Wolf Equation (8.103) — Small-Power Exponent Scope

2026-08-09

Verdict: scope-restriction (resolved).

1 The source statement

Let $X = \Lambda + N \in M_D(\mathbb{C})$ be upper triangular, where Λ is diagonal and N is strictly upper triangular. For an arbitrary submultiplicative norm satisfying $\|\Lambda\| \leq 1$, Equation (8.103) of Ref. [Wol12] states, for every $n \in \mathbb{N}$, that

$$\|X^n\| \leq \|\Lambda^n\| + (D-1)n^{D-1}\|\Lambda\|^{n-D+1} \max\{\|N\|, \|N\|^{D-1}\}. \tag{1}$$

The local source is `Notes/WolfNoteTexSource/ch08_distance_measures.tex`, lines 1189–1210, especially Equations (8.103) and (8.105).

2 The exponent defect

If $n < D-1$, then

$$n - D + 1 < 0. \tag{2}$$

The source specifies no convention for the resulting negative power in (1). In particular, when $\Lambda = 0$, the factor

$$\|\Lambda\|^{n-D+1} = 0^{n-D+1} \tag{3}$$

is undefined under the usual real-power interpretation. Replacing the exponent by truncated natural subtraction would change the printed statement.

The derivation from Equation (8.105) of Ref. [Wol12] factors the diagonal norm as

$$\|\Lambda\|^{n-k} = \|\Lambda\|^{n-(D-1)}\|\Lambda\|^{(D-1)-k}. \tag{4}$$

Both factors use natural exponents only when

$$D - 1 \leq n, \tag{5}$$
$$k \leq D - 1. \tag{6}$$

The source derivation therefore proves the natural-power form of (1) only in the range (5).

3 Formalization scope

Equation (8.105) of Ref. [Wol12], whose terms satisfy $k \le n$, is formalized for every $n \in \mathbb{N}$. Equation (8.103) is formalized with the explicit hypothesis (5). Its refined constant is

$$(D-1)\binom{n}{D-1}, \tag{7}$$

under the source hypothesis

$$2(D-1) \le n, \tag{8}$$

which implies (5). These results retain an arbitrary submultiplicative seminorm; they do not replace it with a particular matrix norm.

The corresponding declarations are:

- `Matrix.wolf_eq_105_seminorm` for the unrestricted Equation (8.105) of Ref. [Wol12] word bound;
- `Matrix.wolf_eq_103` for the natural-power form under $D - 1 \le n$;
- `Matrix.wolf_eq_103_refined` for the refined constant under $2(D-1) \le n$.

Equation (8.111) of Ref. [Wol12] applies Equation (8.103) to the Schur form of $\widehat{T} - \widehat{T}_\varphi$, with matrix dimension $D = d^2$ and $\|\Lambda\|_\infty = \mu$. Under the natural-power interpretation, it inherits the restriction

$$d^2 - 1 \le n. \tag{9}$$

The corresponding coarse and refined Schur-form estimates are `Matrix.wolf_eq_111_schur_form` and `Matrix.wolf_eq_111_schur_form_refined`.

The declarations `Matrix.wolf_eq_111_schur_form_subperipheralModulus` and `Matrix.wolf_eq_111_schur_form_subperipheralModulus_refined` use the shared source-shaped quantity μ from `Module.End.subperipheralModulus` on the same D-dimensional coordinate space as the Schur matrices. They prove

$$\mu \le 1, \tag{10}$$

including the convention for an empty subperipheral spectrum. They retain the hypothesis

$$\|\Lambda\|_\infty = \mu, \tag{11}$$

so they remain independent of a channel realization.

The channel-level declarations `IsPositiveMap.exists_wolf_eq_111` and `IsPositiveMap.exists_wolf_eq_111_refined` construct a unitary Schur representation of $\widehat{T} - \widehat{T}_\varphi$. They prove that the diagonal entries are exactly zero and the subperipheral eigenvalues, establish (11), and show

$$\|N\|_\infty \leq \mu + 2\sqrt{d}. \tag{12}$$

They assume, respectively,

$$d^2 - 1 \leq n, \tag{13}$$

$$2(d^2 - 1) \leq n. \tag{14}$$

When $d = 1$, these ranges allow $n = 0$. Although the auxiliary identity

$$T^0 - T_\varphi^0 = (T - T_\varphi)^0 \tag{15}$$

fails, the final estimate at this boundary is proved directly: its left-hand side is zero and its right-hand side is one.

No declaration uses an inverse power of μ. The case $\mu = 0$ is therefore included, both when the subperipheral spectrum is empty and when it consists only of the zero eigenvalue.

4 Conclusion

This is a source scope restriction, not a missing formal proof. The formal statement records exactly the range in which the natural-exponent derivation printed in Ref. [Wol12] is meaningful. Removing the restriction would require an explicit interpretation of negative powers, including the case $\|\Lambda\| = 0$. Tracking: TNLean issue #5837.

References

[Wol12] Michael M. Wolf. Quantum channels & operations: Guided tour. Lecture notes, 2012.