The Rectangular Choi–Jamiolkowski Correspondence: Resolution of the Square-Dimensional Scope Restriction

2026-08-04

Verdict: scope-restriction (resolved).

1 The source statement

Let $T : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ be linear. Let $\Omega_d \in \mathbb{C}^d \otimes \mathbb{C}^d$ be the normalized maximally entangled vector, and let τ_T denote the normalized Choi matrix. Proposition 2.1 of Ref. [Wol12] defines

$$\tau_T = (T \otimes \mathrm{id}_d)(|\Omega_d\rangle\langle\Omega_d|) \in M_{d'}(\mathbb{C}) \otimes M_d(\mathbb{C}). \tag{1}$$

For $A \in M_{d'}(\mathbb{C})$ and $B \in M_d(\mathbb{C})$, it gives the trace pairing

$$\mathrm{tr}(A\,T(B)) = d\ \mathrm{tr}(\tau_T(A \otimes B^T)). \tag{2}$$

These formulas define mutually inverse correspondences between linear maps and bipartite operators [Wol12, Proposition 2.1].

The same proposition identifies Hermiticity preservation with

$$\tau_T = \tau_T^\dagger, \tag{3}$$

and complete positivity with

$$\tau_T \geq 0. \tag{4}$$

Writing tr_A and tr_B for the partial traces over the output and input factors, respectively, unitality is equivalent to

$$\mathrm{tr}_B(\tau_T) = \frac{\mathbb{1}_{d'}}{d}, \tag{5}$$

and trace preservation is equivalent to

$$\mathrm{tr}_A(\tau_T) = \frac{\mathbb{1}_d}{d}. \tag{6}$$

Without assuming trace preservation, Proposition 2.1 of Ref. [Wol12] gives

$$\operatorname{tr}_A(\tau_T) = \frac{(T^*(\mathbb{1}_{d'}))^T}{d}, \tag{7}$$

$$\operatorname{tr}(\tau_T) = \frac{\operatorname{tr}(T^*(\mathbb{1}_{d'}))}{d}. \tag{8}$$

It also gives the corresponding proportional-identity criterion for a doubly stochastic map [Wol12, Proposition 2.1].

Theorem 2.1 of Ref. [Wol12] states that T is completely positive if and only if it has a rectangular Kraus representation

$$T(X) = \sum_{j=1}^{r} K_j X K_j^\dagger, \tag{9}$$

$$K_j \in M_{d' \times d}(\mathbb{C}). \tag{10}$$

The theorem also identifies the trace-preserving and unital normalization conditions, the minimal Kraus cardinality

$$r = \operatorname{rank}(\tau_T) \le dd', \tag{11}$$

an orthogonal minimal family, and unitary freedom after padding the smaller family with zero operators [Wol12, Theorem 2.1].

2 Resolution

The different-dimension Choi statements are formalized in the namespace `ChoiRectangular`.[1] The development defines `choiMatrix` on $\mathbb{C}^{d'} \otimes \mathbb{C}^d$ and proves:

- the inverse identities `mapOfChoiMatrix_choiMatrix` and `choiMatrix_mapOfChoiMatrix`;
- the trace-pairing formula `trace_pairing`;
- the Hermiticity equivalence `choiMatrix_isHermitian_iff_hermiticityPreserving`;
- the complete-positivity equivalence `isKrausCP_iff_choiMatrix_posSemidef`;
- the unitality equivalence `unital_iff_traceRight_choiMatrix`, obtained from `traceRight_choiMatrix` and (5);
- the trace-preservation equivalence `tracePreserving_iff_traceLeft_choiMatrix`, obtained from `traceLeft_choiMatrix` and (7);
- the normalization formula `trace_choiMatrix`, corresponding to (8);

[1] Formal module: `QICLean/Channel/ChoiRectangular`.

- the doubly-stochastic equivalence `doublyStochastic_iff_partialTraces_proportional`.

The rectangular Kraus development is in `QICLean/Channel/KrausRectangular`, in the same namespace. It proves the normalization equivalences `kraus_tp_iff_sum_conjTranspose_mul` and `kraus_unital_iff_sum_mul_conjTranspose`. It also proves minimality of the Kraus rank through `choiRank_isLeast_hasKrausCard_of_isKrausCP`, the bound `choiRank_le_mul`, and existence of an orthogonal minimal family through `exists_kraus_orthogonal_of_isKrausCP`. The padded unitary freedom is formalized by the dimension-generic declarations `kraus_isometry_freedom_iff` and `kraus_unitary_freedom_iff` in `QICLean/Channel/KrausUnitaryFreedom`.

The complete-positivity predicate is `IsKrausCP`: the existence of rectangular Kraus operators. This is the complete-positivity notion used elsewhere in the development, and the Choi equivalence is stated relative to it.

3 Surviving square-only declarations

The dimension-generic Kraus API states cardinality, rank, and freedom results for rectangular operators $K_j \in M_{d' \times d}(\mathbb{C})$. The square development is the specialization $d = d'$.

The declarations `Channel.HasKrausCard`, `Channel.HasKrausRankLE`, `Channel.choiRank`, `Channel.hasKrausCard_mono`, `Channel.choiMatrix_eq_sum_vecMulVec_of_kraus`, `Channel.choiRank_le_of_hasKrausCard`, and `Channel.hasKrausCard_choiRank_of_cp` are in `QICLean/Channel/KrausRank`. The normalization declarations `kraus_sum_conjTranspose_mul_of_tp`, `kraus_sum_mul_conjTranspose_of_unital`, and `kraus_same_map_of_isometry_combination` are in `QICLean/Channel/KrausRepresentation`. The dual, Gramian, and necessary-direction freedom declarations `kraus_dual_eq_of_map_eq`, `kraus_conjTranspose_mul_eq_of_map_eq`, `kraus_rectangular_freedom`, and `kraus_rectangular_freedom'` are in `QICLean/Channel/KrausFreedom`. The freedom equivalences `kraus_isometry_freedom_iff` and `kraus_unitary_freedom_iff` are in `QICLean/Channel/KrausUnitaryFreedom`. Their former duplicates in the `ChoiRectangular` namespace have been removed.

The following declarations remain square-only. They are specializations of rectangular or dimension-generic results, not independent scope gaps.

- `ChoiJamiolkowski.choiMatrix` and its correspondence clauses remain square. The square Choi matrix is definitionally the rectangular matrix at $d = d'$, as stated by `ChoiRectangular.choiMatrix_eq_choiJamiolkowski`. Each square clause specializes the rectangular clause with the same name.

- `IsTracePreservingMap` and `IsChannel` are stated for endomorphisms of one matrix algebra. The rectangular development expresses trace preservation pointwise as

$$\mathrm{tr}(T(X)) = \mathrm{tr}(X), \tag{12}$$

and represents rectangular complete positivity together with trace preservation by `IsKrausCPTP`.

- `Channel.hasKrausRankLE_choiRank_of_cptp` and `kraus_transition_unitary_of_hs_orthonormal` have no rectangular counterparts. The first uses the square-only `IsChannel` structure. The second is a square linear-algebraic core lemma with square-only call sites.

4 Conclusion

The scope restriction recorded on 2026-07-30 is lifted. Proposition 2.1 and Theorem 2.1 of Ref. [Wol12] are formalized with independent input and output dimensions and with all displayed clauses. No strengthened hypothesis or source-level gap remains in this area. The resolution is tracked in TNLean issue #3987.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.