# Wolf Corollary 6.7: Source-General Resolution Record

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**Verdict:** scope-restriction (resolved).

## 1 The assertion

Let $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be positive and trace preserving, and assume that its trace adjoint satisfies the Schwarz inequality. Define the fixed-point space

$$\mathcal{F}_T = \{X \in M_d(\mathbb{C}) \mid T(X) = X\}\,. \tag{1}$$

Let $\rho$ be any maximum-rank density matrix in $\mathcal{F}_T$, and interpret $\rho^{-1/2}$ as the inverse square root on the support of $\rho$ and as zero on its kernel. Corollary 6.7 of Ref. [Wol12] states that

$$\rho^{-1/2}\mathcal{F}_T\rho^{-1/2} = \left\{\rho^{-1/2}X\rho^{-1/2} \;\middle|\; X \in \mathcal{F}_T\right\} \tag{2}$$

is a $*$-algebra.

The corollary uses the maximal-support property established earlier in Section 6.4 of Ref. [Wol12]: every fixed point of $T$ has support and range contained in the support of a maximum-rank fixed point. Consequently, conjugation by $\rho^{-1/2}$ discards no part of $\mathcal{F}_T$.

## 2 The formalization

The source-general implementation is in `QICLean/Channel/FixedPoint/AbstractWeightedFixedPoints.lean`. It assumes exactly that $T$ is positive and trace preserving and that its trace adjoint is Schwarz. It assumes neither a Kraus representation nor complete positivity.

For an arbitrary positive semidefinite fixed point $\rho$, let $Q$ be its support projection. The declaration `IsPositiveMap.weightedCornerFixedPointsStarSubalgebra` proves that

$$\mathcal{W}_{\rho,T} = \{Y \in QM_d(\mathbb{C})Q \mid T(\sqrt{\rho}\,Y\sqrt{\rho}) = \sqrt{\rho}\,Y\sqrt{\rho}\} \tag{3}$$

is a $*$-subalgebra of $QM_d(\mathbb{C})Q$.

Its multiplication proof follows the density-block decomposition in Wolf's Theorem 6.14 [Wol12]. On each support sector, fixed blocks have the form $\sigma_k \otimes X_k$, while $\rho$ has blocks $\sigma_k \otimes R_k$, with both density factors positive definite. The weighted product remains fixed because

$$\begin{aligned}(\sigma_k \otimes X_k)(\sigma_k^{-1} \otimes R_k^{-1})(\sigma_k \otimes Z_k)\\ = \sigma_k \otimes (X_k R_k^{-1} Z_k).\end{aligned} \tag{4}$$

This is the factor-swapped form of the $\mathcal{M}_{d_k} \otimes \rho_k$ notation in Ref. [Wol12], as already recorded for the formalization of Theorem 6.14.

The maximum-rank step is also source-general. The declaration `IsPositiveMap.maximalSupport_of_maximalRank` in `QICLean/Channel/FixedPoint/AbstractMaximalRank.lean` accepts any positive semidefinite fixed point whose rank is at least the rank of every stationary density. It proves

$$QXQ = X \qquad \text{for every } X \in \mathcal{F}_T. \tag{5}$$

The proof compares the arbitrary maximum-rank choice with $T_\infty(\mathbf{1})$; it does not replace the source quantifier by a canonical witness.

Finally, `IsPositiveMap.mem_weightedCornerFixedPointsStarSubalgebra_iff_inverseSandwich` identifies the carrier (3) in both directions with (2). The support identities are

$$\sqrt{\rho}\rho^{-1/2} = Q, \tag{6}$$

$$\rho^{-1/2}\sqrt{\rho} = Q. \tag{7}$$

They establish the carrier equality without assuming that $\rho$ is invertible on the ambient space. The generic construction also includes the zero-support case as the zero corner algebra. The density-matrix hypothesis in Corollary 6.7 of Ref. [Wol12] has trace one and therefore excludes that case.

## 3 Conclusion

The former Kraus-only scope restriction is eliminated. `IsPositiveMap.wolfCorollary67` states Corollary 6.7 of Ref. [Wol12] for a positive trace-preserving linear map with Schwarz trace adjoint and for every maximum-rank stationary density matrix. It returns a $*$-algebra whose carrier is exactly the inverse-square-root set (2). The older declarations in the `Kraus` namespace remain compatibility specializations. The maximum-rank and surjectivity declarations delegate to the source-general proofs rather than providing a separate route.

## References


[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.