Jordan-Block Norm Estimate: the Shift Norm in One Dimension

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Verdict: local-correction (resolved).

1 Source passage

Fix an integer $D \geq 1$. Let $N \in M_D(\mathbb{C})$ be the strict upper shift, defined by

$$N_{i,j} = \begin{cases} 1, & j = i+1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For $\lambda \in \mathbb{C}$, define the Jordan block $J_D(\lambda) = \lambda \mathbb{1} + N$, and let $\|\cdot\|_\infty$ denote the largest singular value.

Equation (8.104) in Chapter 8 of Ref. [Wol12] states that, for every $n \in \mathbb{N}$ and every natural number $k_0 \leq \min\{n, D-1\}$,

$$|\lambda|^{n-k_0} \binom{n}{k_0} \leq \left\| J_D(\lambda)^n \right\|_\infty, \tag{2}$$

$$\left\| J_D(\lambda)^n \right\|_\infty \leq \sum_{k=0}^{\min\{n,D-1\}} |\lambda|^{n-k} \binom{n}{k}. \tag{3}$$

The local source derives these bounds at lines 1197–1215. The upper bound follows from the binomial expansion

$$J_D(\lambda)^n = \sum_{k=0}^{\min\{n,D-1\}} \binom{n}{k} \lambda^{n-k} N^k \tag{4}$$

and the triangle inequality. At line 1215, the source removes the shift factors by asserting that $\|N\|_\infty = 1$.

2 The deviation

The asserted equality holds when $D \geq 2$, but not when $D = 1$. In one dimension, the strict upper shift defined in (1) is the zero matrix. Consequently,

$$D = 1 \implies N = 0, \tag{5}$$

$$D = 1 \implies \|N\|_\infty = 0. \tag{6}$$

The source states $\|N\|_\infty = 1$ without restricting the dimension. Since the Jordan block and Equation (8.104) of Ref. [Wol12] both allow $D = 1$, this intermediate assertion includes one degenerate case in which it is false.

3 Formalized interpretation

The derivation of the upper bound requires only $\|N\|_\infty \leq 1$. Submultiplicativity gives

$$\|N^k\|_\infty \leq \|N\|_\infty^k, \tag{7}$$

$$\|N\|_\infty \leq 1 \implies \|N^k\|_\infty \leq 1. \tag{8}$$

Applying (8) to the triangle inequality for (4) yields the upper bound (3).

The formalization therefore proves the dimension-uniform inequality $\|N\|_\infty \leq 1$ and derives the corresponding power bound from it.[1] The proof follows the route indicated in Ref. [Wol12]. The Gram matrix satisfies

$$N^\dagger N = \operatorname{diag}(0, 1, \ldots, 1), \tag{9}$$

$$\left\|N^\dagger N\right\|_\infty \leq 1. \tag{10}$$

The first identity is also formalized directly.[2] The C^*-identity then gives

$$\|N\|_\infty^2 = \left\|N^\dagger N\right\|_\infty \leq 1, \tag{11}$$

$$\|N\|_\infty \leq 1. \tag{12}$$

When $D = 1$, the hypothesis $k_0 \leq \min\{n, D-1\}$ forces $k_0 = 0$. The Jordan block is the scalar matrix $J_1(\lambda) = \lambda\mathbb{1}$, so both sides of Equation (8.104) of Ref. [Wol12] reduce to

$$|\lambda|^n \leq \|\lambda^n\|_\infty \leq |\lambda|^n. \tag{13}$$

Thus the source estimate remains valid in one dimension despite the false intermediate equality. The formal lower bound, upper bound, and combined bound retain Wolf's hypotheses, including $D \geq 1$, without any additional dimension restriction.[3]

[1] Formal declarations: `Matrix.l2_opNorm_nilpotentShift_le_one` and `Matrix.l2_opNorm_nilpotentShift_pow_le_one`.

[2] Formal declaration: `Matrix.conjTranspose_nilpotentShift_mul_self`.

[3] Formal declarations: `Matrix.le_l2_opNorm_jordanBlock_pow`, `Matrix.l2_opNorm_jordanBlock_pow_le`, and `Matrix.l2_opNorm_jordanBlock_pow_bounds`.

4 Verdict

Replacing the source's intermediate equality $\|N\|_\infty = 1$ by

$$\|N\|_\infty \leq 1 \tag{14}$$

corrects the degenerate case $D = 1$ without weakening Equation (8.104) of Ref. [Wol12]. The correction is local and resolved; the formalized Jordan-block estimate has exactly the source's hypotheses.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.