{d,m,s}(x) = \lambda_{S^{-1}}(Sx). \tag{19}$$

It supplies an algebraic ingredient used in Lemmas 2–4 and Corollary 5 of Ref. [Yam93, pp. 522–523], together with the corrected range of Lemma 6 on pp. 523–524.[7]

3.2 The projective boundary

The definitions retain Yamagami's forward window and use index negation to obtain the backward Choi window. The simultaneous-singularity argument treats a nonzero nonnegative boundary vector under the hypotheses

$$N \ge 1, \tag{20}$$

$$m < N, \tag{21}$$

$$N \le s. \tag{22}$$

[5] Formal declaration: `Yamagami.norm_symbol_sub_half_lt_half`.

[6] The cyclic matrix development is in `QICLean/Analysis/YamagamiCyclicMatrix`. Formal declaration for the off-character positivity statement: `Yamagami.hessianSymbol_pos`.

[7] Formal declaration: `Yamagami.functional_eq_lambdaT_inverseOperator_mulVec`.

A vanishing denominator produces a first positive coordinate after its zero window. This makes m summands tend to zero, and counting the complementary summands gives

$$\frac{N-m}{s-m} \leq \frac{N}{s}. \tag{23}$$

This is the global count indicated, but not printed, after Yamagami's one-singularity calculation [Yam93, pp. 524–525]. Its completion and attribution are recorded in Ref. [con26b].

The regular-boundary recurrence in Ref. [Yam93, p. 525] is formalized for $N \geq 3$, $2 \leq m \leq N-2$, and $s \geq N$, including reduced vectors with singular denominators under Lean's totalized convention.[8] Strong induction applies this recurrence at regular boundary points. If a positive vector had functional value greater than one, normalization would produce a compact superlevel closure in the standard simplex. The simultaneous-singularity estimate excludes boundary points with a zero denominator; continuity and the inductive estimate exclude the remaining boundary points. Hence a maximizer must be strictly positive. The Fourier–Hessian result and Nowosad's uniqueness theorem force that maximizer to be scalar, where the functional equals one, giving a contradiction [Now68, Yam93].

3.3 The finite-coordinate Nowosad argument

The finite-coordinate input follows the source route in Ref. [Now68, pp. 409–418]. Boundedness is proved for the norm induced by the coordinate-sum functional, which is the finite L^2 norm.[9] Lagrange idempotents for the value classes of the generator show that the coordinate derivation vanishes on the possibly proper Laurent algebra $P(w)$, including the case in which coordinates of w repeat.[10] The genuine first and second variations yield the finite real specialization of Nowosad's Theorem 1.8 [Now68].[11] Its local-maximum corollary follows from

$$\lambda_{-T} = -\lambda_T \tag{24}$$

rather than from reversing the orientation of Nowosad's theorem.

For Yamagami's matrix S, define the change of variables by

$$b = \frac{Sa}{s}, \tag{25}$$

$$x(t) = sS^{-1}(b^t). \tag{26}$$

The matrix H remains the matrix defined in (16). The ordinary Euclidean Hessian satisfies

$$D^2 f_S(\mathbf{1})[h,h] = -s^{-3}\langle h, Hh\rangle. \tag{27}$$

[8] Formal declaration: `Yamagami.functional_le_deleteLastCoordinate`.

[9] Formal declaration: `Nowosad.exists_inducedHilbertNorm_bound`.

[10] Formal declarations: `Nowosad.mem_laurentSubalgebra_of_eq_on_valueClass` and `Nowosad.coordinate_pointDerivation_eq_zero`.

[11] Formal declaration: `Nowosad.finiteCoordinate_theorem_one_eight`.

This identity fixes Yamagami's normalization. It is not formalized separately as a Fréchet-derivative theorem; the development instead uses the corresponding algebraic second-variation identity.[12]

Under (17) and (18), every strictly positive local maximum is scalar.[13] More precisely, a non-scalar maximum would produce the non-scalar tangent

$$sS^{-1}\log\left(\frac{Sa}{s}\right) \in \ker H. \tag{28}$$

This is the uniqueness assertion in Lemma 3 of Ref. [Yam93, p. 522]; it does not assert that the unit vector is itself a local maximum.[14]

3.4 Why the endpoint summand bound does not extend

Two computations explain why elementary summand bounds do not cover the remaining range. First, the cyclic estimate is tight in two distinct regimes. At the constant vector, every summand equals $1/d$. For the geometric family

$$x_i = t^i, \qquad i = 0, \ldots, d-1, \tag{29}$$

as $t \to \infty$, each of the $d-n$ summands whose backward window avoids wrap-around tends to $1/(d-n)$, while every remaining summand tends to zero. The full sum again tends to one. Any summand-wise estimate would therefore need to be sharp in both regimes.

Second, the proof for $n = d-2$ does not extend. Set

$$m = d - n, \tag{30}$$

$$\delta_i = \sum_{k=1}^{m-1} (x_i - x_{i+k}), \tag{31}$$

$$a_i = T + \delta_i. \tag{32}$$

For $T > 0$ and positive weights a_i, the reciprocal estimate is equivalent to

$$\sum_i \frac{x_i \delta_i^2}{a_i} \le \sum_i x_i \delta_i. \tag{33}$$

[12] Formal declaration: `Yamagami.minimumQuadraticForm_normalizedNegativeInverse_eq`.

[13] Formal declaration: `Yamagami.lemma_three_localMax_isScalar`.

[14] Formal declaration: `Yamagami.pulledBackTangent_mem_hessianKernel_and_not_isScalarVector`.

The endpoint proof factors this inequality through

$$\sum_i \frac{x_i \delta_i^2}{a_i} \le \frac{1}{m} \sum_i \delta_i^2, \tag{34}$$

$$\frac{1}{m} \sum_i \delta_i^2 \le \frac{1}{2} \sum_{k=1}^{m-1} \sum_i (x_i - x_{i+k})^2, \tag{35}$$

$$\frac{1}{2} \sum_{k=1}^{m-1} \sum_i (x_i - x_{i+k})^2 = \sum_i x_i \delta_i. \tag{36}$$

The first inequality follows from $a_i \ge m x_i$ and holds for every m. In Fourier variables, the second inequality is

$$\left| \sum_{k=1}^{m-1} (1 - \zeta^k) \right|^2 \le \frac{m}{2} \sum_{k=1}^{m-1} |1 - \zeta^k|^2 \tag{37}$$

for every dth root of unity ζ. It is an equality for $m = 2$, but it fails for $m = 3$ and $\zeta = e^{i\pi/3}$:

$$\left| \sum_{k=1}^{2} (1 - \zeta^k) \right|^2 = 7, \tag{38}$$

$$\frac{3}{2} \sum_{k=1}^{2} |1 - \zeta^k|^2 = 6. \tag{39}$$

This obstruction agrees with the historical development: the case $(d, n) = (5, 2)$ required a separate argument [Osa91], and Yamagami later established the full range by the variational proof [Yam93].

The formalization therefore follows the variational route and resolves QICLean issue #19. The indecomposability assertion in the same paragraph of Ref. [Wol12] is separate and remains tracked by QICLean issue #18.

4 Classification

The scope restriction is resolved. The formal positivity theorem has exactly Wolf's hypotheses

$$d \ge 3, \tag{40}$$

$$1 \le n \le d - 2, \tag{41}$$

and concludes that the Choi-type map (1) is positive.[15] The former endpoint-only restriction has been removed. Wolf's independent indecomposability assertion remains tracked by QICLean issue #18.

[15] Formal declaration: `Matrix.choiTypeMap_isPositiveMap`.

References

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