Source-General Projected Support Corner Star-Algebra

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Verdict: scope-restriction (resolved).

1 The assertion

Let $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be positive and trace preserving. Let T^* denote its trace adjoint, assume that T^* is unital and satisfies the Schwarz inequality, and let $\rho \succeq 0$ be a maximum-rank fixed point of T. If Q is the support projection of ρ, Corollary 6.6 of Ref. [Wol12] states that

$$\mathcal{A}_Q = \{Y \in QM_D(\mathbb{C})Q \mid QT^*(Y)Q = Y\} \tag{1}$$

is a $*$-subalgebra of the corner algebra $QM_D(\mathbb{C})Q$. The local source is `Notes/WolfNoteTexSource/ch06_spectral_properties.tex`, lines 1423–1437.

Complete positivity is an example of a sufficient hypothesis in Ref. [Wol12]; it is not an assumption of Corollary 6.6. For a unital map, complete positivity implies the Schwarz inequality, but the Schwarz condition is strictly weaker because some positive maps that are not completely positive also satisfy it.

2 The former restriction

The earlier formal result was `Kraus.cornerFixedPointsStarSubalgebra`, with membership characterization `Kraus.mem_cornerFixedPointsStarSubalgebra`, in `QICLean/Channel/FixedPoint/CornerFixedPoints.lean`. Its hypotheses are a Kraus family K satisfying `IsTP` K and a positive semidefinite fixed point of the induced map.

A Kraus representation assumes complete positivity. The earlier declarations therefore had strictly stronger hypotheses than Corollary 6.6 of Ref. [Wol12]; they remain completely positive specializations, not the source-facing formalization.

The analogous former restriction for Corollary 6.7 of Ref. [Wol12] is recorded in `wolf_cor67_maximal_support_restriction.tex`.

3 Resolution

The source-general result is `IsPositiveMap.stationaryCornerAdjointFixedPointsStarSubalgebra`, with membership characterization `IsPositiveMap.mem_stationaryCornerAdjointFixedPointsStarSubalgebra`, in `QICLean/Channel/FixedPoint/AbstractCornerFixedPoints.lean`. Its hypotheses are exactly positivity and trace preservation of T together with the Schwarz inequality for T^*. It assumes neither a Kraus representation nor complete positivity. The result holds for every positive semidefinite fixed point, so it includes the maximum-rank fixed point used in Corollary 6.6 of Ref. [Wol12].

The proof follows the compression argument in lines 1433–1437 of the local source for Ref. [Wol12]. Compress T to the support of ρ. The compressed map is positive and trace preserving, has a positive definite fixed point, and has a Schwarz trace adjoint. The abstract fixed-point algebra theorem `SchwarzMap.fixedPointsStarSubalgebra` therefore applies.

Extension through the support isometry transports multiplication from the compressed algebra to $QM_D(\mathbb{C})Q$. The trace-adjoint compression identity identifies the compressed fixed-point equation with

$$QT^*(Y)Q = Y. \tag{2}$$

This proves that the set (1) is a $*$-subalgebra under the source hypotheses.

4 Conclusion

The complete-positivity restriction for Corollary 6.6 of Ref. [Wol12] is eliminated. The blueprint and numbering coverage now point to the source-general declaration. The Kraus declarations remain documented completely positive specializations.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.